

On Nowicki's conjecture: a survey and a new result

Dedicated to the 70th anniversary of Vesselin Drensky

Lucio CENTRONE^{1,2} , Andre DUSHIMIRIMANA³ , Şehmus FINDIK^{3,*} 

¹Dipartimento di Matematica, Università degli Studi di Bari "Aldo Moro", via E. Orabona 4, 70125, Bari, Italy

²IMECC, Universidade Estadual de Campinas, Rua Sergio Buarque de Holanda

651, 13083-859, Campinas (SP), Brazil

³Department of Mathematics, Faculty of Arts and Sciences, Çukurova University, Adana, Turkey

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Abstract: The goal of the paper is twofold: it aims to give an extensive set of tools and bibliography towards Nowicki's conjecture both in for polynomial algebras and in an associative setting; it establishes a new result about Nowicki's conjecture for the free metabelian Poisson algebra.

Key words: Lie algebras, Poisson algebras, the Nowicki conjecture

1. Introduction

In this paper we will handle Nowicki's conjecture from its classical formulation to some related problems. Our intent is producing a survey toward this theme and adding a new brick concerning Nowicki's conjecture in a Poisson algebra setting.

Our first aim is introducing the so-called Weitzenböck derivations. We start considering a field K of characteristic zero and a finite set of variables $X_m = \{x_1, \dots, x_m\}$; we denote by KX_m the vector space over K with basis X_m and we shall denote by $K[X_m]$ the (commutative) unitary polynomial algebra generated by X_m . Every map $\delta : X_m \rightarrow K[X_m]$ can be extended to a derivation of $K[X_m]$. When δ is a nonzero linear map of KX_m , then δ is called Weitzenböck. This name is due to the classical result of Weitzenböck [23], dating back to 1932, which states that the algebra $K[X_m]^\delta = \ker(\delta)$ of constants of the derivation δ in the algebra $K[X_m]$ is finitely generated. If δ is a Weitzenböck derivation, then it is locally nilpotent and the linear map $\exp(\delta)$ acting on the vector space KX_m is unipotent. Hence the algebra $K[X_m]^\delta$ of constants of δ is equal to the algebra of invariants

$$K[X_m]^{\exp(\delta)} = \{x \in K[X_m] \mid \exp(\delta)(x) = x\}.$$

The algebra of invariants of $\exp(\delta)$ is equal to the algebra of invariants $K[X_m]^G$, where the group G consists of all elements $\exp(c\delta)$, $c \in K$. As an abstract group, the group G is isomorphic to the unitriangular group consisting of all 2×2 upper triangular matrices with 1's on the diagonal. The group G can be embedded into the general linear group $GL_m(K)$ via the representation corresponding to the Jordan normal form of δ . Thus we are allowed to study the algebra of constants $K[X_m]^\delta$ with the methods of the classical invariant theory.

*Correspondence: sfindik@cu.edu.tr

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The computational aspects of algebras of constants and invariant theory are depicted in part in the books by Nowicki [20], Derksen and Kemper [7], and Sturmfels [22].

We can say more about Weitzenböck derivations. For instance, take a Weitzenböck derivation, then its Jordan normal form consists of Jordan cells with zero diagonals. Certainly, Weitzenböck derivations are in one-to-one correspondence with the partitions of m , up to a linear change of variables. This means there are essentially finite number of Weitzenböck derivations for a fixed dimension m . In particular, let $m = 2d$, $d \geq 1$, and assume that the Jordan form of δ contains the Jordan cells of size 2×2 only, i.e.

$$J(\delta) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We assume that $\delta(x_{2j}) = x_{2j-1}$, $\delta(x_{2j-1}) = 0$, $j = 1, \dots, d$. Nowicki conjectured in his book [20] (Section 6.9, page 76) that in this case the algebra $K[X_{2d}]^\delta$ is generated by $x_1, x_3, \dots, x_{2d-1}$, and $x_{2i-1}x_{2j} - x_{2i}x_{2j-1}$, $1 \leq i < j \leq d$. The last conjecture will be called throughout the text classical Nowicki's conjecture.

Nowicki's conjecture was proved by several authors with different techniques. The first published proof appeared in 2004 by Khoury [15] in his PhD thesis, followed by his paper [16], where he makes use of a computational approach involving Gröbner basis techniques. Derksen and Panyushev applied ideas of classical invariant theory in order to prove the Nowicki conjecture but their proof remained unpublished. Later in 2009, Drensky and Makar-Limanov [14] confirmed the conjecture by an elementary proof from undergraduate algebra, without involving invariant theory. In this paper we shall highlight the key points of the proof given by Drensky and Makar-Limanov. Moreover, another simple short proof was given by Kuroda [18] in 2009, too. Bedratyuk [3] proved the Nowicki conjecture by reducing it to a well known problem of classical invariant theory. We were able to find three more proofs of Nowicki's conjecture: two of them are cited in the paper by Kuroda [18], and the latest proof was given by Drensky in [11].

In this paper we would like to attempt to give a complete account on the state of the art of Nowicki's conjecture presenting a sketch of the proof obtained by Drensky and Makar-Limanov. We will also treat a bit a generalization of Nowicki's conjecture (see Sections 3 and 4 in the text). We will also focus on recent developments of the following problem, strictly related to the classical Nowicki's conjecture, but with a scent of theory of algebras with polynomial identities (PI-algebras). Let us explain briefly our framework: let δ be a Weitzenböck derivation of the free algebra $F_m(\mathfrak{A})$ of rank m in a given variety of algebras \mathfrak{A} (or relatively free algebra of rank m of \mathfrak{A}), then

give an explicit set of generators of the algebra of constants $F_m^\delta(\mathfrak{A})$.

In [13] Drensky and Gupta studied Weitzenböck derivations δ acting on $F_m(\mathfrak{A})$ proving that if the algebra $UT_2(K)$ of 2×2 upper triangular matrices over a field K of characteristic zero belongs to a variety \mathfrak{A} , then $F_m^\delta(\mathfrak{A})$ is not finitely generated whereas if $UT_2(K)$ does not belong to \mathfrak{A} , then $F_m^\delta(\mathfrak{A})$ is finitely generated by a result of Drensky in [9]. Recently, Dangovski et al. [5, 6] gave some new results in this direction. They showed that if \mathfrak{A} is the variety of metabelian Lie or associative algebras then the algebra of constants $(F_d'(\mathfrak{A}))^\delta$ in the commutator ideal $F_d'(\mathfrak{A})$ of $F_d(\mathfrak{A})$ is a finitely generated module of $K[X_d]^\delta$ and $K[U_d, V_d]^\delta$, respectively. Here

δ acts on U_d and V_d in the same way as on X_d . When δ is as in the Nowicki conjecture, in [4] the authors give an explicit set of generators of the algebra of constants in the case of the variety generated by the infinite dimensional Grassmann algebra and by the free metabelian associative algebra. A good part of those result is presented explicitly in the text (Sections 5 and 6).

In the second part of the paper (Section 7) we will show a substantially new result. We consider the free metabelian Poisson algebra P_{2n} of rank $2n$ over the field K and we give out an explicit set of generators of its algebra of constants under the action of the Weitzenböck derivation δ as in the Nowicki conjecture (see Theorems 7.2 and 7.3 in the text) as a vector space.

We would like to stress the fact that in the study of varieties of (not necessarily associative) algebras over a field of characteristic zero, the metabelian identities play a crucial role. It is well known the identities of $UT_2(K)$ follow from the metabelian identity of order 4, so every variety \mathfrak{V} contains $UT_2(K)$ or satisfies the Engel identity $[x_2, x_1, \dots, x_1] \equiv 0$.

2. Preliminaries

Let K be a field of characteristic 0. We consider the commutative and unitary polynomial algebra $K[X_m]$ with generating set $X_m = \{x_1, \dots, x_m\}$, $m \geq 2$ over K . Of course, $K[X_m]$ is the free algebra of rank m in the variety of unitary commutative algebras over K . Now we consider the general linear group $GL_m(K)$ and $H \leq GL_m(K)$ a subgroup. Usually in invariant theory $GL_m(K)$ acts on the m -dimensional vector space K^m and one considers polynomials as functions $K^m \rightarrow K$. For our purposes it is more convenient to assume that $GL_m(K)$ acts on KX_m and extend this action diagonally on $K[X_m]$. We say a polynomial $p \in K[X_m]$ is H -invariant if it is preserved under the action of each element of H . The vector space $K[X_m]^H$ of all H -invariants turns out to be an algebra that is called the algebra of H -invariants.

The question whether the algebra $K[X_m]^H$ of invariants is finitely generated for every subgroup H of $GL_m(K)$ is a special case of the Hilbert's fourteenth problem suggested by the German mathematician David Hilbert in 1900 at the International Congress of Mathematicians in Paris.

Although the answer to Hilbert's question turned out to be negative in general, as showed by Nagata in 1958, some remarkable affirmative cases have been handled as well. Among them is the approach of Weitzenböck, where he considered the locally nilpotent linear derivations δ of the algebra $K[X_m]$. Then the kernel $\ker(\delta) = K[X_m]^\delta$ of the derivation δ is an algebra called the algebra of constants of $K[X_m]$. He showed that the algebra $K[X_m]^\delta$ is finitely generated as an algebra and is equal to the algebra $K[X_m]^{UT_2(K)}$ of invariants of the unitriangular group

$$UT_2(K) = \{\exp(c\delta) \mid c \in K\},$$

as already said in the introduction.

The next question is

What is the explicit form of the generators of $K[X_m]^\delta$?

Nowicki conjectured in his book that the algebra $K[X_m, Y_m]^\delta$ is generated by

$$x_1, \dots, x_m \text{ and } u_{i,j} := x_i y_j - x_j y_i, \quad 1 \leq i < j \leq m,$$

assuming that $\delta(y_i) = x_i$, $\delta(x_i) = 0$, $i = 1, \dots, m$, where $K[X_m, Y_m]$ denotes the commutative algebra generated by the set $X_m \cup Y_m$.

This is what we shall call Nowicki's conjecture until the end of the paper.

Let us introduce now the noncommutative setting in which we shall generalize Nowicki's conjecture. Let $X = \{x_1, x_2, \dots\}$ be an infinite set of variables and let \mathcal{C} denote a special class of algebras (for instance, associative algebras, Lie algebras, commutative algebras, etc.). We can consider the free \mathcal{C} -algebra $\mathcal{C}\{X\}$ generated by X over K and we shall call its elements \mathcal{C} -polynomials or simply polynomials. A polynomial $f = f(x_1, \dots, x_n)$ is said to be a polynomial identity for the algebra $A \in \mathcal{C}$ if $f(a_1, \dots, a_n) = 0$ for any $a_i \in A$.

The set of all polynomial identities of a given algebra A is denoted by $T_{\mathcal{C}}(A)$ (or $Id_{\mathcal{C}}(A)$) and is invariant under all \mathcal{C} -endomorphisms of $\mathcal{C}\{X\}$, i.e. it is a $T_{\mathcal{C}}$ -ideal called the $T_{\mathcal{C}}$ -ideal of A . The algebra $\mathcal{C}\{X\}/Id_{\mathcal{C}}(A)$ is called relatively free \mathcal{C} -algebra of A .

We can generalize Nowicki's conjecture to relatively free algebras in the following way: let us consider the free algebra $\mathcal{C}\{X_m, Y_m\}$ endowed with the derivation δ such that $\delta(x_i) = 0$ and $\delta(y_i) = x_i$ for any i . Chosen $A \in \mathcal{C}$, $Id(A)$ is, of course, δ -invariant, then δ induces a derivation δ on the relatively free algebra $\mathcal{C}_{2m}(A) := \mathcal{C}\{X_m, Y_m\}/Id(A)$.

Is the algebra of constants of $\mathcal{C}_n(A)$ finitely generated?

Are the generators uniformly looking?

