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Henselian discrete valued stable fields

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Dedicated to Prof. Vesselin S. Drensky on the occasion of his 70th anniversary

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Abstract: Let (K, v) be a Henselian discrete valued field with residue field $\widehat{K}$ of characteristic $q \geq 0$, and $\mathrm{Brd}_p(K)$ be the Brauer p-dimension of K, for each prime p. The present paper shows that if $p = q$, then $\mathrm{Brd}_p(K) \leq 1$ if and only if $\widehat{K}$ is a p-quasilocal field and the degree $[\widehat{K} \colon \widehat{K}^p]$ is $\leq p$. This complements our earlier result that, in case $p \neq q$, we have $\mathrm{Brd}_p(K) \leq 1$ if and only if $\widehat{K}$ is p-quasilocal and $\mathrm{Brd}_p(\widehat{K}) \leq 1$.

Key words: Henselian field, stable field, Brauer p-dimension, p-quasilocal field, almost perfect field

1. Introduction

Let E be a field, E_{sep} its separable closure, $\mathrm{Br}(E)$ the Brauer group of E, $s(E)$ the class of associative finite-dimensional central simple algebras over E, and $d(E)$ the subclass of division algebras $D \in s(E)$. For each $A \in s(E)$, let $\deg(A)$, $\mathrm{ind}(A)$ and $\exp(A)$ be the degree, the Schur index and the exponent of A, respectively. It is well-known (cf. [28], Sect. 14.4) that $\exp(A)$ divides $\mathrm{ind}(A)$ and shares with it the same set of prime divisors; also, $\mathrm{ind}(A) \mid \deg(A)$, and $\deg(A) = \mathrm{ind}(A)$ if and only if $A \in d(E)$. Note that $\mathrm{ind}(B_1 \otimes_E B_2) = \mathrm{ind}(B_1)\mathrm{ind}(B_2)$ if $B_1, B_2 \in s(E)$ and g.c.d.$\{\mathrm{ind}(B_1), \mathrm{ind}(B_2)\} = 1$; equivalently, $B_1' \otimes_E B_2' \in d(E)$ in case $B_j' \in d(E)$, $j = 1, 2$, and g.c.d.$\{\deg(B_1'), \deg(B_2')\} = 1$ (see [28], Sect. 13.4). Since $\mathrm{Br}(E)$ is an abelian torsion group and $\mathrm{ind}(A)$, $\exp(A)$ are invariants both of A and its equivalence class $[A] \in \mathrm{Br}(E)$, these results reduce the study of the restrictions on the pairs $\mathrm{ind}(A)$, $\exp(A)$, $A \in s(E)$, to the special case of p-primary pairs, for an arbitrary fixed prime p. The Brauer p-dimensions $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, where $\mathbb{P}$ is the set of prime numbers, contain essential information on these restrictions. We say that $\mathrm{Brd}_p(E) = n < \infty$, for a given $p \in \mathbb{P}$, if n is the least integer ≥ 0, for which $\mathrm{ind}(A_p) \mid \exp(A_p)^n$ whenever $A_p \in s(E)$ and $[A_p]$ lies in the p-component $\mathrm{Br}(E)_p$ of $\mathrm{Br}(E)$; if no such n exists, we put $\mathrm{Brd}_p(E) = \infty$. For instance, $\mathrm{Brd}_p(E) \leq 1$, for all $p \in \mathbb{P}$, if and only if E is a stable field, i.e. $\deg(D) = \exp(D)$, for each $D \in d(E)$; $\mathrm{Brd}_{p'}(E) = 0$, for some $p' \in \mathbb{P}$, if and only if $\mathrm{Br}(E)_{p'} = \{0\}$. The absolute Brauer p-dimension $\mathrm{abrd}_p(E)$ of E is defined to be the supremum of $\mathrm{Brd}_p(R) \colon R \in \mathrm{Fe}(E)$, where $\mathrm{Fe}(E)$ is the set of finite extensions of E in E_{sep}. We have $\mathrm{abrd}_p(E) \leq 1$, $p \in \mathbb{P}$, if E is an absolutely stable field, i.e. its finite extensions are stable fields. Important fields of this kind have been exhibited by class field theory and the theory of algebraic surfaces, which shows that $\mathrm{Brd}_p(\Phi) = \mathrm{abrd}_p(\Phi) = 1$, $p \in \mathbb{P}$, if Φ is a global or local field (see, e.g., [29], (31.4) and (32.19)), or a finitely-generated extension of transcendence degree 2 over

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2010 *AMS Mathematics Subject Classification:* 16K50, 12J10 (primary), 16K20, 12E15, 11S15 (secondary) an algebraically closed field Φ_0 [22, 24].

Similarly to other topics in the theory of central simple algebras and Brauer groups of fields (see, e.g., [32], Chaps. 9-12), the study of the sequence $\mathrm{Brd}_p(E), \mathrm{abrd}_p(E)$, $p \in \mathbb{P}$, brings useful general results if it restricts at one point or another to certain classes of Henselian (valued) fields or other suitably chosen special fields. The restriction on E allows to find formulae for $\mathrm{Brd}_p(E)$ and $\mathrm{abrd}_p(E)$ and to use them for constructing fields E' with prescribed sequences $\mathrm{Brd}_p(E'), \mathrm{abrd}_p(E')$, $p \in \mathbb{P}$ [13]. This, in turn, provides new information on the behaviour of index-exponent relations under finitely-generated field extensions (see, e.g., the answer to [4], Problem 4.4, given in [10, 11], or else [13], Corollary 5.6 and [11], Remark 5.5, for the ground fields in a sequence of examples disproving Conjecture 2 of [15]). The chosen approach also contributes to better knowledge of Brauer groups of absolutely stable fields, viewed as abstract abelian torsion groups [7], Corollary 4.7. More recently, it has been shown in [6] that some absolutely stable fields (with absolute stability proved in [7]) admit noncyclic division algebras of degree 2^ν, for every integer $\nu \geq 2$.

A nontrivial Krull valuation v of a field K is called Henselian, if it extends uniquely, up-to equivalence, to a valuation v_L on each algebraic extension L of K. The stability condition on a Henselian (valued) field (K_0, v_0) with a residue field $\widehat{K}_0$ of zero characteristic has been fully characterized in [7] by conditions on $\widehat{K}_0$ and the value group $v_0(K_0)$. Also, [12], Proposition 3.5 and results of [7] characterize maximally complete stable fields (K_q, v_q) with $\widehat{K}_q$ perfect and $\mathrm{char}(K_q) = q > 0$. For example, by [7], Corollary 4.5 (ii), the iterated formal (Laurent) power series field $J((X))((Y))$ in 2 variables over a field J is absolutely stable if and only if J is perfect, and the absolute Galois group $\mathcal{G}_J := \mathcal{G}(J_{\mathrm{sep}}/J)$ is metabelian of cohomological dimension $\mathrm{cd}(\mathcal{G}_J) \leq 1$, in the sense of [31]. By Lemma 1.2 of [8], $\mathcal{G}_J$ possesses the noted properties if and only if its Sylow pro-p-groups are topologically isomorphic to the additive group $\mathbb{Z}_p$ of p-adic integers whenever $p \in \mathbb{P}$ and the cohomological p-dimension $\mathrm{cd}_p(\mathcal{G}_J)$ (in the sense of [31]) is nonzero. Therefore, $J((X))((Y))$ is absolutely stable if the field J is quasifinite, i.e., perfect with $\mathcal{G}_J$ isomorphic to the topological group product $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ (see also Remark 5.3 (ii)).

The present paper can be viewed as a continuation of [7]. It completes the characterization of Henselian discrete valued (abbr, HDV) stable fields by properties of their residue fields. Combined with [27], Theorem 2, it determines $\mathrm{Brd}_p(K)$ in case (K, v) is an HDV-field, $\mathrm{char}(\widehat{K}) = p > 0$, and the degree $[\widehat{K} : \widehat{K}^p]$ is at most equal to p, where $\widehat{K}^p = \{\hat{\alpha}^p : \hat{\alpha} \in \widehat{K}\}$.

