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On the expansion of the multiplicity-free plethysms $p_2[s_{(a,b)}]$ and $p_2[s_{(1^r,2^t)}]$

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Abstract: We show how to compute the explicit expansion of the plethysm $p_2[s_\lambda]$ of the power symmetric function p_2 and the Schur function s_λ, where λ has either two rows or two columns, via the well known Littlewood–Richardson coefficients which occur in the decomposition of s_λ^2 .

Key words: Schur functions, multiplicity-free, plethysm

1. Introduction

Schur functions are symmetric polynomials introduced by Schur [15] as characters for irreducible polynomial representations of the general linear group of invertible matrices and form a basis for the ring of symmetric functions. Given two Schur functions $s_\mu(x)$ and $s_\lambda(x)$, where $x = (x_1, x_2, \ldots)$ is an infinite sequence of variables, μ and λ are partitions of weight m and n, respectively, the plethysm $s_\mu[s_\lambda(x)]$ is the symmetric function obtained by substituting the monomials of $s_\lambda(x)$ by the variables of $s_\mu(x)$. Littlewood [10] introduced this operation in the context of the representations of the general linear group and showed that for any partition μ of m,

$$s_\mu[s_\lambda(x)] = \sum_{\gamma \vdash mn} g^\gamma_{\mu,\lambda} s_\gamma(x)$$

where the sum runs over all partitions γ of mn and $g^\gamma_{\mu,\lambda}$ are nonnegative integers.

The problem of computing the coefficients $g^\gamma_{\mu,\lambda}$ is one of the fundamental open problems in the theory of symmetric functions and has proved to be very difficult. Essentially there are explicit formulas for $g^\gamma_{\mu,\lambda}$ in a few special cases.

We say that the plethysm $s_\mu[s_\lambda(x)]$ is multiplicity-free if every coefficient in the resulting Schur function expansion is $0, +1$.

A well known example of multiplicity-free plethysm was given by Littlewood in [11], where he proved the following remarkably simple formulas:

$$s_{(2)}[s_{(n)}] = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ even}} s_\gamma$$

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$$s_{(1^2)}[s_{(n)}] = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ odd}} s_\gamma.$$

Later on, Carbonara, Remmel and Yang (see [2, 3]) generalized Littlewood's formulas by replacing the Schur function of a one row shape (n) by a Schur function of an arbitrary hook shape $s_{(1^a,b)}$ and derived explicit formulas for the Schur function expansion of the multiplicity-free plethysms $s_{(2)}[s_{(1^a,b)}]$ and $s_{(1^2)}[s_{(1^a,b)}]$. Other examples of multiplicity-free plethysms are $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ where s_λ is the Schur function indexed by a rectangular partition. In fact, in case $\lambda = (n^k)$ is a rectangle, surprisingly simple formulas for the Schur function expansion of the plethysms $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ were shown in [5]. Also explicit formulas for the expansion of the plethysms $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$, where λ has either two rows or two columns, have been derived in [5] and subsequently reformulated in [8]. In 2020 Bessenrodt, Bowman and Paget [1] have classified all multiplicity-free plethysms of Schur functions. In particular they have proved that $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ are multiplicity-free if and only if λ is the partition (a^b), $(a^{b-1}, a+1)$, $(1, a^b)$, $(a-1, a^{b-1})$ or a hook. Their approach is based on Carré–Leclerc's "domino-Littlewood–Richardson tableaux" algorithm [6] for calculating the decomposition of the products $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$. A different approach for computing the expansion $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ makes use of the plethysm $s_\lambda[p_2] = p_2[s_\lambda]$ of the Schur function s_λ with the power symmetric function $p_2(x) = \sum_i x_i^2$ and involve multiplication of Schur functions.

More precisely, the approach we use to calculate $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ is the following. First we expand $s_{(2)}$ and $s_{(1^2)}$ in terms of the power symmetric function: $s_{(2)} = \frac{1}{2}(p_1^2 + p_2)$ and $s_{(1^2)} = \frac{1}{2}(p_1^2 - p_2)$. However, $p_1[s_\lambda] = s_\lambda$ so that $p_1^2[s_\lambda] = s_\lambda^2$.