Let $A \in \mathcal{C}$ and consider the class

$$\mathfrak{V}(A) = \{B \in \mathcal{C} \mid Id(A) \subseteq Id(B)\}$$

that is called variety of \mathcal{C} -algebras generated by A . Here is the answer to the first question:

Theorem 2.1 (Drensky [9], Drensky and Gupta [13]) *Let $\mathfrak{V} := \mathfrak{V}(A)$ be a variety of associative or Lie algebras, where A is endowed with the derivation δ . Then the algebra of constants of $\mathcal{C}_2(A)$ is not finitely generated if and only if \mathfrak{V} contains $UT_2(K)$, the algebra of upper triangular 2×2 matrices with entries from K .*

3. Drensky and Makar–Limanov's approach

Herein we shall depict sketches of the proof of Nowicki's conjecture as showed by Drensky and Makar–Limanov in [14].

Let $X_m = \{x_1, \dots, x_m\}$, $Y_m = \{y_1, \dots, y_m\}$ be two disjoint sets of commutative variables. Let $X' = \{x_1, \dots, x_{m-1}\}$, $Y' = \{y_1, \dots, y_{m-1}\}$, $U' = \{u_{i,j} \mid 1 \leq i, j \leq m-1\}$ (where $u_{i,j} = x_i y_j - x_j y_i$), $X = X_m$, $Y = Y_m$ and consider a constant $f = f(X', Y', x_m, y_m)$. Of course f can be taken to be homogeneous in (X, Y) and we carry on by induction on m and on the total degree in x_m, y_m .

Remark 3.1 *For a monomial $v \in K[X, Y]$ of degree (d_1, d_2) with respect to (X, Y) , $\delta(v)$ is of degree $(d_1 + 1, d_2 - 1)$ with respect to (X, Y) .*

If $\underline{m = 1}$, then by the previous remark, we get $K[x_1, y_1]^\delta = K[x_1]$ and we are done because in this case $u_{1,1} = 0$.

We consider $m \geq 1$ and we may suppose $\deg_{x_m, y_m} f > 0$. We write

$$f = (a_p y_m^p + a_{p-1} x_m y_m^{p-1} + \dots + a_1 x_m^{p-1} y_m + a_0 x_m^p) x_m^q,$$

where $a_j \in K[X', Y']$, $a_p \neq 0$.

Remark 3.2 *Since both f and x_m^q are constants, the same holds for f/x_m^q . Hence we may assume $q = 0$.*

Remark 3.3 *Because*

$$\begin{aligned} 0 &= \delta(f) \\ &= \delta(a_p) y_m^p + (p a_p + \delta(a_{p-1})) x_m y_m^{p-1} + \dots + (a_1 + \delta(a_0)) x_m^p \end{aligned}$$

we have a_p is a constant and, by induction, has the form

$$a_p = \sum x_{s_1} \cdots x_{s_k} b_s(U').$$

It is easy to note that $\deg_X(f) \geq p + \deg_{Y'}(a_p)$, then, by the last remark, we may write f as

$$f = \sum x_{s_1} \cdots x_{s_p} c_s(X', U') y_m^p + \sum_{i=0}^{p-1} a_i(X', Y') x_m^{p-i} y_m^i.$$

Remark 3.4 *This is very important:*

$$x_{s_j} y_m = (x_{s_j} y_m - x_m y_{s_j}) + x_m y_{s_j} = u_{s_j, m} + x_m y_{s_j}.$$

Then we get

$$f = \sum c_s(X', U') \prod_{j=1}^p u_{s_j, m} + x_m f_1(X', Y', x_m, y_m)$$

and $\deg_{x_m, y_m}(f_1) < \deg_{x_m, y_m}(f)$ but f , x_m , $c_s(X', U') \prod_{j=1}^p u_{s_j, m}$ are all constants, then f_1 is a constant and we are done by induction.

4. Generalized Nowicki's conjecture

Let us take an integral domain D over K , a set of variables $Y_m := \{y_1, \dots, y_m\}$ and the algebra $A := D[Y_m]$. A derivation Δ on A is said elementary if $\Delta(D) = 0$ and $\Delta(y_i) \in D$ for any i . We consider also the following objects

$$u_{i,j} := \begin{vmatrix} \Delta(y_i) & y_j \\ \Delta(y_j) & y_i \end{vmatrix}.$$

They are called determinants and they belong to $D[y_m]^\Delta$.

Question [Generalized Nowicki's conjecture]: Is the D -algebra $D[Y_m]^\Delta$ generated by the determinants?

Of course the previous question is precisely the Nowicki's conjecture when Δ is such that $\Delta(D) = 0$ and $\Delta(y_i) = x_i$ for any i . In the first proof of Nowicki's conjecture by Khoury, the author considered an elementary derivation Δ sending $\Delta(y_i) = x_i^{m_i}$. Then the algebra of constants $K[X_m, Y_m]^\Delta$ is generated by X_m and the $u_{i,j}$'s. The more general result in this direction is due to Kuroda:

Theorem 4.1 *Let Δ be an elementary derivation of $D[Y_m]$ such that $\Delta(y_i) = f_i$, for any i . Suppose further the f_i 's are algebraically independent over K . Then $D[Y_m]^\Delta$ is generated by the determinants if D is flat over $K[f_1, \dots, f_m]$.*

Notice that this result is deeply different from the original version of Nowicki's conjecture. Nowicki's conjecture can be restated in the language of classical invariant theory but Kuroda's result in case the f_i 's are not linear.

Anyway, using Kuroda's result, Drensky [10] proved the next:

Theorem 4.2 *Let Δ be an elementary derivation of $D[Y_m]$ such that $\Delta(y_i) = f_i = f_i(x_i)$, for any i , where the f_i 's are nonconstant polynomials. Then $D[Y_m]^\Delta$ is generated by X_m and the determinants.*

Notice also that in all of the result above, the determinants generating the algebra of constants are not free generators. A prominent part of those works deals with finding a free set of generators.

5. The free metabelian associative algebras

We will focus on another generalization of Nowicki's conjecture. The environment now is that of relatively free algebras. The complete proofs of the results presented in the next two section can be found in the paper [4].

We start off our investigation with the relatively free algebra of the associative metabelian algebra. Let K be a field of characteristic zero, P_{2d} be the free unitary associative algebra of rank $2d$ over K , $P'_{2d} = P_{2d}[P_{2d}, P_{2d}]P_{2d}$ be its commutator ideal generated by all elements of the form

$$[x, y] = xy - yx, \quad x, y \in P_{2d}.$$

Let us consider the quotient algebra $F_{2d} = P_{2d}/(P'_{2d})^2$. The algebra F_{2d} is the free algebra of rank $2d$ in the variety of all associative algebras satisfying the polynomial identity $[x, y][z, t] \equiv 0$. Let the free associative metabelian algebra F_{2d} be generated by $X_{2d} = \{x_1, \dots, x_{2d}\}$. We assume that all Lie commutators are left normed; i.e.

$$[x, y, z] = [[x, y], z], \quad x, y, z \in F_{2d}.$$

The commutator ideal F'_{2d} of F_{2d} has the following basis (see e.g., [2, 8])

$$x_1^{\xi_1} \cdots x_{2d}^{\xi_{2d}} [x_i, x_j, x_{j_1}, \dots, x_{j_m}], \quad \xi_i \geq 0, \quad i > j \leq j_1 \leq \dots \leq j_m \leq 2d.$$

As a consequence of the metabelian identity in F'_{2d} we have the following identity:

$$x_{i_{\pi(1)}} \cdots x_{i_{\pi(m)}} [x_i, x_j, x_{j_{\sigma(1)}}, \dots, x_{j_{\sigma(n)}}] \equiv x_{i_1} \cdots x_{i_m} [x_i, x_j, x_{j_1}, \dots, x_{j_n}],$$

for any permutation $\pi \in S_m$ and $\sigma \in S_n$. Thus the commutator ideal F'_{2d} can be "seen" as a module of polynomial algebra from both sides via the associative (left side) and Lie (right side) multiplication.

We recall now some of the results and constructions given in [6]. Let $U_{2d} = \{u_1, \dots, u_{2d}\}$ and $V_{2d} = \{v_1, \dots, v_{2d}\}$ be two sets of commuting variables and let $K[U_{2d}, V_{2d}]$ be the polynomial algebra acting on F'_{2d} as follows. If $f \in F'_{2d}$, then

$$fu_i = x_i f, \quad fv_i = [f, x_i], \quad i = 1, \dots, 2d.$$

This action defines a $K[U_{2d}, V_{2d}]$ -module structure on the vector space F'_{2d} .

We are going to construct a wreath product which is the same as the one used in [6]. It is a particular case of the construction of Lewin [19] given in [13] and is similar to the construction of Shmel'kin [21] in the case of free metabelian Lie algebras as appeared in [5].

Let $Y_{2d} = \{y_1, \dots, y_{2d}\}$ and $V'_{2d} = \{v'_1, \dots, v'_{2d}\}$ be sets of commuting variables and let $A_{2d} = \{a_1, \dots, a_{2d}\}$ in $2d$ variables. Now let M_{2d} be the free $K[U_{2d}, V'_{2d}]$ -module generated by A_{2d} equipped with trivial multiplication $M_{2d} \cdot M_{2d} = 0$. We endow M_{2d} with a structure of a free $K[Y_{2d}]$ -bimodule structure via the action

$$y_j a_i = a_i u_j, \quad a_i y_j = a_i v'_j, \quad i, j = 1, \dots, 2d.$$

The wreath product $W_{2d} = K[Y_{2d}] \ltimes M_{2d}$ is an algebra satisfying the metabelian identity. As well as in [19] F_{2d} can be embedded into W_{2d} . In fact we have the following result.

Proposition 5.1 *The mapping $\varepsilon : x_j \rightarrow y_j + a_j, j = 1, \dots, 2d$, extends to an embedding ε of F_{2d} into W_{2d} .*

We identify $v_i = v'_i - u_i, i = 1, \dots, 2d$, and get

$$\varepsilon(x_{i_1} \cdots x_{i_m} [x_i, x_j, x_{j_1}, \dots, x_{j_n}]) = (a_i v_j - a_j v_i) v_{j_1} \cdots v_{j_n} u_{i_1} \cdots u_{i_m}. \tag{5.1}$$

Thus we may assume that M_{2d} is a free $K[U_{2d}, V_{2d}]$ -module. Clearly the commutator ideal F'_{2d} is embedded into M_{2d} , too. An element $\sum a_i f(U_{2d}, V_{2d}) \in M_{2d}$ is an image of some element from F'_{2d} if and only if $\sum v_i f(U_{2d}, V_{2d}) = 0$, as a consequence of (5.1).

Let δ be the Weitzenböck derivation of F_{2d} acting on the variables U_{2d}, V_{2d} , as well as explained in Introduction on X_{2d} . By [6] we know that the vector space M_{2d}^δ of the constants of δ in the $K[U_{2d}, V_{2d}]$ -module M_{2d} is a $K[U_{2d}, V_{2d}]^\delta$ -module. The following results are particular cases of [9] (Proposition 3) and [6].