2. Statements of the main results

It is known that $\mathrm{Brd}_p(\widehat{K}) \leq \mathrm{Brd}_p(K)$, $p \in \mathbb{P}$, for any Henselian field (K, v) (see Theorem 2.8 of [20], or Lemma 3.3 below). Therefore, $\widehat{K}$ is a stable field, provided that so is K. The problem of characterizing Henselian stable fields with p-indivisible value groups is related to the study of p-quasilocal fields. By a p-quasilocal field, for some $p \in \mathbb{P}$, we mean a field E satisfying one of the following two conditions: $\mathrm{Brd}_p(E) = 0$ or $E(p) = E$, where $E(p)$ is the maximal p-extension of E (in E_{sep}); $\mathrm{Brd}_p(E) \neq 0$, $E(p) \neq E$, and every extension of E in $E(p)$ of degree p is embeddable as an E-subalgebra in each $D_p \in d(E)$ of degree p. We say that the field E is quasilocal if its finite extensions are p-quasilocal fields for every $p \in \mathbb{P}$. The class of quasilocal fields contains local fields; in addition, it is essentially larger than the class of HDV-fields with quasifinite residue fields (cf. [30], Ch. XIII, Sect. 3, and [9], Remark 3.7). As to global fields, they are not p-quasilocal, for any $p \in \mathbb{P}$; if

F is a global field, then one obtains from the Grunwald–Wang theorem and the description of $\mathrm{Br}(F)$ by class field theory (see [3], Ch. X, and [34], Ch. XIII, Sections 3 and 6, respectively) that for any $p \in \mathbb{P}$ and each $\Delta_p \in d(F)$ with $\deg(\Delta_p) = p$, there exist infinitely many extensions Φ_p of F in $F(p)$, such that $[\Phi_p \colon F] = p$ and $\Delta_p \otimes_F \Phi_p \in d(\Phi_p)$. Note also that quasilocal fields are absolutely stable [9], Proposition 2.3. This is implied by the fact that a p-quasilocal field E satisfies $\mathrm{Brd}_p(E) \leq 1$ in the following cases: (i) $E(p) \neq E$ [9], Theorem 3.1 (ii); (ii) E contains a primitive p-th root of unity; (iii) $\mathrm{char}(E) = p$. Moreover, in cases (ii) and (iii), the assumption that $\mathrm{Brd}_p(E) \neq 0$ ensures that $E(p) \neq E$ (see [25], (16.2), and [1], Ch. VII, Theorem 28). Henselian stable fields and p-quasilocal fields are related by the following results:

Proposition 2.1 *Let (K, v) be a Henselian field and p be a prime. Then,*

(a) *$\widehat{K}$ is a p-quasilocal field if $v(K) \neq pv(K)$ and $\mathrm{Brd}_p(K) \leq 1$.*

(b) *$\mathrm{Brd}_p(K) \leq 1$, provided that $\widehat{K}$ is p-quasilocal, $p \neq \mathrm{char}(\widehat{K})$, $\mathrm{Brd}_p(\widehat{K}) \leq 1$, and the quotient group $v(K)/pv(K)$ has order p.*

Proposition 2.1 (a) follows from [9], Proposition 2.1 (with its proof). Proposition 2.1 (b) is a special case of [12], Theorem 4.1; it can also be deduced from [7], Theorem 3.1 (a). Using Proposition 2.1 (a), (b) and well-known results about the reduction of Schur indices and exponents under a scalar extension of finite degree over the centre (cf. [28], Sects. 13.4 and 14.4), one sees that the problem of characterizing stable HDV-fields reduces to the one of finding a necessary and sufficient condition that $\mathrm{Brd}_p(K) \leq 1$, where (K, v) is an HDV-field with $\mathrm{char}(\widehat{K}) = p$. The main result of [14], stated below, takes a step towards achieving this goal. It shows that if $\mathrm{Brd}_p(K) \leq 1$, then $[\widehat{K} \colon \widehat{K}^p] \leq p$ (in case K contains a primitive p-th root of unity and $\mathrm{char}(\widehat{K}) = p$, this has been proved in [5], Section 4, and in [7], Section 2):

Proposition 2.2 *Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$. Then:*

(a) *$\mathrm{Brd}_p(K)$ is infinite if and only if $\widehat{K}/\widehat{K}^p$ is an infinite extension:*

(b) *$\mathrm{Brd}_p(K) \geq n$ if $[\widehat{K} \colon \widehat{K}^p] = p^n$, for some $n \in \mathbb{N}$.*

When $\mathrm{char}(\widehat{K}) = p > 0$, the inequality $[\widehat{K} \colon \widehat{K}^p] \leq p$ holds if and only if $\widehat{K}$ is an almost perfect field, i.e., its finite extensions are simple; perfect fields of any characteristic satisfy the condition on the right side and, in this sense, are also almost perfect (see [23], Ch. V, Theorem 4.6 and Corollary 6.10). This allows us to state the main results of the present paper as follows:

Theorem 2.3 *Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$. Then $\mathrm{Brd}_p(K) \leq 1$ if and only if $\widehat{K}$ is p-quasilocal and almost perfect; the equality $\mathrm{Brd}_p(K) = 0$ holds if and only if $\widehat{K}$ is perfect and $\widehat{K}(p) = \widehat{K}$.*

Theorem 2.3 yields $\mathrm{Brd}_p(K) = 1$ in case $\widehat{K}_{\mathrm{sep}} = \widehat{K}$ and $[\widehat{K} \colon \widehat{K}^p] = p$. This result is contained in [35], Proposition 2.1 (see also [5], Proposition 4.5), and it is used for proving the stated theorem in general.

Corollary 2.4 *Assuming that (K, v) and p satisfy the conditions of Proposition 2.2, let $[\widehat{K} \colon \widehat{K}^p] = p$. Then $\mathrm{Brd}_p(K) = 2$ unless $\widehat{K}$ is p-quasilocal.*

Corollary 2.4 follows from Theorem 2.3 and [27], Theorem 2. Theorem 2.3 and this corollary determine $\mathrm{Brd}_p(K)$ when (K,v) is HDV with $\mathrm{char}(\widehat{K}) = p$ and $\widehat{K}$ almost perfect. At the same time, Proposition 2.1 and Theorem 2.3 yield the following characterization of stable HDV-fields:

Corollary 2.5 *Let (K,v) be an* HDV*-field. Then K is stable if and only if $\widehat{K}$ is almost perfect, stable, and p-quasilocal for every $p \in \mathbb{P}$.*

It is presently unknown whether a field E is stable under the condition that it is p-quasilocal, for every $p \in \mathbb{P}$. In view of the above-noted facts on the Brauer p-dimensions of p-quasilocal fields, the stability of E will be proved if the following open problem has an affirmative solution:

Problem 2.6 *Let F be a field not containing a primitive p-th root of unity, for some $p \in \mathbb{P}$ different from char(F). Find whether Brd$_p(F) = 0$ in case $F(p) = F$.*

The basic notation, terminology, and conventions kept in this paper are standard and essentially the same as in [23], [28], [20] and [9]. We refer the reader to [28], for the definition of a cyclic algebra over an arbitrary field; the notions of an inertial algebra, an inertial lift, and a nicely semiramified (briefly, NSR) algebra over a Henselian field are defined in [20]. As in [13], we suppose that, for any discrete valued field (K,v), $v(K)$ is chosen to be a subgroup of the additive group $\mathbb{Q}$ of rational numbers. Throughout, Brauer groups and value groups are written additively, Galois groups are viewed as profinite with respect to the Krull topology, and by a profinite group homomorphism, we mean a continuous one. Given a field E, E^* denotes its multiplicative group, $E^{*n} = \{a^n\colon\ a \in E^*\}$, for each $n \in \mathbb{N}$, and $\mathcal{G}_E = \mathcal{G}(E_{\mathrm{sep}}/E)$ is the absolute Galois group of E. For any $p \in \mathbb{P}$, we denote by ${}_p\mathrm{Br}(E)$ the group $\{b_p \in \mathrm{Br}(E)\colon\ pb_p = 0\}$ and by $r_p(E)$ the rank of $\mathcal{G}(E(p)/E)$ as a pro-p-group, i.e., the cardinality of any minimal system of generators of $\mathcal{G}(E(p)/E)$ as a topological group (we put $r_p(E) = 0$ if $E(p) = E$). As usual, $\mathrm{Br}(E'/E)$ stands for the relative Brauer group of any field extension E'/E (defined to be the kernel of the scalar extension map $\pi_{E\to E'}$ of $\mathrm{Br}(E)$ into $\mathrm{Br}(E')$); also, E' is called a splitting field of every $A \in s(E)$ with $[A] \in \mathrm{Br}(E'/E)$. We write $\mathrm{I}(E'/E)$ for the set of intermediate fields of E'/E; when E'/E is separable of finite degree $[E'\colon E]$, $N(E'/E)$ denotes the norm group of E'/E.