Thus,

$$s_{(2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 + p_2[s_\lambda])$$

$$s_{(1^2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 - p_2[s_\lambda]).$$

If λ is a partition of n, then $p_2[s_\lambda] = \sum_{\gamma \vdash 2n} c_\lambda^\gamma s_\gamma$, where the sum runs over all partitions γ of $2n$ and the coefficients c_λ^γ are integers (see [7]). We say that the plethysm $p_2[s_\lambda]$ of the power symmetric function p_2 and the Schur function s_λ is multiplicity-free if every coefficient in the resulting Schur function expansion is $0, +1, -1$. The multiplicity-free plethysms $p_2[s_\lambda]$ have been studied in [4]. We would like to point out that, for those partitions λ such that $s_{(2)}[s_\lambda]$ and $s_{(1^2)}[s_\lambda]$ are multiplicity-free, i.e. (a^b), $(a^{b-1}, a+1)$, $(1, a^b)$, $(a-1, a^{b-1})$ or a hook, also $p_2[s_\lambda]$ is multiplicity-free and s_λ^2 has maximal multiplicity 2.

Here we will show how to compute the explicit expansion of the plethysm $p_2[s_\lambda]$ of the power symmetric function p_2 and the Schur function s_λ, where λ has either two rows or two columns, directly from the expansion of s_λ^2 which can be done without too much difficulty via one of the existing versions of the Littlewood–Richardson rule [12]. For the history of the rule, we refer the reader to [16, pp. 438].

2. Preliminaries

Throughout this paper, by partition λ of a positive integer n, denoted by $\lambda \vdash n$, we mean a sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

- $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k$;
- $\lambda_1 + \lambda_2 + \ldots + \lambda_k = n$.

Each λ_i is called a part of λ. The length of λ, denoted by $l(\lambda) = k$ is the number of parts of λ and $|\lambda|$, the sum of entries, is called the weight of λ. We will also use the notation $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ to mean that λ has m_1 parts of size 1, m_2 parts of size 2 and so on. The conjugate partition, λ', is the partition obtained by interchanging the rows and columns of λ. Going forward, we require the following terminology. We call the partition λ of n linear if $\lambda = (n)$ or $\lambda = (1^n)$, a rectangle if λ is of the form $\lambda = (a^r)$ for some $a, r \geq 1$. A hook (or proper hook) is a partition of the form $(1^{n-a}, a)$. By a near rectangle we mean a partition λ obtained from a rectangle by adding a single row or column.

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, the diagram of λ is the collection of left-justified boxes (called cells) such that there are λ_i boxes in the ith row from the top. This is known as the English convention for the diagram of a partition. The French convention places λ_i left-justified boxes in the ith row from the bottom. In this paper we will follow the French convention and we will interchangeably use greek letters to denote partitions or diagrams.

Let Λ^n denote the space of homogeneous symmetric functions of degree n, $s_\lambda(x)$ the Schur function and $p_\lambda(x)$, the power symmetric function where $\lambda \vdash n$ and $x = (x_1, x_2, \ldots)$ is an infinite sequence of variables. For $u(x) \in \Lambda^n$ and $\gamma \vdash n$, we use $\langle u(x), s_\gamma(x) \rangle$ to denote the coefficient of $s_\gamma(x)$ in the expansion of $u(x)$. From now on, we will write u instead of $u(x)$, for any $u(x) \in \Lambda^n$. Let u, v and w be symmetric functions. We will make frequent use of the following properties for plethysm (see [9] or [13]).

Distributivity:

- $(u + v)[w] = u[w] + v[w]$ and $(uv)[w] = u[w]v[w]$;

commutativity with the power symmetric function:

- $u[p_k] = p_k[u]$;

conjugation:

- $(s_\mu[s_\lambda])' = s_\mu[s_{\lambda'}]$ if $|\lambda|$ is even
- $(s_\mu[s_\lambda])' = s_{\mu'}[s_{\lambda'}]$ if $|\lambda|$ is odd

where for any sum $\sum c_\nu s_\nu$, $(\sum c_\nu s_\nu)'$ denotes the sum $\sum c_\nu s_{\nu'}$ and ν' is the conjugate partition of ν.

3. The computation of $p_2[s_{(a,b)}]$

Let λ and μ partitions of weight n and m, respectively.

The famous Littlewood–Richardson rule [12] gives a combinatorial interpretation for computing $c^\gamma_{\lambda,\mu}$ where

$$s_\lambda s_\mu = \sum_{\gamma \vdash n+m} c^\gamma_{\lambda,\mu} s_\gamma$$

where the sum runs over all partitions γ of $n + m$ and $c^\gamma_{\lambda,\mu}$ are nonnegative integers.

Let $0 \leq a \leq b$, $n = a + b$ and $\lambda = \mu = (a, b) \vdash n$, then

$$s^2_{(a,b)} = \sum_{\gamma \vdash 2n} c^{\gamma}_{(a,b)} s_{\gamma}$$

where $(a, b) \subset \gamma$ and $c^{\gamma}_{(a,b)} = 0$ if γ has more than four parts.