Theorem 5.2 *The vector spaces $(F'_{2d})^\delta$ and M_{2d}^δ are finitely generated $K[U_{2d}, V_{2d}]^\delta$ -modules.*

In the sequel we shall write down an explicit set of generators for the algebra of constants $K[U_{2d}, V_{2d}]^\delta$ as a consequence of the Nowicki conjecture [20] (proved in [3, 14–16, 18]), and one of the results by Drensky and Makar–Limanov [14]. Let the Weitzenböck derivation δ act on $X_{2d}, A_{2d}, U_{2d}, V_{2d}$ by the rule

$$\delta(x_{2i}) = x_{2i-1}, \delta(x_{2i-1}) = 0, \quad \delta(a_{2i}) = a_{2i-1}, \delta(a_{2i-1}) = 0,$$

$$\delta(u_{2i}) = u_{2i-1}, \delta(u_{2i-1}) = 0, \quad \delta(v_{2i}) = v_{2i-1}, \delta(v_{2i-1}) = 0,$$

for $i = 1, \dots, d$. Then the algebra of constants $K[U_{2d}, V_{2d}]^\delta$ is generated by

$$u_1, u_3, \dots, u_{2d-1}, v_1, v_3, \dots, v_{2d-1}$$

and the determinants

$$\alpha_{pq} = u_{2p-1} u_{2q} - u_{2p} u_{2q-1} = \begin{vmatrix} u_{2p-1} & u_{2p} \\ u_{2q-1} & u_{2q} \end{vmatrix}, \quad 1 \leq p < q \leq d,$$

$$\beta_{pq} = v_{2p-1}v_{2q} - v_{2p}v_{2q-1} = \begin{vmatrix} v_{2p-1} & v_{2p} \\ v_{2q-1} & v_{2q} \end{vmatrix}, \quad 1 \leq p < q \leq d,$$

$$\gamma_{pq} = u_{2p-1}v_{2q} - u_{2p}v_{2q-1} = \begin{vmatrix} u_{2p-1} & u_{2p} \\ v_{2q-1} & v_{2q} \end{vmatrix}, \quad p, q = 1, \dots, d,$$

with the following defining relations

$$u_{2i-1}\alpha_{jk} - u_{2j-1}\alpha_{ik} + u_{2k-1}\alpha_{ij} = 0, \quad 1 \leq i < j < k \leq d, \tag{5.2}$$

$$u_{2i-1}\gamma_{jk} - u_{2j-1}\gamma_{ik} + v_{2k-1}\alpha_{ij} = 0, \quad 1 \leq i < j \leq d, 1 \leq k \leq d, \tag{5.3}$$

$$u_{2i-1}\beta_{jk} - v_{2j-1}\gamma_{ik} + v_{2k-1}\gamma_{ij} = 0, \quad 1 \leq i \leq d, 1 \leq j < k \leq d, \tag{5.4}$$

$$v_{2i-1}\beta_{jk} - v_{2j-1}\beta_{ik} + v_{2k-1}\beta_{ij} = 0, \quad 1 \leq i < j < k \leq d, \tag{5.5}$$

$$\alpha_{ij}\alpha_{kl} - \alpha_{ik}\alpha_{jl} + \alpha_{il}\alpha_{jk} = 0, \quad 1 \leq i < j < k < l \leq d, \tag{5.6}$$

$$\alpha_{ij}\gamma_{kl} - \alpha_{ik}\gamma_{jl} + \gamma_{il}\alpha_{jk} = 0, \quad 1 \leq i < j < k \leq d, 1 \leq l \leq d, \tag{5.7}$$

$$\alpha_{ij}\beta_{kl} - \gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} = 0, \quad 1 \leq i < j \leq d, 1 \leq k < l \leq d, \tag{5.8}$$

$$\gamma_{ij}\beta_{kl} - \gamma_{ik}\beta_{jl} + \gamma_{il}\beta_{jk} = 0, \quad 1 \leq i \leq d, 1 \leq j < k < l \leq d, \tag{5.9}$$

$$\beta_{ij}\beta_{kl} - \beta_{ik}\beta_{jl} + \beta_{il}\beta_{jk} = 0, \quad 1 \leq i < j < k < l \leq d. \tag{5.10}$$

The vector space $K[U_{2d}, V_{2d}]^\delta$ has a canonical linear basis consisting of the elements of the form

$$v_{2i_1-1} \cdots v_{2i_m-1} \beta_{p_1 q_1} \cdots \beta_{p_r q_r} \gamma_{p'_1 q'_1} \cdots \gamma_{p'_s q'_s} \alpha_{p''_1 q''_1} \cdots \alpha_{p''_t q''_t} u_{2j_1-1} \cdots u_{2j_n-1} \tag{5.11}$$

such that among the generators β_{pq} , $\gamma_{p'q'}$, and $\alpha_{p''q''}$ there is no intersection, and no one covers v_{2i_k-1} or u_{2j_l-1} .

Note that each β_{pq} , $\gamma_{p'q'}$, and $\alpha_{p''q''}$ is identified with the open interval $(p + d, q + d)$, $(p', q' + d)$, and (p'', q'') , respectively, on the real line. The generators intersect each other if the corresponding open intervals have a nonempty intersection and are not contained in each other. On the other hand, the generators v_{2i-1} and u_{2j-1} are identified with the points $i + d$ and j , respectively. We say also that a generator among β_{pq} , $\gamma_{p'q'}$, or $\alpha_{p''q''}$ covers v_{2i-1} or u_{2j-1} if the corresponding open interval covers the corresponding point.

The pairs of indices are ordered in the following way: $p_1 \leq \cdots \leq p_r$ and if $p_\xi = p_{\xi+1}$, then $q_\xi \leq q_{\xi+1}$; $p'_1 \leq \cdots \leq p'_s$ and if $p'_\mu = p'_{\mu+1}$, then $q'_\mu \leq q'_{\mu+1}$; $p''_1 \leq \cdots \leq p''_t$ and if $p''_\sigma = p''_{\sigma+1}$, then $q''_\sigma \leq q''_{\sigma+1}$; and $i_1 \leq \cdots \leq i_m$, $j_1 \leq \cdots \leq j_n$.

In order to detect the constants in the commutator ideal F'_{2d} it is sufficient to work in the $K[U_{2d}, V_{2d}]^\delta$ -submodule C_{2d}^δ of M_{2d}^δ , which is generated by

$$a_1, a_3, \dots, a_{2d-1}$$

and the determinants

$$w_{pq} = a_{2p-1}v_{2q} - a_{2q}v_{2p-1} = \begin{vmatrix} a_{2p-1} & a_{2q} \\ v_{2p-1} & v_{2q} \end{vmatrix}, \quad p, q = 1, \dots, d.$$

and spanned as a vector space on the elements of the form

$$a_{2i_0-1}v_{2i_1-1} \cdots v_{2i_m-1}\beta_{p_1q_1} \cdots \beta_{p_rq_r}\gamma_{p'_1q'_1} \cdots \gamma_{p'_sq'_s}\alpha_{p''_1q''_1} \cdots \alpha_{p''_tq''_t}u_{2j_1-1} \cdots u_{2j_n-1} \tag{5.12}$$

$$w_{p_0q_0}v_{2i_1-1} \cdots v_{2i_m-1}\beta_{p_1q_1} \cdots \beta_{p_rq_r}\gamma_{p'_1q'_1} \cdots \gamma_{p'_sq'_s}\alpha_{p''_1q''_1} \cdots \alpha_{p''_tq''_t}u_{2j_1-1} \cdots u_{2j_n-1} \tag{5.13}$$

for each $i_0, p_0, q_0 = 1 \dots, d$.

We also have the following relations in the algebra C_{2d}^δ as a consequence of [14].

$$a_{2i-1}\beta_{jk} - w_{ik}v_{2j-1} + w_{ij}v_{2j-1} = 0, \quad 1 \leq i \leq d, 1 \leq j < k \leq d, \tag{5.14}$$

$$w_{ij}\beta_{kl} - w_{ik}\beta_{jl} + w_{il}\beta_{jk} = 0, \quad 1 \leq i \leq d, 1 \leq j < k < l \leq d. \tag{5.15}$$

We denote by L the $K[U_{2d}, V_{2d}]^\delta$ -submodule of C_{2d}^δ generated by the following elements

$$w_{ii}, \quad 1 \leq i \leq d, \tag{5.16}$$

$$w_{ij} + w_{ji}, \quad 1 \leq i < j \leq d, \tag{5.17}$$

$$a_{2i-1}v_{2j-1} - a_{2j-1}v_{2i-1}, \quad 1 \leq i < j \leq d, \tag{5.18}$$

$$a_{2i-1}\beta_{pq} - w_{pq}v_{2i-1}, \quad 1 \leq i \leq d, 1 \leq p < q \leq d, \tag{5.19}$$

$$w_{ij}\beta_{pq} - w_{pq}\beta_{ij}, \quad 1 \leq i < j \leq d, 1 \leq p < q \leq d, \tag{5.20}$$

$$a_{2i-1}\beta_{jk} - a_{2j-1}\beta_{ik} + a_{2k-1}\beta_{ij}, \quad 1 \leq i < j < k \leq d, \tag{5.21}$$

$$w_{kl}\gamma_{ij} - w_{jl}\gamma_{ik} + w_{jk}\gamma_{il}, \quad 1 \leq i \leq d, 1 \leq j < k < l \leq d. \tag{5.22}$$

$$w_{jk}u_{2i-1} - a_{2j-1}\gamma_{ik} + a_{2k-1}\gamma_{ij}, \quad 1 \leq i \leq d, 1 \leq j < k \leq d. \tag{5.23}$$

One can observe that the generating elements (5.16)–(5.23) of L are the images of some elements in the commutator ideal F'_{2d} of the free associative algebra F_{2d} .

Theorem 5.3 *The $K[U_{2d}, V_{2d}]^\delta$ -submodule L of C_{2d}^δ consists of all commutator elements in C_{2d}^δ ; i.e. the images in $(F'_{2d})^\delta \subset F'_{2d}$.*

As a consequence of the previous result it is possible to get the full list of generators of $(F'_{2d})^\delta$ as a $K[U_{2d}, V_{2d}]$ -module.

Corollary 5.4 *The $K[U_{2d}, V_{2d}]^\delta$ -module $(F'_{2d})^\delta$ is generated by the following polynomials:*

$$\begin{aligned}
 g_1(i) &= [x_{2i-1}, x_{2i}], & 1 \leq i \leq d, \\
 g_2(i, j) &= [x_{2i-1}, x_{2j-1}], & 1 \leq i < j \leq d, \\
 g_3(i, j) &= [x_{2i-1}, x_{2j}] + [x_{2j-1}, x_{2i}], & 1 \leq i < j \leq d, \\
 g_4(i, p, q) &= [x_{2i-1}, x_{2p-1}, x_{2q}] - [x_{2i-1}, x_{2p}, x_{2q-1}], & 1 \leq i \leq d, 1 \leq p < q \leq d, \\
 g_5(i, j, k) &= [x_{2i-1}, x_{2j-1}, x_{2k}] - [x_{2i-1}, x_{2k-1}, x_{2j}] + [x_{2j-1}, x_{2k-1}, x_{2i}], \\
 & & 1 \leq i < j < k \leq d, \\
 g_6(i, j, p, q) &= [x_{2i-1}, x_{2p-1}, x_{2j}, x_{2q}] + [x_{2i}, x_{2p}, x_{2j-1}, x_{2q-1}] \\
 & - [x_{2i-1}, x_{2p}, x_{2j}, x_{2q-1}] - [x_{2i}, x_{2p-1}, x_{2j-1}, x_{2q}], \\
 & & 1 \leq i < j \leq d, 1 \leq p < q \leq d, \\
 g_7(i, j, k, l) &= x_{2i}[x_{2j-1}, x_{2k-1}, x_{2l}] + x_{2i-1}[x_{2j}, x_{2k}, x_{2l-1}] \\
 & - x_{2i}[x_{2j-1}, x_{2k}, x_{2l-1}] - x_{2i-1}[x_{2j}, x_{2k-1}, x_{2l}] \\
 & & 1 \leq i \leq d, 1 \leq j < k < l \leq d, \\
 g_8(i, j, k) &= x_{2i}[x_{2j-1}, x_{2k-1}] - x_{2i-1}[x_{2j}, x_{2k-1}], \\
 & & 1 \leq i \leq d, 1 \leq j < k \leq d.
 \end{aligned}$$

We have to add for the generating set of the whole algebra $(F_{2d})^\delta$ the constants x_{2i-1} and $x_{2i-1}x_{2j} - x_{2i}x_{2j-1}$, $1 \leq i < j \leq d$, which are needed for the generation of the factor algebra of $(F_{2d})^\delta$ modulo the commutator ideal of F_{2d} . These generators are the ones lifted from the algebra $K[X_{2d}]^\delta$ of constants of the polynomial algebra to the algebra $(F_{2d})^\delta$ by the fact stated in Corollary 4.3 of the paper [13] by Drensky and Gupta.

The following result gives an infinite generating set of the subalgebra of constants $(F_{2d})^\delta$ of the free metabelian associative algebra F_{2d} as an algebra.

Corollary 5.5 *The algebra $(F_{2d})^\delta$ of the constants is generated by*

$$\begin{aligned}
 &x_1, x_3, \dots, x_{2d-1}, \\
 &x_{2i-1}x_{2j} - x_{2i}x_{2j-1}, \\
 &g_1f_1, \dots, g_8f_8,
 \end{aligned}$$

where $1 \leq i < j \leq d$, $f_1, \dots, f_8 \in K[U_{2d}, V_{2d}]^\delta$, and g_i s are as in Corollary 5.4.