Here is an overview of the paper: Section 3 includes preliminaries on Henselian fields used in the sequel. Theorem 2.3 is proved in Section 4. Absolutely stable HDV-fields are characterized in Section 5, where some special fields of this kind are also presented. Specifically, we show that an m-dimensional local field K_m (i.e., a complete m-discretely valued field (see [33], [19], [37]) with a quasifinite m-th residue field) is absolutely stable, if $m \le 2$, and K_m is not stable, otherwise. When $\mathrm{char}(K_m) > 0$, this is contained in [7], Corollaries 4.5, 4.6.

3. Preliminaries

Let K be a field with a nontrivial valuation v, $O_v(K) = \{a \in K\colon\ v(a) \ge 0\}$ the valuation ring of (K,v), $M_v(K) = \{\mu \in K\colon\ v(\mu) > 0\}$ the maximal ideal of $O_v(K)$, $O_v(K)^* = \{u \in K\colon\ v(u) = 0\}$ the group of units of $O_v(K)$, $v(K)$ and $\widehat{K} = O_v(K)/M_v(K)$ the value group and the residue field of (K,v), respectively. For each $\gamma \in v(K)$, $\gamma \ge 0$, $\nabla_\gamma(K)$ denotes the set $\{\lambda \in K\colon\ v(\lambda - 1) > \gamma\}$. Note that v is Henselian if the following conditions hold: K is complete relative to the topology of v; $v(K)$ is an Archimedean group, i.e., it embeds as an ordered subgroup in the additive group $\mathbb{R}$ of real numbers (cf. [23], Ch. XII, and see Lemma

3.4 below). In order that v be Henselian, it is necessary and sufficient that any of the following two equivalent conditions holds (cf. [18], Sect. 18.1, and [23], Ch. XII, Sect. 4):

(3.1) (a) Given a polynomial $f(X) \in O_v(K)[X]$ and an element $a \in O_v(K)$, such that $2v(f'(a)) < v(f(a))$, where f' is the formal derivative of f, there is a zero $c \in O_v(K)$ of f satisfying the equality $v(c-a) = v(f(a)/f'(a))$;

(b) For each normal extension Ω/K, $v'(\tau(\mu)) = v'(\mu)$ whenever $\mu \in \Omega$, v' is a valuation of Ω extending v, and τ is a K-automorphism of Ω.

When v is Henselian, so is v_L, for every algebraic field extension L/K. In this case, we denote by $\widehat{L}$ the residue field of (L, v_L), and put $O_v(L) = O_{v_L}(L)$, $M_v(L) = M_{v_L}(L)$ and $v(L) = v_L(L)$. Clearly, $\widehat{L}$ is an algebraic extension of $\widehat{K}$, and $v(K)$ is an ordered subgroup of $v(L)$; the index of $v(K)$ in $v(L)$ is denoted by $e(L/K)$. By Ostrowski's theorem, if $[L\colon K]$ is finite, then $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ has no divisor $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$; in particular, v_L can be chosen so that $v(L)$ be an ordered subgroup of a fixed divisible hull of $v(K)$. We say that L/K is defectless, if $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$. The defectlessness of L/K is guaranteed, if $\mathrm{char}(\widehat{K}) \nmid [L\colon K]$ as well as in the following two cases:

(3.2) (a) If (K, v) is HDV and L/K is separable (see [18], Sect. 17.4);

(b) When (K, v) is a complete discrete valued field (cf. [23], Ch. XII, Proposition 6.1).

Assume as above that (K, v) is Henselian. We say that a finite extension R of K is inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}/\widehat{K}$ is separable; R/K is called totally ramified if $e(R/K) = [R\colon K]$. Inertial extensions are separable and have the following properties (see [32], Theorem A.23):

Lemma 3.1 (a) *An inertial extension R'/K is Galois if and only if $\widehat{R}'/\widehat{K}$ is Galois. When this holds, $\mathcal{G}(R'/K)$ and $\mathcal{G}(\widehat{R}'/\widehat{K})$ are canonically isomorphic.*

(b) *The compositum K_{ur} of inertial extensions of K in K_{sep} is a Galois extension of K with $\mathcal{G}(K_{\mathrm{ur}}/K)$ isomorphic to $\mathcal{G}_{\widehat{K}}$. Finite extensions of K in K_{ur} are inertial, and the natural map $I(K_{\mathrm{ur}}/K) \to I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$ is bijective.*

(c) *The group $N(I/K)$ includes $\nabla_0(K)$, for every inertial extension I/K.*

The Henselian property of (K, v) guarantees that v extends on each $D \in d(K)$ to a unique, up-to equivalence, valuation v_D (cf. [20], pp. 131-132; [32], Exercise 3.11). Put $v(D) = v_D(D)$ and let $\widehat{D}$ be the residue division ring of (D, v_D). It is known that $\widehat{D}$ is a division $\widehat{K}$-algebra, $[\widehat{D}\colon \widehat{K}] < \infty$, $v(D)$ is an ordered abelian group and $v(K)$ is an ordered subgroup of $v(D)$ of finite index $e(D/K)$. Moreover, by Ostrowski-Draxl's theorem [17], $[\widehat{D}\colon \widehat{K}]e(D/K) \mid [D\colon K]$ and $[D\colon K][\widehat{D}\colon \widehat{K}]^{-1}e(D/K)^{-1}$ has no prime divisor $p \neq \mathrm{char}(\widehat{K})$. In addition, Proposition 2.2 of [33], states the following:

Lemma 3.2 *Let (K, v) be an* HDV*-field and $D \in d(K)$. Then D/K is defectless, i.e. $[D\colon K] = [\widehat{D}\colon \widehat{K}]e(D/K)$.*

Next we state results on inertial and central K-algebras (contained in [20], Theorem 2.8), which are used for studying the sequence $\mathrm{Brd}_p(K)$, $p \in \mathbb{P}$:

Lemma 3.3 *Let (K,v) be a Henselian field. Then, the set* $\mathrm{IBr}(K) = \{[S'] \in \mathrm{Br}(K)\colon S' \in d(K)$ *is inertial over* $K\}$ *is a subgroup of* $\mathrm{Br}(K)$, *and the natural map* $\mathrm{IBr}(K) \to \mathrm{Br}(\widehat{K})$ *is an index-preserving group isomorphism;* $\mathrm{Brd}_p(\widehat{K}) \leq \mathrm{Brd}_p(K)$, *for all* $p \in \mathbb{P}$, *and equality holds if* $\mathrm{Brd}_p(\widehat{K}) = \infty$.

The following lemma shows that a nontrivially valued field (K,v) with $v(K)$ Archimedean is Henselian if and only if K does not admit a separable proper extension in its completion K_v with respect to the topology induced by v. This is a known consequence of basic properties of valuation prolongations on finite separable extensions (cf. [23], Ch. XII, Sections 2, 3, and 6). When (K,v) is Henselian, the lemma characterizes finite extensions of K_v in $K_{v,\mathrm{sep}}$. As it seems to be difficult to find a standard reference to these results, we refer the reader to [14], Lemma 3.1, for a proof of the lemma.