The coefficients $c^{\gamma}_{(a,b)}$ have been explicitly computed in [5]. Given a partition $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \vdash 2n$ with four parts and $0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \gamma_4$, we say that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have the same parity if either $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are all even or $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are all odd otherwise we say that $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have different parity. For example, in the partition $\gamma = (0, 2, 2, 6) \vdash 10$, $\gamma_1 = 0$, $\gamma_2 = 2$, $\gamma_3 = 2$, $\gamma_4 = 6$ have the same parity and also in the partition $\gamma = (1, 3, 3, 3) \vdash 10$, $\gamma_1 = 1$, $\gamma_2 = \gamma_3 = \gamma_4 = 3$ are all odd so they have the same parity. Instead the four parts of $\mu = (0, 1, 2, 7) \vdash 10$, have different parity.
We can derive the expansion $p_2[s_{(a,b)}]$ directly from the expansion of $s^2_{(a,b)}$ according to the following result:

Theorem 3.1 *Let $\lambda = (a, b) \vdash n$ and $s^2_{(a,b)} = \sum_{\gamma \vdash 2n} c^{\gamma}_{(a,b)} s_{\gamma}$.*
Assume that P is the set of all the partitions $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \vdash 2n$ such that $c^{\gamma}_{(a,b)}$ is odd, A is the set of partitions $\gamma \in P$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have the same parity and B is the set of partitions $\gamma \in P$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have different parity.

Then

$$p_2[s_{(a,b)}] = \sum_{\gamma \in A} s_{\gamma} - \sum_{\gamma \in B} s_{\gamma}.$$

Proof. Let $s_{(2)}[s_{(a,b)}] = \sum_{\gamma \vdash 2n} g^{\gamma}_{(a,b)} s_{\gamma}$, where the $g^{\gamma}_{(a,b)}$ are nonnegative integers. By [4], $p_2[s_{(a,b)}]$ is multiplicity free, i.e. every coefficient in the resulting Schur function expansion is $0, 1, -1$. Therefore, it follows from the formula $s_{(2)}[s_{(a,b)}] = \frac{1}{2}(s^2_{(a,b)} + p_2[s_{(a,b)}])$ that if, for a given γ, $c^{\gamma}_{(a,b)} = \langle s^2_{(a,b)}, s_{\gamma} \rangle$ is even, then the coefficient $\langle p_2[s_{(a,b)}], s_{\gamma} \rangle$ must be zero and $\langle s_{(2)}[s_{(a,b)}], s_{\gamma} \rangle = \frac{c^{\gamma}_{(a,b)}}{2}$. If $c^{\gamma}_{(a,b)} = \langle s^2_{(a,b)}, s_{\gamma} \rangle$ is odd then the coefficient $\langle p_2[s_{(a,b)}], s_{\gamma} \rangle$ is either 1 or -1 and $\langle s_{(2)}[s_{(a,b)}], s_{\gamma} \rangle$ is either equal to $\frac{c^{\gamma}_{(a,b)}+1}{2}$ or to $\frac{c^{\gamma}_{(a,b)}-1}{2}$. Thus if $c^{\gamma}_{(a,b)}$ is odd, in order to compute the coefficient $\langle s_{(2)}[s_{(a,b)}], s_{\gamma} \rangle$, we only need to determine the sign of $\langle p_2[s_{(a,b)}], s_{\gamma} \rangle$. This sign has been computed in [5] via the SXP-algorithm by Chen, Garsia, and Remmel [7]. Let P be the set of all the partitions $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \vdash 2n$ such that $c^{\gamma}_{(a,b)}$ is odd, A the set of partitions $\gamma \in P$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have the same parity and B the set of partitions $\gamma \in P$ where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ have different parity, then by Theorem 5 of [5], it follows that:

$$p_2[s_{(a,b)}] = \sum_{\gamma \in A} s_{\gamma} - \sum_{\gamma \in B} s_{\gamma}.$$

Example 3.2 *Let $\lambda = (1, 2) \vdash 3$. Then*

$$s^2_{(1,2)} = s_{(2,4)} + s_{(1,1,4)} + s_{(3,3)} + 2s_{(1,2,3)} + s_{(1,1,1,3)} + s_{(2,2,2)} + s_{(1,1,2,2)}.$$

$$p_2[s_{(1,2)}] = s_{(2,4)} - s_{(1,1,4)} - s_{(3,3)} + s_{(1,1,1,3)} + s_{(2,2,2)} - s_{(1,1,2,2)}.$$