6. The variety \mathcal{G} generated by the Grassmann algebra

In this section we shall show results concerning Nowicki’s conjecture of the relatively free algebra of the infinite dimensional Grassmann algebra.

The variety \mathcal{G} consists of all associative unitary algebras satisfying the polynomial identity $[z_1, z_2, z_3] \equiv 0$. We shall set $F_{2l} := F_{2l}(\mathcal{G})$ for $0 \leq l \leq d$. As noted in Introduction, \mathcal{G} is the variety generated by the infinite dimensional Grassmann algebra G and F_{2d} coincides with the $2d$ -generated relatively free algebra in the variety \mathcal{G} . The identities of G and related topics have been studied by several authors. See for example the paper [17] by Krakovski and Regev about the ideal of polynomial identities of G .

Let $X = \{x_1, x_2, \dots, x_d\}$, $Y = \{y_1, y_2, \dots, y_d\}$ be two disjoint sets of variables. Of course the relatively free algebra of \mathfrak{G} of rank $2d$ in the variables from $U := X \cup Y$ is isomorphic to F_{2d} and we consider the following order inside U :

$$x_1 < y_1 < x_2 < \dots < x_d < y_d.$$

Moreover we say a polynomial f is homogeneous in the set of variables $S = \{u_1, \dots, u_s\}$ if for each monomial m appearing in f we have $\sum_{i=1}^s \deg_{u_i} m$ is the same.

It is well known (see for example Theorem 5.1.2 of [8]) F_{2d} has a basis consisting of all

$$x_1^{a_1} y_1^{b_1} \dots x_d^{a_d} y_d^{b_d} [u_{i_1}, u_{i_2}] \dots [u_{i_{2c-1}}, u_{i_{2c}}],$$

$$a_i, b_j \geq 0, u_{i_l} \in U, u_{i_1} < u_{i_2} < \dots < u_{i_{2c}}, c \geq 0.$$

We recall the identity $[z_1, z_2, z_3] \equiv 0$ implies the identity

$$[z_1, z_2][z_3, z_4] \equiv -[z_1, z_3][z_2, z_4] \tag{6.1}$$

in F_{2d} .

Consider the following Weitzenböck derivation δ of F_{2d} acting on U such that

$$\delta(y_i) = x_i, \delta(x_i) = 0, \quad 1 \leq i \leq d.$$

Let $\alpha = (\alpha_1, \dots, \alpha_{d-1}) \in K^{d-1}$ and consider the algebra endomorphism φ_α of F_{2d} such that

$$\varphi_\alpha(x_i) = x_i, \varphi_\alpha(y_i) = y_i, \quad i = 1, \dots, d-1,$$

$$\varphi_\alpha(x_d) = \sum_{i=1}^{d-1} \alpha_i x_i, \quad \varphi_\alpha(y_d) = \sum_{i=1}^{d-1} \alpha_i y_i.$$

Notice that φ_α commutes with δ . Hence if $f \in F_{2d}^\delta$, then $\varphi_\alpha(f) \in F_{2d-2}^\delta$ too. Moreover, if $f \in F_{2d}$ is homogeneous with respect to the set $\{x_d, y_d\}$ and $\varphi_\alpha(f) = 0$ for some nonzero $\alpha \in K^{d-1}$, then f is in the left ideal generated by

$$\omega_\alpha := \left(\sum_{i=1}^{d-1} \alpha_i x_i \right) y_d - \left(\sum_{i=1}^{d-1} \alpha_i y_i \right) x_d, \quad [\omega_\alpha, u], \quad u \in U,$$

$$\mu_\alpha := \left[x_d, \sum_{i=1}^{d-1} \alpha_i x_i \right], \quad \nu_\alpha := \left[y_d, \sum_{i=1}^{d-1} \alpha_i y_i \right].$$

We define the following objects inside F_{2d} :

$$v_{ij} := x_i y_j - y_i x_j, \quad 1 \leq i, j \leq d,$$

$$w_{ijk} := y_i[x_j, x_k] - x_i[y_j, x_k], \quad 1 \leq i, j, k \leq d,$$

$$z_{ijkl} := y_i[x_j, v_{k,l}] - x_i[y_j, v_{k,l}], \quad 1 \leq i, j, k, l \leq d,$$

Notice that the v_{ij} 's, the w_{ijk} 's and the z_{ijkl} 's are constants of F_{2d} and starting from these objects we shall construct three subsets of elements of F_{2d} . We set

$$V := \{v_{ij} | 1 \leq i, j \leq d\},$$

$$W_0 := \{w_{ijk} | 1 \leq i, j \leq k \leq d\}$$

and

$$Z_0 := \{z_{ijkl} | 1 \leq i \leq j \leq k \leq l \leq d\}.$$

Suppose $s > 0$ and we set

$$W_s := \{y[x, w] - x[y, w] | w \in W_{s-1}, x \in X, y \in Y\},$$

$$Z_s := \{y[x, z] - x[y, z] | z \in Z_{s-1}, x \in X, y \in Y\}.$$

As above, both the W_s 's and the Z_s 's are subsets of constants. Moreover it can be easily seen for $s \geq d$ both V_s and W_s are 0. We shall denote by \mathcal{C} the algebra generated by the nonzero X, V, W_l, Z_l . Here we have the main result. We recall the proof of the following follows the main steps of the proof of the classical Nowicki's conjecture presented in Section 3. We also point out this proof cannot be adapted to algebras having a non- T -prime ideal of their polynomial identities.

Theorem 6.1 *The algebra of constants F_{2d}^δ is finitely generated as an algebra, and its generators are:*

$$X; \tag{6.2}$$

$$V, \tag{6.3}$$

$$W_l, \quad l \leq d - 1; \tag{6.4}$$

$$Z_l, \quad l \leq d - 1. \tag{6.5}$$

Proof It is easy to see that the elements from (6.2)–(6.5) are constants of F_{2d} .

Let $f = f(X', Y', x_d, y_d) \in F_{2d}^\delta$, then we may assume f being homogeneous in U and in $\{x_d, y_d\}$. We shall prove the theorem by induction on d and on the total degree with respect to the set $\{x_d, y_d\}$. Suppose $d = 1$, then

$$f = \sum_{k=0}^n \alpha_k x^k y^{n-k} + \sum_{j=1}^{n-1} \beta_j x^{j-1} y^{n-j-1} [x, y].$$

This means

$$0 = \delta(f) = \sum_{k=0}^{n-1} \alpha_k(n-k)x^{k+1}y^{n-k-1} + [x, y] \sum_{k=1}^{n-2} (n-k-1) \left(\alpha_k \frac{n-k}{2} x^{k-1} y^{n-k-1} + \beta_k x^k y^{n-k-2} \right).$$

The previous relation gives us $\alpha_k = 0$ for $k \leq n-1$, then $\beta_j = 0$ for $1 \leq j \leq n-2$. Thus the only possibility is $f = \alpha_n x^n + \beta_{n-1} x^{n-2} [x, y]$ and we are done.

Assume now $d > 1$ and the result true for F_{2d-2} . We set $U' = X' \cup Y'$. If $v \in F_{2d-2}^\delta$ is homogeneous in U , then we shall study only the case $\deg_{x_d, y_d}(f) > 0$.

Remark we can write f in the following way:

$$f = a_p y_d^p + \sum_{u_0} a'_{p, u_0} [u_0, y_d] y_d^{p-1} + a_{p-1} x_d y_d^{p-1} + \sum_{u_1^1} a'_{p-1, u_1^1} [u_1^1, x_d] y_d^{p-1} + \sum_{u_1^2} a''_{p-1, u_1^2} [u_1^2, y_d] x_d y_d^{p-2} + a'''_{p-1} [x_d, y_d] y_d^{p-2} + \dots + a_0 x_d^p + \sum_{u_p} a'_0 [u_p, x_d] x_d^{p-1},$$

where the u_j^i 's are in U whereas the a_j 's, the a'_{r, u_j^i} 's, the a''_{s, u_j^i} and the a'''_{t, u_j^i} 's all belong to F_{2d-2} . Deriving f we get $\delta(a_p) = 0$, hence:

$$a_p \in F_{2d}^\delta. \tag{6.6}$$

We also have $\frac{p(p-1)}{2} a_p + \delta(a'''_{p-1}) = 0$, then by (6.6) we get $\delta(a'''_{p-1}) = 0$. This means:

$$a'''_{p-1} \in F_{2d}^\delta. \tag{6.7}$$

Suppose now $u_0 = y_s$, $s \leq d-1$, then $a'_{p, u_0} = 0$. So we can assume $u_0 = x_s$, $s \leq d-1$. In this case we get $\delta(a'_{p, x_s}) = 0$, i.e.:

$$a'_{p, x_s} \in F_{2d}^\delta. \tag{6.8}$$

By (6.6), (6.7) and (6.8), using induction, we get

$$a_p = \sum b_1(X', V', W', Z'),$$

$$a'''_{p-1} = \sum b_2(X', V', W', Z'),$$

$$a'_{p, x_i} = \sum b_{3,i}(X', V', W', Z').$$

If $\varphi_\alpha(f) = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_{d-1})$, then we get $d = 2$ and f is in the left ideal generated by v_{12} and $[v_{12}, u]$, $u \in U$. Hence $f = g v_{12} + \sum_{u \in U} g_u [v_{12}, u]$ and we apply inductive arguments to g and g_u 's and we are done. Now we consider the case $\varphi_\alpha(f) \neq 0$ for some non-zero $\alpha \in K^{n-1}$. The next two remarks are crucial. First we have

$$\deg_X(f) = \deg_{X'}(\varphi_\alpha(f)) \geq \deg_{Y'}(\varphi_\alpha(f)) = \deg_Y(f).$$

Hence

$$\deg_X(f) = \deg_{X'}(a_p) \geq \deg_{Y'}(f) = p + \deg_{Y'}(a_p)$$

so

$$a_p = \sum b'_1(X', V', W', Z')x_{k_1} \cdots x_{k_p}.$$

We can argue analogously and obtain

$$a'''_{p-1} = \sum b'_2(X', V', W', Z')x_{l_1} \cdots x_{l_p},$$

$$a'_{p,x_i} = \sum b_{3,i}(X', V', W', Z')x_{m_1} \cdots x_{m_p}.$$

Now we rewrite $a_p y_d^p$, $a'''_{p-1}[x_d, y_d]y_d^{p-2}$ and $a'_{p,x_i}[x_i, y_d]y_d^{p-1}$ and we get

$$f = c + g x_d + \sum_{i=1}^{d-1} m_{p,i} y_d [x_i, x_d],$$

where $c \in \mathcal{C}$. We can rewrite the last summand as $g_1 x_d + \sum_{i=1}^{d-1} h_i [x_i, x_d]$, where $\deg_{x_d} h_i = 0$ and we get finally

$$f = c + g' x_d + \sum_{i=1}^{d-1} h_i [x_i, x_d].$$

Hence g' and the h_i 's are constants too because x_d and $[x_i, x_d]$ are. Now we are allowed to apply induction on g' and the h_i 's because their degrees with respect to the set $\{x_d, y_d\}$ are strictly smaller than the one of f and we conclude the proof. \square

7. The Nowicki conjecture for the free metabelian Poisson algebras

This section presents a new result and is devoted to the study of Nowicki's conjecture in a nonassociative setting. In particular, we will handle the case of a free metabelian Poisson algebra and we shall give an explicit set of basis elements of the vector space of its constants.

Let K be a field of characteristic zero. Let (P, \cdot) and $(P, [,])$ be a commutative associative algebra and a Lie algebra structure on the K -vector space P , respectively, over K such that the identity

$$[a, b \cdot c] = [a, b] \cdot c + b \cdot [a, c] = [a, b] \cdot c + [a, c] \cdot b$$

holds in P . Then $(P, \cdot, [,])$ is called a Poisson algebra.