Lemma 3.4 *Assume that (K,v) is a nontrivially valued field with $v(K)$ Archimedean. Then, (K,v) is Henselian if and only if K is separably closed in K_v. When (K,v) is Henselian, the following conditions are fulfilled:*

(a) *Every* $L \in \mathrm{Fe}(K_v)$ *is K_v-isomorphic to* $\widetilde{L} \otimes_K K_v$ *and* $\widetilde{L}_v$, *where* $\widetilde{L}$ *is the separable closure of K in L. The extension L/K_v is Galois if and only if so is $\widetilde{L}/K$; in case this holds, $\mathcal{G}(L/K_v)$ and $\mathcal{G}(\widetilde{L}/K)$ are isomorphic.*

(b) $K_{\mathrm{sep}} \otimes_K K_v$ *is a field, and there exist canonical isomorphisms* $K_{\mathrm{sep}} \otimes_K K_v \cong K_{v,\mathrm{sep}}$ *and* $\mathcal{G}_K \cong \mathcal{G}_{K_v}$.

The proof of Theorem 2.3 relies on the following well-known results:

Proposition 3.5 *Let (K,v) be an* HDV*-field, and $\bar{v}$ the valuation of K_v continuously extending v. Then:*

(a) *The map $\pi_{K\to K_v}$ is an injective homomorphism preserving Schur indices and exponents (cf. [16], Theorem 1), so* $\mathrm{Brd}_{p'}(K) \leq \mathrm{Brd}_{p'}(K_v)$, *for every* $p' \in \mathbb{P}$;

(b) *The valued field $(K_v,\bar{v})$ is an immediate extension of (K,v), i.e., it is a valued extension with $\widehat{K}_v = \widehat{K}$ and $\bar{v}(K_v) = v(K)$ (cf. [18], Theorem 9.3.2);*

(c) *If $char(K) = p > 0$, $[\widehat{K}\colon \widehat{K}^p] = p^n$, for some integer $n \geq 0$, and $\mathcal{K}/K_v$ is a finite extension, then $(\mathcal{K},\bar{v}_{\mathcal{K}})$ is a complete discrete valued field with $[\mathcal{K}\colon \mathcal{K}^p] = p^{n+1}$ and $[\widehat{\mathcal{K}}\colon \widehat{\mathcal{K}}^p] = p^n$ (apply (3.2) (b), [5], Lemma 2.12, and, for the completeness of $(\mathcal{K},\bar{v}_{\mathcal{K}})$, see [23], Ch. XII, Proposition 2.5).*

At the end of this Section, we prove the latter assertion of Theorem 2.3. Let (K,v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p > 0$. Proposition 2.2 allows to consider only the case where $\widehat{K}$ is perfect. In this case, $\mathrm{Br}(\widehat{K})_p = \{0\}$ (cf. [1], Ch. VII, Theorem 22), so Lemma 3.2 and [20], Lemma 5.14, imply that if $r_p(\widehat{K}) = 0$, then $\mathrm{Br}(K)_p = \{0\}$, i.e. $\mathrm{Brd}_p(K) = 0$. Suppose that $r_p(\widehat{K}) > 0$. Then, there is an NSR-algebra $\Delta_p \in d(K)$ of degree p, whence $\mathrm{Brd}_p(K) \geq 1$. The inequality $\mathrm{Brd}_p(K) \leq 1$ is known (see, e.g., [27], Corollary 2.5); it is also a part of the next lemma which we prove for convenience of the reader.

Lemma 3.6 *Assume that (K,v) is an* HDV*-field, such that $\widehat{K}$ is perfect,* $\mathrm{char}(\widehat{K}) = p > 0$ *and* $r_p(\widehat{K}) > 0$. *Then, every $D_p \in d(K)$ of p-primary degree* $\deg(D_p) \neq 1$ *is an* NSR*-algebra over K with* $\exp(D_p) = \deg(D_p)$.

Proof Take any K-algebra $D_p \in d(K)$, $D_p \neq K$, and let $\deg(D_p) = p^n$. As $\widehat{K}$ is perfect, whence $\mathrm{Br}(\widehat{K})_p = \{0\}$, $\widehat{D}_p/\widehat{K}$ is a separable field extension, so it is simple, which implies $\widehat{D}_p = \widehat{D}'_p$ and $[D'_p\colon K] =$

$[\widehat{D}_p\colon \widehat{K}]$ divides p^n, for some inertial extension D'_p of K included in D_p as a (commutative) K-subalgebra (see [28], Sect. 13.1). Since $v(K)$ is cyclic, it follows similarly that $e(D_p/K)$ divides p^n. Hence, by Lemma 3.2, $[\widehat{D}_p\colon \widehat{K}] = e(D_p/K) = p^n$. Therefore, by [20], Corollary 6.10 (a result of Platonov and Yanchevskij) and the cyclicity of the group $v(D_p)/v(K)$, $p^n \mid \exp(D_p)$, proving that $\exp(D_p) = p^n$. Finally, one obtains from [20], Lemma 5.14, that $[D_p] = [N_p]$, where $N_p \in d(K)$ is NSR. By Wedderburn's structure theorem (see, e.g., [28], Sect. 3.5), this means that $D_p \cong N_p$ as K-algebras, so Lemma 3.6 is proved. □

4. Proof of the main theorem

The purpose of this Section is to prove the former assertion of Theorem 2.3. Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p$. Using Propositions 2.1 (a), 2.2 and the latter assertion of Theorem 2.3, one sees that it suffices to deduce the inequality $\mathrm{Brd}_p(K) \leq 1$, provided that $[\widehat{K}\colon \widehat{K}^p] = p$ and $\widehat{K}$ is p-quasilocal. Proposition 3.5 (c) and the equality $[\widehat{K}\colon \widehat{K}^p] = p$ imply finite extensions of $\widehat{K}$ are almost perfect fields and $\widehat{K}$ has a unique, up-to a $\widehat{K}$-isomorphism, purely inseparable extension $\widehat{K}_n$ of degree p^n, for each $n \in \mathbb{N}$. Therefore, using [1], Ch. VII, Theorem 32, one obtains the following:

(4.1) (a) Each $\widetilde{D}_p \in d(\widehat{K})$ with $[\widetilde{D}_p] \in \mathrm{Br}(\widehat{K})_p$ has a splitting field that is a purely inseparable extension of $\widehat{K}$ of degree equal to $\exp(\widetilde{D}_p)$; in particular, $\deg(\widetilde{D}_p) = \exp(\widetilde{D}_p)$, i.e. $\mathrm{Brd}_p(\widehat{K}) \leq 1$;

(b) If $\widetilde{\Delta}_p \in d(\widehat{K})$ and $\exp(\widetilde{\Delta}_p) = p$, then $\widetilde{\Delta}_p$ is a cyclic $\widehat{K}$-algebra (cf. [28], Sect. 15.5);

(c) The inner group product $Y^{*g}\nabla_0(Y)$ includes $O_v(K)^*$ in case Y/K is a finite extension, $[Y\colon K] = [\widehat{Y}\colon \widehat{K}] = g$ and $\widehat{Y}/\widehat{K}$ is purely inseparable.

The proof of Theorem 2.3 also relies on the following lemma.