Example 3.3 *Let* $\lambda = (2,4) \vdash 6$. *Then*

$$s^2_{(2,4)} = s_{(4,8)} + s_{(1,3,8)} + s_{(2,2,8)} + s_{(5,7)} + 2s_{(1,4,7)} + 2s_{(2,3,7)} + s_{(1,1,3,7)} + s_{(1,2,2,7)} +$$

$$+ s_{(6,6)} + 2s_{(1,5,6)} + 3s_{(2,4,6)} + s_{(1,1,4,6)} + s_{(3,3,6)} + 2s_{(1,2,3,6)} + s_{(2,2,2,6)} + s_{(2,5,5)} +$$

$$+ s_{(1,1,5,5)} + 2s_{(3,4,5)} + 2s_{(1,2,4,5,)} + s_{(1,3,3,5)} + s_{(2,2,3,5)} + s_{(4,4,4)} + 2s_{(1,3,4,4)} + s_{(2,2,4,4)}.$$

$$p_2[s_{(2,4)}] = s_{(4,8)} - s_{(1,3,8)} + s_{(2,2,8)} - s_{(5,7)} + s_{(1,1,3,7)} - s_{(1,2,2,7)} + s_{(6,6)} + s_{(2,4,6)}$$

$$- s_{(1,1,4,6)} - s_{(3,3,6)} + s_{(2,2,2,6)} - s_{(2,5,5)} + s_{(1,1,5,5)} + s_{(1,3,3,5)} - s_{(2,2,3,5)} + s_{(4,4,4)} + s_{(2,2,4,4)}.$$

Example 3.4 *Let* $\lambda = (2,5) \vdash 7$. *Then*

$$s^2_{(2,5)} = s_{(4,10)} + s_{(1,3,10)} + s_{(2,2,10)} + s_{(5,9)} + 2s_{(1,4,9)} + 2s_{(2,3,9)} + s_{(1,1,3,9)} + s_{(1,2,2,9)} +$$

$$+ s_{(6,8)} + 2s_{(1,5,8)} + 3s_{(2,4,8)} + s_{(1,1,4,8)} + s_{(3,3,8)} + 2s_{(1,2,3,8)} + s_{(2,2,2,8)} + s_{(7,7)} +$$

$$+ 2s_{(1,6,7)} + 3s_{(2,5,7)} + s_{(1,1,5,7)} + 2s_{(3,4,7)} + 2s_{(1,2,4,7)} + s_{(1,3,3,7)} + s_{(2,2,3,7)} + s_{(2,6,6)} +$$

$$+ s_{(1,1,6,6)} + 2s_{(3,5,6)} + 2s_{(1,2,5,6)} + s_{(4,4,6)} + s_{(1,3,4,6)} + s_{(2,2,4,6)} + s_{(4,5,5)} + s_{(1,3,5,5)} + s_{(2,2,5,5)}.$$

$$p_2[s_{(2,5)}] = s_{(4,10)} - s_{(1,3,10)} + s_{(2,2,10)} - s_{(5,9)} + s_{(1,1,3,9)} - s_{(1,2,2,9)} + s_{(6,8)} + s_{(2,4,8)}$$

$$- s_{(1,1,4,8)} - s_{(3,3,8)} + s_{(2,2,2,8)} - s_{(7,7)} - s_{(2,5,7)} + s_{(1,1,5,7)} + s_{(1,3,3,7)} - s_{(2,2,3,7)} +$$

$$+ s_{(2,6,6)} - s_{(1,1,6,6)} + s_{(4,4,6)} - s_{(1,3,4,6)} + s_{(2,2,4,6)} - s_{(4,5,5)} + s_{(1,3,5,5)} - s_{(2,2,5,5)}.$$

Example 3.5 *Let* $\lambda = (3,5) \vdash 8$. *Then*

$$s^2_{(3,5)} = s_{(6,10)} + s_{(1,5,10)} + s_{(2,4,10)} + s_{(3,3,10)} + s_{(7,9)} + 2s_{(1,6,9)} + 2s_{(2,5,9)} + s_{(1,1,5,9)}$$

$$+ 2s_{(3,4,9)} + s_{(1,2,4,9)} + s_{(1,3,3,9)} + s_{(8,8)} + 2s_{(1,7,8)} + 3s_{(2,6,8)} + s_{(1,1,6,8)} + 3s_{(3,5,8)}$$