We will consider throughout the section only left normed commutators, i.e. $[a, b, c] := [[a, b], c]$. If $a \cdot b = [a, b] = 0$ holds in P , then the Poisson algebra P is called abelian and any extension of P by an abelian Poisson algebra is called metabelian (see e.g., [1]).

Let us consider the free metabelian Poisson algebra P_{2n} over K of rank $2n$ generated by the set

$$Z_{2n} = \{z_1, \dots, z_{2n}\} = \{x_1, \dots, x_n, y_1, \dots, y_n\} = X_n \cup Y_n.$$

It is well known (see e.g., [24]) that the following identities hold in P_{2n} :

$$\begin{aligned} 0 &= a \cdot b \cdot c \cdot d = [a \cdot b, c \cdot d] = [[a, b], c \cdot d] \\ &= [a, b] \cdot c \cdot d = [a, b] \cdot [c, d] = [[a, b], [c, d]], \end{aligned}$$

which imply the following ones:

$$\begin{aligned} [a, b, c] &= [a, c, b] - [b, c, a] \\ [a, b, c] \cdot d &= [d, b, c] \cdot a - [c, b, d] \cdot a \\ [a, b] + [b, a] &= 0, \quad [a, b, c, d] = [a, b, d, c] \\ [a, b, c] \cdot d &= -[a, b, d] \cdot c, \quad [a, b, c] \cdot c = 0. \end{aligned}$$

We put an order " $<$ " in the generating set so that

$$z_1 < \dots < z_{2n}$$

or

$$x_1 < \dots < x_n < y_1 < \dots < y_n.$$

It is well known (see [24]) $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ forms a K -basis for the free metabelian Poisson algebra P_{2n} , where

$$\begin{aligned} \mathcal{B}_1 &= Z_{2n} \cup \{[z_{i_1}, \dots, z_{i_k}] \mid i_1 > i_2 \leq i_3 \leq \dots \leq i_k, 2 \leq k\}, \\ \mathcal{B}_2 &= \{z_i \cdot z_j, z_i \cdot z_j \cdot z_k \mid 1 \leq i \leq j \leq k \leq 2n\}, \\ \mathcal{B}_3 &= \{[z_i, z_j] \cdot z_k \mid 1 \leq j \leq i \leq 2n, 1 \leq k \leq 2n\}, \\ \mathcal{B}_4 &= \{[z_i, z_j, z_k] \cdot z_l \mid 1 \leq j < i \leq l \leq 2n, 1 \leq j \leq k < l \leq 2n\}. \end{aligned}$$

Now consider the linear nilpotent derivation $\delta : y_i \rightarrow x_i \rightarrow 0, i = 1, \dots, n$. Clearly the spaces $K\mathcal{B}_i, i = 1, 2, 3, 4$, are δ -invariant, i.e. $(K\mathcal{B}_i)^\delta \subset K\mathcal{B}_i$, then

$$P_{2n}^\delta = (K\mathcal{B}_1)^\delta \oplus (K\mathcal{B}_2)^\delta \oplus (K\mathcal{B}_3)^\delta \oplus (K\mathcal{B}_4)^\delta.$$

Note that $K\mathcal{B}_1 = L_{2n}$ is the free metabelian Lie algebra; moreover, by [12], we know the generators of the algebra $(L'_{2n})^\delta$ as a $K[X_n, Y_n]^\delta$ -module:

Theorem 7.1 *The $K[X_n, Y_n]^\delta$ -module $(L'_{2n})^\delta$ is generated by the following elements*

$$\begin{aligned} [x_i, y_i], \quad & 1 \leq i \leq n, \\ [x_i, x_j], \quad & 1 \leq i < j \leq n, \\ [x_i, y_j] + [x_j, y_i], \quad & 1 \leq i < j \leq n, \\ [x_i, x_p, y_q] - [x_i, y_p, x_q], \quad & 1 \leq i \leq n, 1 \leq p < q \leq n, \\ [x_i, x_j, y_k] - [x_i, x_k, y_j] + [x_j, x_k, y_i], \quad & 1 \leq i < j < k \leq n, \end{aligned}$$

and

$$[x_i, x_p, y_j, y_q] + [y_i, y_p, x_j, x_q] - [x_i, y_p, y_j, x_q] - [y_i, x_p, x_j, y_q],$$

where $1 \leq i < j \leq n, 1 \leq p < q \leq n$.

By Nowicki's conjecture we get $(KB_2)^\delta$ is spanned by the elements of the form

$$u_{i,j} = x_i \cdot y_j - x_j \cdot y_i, \quad 1 \leq i < j < k \leq n$$

and

$$u_{i,j} \cdot x_k, \quad 1 \leq i < j < k \leq n, \quad 1 \leq k \leq n$$

with the following relations:

$$x_i \cdot u_{j,k} - x_j \cdot u_{i,k} + x_k \cdot u_{i,j} = 0, \quad 1 \leq i < j < k \leq n.$$

We want to highlight the fact that once again some "determinants" (the $u_{i,j}$ above) appear as generators of the algebra of constants as in the classical Nowicki's conjecture and its generalization.

In the sequel we are going to show an explicit basis of KB_3^δ and KB_4^δ as vector spaces. Then the explicit basis of P_{2n}^δ will be an obvious corollary.

Theorem 7.2 *The K -vector space $(KB_3)^\delta$ is of a basis consisting of elements of the form*

$$[x_i, x_j] \cdot x_k, \quad j < i,$$

$$[y_i, x_i] \cdot x_j, \quad i, j,$$

$$[x_i, x_j] \cdot y_k + [y_j, x_i] \cdot x_k, \quad j < i < k,$$

$$[y_i, x_j] \cdot x_k + [y_j, x_i] \cdot x_k, \quad j < i < k.$$

Proof Let us set

$$A_1 = \{[x_i, x_j] \cdot x_k \mid 1 \leq j < i \leq n, \quad 1 \leq k \leq n\},$$

$$A_2 = \{[y_i, y_j] \cdot y_k \mid 1 \leq j < i \leq n, \quad 1 \leq k \leq n\},$$

$$A_3 = \{[x_i, x_j] \cdot y_k \mid 1 \leq j < i \leq n, \quad 1 \leq k \leq n\},$$

$$A_4 = \{[y_i, y_j] \cdot x_k \mid 1 \leq j < i \leq n, \quad 1 \leq k \leq n\},$$

$$A_5 = \{[y_i, x_j] \cdot y_k \mid 1 \leq i, j, k \leq n\},$$

$$A_6 = \{[y_i, x_j] \cdot x_k \mid 1 \leq i, j, k \leq n\}.$$

Indeed $KB_3 = KA_1 \oplus KA_2 \oplus KA_3 \oplus KA_4 \oplus KA_5 \oplus KA_6$, while $(KA_1)^\delta = KA_1$,

$$(KA_2)^\delta \subset KA_4 \oplus KA_5, \quad (KA_3)^\delta \subset KA_1,$$

$$(KA_4)^\delta \subset KA_6, \quad (KA_5)^\delta \subset KA_3 \oplus KA_6, \quad (KA_6)^\delta \subset KA_1.$$

Thus

$$(KB_3)^\delta = KA_1 \oplus (KA_2)^\delta \oplus (KA_3 \oplus KA_6)^\delta \oplus (KA_4 \oplus KA_5)^\delta.$$

Now let

$$p_2 = \sum_{j < i} \alpha_{ijk} [y_i, y_j] \cdot y_k$$

be an element in $(KA_2)^\delta$, then $\delta(p_2) = 0$ and

$$\begin{aligned} 0 &= \sum_{j < i} \alpha_{ijk}([x_i, y_j] \cdot y_k + [y_i, x_j] \cdot y_k + [y_i, y_j] \cdot x_k) \\ &= \sum_{j < i} \alpha_{ijk}(-[y_j, x_i] \cdot y_k + [y_i, x_j] \cdot y_k) + \sum_{j < i} \alpha_{ijk}[y_i, y_j] \cdot x_k. \end{aligned}$$

Notice that $\sum_{j < i} \alpha_{ijk}[y_i, y_j] \cdot x_k$ belongs to KA_4 , then

$$0 = \sum_{j < i} \alpha_{ijk}[y_i, y_j] \cdot x_k.$$

We immediately get any α_{ijk} is 0 and p_2 turns out to be 0, too. This proves that there is no nonzero constant in KA_2 .

Suppose now $p_3 + p_6 \in (KA_3 \oplus KA_6)^\delta$ for some $p_3 \in KA_3$ and $p_6 \in KA_6$. First observe that elements of the form $[y_i, x_i] \cdot x_k$ are constants in KA_6 . Then we may assume

$$p_3 = \sum_{j < i} \alpha_{ijk}[x_i, x_j] \cdot y_k \in KA_3 \quad \text{and} \quad p_6 = \sum_{i \neq j} \beta_{ijk}[y_i, x_j] \cdot x_k \in KA_6.$$

Hence

$$\begin{aligned} \delta(p_3 + p_6) = 0 &= \sum_{j < i} \alpha_{ijk}[x_i, x_j] \cdot x_k + \sum_{j < i} \beta_{ijk}[x_i, x_j] \cdot x_k + \sum_{i < j} \beta_{ijk}[x_i, x_j] \cdot x_k \\ &= \sum_{j < i} (\alpha_{ijk} + \beta_{ijk} - \beta_{jik})[x_i, x_j] \cdot x_k \end{aligned}$$

and $\beta_{jik} = \alpha_{ijk} + \beta_{ijk}$ for $j < i$. This gives us $(KA_3 \oplus KA_6)^\delta$ is spanned on the elements of the form

$$[x_i, x_j] \cdot y_k + [y_j, x_i] \cdot x_k \quad \text{and} \quad [y_i, x_j] \cdot x_k + [y_j, x_i] \cdot x_k$$

for $j < i$.

We simply need to show $(KA_4 \oplus KA_5)^\delta = \{0\}$. Let $p_4 + p_5 \in (KA_4 \oplus KA_5)^\delta$ such that

$$p_4 = \sum_{j < i} \alpha_{ijk}[y_i, y_j] \cdot x_k$$

and

$$p_5 = \sum \beta_{ijk}[y_i, x_j] \cdot y_k = \sum_{j < i} (\beta_{ijk}[y_i, x_j] \cdot y_k + \beta_{jik}[y_j, x_i] \cdot y_k) + \sum \beta_{iik}[y_i, x_i] \cdot y_k.$$

We have

$$\begin{aligned} \delta(p_4) &= \sum_{j < i} \alpha_{ijk}([x_i, y_j] \cdot x_k + [y_i, x_j] \cdot x_k), \\ \delta(p_5) &= \sum_{j < i} (\beta_{ijk}([x_i, x_j] \cdot y_k + [y_i, x_j] \cdot x_k) - \beta_{jik}([x_j, x_i] \cdot y_k + [y_j, x_i] \cdot x_k)) \\ &\quad + \sum \beta_{iik}[y_i, x_i] \cdot x_k. \end{aligned}$$

Then $\delta(p_4 + p_5) = 0$ gives that

$$0 = \sum_{j < i} (\alpha_{ijk} + \beta_{ijk}) [y_i, x_j] \cdot x_k + (-\alpha_{ijk} + \beta_{jik}) [y_j, x_i] \cdot x_k \in KA_6$$

$$0 = \sum \beta_{iik} [y_i, x_i] \cdot x_k \in KA_6, \quad 0 = \sum_{j < i} (\beta_{ijk} - \beta_{jik}) [x_i, x_j] \cdot y_k \in KA_3$$

that is

$$\alpha_{ijk} + \beta_{ijk} = 0, \quad -\alpha_{ijk} + \beta_{jik} = 0, \quad \beta_{iik} = 0, \quad \beta_{ijk} - \beta_{jik} = 0;$$

hence

$$\alpha_{ijk} = \beta_{ijk} = \beta_{jik} = \beta_{iik} = 0$$

for $j < i$. Therefore, $p_4 = p_5 = 0$. Now the proof follows by direct computations showing that the constants found are linearly independent. □