Lemma 4.1 *Let (K, v) be an HDV-field with $\mathrm{char}(\widehat{K}) = p$ and $[\widehat{K}\colon \widehat{K}^p] = p$, and let Y/K be a field extension, such that $[Y\colon K] = [\widehat{Y}\colon \widehat{K}] = p$. Suppose that $\widehat{K}$ is p-quasilocal and $\widehat{Y}$ is normal over $\widehat{K}$. Then, $\mathrm{Br}(Y/K)$ includes the group ${}_p\mathrm{Br}(K) \cap \mathrm{IBr}(K)$, and the homomorphism $\pi_{K\to Y}\colon \mathrm{Br}(K) \to \mathrm{Br}(Y)$ maps $\mathrm{Br}(K)_p \cap \mathrm{IBr}(K)$ surjectively upon $\mathrm{Br}(Y)_p \cap \mathrm{IBr}(Y)$.*

Proof It follows from [9], Theorem 4.1, and Albert–Hochschild's theorem (cf. [31], Ch. II, 2.2) that $\pi_{\widehat{K}\to\widehat{Y}}$ maps $\mathrm{Br}(\widehat{K})_p$ surjectively upon $\mathrm{Br}(\widehat{Y})_p$. At the same time, we have $\mathrm{Br}(\widehat{Y}/\widehat{K}) = {}_p\mathrm{Br}(\widehat{K})$, by [9], Theorem 4.1, if $\widehat{Y}/\widehat{K}$ is separable, and by (4.1) (a), when $\widehat{Y}/\widehat{K}$ is inseparable. Note further that $\mathrm{IBr}(Y)$ includes the image of $\mathrm{IBr}(K)$ under $\pi_{K\to Y}$, and the natural maps $r_{K\to\widehat{K}}\colon \mathrm{IBr}(K) \to \mathrm{Br}(\widehat{K})$ and $r_{Y\to\widehat{Y}}\colon \mathrm{IBr}(Y) \to \mathrm{IBr}(\widehat{Y})$, are index-preserving group isomorphisms (see [20], Theorems 5.6 and 2.8). Since $(\pi_{\widehat{K}\to\widehat{Y}} \circ r_{K\to\widehat{K}})([D]) = (r_{Y\to\widehat{Y}} \circ \pi_{K\to Y})([D])$ (in $\mathrm{Br}(\widehat{Y})$) whenever $D \in d(K)$ is inertial over K, this enables one to prove the latter part of the assertion of Lemma 4.1, and the fact that $\mathrm{ind}(D_p \otimes_K Y) = \deg(D_p)/p$, for each $D_p \in d(K)$ with $[D_p] \neq 0$ and $[D_p] \in (\mathrm{Br}(K)_p \cap \mathrm{IBr}(K))$ (the stated equality is also implied by (4.1), Lemma 3.1 and [28], Section 15.1, Proposition b). In view of the Corollary in [28], Section 13.4, these results complete our proof. □

Next we show that Theorem 2.3 will be proved, if we deduce the equality $\deg(\Delta) = p$, assuming that $\Delta \in d(K)$ and $\exp(\Delta) = p$. It follows from Lemma 3.2 and [20], Proposition 1.7, that each $D \in d(K)$ with $\deg(D) = p$ possesses a maximal subfield Y satisfying the conditions of Lemma 4.1. Hence, $\widehat{Y}$ is p-quasilocal (cf. [9], Theorem 4.1 and Proposition 4.4), which enables one to obtain from the claimed property of Δ, by the method of proving [10], Lemma 4.1, that if $\Delta_n \in d(K)$ and $\exp(\Delta_n) = p^n$, then Δ_n has a splitting field Y_n with $[Y_n\colon K] = p^n$, $v(Y_n) = v(K)$ and $\widehat{Y}_n \in I(\widehat{Y}'/\widehat{K})$, where $\widehat{Y}'$ is a perfect closure of $\widehat{K}(p)$. This result gives the desired reduction. Since, by Merkur'ev's theorem [26], Sect. 4, Theorem 2, each $\Delta \in d(K)$ with $\exp(\Delta) = p$ is Brauer equivalent to a tensor product of degree p algebras from $d(K)$, we need only prove that if $D_j \in d(K)$ and $\deg(D_j) = p$, $j = 1, 2$, then $D_1 \otimes_K D_2 \notin d(K)$. This can be deduced from the next lemma.

Lemma 4.2 *Let (K, v) be an* HDV*-field with* $\mathrm{char}(\widehat{K}) = p$, $\widehat{K}$ *p-quasilocal and* $[\widehat{K}\colon \widehat{K}^p] = p$. *Then* $\exp(\Delta) = p^2$, *for any* $\Delta \in d(K)$ *of degree* p^2.

Proof Let Δ be a K-algebra satisfying the conditions of the lemma. As $\widehat{K}$ is almost perfect, this implies p^2 is divisible by the dimension of any commutative $\widehat{K}$-subalgebra of $\widehat{\Delta}$. At the same time, it follows from Lemma 3.2 and the cyclicity of $v(\Delta)$ that $e(\Delta/K) \mid p^2$. Suppose first that $e(\Delta/K) = 1$. Then $[\widehat{\Delta}\colon \widehat{K}] = p^4$, by Lemma 3.2, so the observation on commutative $\widehat{K}$-subalgebras of $\widehat{\Delta}$ indicates that $\widehat{\Delta} \in d(\widehat{K})$ and $\deg(\widehat{\Delta}) = p^2$. Applying [1], Ch. VII, Theorem 28, and [9], Theorem 3.1, one concludes that $\exp(\widehat{\Delta}) = p^2$. It is now easily obtained from [20], Theorems 2.8 and 2.9, that Δ/K is inertial and $\deg(\Delta) = \exp(\Delta)$, as claimed by Lemma 4.2.

Henceforth, we assume that $e(\Delta/K) \neq 1$. Our first goal is to prove that

(4.2) (a) If U is a central K-subalgebra of Δ of degree p, then U is neither an inertial nor an NSR-algebra over K;

(b) If $e(\Delta/K) = p$, then totally ramified extensions of K of degree p are not embeddable in Δ as K-subalgebras.

The proof of (4.2) (a) relies on the double centralizer theorem (see [28], Section 12.7), which implies Δ is K-isomorphic to $U \otimes_K U'$, for some $U' \in d(K)$ with $\deg(U') = p$. Suppose for a moment that U/K is inertial. Applying (3.2) (a), Lemma 3.2 and [20], Theorem 2.8 and Proposition 1.7, one concludes that $e(U'/K) = p$, $\widehat{U}'/\widehat{K}$ is a normal field extension of degree p, and U' contains as a K-subalgebra an extension Y of K with $\widehat{Y} = \widehat{U}'$. Therefore, by Lemma 4.1, Y is embeddable in U as a K-subalgebra, which means that $U \otimes_K Y \notin d(Y)$. As $\Delta \in d(K)$ and $U \otimes_K Y$ is a K-subalgebra of Δ, this is a contradiction proving that U/K cannot be inertial. The non-existence of an NSR-subalgebra of Δ of degree p is merely a consequence of (4.2) (b).

We turn to the proof of (4.2) (b), so we assume that $e(\Delta/K) = p$. Suppose that our assertion is false, i.e., Δ contains as a K-subalgebra a totally ramified extension T of K of degree p, and let W' be the centralizer of T in Δ. It is clear from the double centralizer theorem that $W' \in d(T)$ and $\deg(W') = p$, and it follows from Lemma 3.2 and the assumptions on Δ/K and T/K that $[\widehat{W}'\colon \widehat{T}] = p^2$. As $\widehat{T} = \widehat{K}$ is almost perfect, each commutative $\widehat{T}$-subalgebra $\widehat{\Theta}'$ of $\widehat{W}'$ embeds as a $\widehat{T}$-subalgebra in $\widehat{\Theta}$, for some commutative T-subalgebra Θ of W', so the noted facts show (similarly to the proof of the equality $[\widehat{D}_p\colon \widehat{T}] = \deg(D_p)$, in the setting of Lemma 3.6) that $[\widehat{\Theta}'\colon \widehat{T}]$ equals 1 or p. Thus, they prove that $\widehat{W}' \in d(\widehat{T})$ and $\deg(\widehat{W}') = p$. Taking again into account that $\widehat{T} = \widehat{K}$, and using [20], Theorem 2.8, one concludes that $W' \cong W \otimes_K T$ as a T-algebra, where $W \in d(K)$ is an inertial lift of $\widehat{W}'$ over K. This leads to the conclusion that W is embeddable in Δ as a K-subalgebra, which contradicts the nonexistence of inertial central K-subalgebras of Δ of degree p. The obtained contradiction proves (4.2) (b) and completes the proof of (4.2) (a).