$$+ 2s_{(1,2,5,8)} + s_{(4,4,8)} + 2s_{(1,3,4,8)} + s_{(2,2,4,8)} + s_{(2,3,3,8)} + s_{(2,7,7)} + s_{(1,1,7,7)} + 2s_{(3,6,7)}$$

$$+ 2s_{(1,2,6,7)} + 2s_{(4,5,7)} + 3s_{(1,3,5,7)} + s_{(2,2,5,7)} + s_{(1,4,4,7)} + 2s_{(2,3,4,7)} + s_{(3,3,3,7)} + s_{(4,6,6)}$$

$$+ s_{(1,3,6,6)} + s_{(2,2,6,6)} + s_{(5,5,6)} + 2s_{(1,4,5,6)} + 2s_{(2,3,5,6)} + s_{(2,4,4,6)} + s_{(3,3,4,6)} + s_{(1,5,5,5)}$$

$$+ s_{(2,4,5,5)} + s_{(3,3,5,5)}.$$

$$p_2[s_{(3,5)}] = s_{(6,10)} - s_{(1,5,10)} + s_{(2,4,10)} - s_{(3,3,10)} - s_{(7,9)} + s_{(1,1,5,9)} - s_{(1,2,4,9)} + s_{(1,3,3,9)}$$

$$+ s_{(8,8)} + s_{(2,6,8)} - s_{(1,1,6,8)} - s_{(3,5,8)} + s_{(4,4,8)} + s_{(2,2,4,8)} - s_{(2,3,3,8)} - s_{(2,7,7)}$$

$$+ s_{(1,1,7,7)} + s_{(1,3,5,7)} - s_{(2,2,5,7)} - s_{(1,4,4,7)} + s_{(3,3,3,7)} + s_{(4,6,6)} - s_{(1,3,6,6)} + s_{(2,2,6,6)} - s_{(5,5,6)}$$

$$+ s_{(2,4,4,6)} - s_{(3,3,4,6)} + s_{(1,5,5,5)} - s_{(2,4,5,5)} + s_{(3,3,5,5)}.$$

Conjecture

The previous examples show that the products $s^2_{(2,4)}$, $s^2_{(2,5)}$ and $s^2_{(3,5)}$ have maximal multiplicity 3. By some computer calculation it looks like that the only partitions λ such that the products s^2_λ have maximal multiplicity 3 are: $(2,b)$ where $b \geq 4$, $(b-2,b)$ where $b \geq 5$ and their conjugates $(2,b)' = (1^{b-2}, 2^2)$ and $(b-2,b)' = (1^2, 2^{b-2})$. In fact, it follows from Theorem 2 of [5], that if either $\lambda = (2,b)$ or $\lambda = (b-2,b)$, then s^2_λ has maximal multiplicity 3. Also, if we consider

$$s^2_{(a,b)} = \sum_{\gamma \vdash 2n} c^\gamma_{(a,b)} s_\gamma$$

by the conjugation symmetry of the Littlewood–Richardson coefficients [17], it follows that

$$s^2_{(a,b)'} = \sum_{\gamma' \vdash 2n} c^{\gamma'}_{(a,b)'} s_{\gamma'}$$

where $c^{\gamma'}_{(a,b)'} = c^\gamma_{(a,b)}$. Therefore s^2_λ has maximal multiplicity 3 also in the case $\lambda = (2,b)'$ or $\lambda = (b-2,b)'$. I claim that there are no other partitons λ such that s^2_λ has maximal multiplicity 3.

Corollary 3.6 *Let $\lambda = (n) \vdash n$. Then $s^2_{(n)} = \sum_{i=0}^n s_{(n-i,n+i)}$ and*

$$s_{(2)}[s_{(n)}] = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ \ even}} s_\gamma$$

$$s_{(1^2)}[s_{(n)}] = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ \ odd}} s_\gamma.$$

Proof.

As a consequence of Theorem 3.1, we get the classical formulas of Littlewood for the Schur function expansions of the plethysms $s_{(2)}[s_{(n)}]$ and $s_{(1^2)}[s_{(n)}]$. In fact, in the special case $\lambda = (a,b) = (n)$, then $a = 0$, $b = n$ and by Pieri's rule [14] it follows that

$$s^2_{(a,b)} = \sum_{\gamma \vdash 2n} c^\gamma_{(a,b)} s_\gamma = s^2_{(n)} = \sum_{i=0}^{n} s_{(n-i,n+i)}.$$