Theorem 7.3 *The K -vector space $(KB_4)^\delta$ is of a basis consisting of elements of the form*

$$[x_i, x_j, x_k] \cdot x_l, \quad j < i \leq l, \quad j \leq k < l,$$

$$[x_i, x_j, x_j] \cdot y_j, \quad [x_i, x_j, x_i] \cdot y_i, \quad [x_i, x_j, x_i] \cdot y_j + [x_i, x_j, x_j] \cdot y_i, \quad j < i,$$

$$[x_i, x_j, x_k] \cdot y_k, \quad [x_k, x_j, x_i] \cdot y_i, \quad [x_i, x_j, x_i] \cdot y_k + [x_i, x_j, x_k] \cdot y_i, \quad j < i < k,$$

$$[x_i, x_j, x_i] \cdot y_k + [x_k, x_i, x_i] \cdot y_j, \quad [x_i, x_j, x_k] \cdot y_j - [x_k, x_j, x_i] \cdot y_j, \quad j < i < k,$$

$$[x_i, x_j, x_k] \cdot y_j + [x_i, x_j, x_j] \cdot y_k, \quad [x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_j] \cdot y_i, \quad j < i < k,$$

$$[x_k, x_j, x_k] \cdot y_i + [x_k, x_j, x_i] \cdot y_k, \quad [x_k, x_j, x_k] \cdot y_i - [x_k, x_i, x_k] \cdot y_j, \quad j < i < k,$$

$$[x_k, x_j, x_l] \cdot y_i + [x_i, x_j, x_k] \cdot y_l - [x_k, x_i, x_l] \cdot y_j, \quad j < i < k < l,$$

$$[x_k, x_j, x_i] \cdot y_l - [x_i, x_j, x_k] \cdot y_l + [x_k, x_i, x_l] \cdot y_j, \quad j < i < k < l,$$

$$[x_l, x_i, x_k] \cdot y_j + [x_i, x_j, x_k] \cdot y_l - [x_k, x_i, x_l] \cdot y_j, \quad j < i < k < l,$$

$$[x_i, x_j, x_l] \cdot y_k + [x_i, x_j, x_k] \cdot y_l, \quad [x_l, x_j, x_i] \cdot y_k + [x_k, x_i, x_l] \cdot y_j, \quad j < i < k < l,$$

$$[x_l, x_j, x_k] \cdot y_i - [x_k, x_i, x_l] \cdot y_j, \quad j < i < k < l,$$

$$[y_i, x_i, x_i] \cdot y_i, \quad [y_j, x_j, x_i] \cdot y_i, \quad [y_j, x_i, x_i] \cdot y_i + [y_i, x_j, x_i] \cdot y_i, \quad j < i,$$

$$[y_j, x_j, x_i] \cdot y_j + [y_j, x_j, x_j] \cdot y_i, \quad [y_j, x_i, x_i] \cdot y_j - 2[x_i, x_j, y_j] \cdot y_i + [y_i, x_j, x_j] \cdot y_i, \quad j < i,$$

$$[y_i, x_j, x_k] \cdot y_i + [x_i, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_i + [y_j, x_i, x_i] \cdot y_k, \quad j < i < k,$$

$$[y_j, x_i, x_k] \cdot y_i - [x_i, x_j, y_i] \cdot y_k - [x_k, x_i, y_j] \cdot y_i + [y_i, x_j, x_i] \cdot y_k, \quad j < i < k,$$

$$[x_k, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_k - [y_j, x_k, x_k] \cdot y_i - [y_k, x_j, x_i] \cdot y_k, \quad j < i < k,$$

$$[y_i, x_j, x_k] \cdot y_k + [y_j, x_i, x_k] \cdot y_k, \quad [y_j, x_j, x_k] \cdot y_i + [y_j, x_j, x_i] \cdot y_k, \quad j < i < k,$$

$$[x_i, x_j, y_j] \cdot y_k + [x_k, x_j, y_j] \cdot y_i - [y_i, x_j, x_j] \cdot y_k - [y_j, x_i, x_k] \cdot y_j, \quad j < i < k,$$

$$\begin{aligned}
 & [x_l, x_k, y_j] \cdot y_i + [x_i, x_j, y_k] \cdot y_l + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k, \quad j < i < k < l, \\
 & [y_j, x_k, x_l] \cdot y_i + [y_k, x_j, x_i] \cdot y_l - [x_l, x_i, y_j] \cdot y_k - [x_k, x_j, y_i] \cdot y_l, \quad j < i < k < l, \\
 & [y_j, x_i, x_l] \cdot y_k + [y_i, x_j, x_k] \cdot y_l + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k, \quad j < i < k < l, \\
 & [x_l, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_l - [y_j, x_i, x_k] \cdot y_l - [y_i, x_j, x_l] \cdot y_k - [x_l, x_i, y_j] \cdot y_k - [x_k, x_j, y_i] \cdot y_l.
 \end{aligned}$$

Proof It is easy to see

$$K\mathcal{B}_4 = KB_1 \oplus KB_2 \oplus KB_3 \oplus KB_4 \oplus KB_5 \oplus KB_6,$$

where

$$\begin{aligned}
 B_1 &= \{[x_i, x_j, x_k] \cdot x_l \mid 1 \leq j < i \leq l \leq n, 1 \leq j \leq k < l \leq n\}, \\
 B_2 &= \{[y_i, y_j, y_k] \cdot y_l \mid 1 \leq j < i \leq l \leq n, 1 \leq j \leq k < l \leq n\}, \\
 B_3 &= \{[x_i, x_j, x_k] \cdot y_l \mid 1 \leq j < i \leq n, 1 \leq j \leq k \leq n\}, \\
 B_4 &= \{[x_i, x_j, y_k] \cdot y_l \mid 1 \leq j < i \leq n, 1 \leq k < l \leq n\}, \\
 B_5 &= \{[y_i, x_j, x_k] \cdot y_l \mid 1 \leq i \leq l \leq n, 1 \leq j \leq k \leq n\}, \\
 B_6 &= \{[y_i, x_j, y_k] \cdot y_l \mid 1 \leq i \leq l \leq n, 1 \leq k < l \leq n\}.
 \end{aligned}$$

We have $(KB_1)^\delta = KB_1$,

$$\begin{aligned}
 & (KB_2)^\delta \subset KB_6, \quad (KB_3)^\delta \subset KB_1, \\
 & (KB_4)^\delta \subset KB_3, \quad (KB_5)^\delta \subset KB_3, \quad (KB_6)^\delta \subset KB_4 \oplus KB_5
 \end{aligned}$$

and we get

$$(K\mathcal{B}_4)^\delta = KB_1 \oplus (KB_2)^\delta \oplus (KB_3)^\delta \oplus (KB_4 \oplus KB_5)^\delta \oplus (KB_6)^\delta.$$

We divide the proof in four cases: we study separately $(KB_2)^\delta$, $(KB_3)^\delta$, $(KB_4 \oplus KB_5)^\delta$, and finally $(KB_6)^\delta$.

Case 1 Let $p_2 \in (KB_2)^\delta$ be of the form

$$p_2 = \sum_{j < i \leq l, j \leq k < l} \alpha_{ijkl} [y_i, y_j, y_k] \cdot y_l.$$

Note that the condition $j < i \leq l, j \leq k < l$ can be divided into six parts:

$$\begin{aligned}
 & j = k < i = l, \quad j < k < i < l, \quad j < i < k < l, \\
 & j < i = k < l, \quad j < k < i = l, \quad j = k < i < l.
 \end{aligned}$$

Hence

$$\begin{aligned}
 p_2 &= \sum_{j < i} \alpha_{ijji} [y_i, y_j, y_j] \cdot y_i + \sum_{j < k < i < l} \alpha_{ijkl} [y_i, y_j, y_k] \cdot y_l + \sum_{j < i < k < l} \alpha_{ijkl} [y_i, y_j, y_k] \cdot y_l \\
 &+ \sum_{j < i < l} \alpha_{ijil} [y_i, y_j, y_i] \cdot y_l + \sum_{j < k < i} \alpha_{ijkil} [y_i, y_j, y_k] \cdot y_i + \sum_{j < i < l} \alpha_{ijjl} [y_i, y_j, y_j] \cdot y_l \\
 &= \sum_{j < i} \alpha_{ijji} [y_i, y_j, y_j] \cdot y_i + \sum_{j < i < k < l} (\alpha_{kjil} [y_k, y_j, y_i] \cdot y_l + \alpha_{ijkl} [y_i, y_j, y_k] \cdot y_l) \\
 &+ \sum_{j < i < k} (\alpha_{ijik} [y_i, y_j, y_i] \cdot y_k + \alpha_{kjik} [y_k, y_j, y_i] \cdot y_k + \alpha_{ijjk} [y_i, y_j, y_j] \cdot y_k).
 \end{aligned}$$

Let us rewrite p_2 as

$$p_2 = p_{22} + p_{23} + p_{24},$$

where

$$p_{22} = \sum_{j < i} \alpha_{ijji} [y_i, y_j, y_j] \cdot y_i, \quad p_{24} = \sum_{j < i < k < l} (\alpha_{kjil} [y_k, y_j, y_i] \cdot y_l + \alpha_{ijkl} [y_i, y_j, y_k] \cdot y_l),$$

$$p_{23} = \sum_{j < i < k} (\alpha_{ijik} [y_i, y_j, y_i] \cdot y_k + \alpha_{kjik} [y_k, y_j, y_i] \cdot y_k + \alpha_{ijjk} [y_i, y_j, y_j] \cdot y_k).$$

Let V_{2m} , $m = 2, 3, 4$, be the subspace of KB_6 spanned on monomials $[y_i, x_j, y_k] \cdot y_l$ with $|\{i, j, k, l\}| = m$. Then $V_{22} \cap V_{23} = \{0\}$, $V_{22} \cap V_{24} = \{0\}$, $V_{23} \cap V_{24} = \{0\}$, and $\delta(p_{22}) \in V_{22}$, $\delta(p_{23}) \in V_{23}$, $\delta(p_{24}) \in V_{24}$. Therefore $\delta(p_2) = 0$ implies that $\delta(p_{22}) = \delta(p_{23}) = \delta(p_{24}) = 0$.

Remark that $\delta([y_i, y_j, y_j] \cdot y_i)$ can be written as a linear combination of basis elements as follows:

$$\begin{aligned} \delta([y_i, y_j, y_j] \cdot y_i) &= [x_i, y_j, y_j] \cdot y_i + [y_i, x_j, y_j] \cdot y_i + [y_i, y_j, x_j] \cdot y_i + [y_i, y_j, y_j] \cdot x_i \\ &= -[y_j, x_i, y_j] \cdot y_i + [y_i, x_j, y_j] \cdot y_i - [y_j, x_j, y_i] \cdot y_i + [y_i, x_j, y_j] \cdot y_i \\ &\quad + [x_i, y_j, y_j] \cdot y_i - [y_j, y_j, x_i] \cdot y_i \\ &= -2[y_j, x_i, y_j] \cdot y_i + 2[y_i, x_j, y_j] \cdot y_i. \end{aligned}$$

Thus

$$\delta(p_{22}) = 0 = 2 \sum_{j < i} \alpha_{ijji} (-[y_j, x_i, y_j] \cdot y_i + [y_i, x_j, y_j] \cdot y_i)$$

and consequently

$$\sum_{j < i} \alpha_{ijji} [y_j, x_i, y_j] \cdot y_i = 0 = \sum_{j < i} \alpha_{ijji} [y_i, x_j, y_j] \cdot y_i$$

since the basis elements belong to different multilinear components for $j < i$. We get $\alpha_{ijji} = 0$ if $j < i$ that means $p_{22} = 0$.