We continue with the proof of Lemma 4.2 in the case of $e(\Delta/K) = p$. Clearly, Lemma 3.2 yields $[\widehat{\Delta}\colon \widehat{K}] = p^3$, so the assumption that $[\widehat{K}\colon \widehat{K}^p] = p$ implies $\widehat{\Delta}$ is noncommutative. This means that $[\widehat{\Delta}\colon Z(\widehat{\Delta})] = p^2$ and $[Z(\widehat{\Delta})\colon \widehat{K}] = p$, where $Z(\widehat{\Delta})$ is the centre of $\widehat{\Delta}$. First we prove that $\exp(\Delta) = p^2$, under the extra hypothesis that Δ possesses a K-subalgebra Δ_0, such that $[\Delta_0\colon K] = p^3$ and $\widehat{\Delta}_0$ is $\widehat{K}$-isomorphic to $\widehat{\Delta}$; by [20], Theorem 2.9, this holds in the special case where $Z(\widehat{K})$ is a separable extension of $\widehat{K}$. It follows from [20], Proposition 1.7, our extra hypothesis and the cyclicity of $v(K)$ that $Z(\widehat{\Delta})/\widehat{K}$ is a normal extension of degree p. Hence, by Lemma 4.1, $[\Delta_0] = [D \otimes_K Z(\Delta_0)]$ (in $\mathrm{Br}(Z(\Delta_0))$), for some $D \in d(K)$ inertial over K. The obtained result shows that $[\Delta \otimes_K D^{\mathrm{op}}] \in \mathrm{Br}(Z(\Delta_0)/K)$, where D^{op} is the K-algebra opposite to D. This requires that $\exp(\Delta \otimes_K D^{\mathrm{op}}) \mid p$. Since $\deg(D) = \exp(D) = p^2$, it follows now that $\exp(\Delta) = p^2$, as claimed.

We are now prepared to consider the case of $e(\Delta/K) = p$ in general. The preceding part of our proof allows us to assume that $Z(\widehat{\Delta})$ is a purely inseparable extension of $\widehat{K}$. Note also that $[Z(\widehat{\Delta})\colon \widehat{K}] = p$, and it follows from [9], Theorem 3.1, and [1], Ch. VII, Theorem 28, that $\widehat{\Delta}$ is a cyclic $Z(\widehat{\Delta})$-algebra of degree p. Therefore, there exists $\eta \in \Delta$, which generates an inertial cyclic extension of K of degree p. Hence, by the Skolem–Noether theorem (cf. [28], Sect. 12.6), there is $\xi \in \Delta^*$, such that $\xi\eta'\xi^{-1} = \varphi(\eta')$, for every $\eta' \in K(\eta)$, where φ is a generator of $\mathcal{G}(K(\eta)/K)$. Denote by B the K-subalgebra of Δ generated by η and ξ. It is easy to see that $K(\xi^p) = Z(B)$, $\deg(B) = p$ and B is either an inertial or an NSR-algebra over $K(\xi^p)$. In view of (4.2) (a), this means that $\xi^p \notin K$ which gives $[K(\xi^p)\colon K] = p$, and combined with (4.2) (b), proves that $v(K(\xi^p)) = v(K)$. In other words, $K(\xi^p)^* = O_v(K(\xi^p))^*.K^*$. As $e(\Delta/K) = p$, the obtained properties of B and $K(\xi^p)$ indicate that if $B/K(\xi^p)$ is inertial (equivalently, if $v_\Delta(\xi) \in v(K)$, see [20], Theorem 5.6 (a)), then $\widehat{B} \cong \widehat{\Delta}$ over $\widehat{K}$. This means that Δ/K is subject to the extra hypothesis, which yields $\exp(\Delta) = p^2$. When $B/K(\xi^p)$ is NSR, these properties imply with (4.2) (b) and [28], Section 15.1, Proposition b, the existence of an algebra $\Sigma \in d(K)$ satisfying the following conditions:

(4.3) (a) Σ is isomorphic to the cyclic K-algebra $(K(\eta)/K, \varphi, \pi')$, for some $\pi' \in K^*$; Σ/K is NSR, whence Σ does not embed in Δ as a K-subalgebra;

(b) $\mathrm{ind}(\Delta \otimes_K \Sigma) = p^2$ (see also [28], Section 13.4, and [11], (1.1)(b)), the underlying division K-algebra Δ' of $\Delta \otimes_K \Sigma$ has a K-subalgebra Z' isomorphic to $Z(B)$, and the centralizer $C_{\Delta'}(Z') := C$ is an inertial Z'-algebra. Note here that $[\Delta'] \in \mathrm{Br}(K(\xi^p, \eta)/K)$. Using (3.2) (a), (4.3) and Lemma 3.2 (and also, the double centralizer theorem), one concludes that $[C\colon K] = p^3$ and either Δ'/K is inertial or $e(\Delta'/K) = p$ and $\widehat{C} \cong \widehat{\Delta}'$ as a $\widehat{K}$-algebra. As shown above, this requires that $\exp(\Delta') = p^2$. In view of (4.3) (b) and the equality $\deg(\Sigma) = \exp(\Sigma) = p$, it thereby proves that $\exp(\Delta) = p^2$ as well.

It remains to consider the case where $e(\Delta/K) = p^2$. We first show that one may assume without loss of generality that $\mathrm{Brd}_p(\widehat{K}) = 0$. It follows from (4.2) (a), Lemma 3.2 and the equality $e(\Delta/K) = p^2$ that $\widehat{\Delta}/\widehat{K}$ is a field extension of degree p^2. Using [20], Theorem 3.1, one obtains that $\Delta \otimes_K U \in d(U)$, $v(\Delta \otimes_K U) = v(\Delta)$ and $e((\Delta \otimes_K U)/U) = p^2$, provided U is an extension of K in $K(p) \cap K_{\mathrm{ur}}$, such that no proper extension of $\widehat{K}$ in $\widehat{U}$ is embeddable in $\widehat{\Delta}$ as a $\widehat{K}$-subalgebra. Note also that $\widehat{\Delta} \otimes_{\widehat{K}} \widehat{U}$ is $\widehat{U}$-isomorphic to the residue field of $\Delta \otimes_K U$, which enables one to prove (by applying Galois theory and Zorn's lemma) that U can be chosen so that $r_p(\widehat{U}) \leq 1$. Then, by [21], Proposition 4.4.8, $\mathrm{Br}(\widehat{U})_p = \{0\}$, which leads to the desired reduction.

We suppose further that $\mathrm{Brd}_p(\widehat{K}) = 0$ and prove the following assertion:

(4.4) If Δ possesses a K-subalgebra Z, such that $[Z\colon K] = [\widehat{Z}\colon \widehat{K}] = p$ and $\widehat{Z}$ is purely inseparable over $\widehat{K}$, then $\widehat{\Delta}/\widehat{K}$ is purely inseparable.