Therefore the set P in Theorem 3.1 of all the partitions $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of $2n$ such that $c^\gamma_{(a,b)}$ is odd reduces to the set of partitions $\gamma = (0, 0, \gamma_3, \gamma_4) \vdash 2n$ such that $c^\gamma_{(a,b)}$ is 1. If we denote $k = \gamma_3$ and $l = \gamma_4$, then, by Pieri's rule, P reduces to the set of all the partitions $(k,l) \vdash 2n$ where either k, l are even or k, l are odd. By Theorem 3.1 it follows

$$p_2[s_{(n)}] = \sum_{\gamma \in A} s_\gamma - \sum_{\gamma \in B} s_\gamma$$

and

$$s_{(n)}^2 = \sum_{\gamma \in A} s_\gamma + \sum_{\gamma \in B} s_\gamma$$

where A is the set of partitions $(k,l) \in P$ where k,l are even and B is the set of partitions $(k,l) \in P$ where k,l are odd.

Thus, by

$$s_{(2)}[s_{(n)}] = \frac{1}{2}(s_{(n)}^2 + p_2[s_{(n)}])$$

$$s_{(1^2)}[s_{(n)}] = \frac{1}{2}(s_{(n)}^2 - p_2[s_{(n)}])$$

we get Littlewood's formulas:

$$s_{(2)}[s_{(n)}] = \sum_{\gamma \in A} s_\gamma = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ even}} s_\gamma$$

$$s_{(1^2)}[s_{(n)}] = \sum_{\gamma \in B} s_\gamma = \sum_{\substack{\gamma=(k,l)\vdash 2n \\ k,l \ odd}} s_\gamma .$$

Example 3.7 *Let* $n = 3$. *Then*
$s_{(3)}^2 = s_{(6)} + s_{(1,5)} + s_{(2,4)} + s_{(3,3)}$
$p_2[s_{(3)}] = s_{(6)} + s_{(2,4)} - s_{(1,5)} - s_{(3,3)}$
$s_{(2)}[s_{(3)}] = s_{(6)} + s_{(2,4)}$
$s_{(1^2)}[s_{(3)}] = s_{(1,5)} + s_{(3,3)}$.

Example 3.8 *Let* $n = 4$. *Then*
$s_{(4)}^2 = s_{(8)} + s_{(1,7)} + s_{(2,6)} + s_{(3,5)} + s_{(4,4)}$
$p_2[s_{(4)}] = s_{(8)} + s_{(2,6)} + s_{(4,4)} - s_{(3,5)} - s_{(1,7)}$
$s_{(2)}[s_{(4)}] = s_{(8)} + s_{(2,6)} + s_{(4,4)}$
$s_{(1^2)}[s_{(4)}] = s_{(1,7)} + s_{(3,5)}$.

Corollary 3.9 *If* $\lambda \vdash n$ *and* λ *is* $(1,a)$, $(a-1,a)$, (a,a), $s_\lambda^2 = \sum_{\gamma \vdash 2n} c_\lambda^\gamma s_\gamma$ *and* P *the set of all the partitions* $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \vdash 2n$ *such that* c_λ^γ *is* 1, *then*

$$p_2[s_\lambda] = \sum_{\gamma \in A} s_\gamma - \sum_{\gamma \in B} s_\gamma$$

where A *is the set of partitions* $\gamma \in P$ *where* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *have the same parity and* B *the set of partitions* $\gamma \in P$ *where* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ *have different parity.*

Proof. In [1], it has been shown that, in case λ is either $(1,a)$ or $(a-1,a)$, s_λ^2 has maximal multiplicity 2. Also by [17], if λ is (a,a), s_λ^2 is multiplicity-free. Therefore, by Theorem 3.1, the set P of all the partitions

$\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of $2n$ such that c_λ^γ is odd reduces to the set of partitions $\gamma = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of $2n$ such that c_λ^γ is 1 and now the corollary follows.

Example 3.10 *Let* $\lambda = (2,3) \vdash 5$. *Then*

$$s_{(2,3)}^2 = s_{(4,6)} + s_{(1,3,6)} + s_{(2,2,6)} + s_{(5,5)} + 2s_{(1,4,5)} + 2s_{(2,3,5)} + s_{(1,1,3,5)} + s_{(1,2,2,5)}$$
$$+ s_{(2,4,4)} + s_{(1,1,4,4)} + s_{(3,3,4)} + 2s_{(1,2,3,4)} + s_{(2,2,2,4)} + s_{(1,3,3,3)} + s_{(2,2,3,3)}$$

$$p_2[s_{(2,3)}] = s_{(4,6)} - s_{(1,3,6)} + s_{(2,2,6)} - s_{(5,5)} + s_{(1,1,3,5)} - s_{(1,2,2,5)}$$
$$+ s_{(2,4,4)} - s_{(1,1,4,4)} - s_{(3,3,4)} + s_{(2,2,2,4)} + s_{(1,3,3,3)} - s_{(2,2,3,3)}.$$

4. The computation of $p_2[s_{(1^r,2^t)}]$

In this section we derive formulas for the Schur function expansion of the plethysm $p_2[s_\lambda]$ when λ has two columns.