Similarly, because $\delta(p_{23}) = 0$ we have the sum

$$\sum_{j < i < k} \left(\begin{aligned} &\alpha_{ijik} (-2[y_j, x_i, y_i] \cdot y_k + [y_i, x_j, y_i] \cdot y_k + [y_i, x_i, y_j] \cdot y_k + [y_i, x_k, y_j] \cdot y_i) \\ &+ \alpha_{kjik} ([y_k, x_j, y_i] \cdot y_k + [y_k, x_i, y_j] \cdot y_k - [y_j, x_k, y_i] \cdot y_k - [y_i, x_l, y_j] \cdot y_l) \\ &+ \alpha_{ijjk} (2[y_i, x_j, y_j] \cdot y_k - [y_j, x_i, y_j] \cdot y_k - [y_j, x_j, y_i] \cdot y_k - [y_j, x_k, y_j] \cdot y_i) \end{aligned} \right)$$

is zero, from which we deduce that each summand is zero; moreover,

$$0 = \sum_{j < i < k} \alpha_{ijik} [y_j, x_i, y_i] \cdot y_k = \sum_{j < i < k} \alpha_{kjik} [y_k, x_j, y_i] \cdot y_k = \sum_{j < i < k} \alpha_{ijjk} [y_i, x_j, y_j] \cdot y_k$$

because the basis elements belong to different multilinear components. As above, we get $0 = \alpha_{ijik} = \alpha_{kjik} = \alpha_{ijjk}$ for $j < i < k$ implying that $p_{23} = 0$.

Again, writing $\delta(p_{24})$ as a linear combination of basis elements, we get

$$\begin{aligned} 0 &= \sum_{j < i < k < l} \alpha_{kjil} \left(\begin{aligned} &-[y_j, x_k, y_i] \cdot y_l + [y_k, x_j, y_i] \cdot y_l - [y_j, x_i, y_k] \cdot y_l \\ &+ [y_k, x_i, y_j] \cdot y_l - [y_i, x_l, y_j] \cdot y_k \end{aligned} \right) \\ &+ \sum_{j < i < k < l} \alpha_{ijkl} \left(\begin{aligned} &-[y_j, x_i, y_k] \cdot y_l + [y_i, x_j, y_k] \cdot y_l - [y_j, x_k, y_i] \cdot y_l \\ &+ [y_i, x_k, y_j] \cdot y_l - [y_i, x_l, y_j] \cdot y_k + [y_j, x_l, y_i] \cdot y_k \end{aligned} \right) \end{aligned}$$

and we get $\alpha_{kjil} = \alpha_{ijkl} = 0$ for $j < i < k < l$. Thus $p_{24} = 0$, from which we get $p_2 = 0$.

Case 2 We study $(KB_3)^\delta$. Let

$$p_3 = \sum_{j < i, j < k} \alpha_{ijkl}[x_i, x_j, x_k] \cdot y_l = p_{32} + p_{33} + p_{34}$$

be a constant, where

$$p_{32} = \sum_{j < i} \alpha_{ijii}[x_i, x_j, x_i] \cdot y_i + \alpha_{ijij}[x_i, x_j, x_i] \cdot y_j + \alpha_{ijji}[x_i, x_j, x_j] \cdot y_i + \alpha_{ijjj}[x_i, x_j, x_j] \cdot y_j,$$

$$p_{33} = \sum_{j < i < k} \left(\begin{array}{l} \alpha_{ijik}[x_i, x_j, x_i] \cdot y_k + \alpha_{ijik}[x_i, x_j, x_i] \cdot y_k + \alpha_{ijik}[x_i, x_j, x_i] \cdot y_k \\ + \alpha_{ijkk}[x_i, x_j, x_k] \cdot y_k + \alpha_{ijki}[x_i, x_j, x_k] \cdot y_i + \alpha_{ijkj}[x_i, x_j, x_k] \cdot y_j \\ + \alpha_{kjik}[x_k, x_j, x_i] \cdot y_k + \alpha_{kjii}[x_k, x_j, x_i] \cdot y_i + \alpha_{kji j}[x_k, x_j, x_i] \cdot y_j \\ + \alpha_{ijjk}[x_i, x_j, x_j] \cdot y_k + \alpha_{kjjj}[x_k, x_j, x_k] \cdot y_i + \alpha_{kijj}[x_k, x_i, x_i] \cdot y_j \end{array} \right),$$

$$p_{34} = \sum_{j < i < k < l} \left(\begin{array}{l} \alpha_{ijkl}[x_i, x_j, x_k] \cdot y_l + \alpha_{ijlk}[x_i, x_j, x_i] \cdot y_k + \alpha_{kjl i}[x_k, x_j, x_l] \cdot y_i \\ + \alpha_{kilj}[x_k, x_i, x_l] \cdot y_j + \alpha_{kjil}[x_k, x_j, x_i] \cdot y_l + \alpha_{ljik}[x_l, x_j, x_i] \cdot y_k \\ + \alpha_{ljk i}[x_l, x_j, x_k] \cdot y_i + \alpha_{likj}[x_l, x_i, x_k] \cdot y_j \end{array} \right).$$

As well as in the previous case, $\delta(p_{32}) = \delta(p_{33}) = \delta(p_{34}) = 0$. Let us study the case $\delta(p_{32}) = 0$. An easy computation shows the elements

$$[x_i, x_j, x_i] \cdot y_i, [x_i, x_j, x_j] \cdot y_j, j < i$$

are constants, then we get

$$0 = \sum_{j < i} (-\alpha_{ijjj} + \alpha_{ijji})[x_i, x_j, x_j] \cdot x_i$$

and consequently $\alpha_{ijij} = \alpha_{ijji}$. Moreover, we get the elements of the form

$$[x_i, x_j, x_i] \cdot y_j + [x_i, x_j, x_j] \cdot y_i, j < i$$

are constants, too.

Notice also

$$[x_i, x_j, x_k] \cdot y_k, [x_k, x_j, x_i] \cdot y_i, j < i < k$$

are constants, then $\delta(p_{33}) = 0$ may be rewritten as

$$0 = \sum_{j < i < k} \left(\begin{array}{l} (\alpha_{ijik} - \alpha_{ijki} - \alpha_{kijj})[x_i, x_j, x_i] \cdot x_k \\ (-\alpha_{kjk i} - \alpha_{kikj} + \alpha_{kji k})[x_k, x_j, x_i] \cdot x_k \\ (-\alpha_{ijkj} - \alpha_{kji j} + \alpha_{ijjk} + \alpha_{kjj i})[x_i, x_j, x_j] \cdot x_k \end{array} \right).$$

Hence we get the following system of equations for $j < i < k$:

$$\begin{cases} \alpha_{ijik} = \alpha_{ijki} + \alpha_{kijj} \\ \alpha_{kjk i} = \alpha_{kji k} - \alpha_{kikj} \\ \alpha_{ijkj} = -\alpha_{kji j} + \alpha_{ijjk} + \alpha_{kjj i} \end{cases}$$

By the previous system, we get the next polynomials are constants, too.

$$\begin{aligned}
 & [x_i, x_j, x_i] \cdot y_k + [x_i, x_j, x_k] \cdot y_i, \quad [x_i, x_j, x_i] \cdot y_k + [x_k, x_i, x_i] \cdot y_j \\
 & [x_k, x_j, x_k] \cdot y_i + [x_k, x_j, x_i] \cdot y_k, \quad [x_k, x_j, x_k] \cdot y_i - [x_k, x_i, x_k] \cdot y_j \\
 & [x_i, x_j, x_k] \cdot y_j - [x_k, x_j, x_i] \cdot y_j, \quad [x_i, x_j, x_k] \cdot y_j + [x_i, x_j, x_j] \cdot y_k \\
 & [x_i, x_j, x_k] \cdot y_j + [x_k, x_j, x_j] \cdot y_i, \quad j < i < k.
 \end{aligned}$$

Let us rewrite now the relation $0 = \delta(p_{34})$ as a linear combination of the basis elements.

$$\begin{aligned}
 0 &= \sum_{j < i < k < l} \left(\begin{aligned} & \alpha_{ijkl}[x_i, x_j, x_k] \cdot x_l + \alpha_{ijlk}[x_i, x_j, x_l] \cdot x_k + \alpha_{kjli}[x_k, x_j, x_l] \cdot x_i \\ & + \alpha_{kilj}[x_k, x_i, x_l] \cdot x_j + \alpha_{kjil}[x_k, x_j, x_i] \cdot x_l + \alpha_{ljik}[x_l, x_j, x_i] \cdot x_k \\ & + \alpha_{ljki}[x_l, x_j, x_k] \cdot x_i + \alpha_{likj}[x_l, x_i, x_k] \cdot x_j \end{aligned} \right) \\
 &= \sum_{j < i < k < l} \left(\begin{aligned} & (\alpha_{ijkl} - \alpha_{ijlk} + \alpha_{kilj} - \alpha_{ljik} + \alpha_{ljki})[x_i, x_j, x_k] \cdot x_l \\ & + (-\alpha_{kjli} - \alpha_{kilj} + \alpha_{kjil} + \alpha_{ljik} - \alpha_{ljki} - \alpha_{likj})[x_k, x_j, x_i] \cdot x_l \end{aligned} \right).
 \end{aligned}$$

Hence we get

$$\begin{aligned}
 & \alpha_{ijkl} - \alpha_{ijlk} + \alpha_{kilj} - \alpha_{ljik} + \alpha_{ljki} = 0 \\
 & -\alpha_{kjli} - \alpha_{kilj} + \alpha_{kjil} + \alpha_{ljik} - \alpha_{ljki} - \alpha_{likj} = 0
 \end{aligned}$$

for $j < i < k < l$, from which we have the following relations:

$$\begin{aligned}
 & \alpha_{ijkl} = \alpha_{ijlk} + \alpha_{kjli} - \alpha_{kjil} + \alpha_{likj} \\
 & \alpha_{kilj} = \alpha_{ljik} - \alpha_{ljki} - \alpha_{kjli} + \alpha_{kjil} - \alpha_{likj}.
 \end{aligned}$$

Substituting the relations above in the expression of p_{34} , we get the following constants:

$$\begin{aligned}
 & [x_i, x_j, x_l] \cdot y_k + [x_i, x_j, x_k] \cdot y_l, \\
 & [x_l, x_j, x_i] \cdot y_k + [x_k, x_i, x_l] \cdot y_j, \\
 & [x_l, x_j, x_k] \cdot y_i - [x_k, x_i, x_l] \cdot y_j, \\
 & [x_k, x_j, x_l] \cdot y_i + [x_i, x_j, x_k] \cdot y_l - [x_k, x_i, x_l] \cdot y_j, \\
 & [x_k, x_j, x_i] \cdot y_l - [x_i, x_j, x_k] \cdot y_l + [x_k, x_i, x_l] \cdot y_j, \\
 & [x_l, x_i, x_k] \cdot y_j + [x_i, x_j, x_k] \cdot y_l - [x_k, x_i, x_l] \cdot y_j.
 \end{aligned}$$

Case 3 We study $(KB_4 \oplus KB_5)^\delta$. Let

$$p_4 = \sum_{j < i, k < l} \alpha_{ljki}[x_i, x_j, y_k] \cdot y_l \in KB_4, \quad p_5 = \sum_{i \leq l, j \leq k} \beta_{ljki}[y_i, x_j, x_k] \cdot y_l \in KB_5$$

such that $\delta(p_4 + p_5) = 0$. We point out $[y_i, x_i, x_i] \cdot y_i, [y_j, x_j, x_i] \cdot y_i \in (KB_4 \oplus KB_5)^\delta, 1 \leq j < i \leq n$, since

$$\begin{aligned}
 & \delta([y_i, x_i, x_i] \cdot y_i) = [x_i, x_i, x_i] \cdot y_i + [y_i, x_i, x_i] \cdot x_i = 0, \\
 & \delta([y_j, x_j, x_i] \cdot y_i) = [x_j, x_j, x_i] \cdot y_i + [y_j, x_j, x_i] \cdot x_i = 0.
 \end{aligned}$$