Assuming the opposite and using (3.2) (a) and Lemma 3.2, one obtains that Z has an inertial extension M, which is a maximal subfield of Δ. As v is Henselian, the assumptions on Z and M ensure that $M = LZ$, for some inertial extension L of K in M of degree p. Note further that $[M\colon K]$, $[\widehat{M}\colon \widehat{K}]$ and $[\widehat{\Delta}\colon \widehat{K}]$ are equal to p^2, which means that $\widehat{M} = \widehat{\Delta}$. This enables one to deduce from Lemma 3.1 and [20], Proposition 1.7, that L/K is a cyclic extension. At the same time, the equality $\mathrm{Brd}_p(\widehat{K}) = 0$ and Albert–Hochschild's theorem, applied to the extension $\widehat{Z}/\widehat{K}$, indicate that $\mathrm{Brd}_p(\widehat{Z}) = 0$. Therefore, the group $N(M/Z)$ includes $O_v(Z)^*$ (cf. [28], Sect. 15.1, Proposition b), which allows to obtain from the Skolem-Noether theorem and the double centralizer theorem that there is a Z-isomorphism $C_\Delta(Z) \cong (M/Z, \psi', \gamma)$, for some $\gamma \in K^*$ and a generator ψ' of $\mathcal{G}(M/Z)$. This implies $\Delta \cong D_1 \otimes_K D_2$ as K-algebras, where $D_1 = (L/K, \psi, \gamma)$, ψ is the K-automorphism of L induced by ψ', $D_2 \in d(K)$ and $[D_2] \in \mathrm{Br}(Z/K)$. As $\mathrm{Brd}_p(K) = 0$, $\widehat{K}$ is almost perfect and $\deg(D_2) = p$, one obtains further that D_2 has a K-subalgebra T that is a totally ramified extension of K of degree p. It is easy to see that the T-algebra $L \otimes_K T$ is a field. More precisely, $(L \otimes_K T)/T$ is an inertial and cyclic extension of degree p, which allows to deduce consecutively that the norm group $N((L \otimes_K T)/T)$ includes $O_v(T)^*$ and K^*. Observing also that $D_1 \otimes_K T$ is T-isomorphic to $((L \otimes_K T)/T, \psi_T, \gamma)$, where ψ_T is the T-isomorphism of $L \otimes_K T$ extending ψ, one obtains from [28], Sect. 15.1, Proposition b, that $D_1 \otimes_K T \in s(T) \setminus d(T)$, whence, $D_1 \otimes_K T$ contains zero-divisors. As $D_1 \otimes_K T$ is a K-subalgebra of $D_1 \otimes_K D_2$ and $D_1 \otimes_K D_2 \cong \Delta \in d(K)$, this is a contradiction proving (4.4).

We are now in a position to prove Lemma 4.2. If $\widehat{\Delta}/\widehat{K}$ is a purely inseparable field extension, then it follows from Proposition 3.5 and [35], Proposition 2.1, that $\exp(\Delta) = p^2$. Suppose finally that $\widehat{\Delta}$ is a field and $\widehat{\Delta}/\widehat{K}$ is not purely inseparable. In view of [20], Proposition 1.7 and Theorem 2.9, this ensures the existence of an inertial cyclic extension Λ of K of degree p, which embeds in Δ as a K-subalgebra. Our goal is to show that there is an infinite extension W of K in an algebraic closure $\overline{K}$, satisfying the following:

(4.5) $v(W) = v(K)$, $\widehat{W}$ is purely inseparable over $\widehat{K}$ and $\Delta \otimes_K W \in d(W)$.

Note that (4.5) implies $\exp(\Delta) = p^2$. Indeed, $[\widehat{K}\colon \widehat{K}^p] = p$, so it follows from (3.2) (a) and (4.5) that $\widehat{W}$ is perfect; hence, by Lemma 3.6, $(\Delta \otimes_K W)/W$ is NSR and $\exp(\Delta \otimes_K W) = \deg(\Delta \otimes_K W) = p^2$. Since $\exp(\Delta \otimes_K W) \mid \exp(\Delta)$ and $\exp(\Delta) \mid \deg(\Delta) = p^2$, this gives $\exp(\Delta) = p^2$, as required.

Finally, we prove (4.5). Fix an element $a_0 \in O_v(K)^*$ so that $\hat{a}_0 \notin \widehat{K}^p$, take a system $a_n \in \overline{K}$, $n \in \mathbb{N}$, satisfying $a_n^p = a_{n-1}$, for each n, and let W be the union of the fields $W_n = K(a_n)$, $n \in \mathbb{N}$. It is easily verified that $[W_n\colon K] = [\widehat{W}_n\colon \widehat{K}] = p^n$ and $\widehat{W}_n/\widehat{K}$ is purely inseparable, for every $n \in \mathbb{N}$, so it follows from (3.2) (a), the equality $[\widehat{K}\colon \widehat{K}^p] = p$ and the inclusions $W_n \subset W_{n+1}$, $n \in \mathbb{N}$, that W is a field, $v(W) = v(K)$ and $\widehat{W}$ is a perfect closure of $\widehat{K}$. Arguing by induction on n, taking into account that $\Delta \otimes_K W_{n+1}$ and $(\Delta \otimes_K W_n) \otimes_{W_n} W_{n+1}$ are isomorphic as W_{n+1}-algebras, and using (4.4), the noted properties of W_n, and the behaviour of Schur indices under scalar extensions of finite degrees (cf. [28], Sect. 13.4), one obtains that, for each $n \in \mathbb{N}$, $\Delta \otimes_K W_n \in d(W_n)$, and $\Lambda \otimes_K W_n$ is an inertial cyclic extension of W_n of degree p, embeddable in $\Delta \otimes_K W_n$ as a W_n-subalgebra. Therefore, $\Delta \otimes_K W \in d(W)$, so (4.5), Lemma 4.2 and Theorem 2.3 are proved. □

5. Absolutely stable HDV-fields

The first result of this Section states the following:

Corollary 5.1 *Let (K, v) be an* HDV*-field. Then, K is absolutely stable if and only if $\widehat{K}$ is quasilocal and almost perfect; for instance, this holds when $\widehat{K}$ is a complete discrete valued field with a quasifinite residue field.*

Proof Our former conclusion follows from Corollary 2.4, the absolute stability of quasilocal fields, and the fact that the class of almost perfect fields is closed under the formation of finite extensions. Since complete discrete valued fields with quasifinite residue fields are quasilocal and almost perfect (see [30], Ch. XIII, Section 3, and [18], Theorem 12.2.3), our latter conclusion is an immediate consequence of the former one. □

The latter part of Corollary 5.1 can be restated by saying that 2-dimensional local fields K_2 with quasifinite second residue fields K_0 are absolutely stable. This result can be specified as follows:

Proposition 5.2 *An m-dimensional local field K_m with a quasifinite m-th residue field K_0 is stable if and only if $m \leq 2$. When $m \leq 2$, K_m is absolutely stable and* $\mathrm{Brd}_{p'}(K'_m) = 1$*, $p' \in \mathbb{P}$, for every finite extension K'_m/K_m.*

Proof It is known (cf. [30], Ch. XIII, Sect. 3) that if $m = 1$, then K_m is a quasilocal field with $\mathrm{Br}(K_m)$ isomorphic to the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers by the subgroup of integers. This implies K_m is absolutely stable, and $\mathrm{Brd}_{p'}(K'_m) = 1$, for all $p' \in \mathbb{P}$ and $K'_m \in \mathrm{Fe}(K_m)$, as claimed. We assume further that $m \geq 2$. Then, K_m is complete with respect to a discrete valuation w_m whose residue field K_{m-1} is an $(m-1)$-dimensional local field with last residue field isomorphic to K_0. Therefore, (K_m, w_m) is HDV, and by Lemma 3.3, $\mathrm{Brd}_{p'}(K_{m-1}) \leq \mathrm{Brd}_p(K_m)$, for each $p' \in \mathbb{P}$. Suppose now that $m = 2$. As noted above, K_1 is quasilocal with $\mathrm{Br}(K_1) \cong \mathbb{Q}/\mathbb{Z}$; in addition, if $\mathrm{char}(K_1) = \mathrm{char}(K_0)$, then K_1 is isomorphic to the formal power series field $K_0((X_1))$ (see [18], Theorem 12.2.3), which is almost perfect. Hence, by Corollary 5.1, K_2 is absolutely stable and $\mathrm{Brd}_{p'}(K_u) = 1$, $u = 1, 2$, $p' \in \mathbb{P}$. This proves Proposition 5.2, for $m = 2$ (since finite extensions of K_2 are 2-dimensional local fields with quasifinite 2nd residue fields). Next, we prove that $r_2(K_1) \geq 2$. Firstly, if $\mathrm{char}(K_0) = 2$, then [14], Lemma 2.2 and [11], Lemma 4.2, show that $r_2(K_1) = \infty$ unless $\mathrm{char}(K_1) = 0$ and K_0 is finite. Secondly, if $\mathrm{char}(K_1) = 0$, $\mathrm{char}(K_0) = 2$ and K_0 is finite, then it follows from Lemma 3.4 and [31], Ch. II, Theorem 4, that $r_2(K_1) \geq 3$. When $\mathrm{char}(K_0) \neq 2$, K_1^*/K_1^{*2} is a noncyclic group of order 4 (it is isomorphic to the direct sum $K_0^*/K_0^{*2} \oplus w_1(K_1)/2w_1(K_1)$, w_1 being the discrete Henselian valuation of K_1 with $\widehat{K}_1 = K_0$), so it is clear from Kummer theory that $r_2(K_1) = 2$. Lemma 3.1 and the inequality $r_2(K_1) \geq 2$ imply there exist an algebra $\Delta_2 \in d(K_2)$ and a field extension L_2/K_2, such that $\deg(\Delta_2) = [L_2 \colon K_2] = 2$, Δ_2/K_2 is NSR, L_2/K_2 is inertial relative to w_2, and $\Delta_2 \otimes_K L_2 \in d(L_2)$. Thus it follows that K_2 is not 2-quasilocal. Using this result, Proposition 2.1 (a) and Lemma 3.3, one obtains that if $m \geq 3$, then $\mathrm{Brd}_2(K_j) \geq 2$, $j = 3, \dots, m$, which completes our proof. □