Proposition 4.1 *Let* $\lambda \vdash n$, λ' *the conjugate partition of* λ *and* $p_2[s_\lambda] = \sum_{\gamma \vdash 2n} c_\lambda^\gamma s_\gamma$. *Then*

- $p_2[s_{\lambda'}] = \sum_{\gamma' \vdash 2n} c_\lambda^\gamma s_{\gamma'}$ *if* $|\lambda|$ *is even*
- $p_2[s_{\lambda'}] = \sum_{\gamma' \vdash 2n} (-c_\lambda^\gamma) s_{\gamma'}$ *if* $|\lambda|$ *is odd.*

Proof. Since $s_{(2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 + p_2[s_\lambda])$ and $s_{(1^2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 - p_2[s_\lambda])$, then

$$p_2[s_\lambda] = s_{(2)}[s_\lambda] - s_{(1^2)}[s_\lambda]$$

and

$$p_2[s_{\lambda'}] = s_{(2)}[s_{\lambda'}] - s_{(1^2)}[s_{\lambda'}].$$

Let $s_{(2)}[s_\lambda] = \sum_{\gamma \vdash 2n} a_\lambda^\gamma s_\gamma$ and $s_{(1^2)}[s_\lambda] = \sum_{\gamma \vdash 2n} b_\lambda^\gamma s_\gamma$. Then, by the conjugation property:

- $(s_\mu[s_\lambda])' = s_\mu[s_{\lambda'}]$ if $|\lambda|$ is even

it follows that if, $|\lambda|$ is even, $(s_{(2)}[s_\lambda])' = \sum_{\gamma \vdash 2n} a_\lambda^\gamma s_{\gamma'} = s_{(2)}[s_{\lambda'}]$ and $(s_{(1^2)}[s_\lambda])' = \sum_{\gamma \vdash 2n} b_\lambda^\gamma s_{\gamma'} = s_{(1^2)}[s_{\lambda'}]$. Since

$$p_2[s_\lambda] = s_{(2)}[s_\lambda] - s_{(1^2)}[s_\lambda] = \sum_{\gamma \vdash 2n} a_\lambda^\gamma s_\gamma - \sum_{\gamma' \vdash 2n} b_\lambda^\gamma s_\gamma = \sum_{\gamma' \vdash 2n} c_\lambda^\gamma s_\gamma$$

in case $|\lambda|$ is even we get

$$p_2[s_{\lambda'}] = s_{(2)}[s_{\lambda'}] - s_{(1^2)}[s_{\lambda'}] = \sum_{\gamma' \vdash 2n} a_\lambda^\gamma s_{\gamma'} - \sum_{\gamma' \vdash 2n} b_\lambda^\gamma s_{\gamma'} = \sum_{\gamma' \vdash 2n} c_\lambda^\gamma s_{\gamma'}.$$

In case $|\lambda|$ is odd we have to consider the conjugation property:

- $(s_\mu[s_\lambda])' = s_{\mu'}[s_{\lambda'}]$ if $|\lambda|$ is odd

Therefore, in case $\mu = 2$ and $|\lambda|$ is odd,

$$(s_{(2)}[s_\lambda])' = s_{(1^2)}[s_{\lambda'}]$$

$$(s_{(1^2)}[s_\lambda])' = s_{(2)}[s_{\lambda'}]$$

and consequently

$$p_2[s_{\lambda'}] = s_{(2)}[s_{\lambda'}] - s_{(1^2)}[s_{\lambda'}] = s_{(1^2)}[s_{\lambda'}] - s_{(2)}[s_{\lambda'}] = \sum_{\gamma' \vdash 2n} b_\lambda^\gamma s_{\gamma'} - \sum_{\gamma' \vdash 2n} a_\lambda^\gamma s_{\gamma'} = \sum_{\gamma' \vdash 2n} (-c_\lambda^\gamma) s_{\gamma'}.$$

Corollary 4.2 *Let $\lambda = (a,b) \vdash n$, λ' the conjugate partition of λ, $p_2[s_{(a,b)}] = \sum_{\gamma \vdash 2n} c_\lambda^\gamma s_\gamma$. Then*