Thus we may assume $p_4 + p_5 = p_{452} + p_{453} + p_{454}$, then

$$\delta(p_4 + p_5) = \delta(p_{452}) + \delta(p_{453}) + \delta(p_{454}) = 0,$$

where

$$p_{452} = \sum_{j < i} \left(\begin{array}{l} \alpha_{ijji}[x_i, x_j, y_j] \cdot y_i + \beta_{ijii}[y_i, x_j, x_i] \cdot y_i + \beta_{ijji}[y_i, x_j, x_j] \cdot y_i \\ + \beta_{jjji}[y_j, x_j, x_j] \cdot y_i + \beta_{jjij}[y_j, x_j, x_i] \cdot y_j + \beta_{jiii}[y_j, x_i, x_i] \cdot y_i \\ + \beta_{jiiij}[y_j, x_i, x_i] \cdot y_j \end{array} \right),$$

$$p_{453} = \sum_{j < i < k} \left(\begin{array}{l} \alpha_{ijik}[x_i, x_j, y_i] \cdot y_k + \alpha_{kiji}[x_k, x_i, y_j] \cdot y_i + \alpha_{ijjk}[x_i, x_j, y_j] \cdot y_k \\ + \alpha_{kjjj}[x_k, x_j, y_j] \cdot y_i + \alpha_{kijk}[x_k, x_i, y_j] \cdot y_k + \alpha_{kjik}[x_k, x_j, y_i] \cdot y_k \\ + \beta_{ijik}[y_i, x_j, x_i] \cdot y_k + \beta_{ijki}[y_i, x_j, x_k] \cdot y_i + \beta_{jiki}[y_j, x_i, x_k] \cdot y_i \\ + \beta_{jiiik}[y_j, x_i, x_i] \cdot y_k + \beta_{ijjk}[y_i, x_j, x_j] \cdot y_k + \beta_{jjik}[y_j, x_j, x_i] \cdot y_k \\ + \beta_{jjki}[y_j, x_j, x_k] \cdot y_i + \beta_{jikj}[y_j, x_i, x_k] \cdot y_j + \beta_{ijkk}[y_i, x_j, x_k] \cdot y_k \\ + \beta_{kjik}[y_k, x_j, x_i] \cdot y_k + \beta_{jikj}[y_j, x_i, x_k] \cdot y_k + \beta_{jkki}[y_j, x_k, x_k] \cdot y_i \end{array} \right),$$

$$p_{454} = \sum_{j < i < k < l} \left(\begin{array}{l} \alpha_{ijkl}[x_i, x_j, y_k] \cdot y_l + \alpha_{kjil}[x_k, x_j, y_i] \cdot y_l + \alpha_{ljik}[x_l, x_j, y_i] \cdot y_k \\ + \alpha_{kijl}[x_k, x_i, y_j] \cdot y_l + \alpha_{lijl}[x_l, x_i, y_j] \cdot y_k + \alpha_{lkji}[x_l, x_k, y_j] \cdot y_i \\ + \beta_{ijkl}[y_i, x_j, x_k] \cdot y_l + \beta_{kjil}[y_k, x_j, x_i] \cdot y_l + \beta_{ijlk}[y_i, x_j, x_l] \cdot y_k \\ + \beta_{jikl}[y_j, x_i, x_k] \cdot y_l + \beta_{jilk}[y_j, x_i, x_l] \cdot y_k + \beta_{jkli}[y_j, x_k, x_l] \cdot y_i \end{array} \right).$$

Rewriting $\delta(p_{452}) \in KB_3$ as a linear combination of basis elements, we have

$$0 = \sum_{j < i} \left(\begin{array}{l} (\alpha_{ijji} + 2\beta_{ijji})[x_i, x_j, x_j] \cdot y_i + (-\alpha_{ijji} - 2\beta_{jiiij})[x_i, x_j, x_i] \cdot y_j \\ (\beta_{jiii} - \beta_{jiiij})[x_i, x_j, x_i] \cdot y_i + (\beta_{jjji} - \beta_{jjij})[x_i, x_j, x_j] \cdot y_j \end{array} \right),$$

then $\alpha_{ijji} = -2\beta_{jiiij}$, $\beta_{ijji} = \beta_{jiiij}$, $\beta_{jiiij} = \beta_{jiiij}$, $\beta_{jjji} = \beta_{jjij}$. Thus, for $j < i$, the next are constants:

$$\begin{aligned} & [y_j, x_i, x_i] \cdot y_j - 2[x_i, x_j, y_j] \cdot y_i + [y_i, x_j, x_j] \cdot y_i, \\ & [y_j, x_i, x_i] \cdot y_i + [y_i, x_j, x_i] \cdot y_i, [y_j, x_j, x_i] \cdot y_j + [y_j, x_j, x_j] \cdot y_i. \end{aligned}$$

A careful check of the coefficients of $[x_i, x_j, x_i] \cdot y_k$, $[x_i, x_j, x_k] \cdot y_i$, $[x_k, x_j, x_i] \cdot y_i$, $[x_k, x_i, x_i] \cdot y_j$, $[x_k, x_j, x_i] \cdot y_k$, $[x_k, x_j, x_k] \cdot y_i$, $[x_i, x_j, x_k] \cdot y_k$, $[x_k, x_i, x_k] \cdot y_j$, $[x_i, x_j, x_j] \cdot y_k$, $[x_i, x_j, x_k] \cdot y_j$, $[x_k, x_j, x_j] \cdot y_i$, $[x_k, x_j, x_i] \cdot y_j$, respectively, in the expression of $\delta(p_{453}) = 0$, gives us the following homogeneous system of equation:

$$\begin{array}{ll} 0 = \alpha_{ijik} + \beta_{ijik} - \beta_{jiiik} & 0 = -\alpha_{kijk} + \beta_{ijkk} - \beta_{jikj} - \beta_{kjik} \\ 0 = -\alpha_{ijik} - \alpha_{kiji} - \beta_{ijik} + 2\beta_{ijki} - \beta_{jiki} & 0 = -\alpha_{kijk} - \beta_{jkki} \\ 0 = \alpha_{kiji} + \beta_{ijik} - \beta_{ijki} & 0 = \alpha_{ijjk} + \beta_{ijjk} \\ 0 = -\alpha_{kiji} - \beta_{jiki} + \beta_{jiiik} & 0 = -\alpha_{ijjk} - \beta_{jjik} + \beta_{jjki} - \beta_{jikj} \\ 0 = \alpha_{kjik} + \alpha_{kijk} + 2\beta_{kjik} & 0 = \alpha_{kjjj} + \beta_{ijjk} \\ 0 = -\alpha_{kjik} - \beta_{jkki} & 0 = -\alpha_{kjjj} + \beta_{jjik} - \beta_{jjki} - \beta_{jikj} \end{array}$$

We have the next relations:

$$\begin{aligned} \alpha_{ijik} &= \alpha_{kiji} = \beta_{ijki} - \beta_{jiki}, \quad \beta_{ijik} = \beta_{jiki}, \quad \beta_{jiiik} = \beta_{ijki} \\ \alpha_{kjik} &= \alpha_{kijk} = -\beta_{jkki} = -\beta_{kjik}, \quad \beta_{ijkk} = \beta_{jikj} \\ \alpha_{ijjk} &= \alpha_{kjjj} = -\beta_{ijjk} = -\beta_{jikj}, \quad \beta_{jjki} = \beta_{jjik}. \end{aligned}$$

Now substituting $\beta_{ijk}, \beta_{jki}, \alpha_{kjk}, \beta_{jkk}, \beta_{jjk}, \alpha_{ijj}$, respectively, in the expression of p_{453} , we obtain the following constants for $j < i < k$:

$$\begin{aligned} & [y_i, x_j, x_k] \cdot y_i + [x_i, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_i + [y_j, x_i, x_i] \cdot y_k \\ & [y_j, x_i, x_k] \cdot y_i - [x_i, x_j, y_i] \cdot y_k - [x_k, x_i, y_j] \cdot y_i + [y_i, x_j, x_i] \cdot y_k \\ & [x_k, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_k - [y_j, x_k, x_k] \cdot y_i - [y_k, x_j, x_i] \cdot y_k \\ & [y_i, x_j, x_k] \cdot y_k + [y_j, x_i, x_k] \cdot y_k, [y_j, x_j, x_k] \cdot y_i + [y_j, x_j, x_i] \cdot y_k \\ & [x_i, x_j, y_j] \cdot y_k + [x_k, x_j, y_j] \cdot y_i - [y_i, x_j, x_j] \cdot y_k - [y_j, x_i, x_k] \cdot y_j \end{aligned}$$

Similar arguments for $\delta(p_{454}) = 0$ give out the following relations:

$$\begin{aligned} \alpha_{ijkl} &= \alpha_{lkji}, \alpha_{kijl} = \alpha_{ljik}, \beta_{ijkl} = \beta_{jilk}, \beta_{kjil} = \beta_{jkli} \\ \beta_{jikl} &= \beta_{jilk} + \alpha_{lkji} - \alpha_{ljik} = \beta_{ijlk}, \alpha_{lijk} = -\beta_{jkli} - \alpha_{ljik} = \alpha_{kjil} \end{aligned}$$

and substituting $\alpha_{lkji}, \beta_{jkli}, \beta_{jilk}, \alpha_{ljik}$, in the expression of p_{454} ,

$$\begin{aligned} & [x_l, x_k, y_j] \cdot y_i + [x_i, x_j, y_k] \cdot y_l + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k \\ & [y_j, x_k, x_l] \cdot y_i + [y_k, x_j, x_i] \cdot y_l - [x_l, x_i, y_j] \cdot y_k - [x_k, x_j, y_i] \cdot y_l \\ & [y_j, x_i, x_l] \cdot y_k + [y_i, x_j, x_k] \cdot y_l + [y_j, x_i, x_k] \cdot y_l + [y_i, x_j, x_l] \cdot y_k \\ & [x_l, x_j, y_i] \cdot y_k + [x_k, x_i, y_j] \cdot y_l - [y_j, x_i, x_k] \cdot y_l - [y_i, x_j, x_l] \cdot y_k - [x_l, x_i, y_j] \cdot y_k - [x_k, x_j, y_i] \cdot y_l \end{aligned}$$

are constants, where $j < i < k < l$.

Case 4 Finally we have to show $(KB_6)^\delta = \{0\}$. Let

$$p_6 = \sum_{i \leq l, k < l} \alpha_{ijkl} [y_i, x_j, y_k] \cdot y_l$$

be a constant of KB_6 . We have $p_6 = p_{62} + p_{63} + p_{64}$, where

$$\begin{aligned} p_{62} &= \sum_{j < i} \alpha_{iijj} [y_i, x_i, y_j] \cdot y_i + \alpha_{ijji} [y_i, x_j, y_j] \cdot y_i + \alpha_{jiji} [y_j, x_i, y_j] \cdot y_i + \alpha_{jjji} [y_j, x_j, y_j] \cdot y_i, \\ p_{63} &= \sum_{j < i < k} \left(\begin{aligned} & \alpha_{ikji} [y_i, x_k, y_j] \cdot y_i + \alpha_{kijk} [y_k, x_i, y_j] \cdot y_k + \alpha_{kjik} [y_k, x_j, y_i] \cdot y_k \\ & + \alpha_{ikjk} [y_i, x_k, y_j] \cdot y_k + \alpha_{iijk} [y_i, x_i, y_j] \cdot y_k + \alpha_{ijjk} [y_i, x_j, y_j] \cdot y_k \\ & + \alpha_{jkji} [y_j, x_k, y_j] \cdot y_i + \alpha_{jijk} [y_j, x_i, y_j] \cdot y_k + \alpha_{ijik} [y_i, x_j, y_i] \cdot y_k \\ & + \alpha_{jkik} [y_j, x_k, y_i] \cdot y_k + \alpha_{jiik} [y_j, x_i, y_i] \cdot y_k + \alpha_{jjik} [y_j, x_j, y_i] \cdot y_k \end{aligned} \right), \\ p_{64} &= \sum_{j < i < k < l} \left(\begin{aligned} & \alpha_{iljk} [y_i, x_l, y_j] \cdot y_k + \alpha_{ikjl} [y_i, x_k, y_j] \cdot y_l + \alpha_{kijl} [y_k, x_i, y_j] \cdot y_l \\ & + \alpha_{kjil} [y_k, x_j, y_i] \cdot y_l + \alpha_{jlik} [y_j, x_l, y_i] \cdot y_k + \alpha_{jkil} [y_j, x_k, y_i] \cdot y_l \\ & + \alpha_{jikl} [y_j, x_i, y_k] \cdot y_l + \alpha_{ijkl} [y_i, x_j, y_k] \cdot y_l \end{aligned} \right). \end{aligned}$$

Arguing as in the previous cases, we get all coefficients are 0 and we are done with the spanning elements. The final step proving that these spanning elements, all written as linear combinations of basis elements of the free metabelian Poisson algebra, is straightforward. □

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