- $p_2[s_{(1^{b-a},2^a)}] = \sum_{\gamma' \vdash 2n} c_\lambda^\gamma s_{\gamma'}$ *if $|\lambda|$ is even*
- $p_2[s_{(1^{b-a},2^a)}] = \sum_{\gamma' \vdash 2n} (-c_\lambda^\gamma) s_{\gamma'}$ *if $|\lambda|$ is odd.*

Proof. In case $\lambda = (a,b) \vdash n$, the conjugate partition λ' is equal to $(1^{b-a}, 2^a)$ and now the Corollary follows from Proposition 4.1.

Corollary 4.3 *Let $\lambda = (a,b) \vdash n$, λ' the conjugate partition of λ, $s^2_{(a,b)} = \sum_{\gamma \vdash 2n} c^\gamma_{(a,b)} s_\gamma$. Then*

$$s^2_{(1^{b-a},2^a)} = \sum_{\gamma' \vdash 2n} c^\gamma_{(a,b)} s_{\gamma'}$$

Proof.

Since $s_{(2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 + p_2[s_\lambda])$ and $s_{(1^2)}[s_\lambda] = \frac{1}{2}(s_\lambda^2 - p_2[s_\lambda])$, then

$$s_\lambda^2 = s_{(2)}[s_\lambda] + s_{(1^2)}[s_\lambda].$$

Similarly as in Proposition 4.1 we can get the conjugation symmetry of the Littlewood–Richardson coefficients which implies that if $s_\lambda^2 = \sum_{\gamma \vdash 2n} c_\lambda^\gamma s_\gamma$ then $s_{\lambda'}^2 = \sum_{\gamma' \vdash 2n} c_{\lambda'}^{\gamma'} s_{\gamma'}$ where $c_{\lambda'}^{\gamma'} = c_\lambda^\gamma$. Therefore in case $\lambda = (a,b)$, the conjugate partition λ' is equal to $(1^{b-a}, 2^a)$ and

$$s^2_{(1^{b-a},2^a)} = \sum_{\gamma' \vdash 2n} c^\gamma_{(a,b)} s_{\gamma'}.$$

Example 4.4 *In case $\lambda = (2,3) \vdash 5$, then $|\lambda|$ is odd. By Example 3.10, Corollaries 4.2 and 4.3 we get:*

$$s^2_{(1,2^2)} = s^2_{(2,3)'} = s_{(4,6)'} + s_{(1,3,6)'} + s_{(2,2,6)'} + s_{(5,5)'} + 2s_{(1,4,5)'} + 2s_{(2,3,5)'} + s_{(1,1,3,5)'} + s_{(1,2,2,5)'}$$

$$+ s_{(2,4,4)'} + s_{(1,1,4,4)'} + s_{(3,3,4)'} + 2s_{(1,2,3,4)'} + s_{(2,2,2,4)'} + s_{(1,3,3,3)'} + s_{(2,2,3,3)'} =$$

$$= s_{(1^2,2^4)} + s_{(1^3,2^2,3)} + s_{(1^4,3^2)} + s_{(2^5)} + 2s_{(1,2^3,3)} + 2s_{(1^2,2,3^2)} + s_{(1^2,2^2,4)} + s_{(1^3,3,4)}$$

$$+ s_{(2^2,3^2)} + s_{(2^3,4)} + s_{(1,3^3)} + 2s_{(1,2,3,4)} + s_{(1^2,4^2)} + s_{(3^2,4)} + s_{(2,4^2)}.$$

$$p_2[s_{(1,2^2)}] = p_2[s_{(2,3)'}] = -s_{(4,6)'} + s_{(1,3,6)'} - s_{(2,2,6)'} + s_{(5,5)'} - s_{(1,1,3,5)'} + s_{(1,2,2,5)'} - s_{(2,4,4)'} +$$

$$+ s_{(1,1,4,4)'} + s_{(3,3,4)'} - s_{(2,2,2,4)'} - s_{(1,3,3,3)'} + s_{(2,2,3,3)'} =$$

$$= -s_{(1^2,2^4)} + s_{(1^3,2^2,3)} - s_{(1^4,3^2)} + s_{(2^5)} - s_{(1^2,2^2,4)} + s_{(1^3,3,4)}$$

$$-s_{(2^2,3^2)} + s_{(2^3,4)} + s_{(1,3^3)} - s_{(1^2,4^2)} - s_{(3^2,4)} + s_{(2,4^2)}.$$

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