In the setting of Proposition 5.2, K_m is p-quasilocal with $\mathrm{Brd}_p(K_m) = 1$, provided that $p \in \mathbb{P}$, $p \neq \mathrm{char}(K_0)$ and K_0 does not contain a primitive p-th root of unity (apply Proposition 2.1 (a) and [12], Corollary 4.3, to the HDV-field (K_m, w_m) with $\widehat{K}_m \cong K_{m-1}$, for $m \geq 2$). When K_0 is finite, this means that $\mathrm{Brd}_p(K_m) = 1$, for all $p \in \mathbb{P}$, with finitely many exceptions in case $m \geq 3$, such as $p = 2$ and $p = \mathrm{char}(K_0)$ (see [12], Proposition 4.4).

Remark 5.3 *Here are two special cases of Corollary 5.1 obtained in [7]:*

(i) *An* HDV*-field* (K, v) *with* $\widehat{K}$ *perfect is absolutely stable if and only if* $\widehat{K}$ *is quasilocal [7], Corollary 4.6;*

(ii) *For any complete discrete valued field* (L, ω) *with a quasifinite residue field* $\widehat{L}$ *(specifically, for any local field* L*), the formal power series field* $L((T))$ *is absolutely stable [7], Corollary 4.5 (ii). If char*$(L) = 0$*, this is also contained in [7], Corollary 4.6. When char*$(L) = p > 0$*,* $L((T))$ *is isomorphic to the iterated formal power series field* $\widehat{L}((Z))((T))$ *(apply [18], Theorem 12.2.3), so the inequality abrd*$_p(L((T))) \leq 1$*, used for proving [7], Corollary 4.5 (ii), follows from [1], Ch. XI, Theorem 3, and results of Aravire, Jacob, Merkurjev and Tignol (see [2], Sect. 3 and the Appendix).*

The concluding result of this paper is new if $\mathrm{char}(K) \neq \mathrm{char}(\widehat{K})$ and $\widehat{K}$ is an imperfect field of type C_1, in the sense of Lang and [31], Ch. II. Under the same hypotheses on $\widehat{K}$, if $\mathrm{char}(K) = \mathrm{char}(\widehat{K})$, then the result is contained in [36], Theorem 2, and in case $\widehat{K}$ is perfect, it follows from [7], Corollary 4.6.

Corollary 5.4 *An* HDV*-field* (K, v) *is absolutely stable, if* $\widehat{K}$ *has type* C_1*.*

Proof The field $\widehat{K}$ is almost perfect with $\mathrm{abrd}_p(\widehat{K}) = 0\colon p \in \mathbb{P}$ (cf. [31], Ch. II, 3.2), so $\widehat{K}$ is quasilocal, and by Corollary 5.1, K is absolutely stable. □

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References

[1] Albert AA. Structure of Algebras. American Mathematical Society Colloquium Publications, XXIV, New York: American Mathematical Society, 1939.

[2] Aravire R, Jacob B. p-algebras over maximally complete fields. With an Appendix by Jean-Pierre Tignol. Proceedings of Symposia in Pure Mathematics 1995; 58 (Part 2, K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, USA, July 6-24, 1992)): 27-49, Providence, RI: American Mathematical Society, 1995.

[3] Artin E, Tate J. Class Field Theory. Reprint of the 1990 second edition. AMS Chelsea Publishing, 366, Providence, RI: American Mathematical Society, 2008.

[4] Auel A, Brussel E, Garibaldi S, Vishne U. Open problems on central simple algebras. Transformation Groups 2011; 16 (1): 219-264. doi: 10.1007/s00031-011-9119-8

[5] Bhaskhar N, Haase B. Brauer p-dimension of complete discretely valued fields. Transactions of the American Mathematical Society 2020; 373 (5), 3709-3732. doi: 10.1090/tran/8038

[6] Brussel E. Noncyclic division algebras over fields of Brauer dimension one. Advances in Mathematics 2020; 366: 107058, ten pp. doi: 10.1016/j.aim.2020.107058

[7] Chipchakov ID. Henselian valued stable fields. Journal of Algebra 1998; 206 (1): 344-369. doi: 10.1006/jabr.1997.7396

[8] Chipchakov ID. Henselian valued quasi-local fields with totally indivisible value groups. Communications in Algebra 1999; 27 (7): 3093-3108. doi: 10.1080/00927879908826612

[9] Chipchakov ID. On the residue fields of Henselian valued stable fields. Journal of Algebra 2008; 319 (1): 16-49. doi: 10.1016/j.jalgebra.2007.08.034

[10] Chipchakov ID. On the behaviour of Brauer p-dimensions under finitely-generated field extensions. Journal of Algebra 2015; 428: 190-204. doi: 10.1016/j.jalgebra.2014.12.035

[11] Chipchakov ID. On Brauer p-dimensions and index-exponent relations over finitely-generated field extensions. Manuscripta Mathematica 2015; 148 (3-4): 485-500. doi: 10.1007/s00229-015-0745-7

[12] Chipchakov ID. On index-exponent relations over Henselian fields with local residue fields. Serdica Mathematical Journal 2018; 44 (3-4): 303-328.

[13] Chipchakov ID. On Brauer p-dimensions and absolute Brauer p-dimensions of Henselian fields. Journal of Pure and Applied Algebra 2019; 223 (1): 10-29. doi: 10.1016/j.jpaa.2018.02.032

[14] Chipchakov ID. On the Brauer p-dimension of Henselian discrete valued fields of residual characteristic $p > 0$. Journal of Pure and Applied Algebra (in press; to appear in 2022). doi: 10.1016/j.jpaa.2021.106948

[15] Clark PL. The period-index problem in WC-groups IV: a local transition theorem. Journal de Théorie des Nombres de Bordeaux 2010; 22 (3): 583-606. doi: 10.5802/jtnb.734

[16] Cohn PM. On extending valuations in division algebras. Studia Scientiarum Mathematicarum Hungarica 1981; 16 (1-2): 65-70.

[17] Draxl P. Ostrowski's theorem for Henselian valued skew fields. Journal für die Reine und Angewandte Mathematik 1984; 354: 213-218. doi: 10.1515/crll.1984.354.213

[18] Efrat I. Valuations, Orderings, and Milnor K-Theory. Mathematical Surveys and Monographs 124, Providence, RI: American Mathematical Society, 2006.

[19] Fesenko IB. Theory of local fields. Local class field theory. Multidimensional local class field theory. (English. Russian original) St. Petersburg Mathematical Journal 1993; 4 (3): 403-438; translation from Algebra i Analiz 1992; 4 (3): 1-41.

[20] Jacob B, Wadsworth AR. Division algebras over Henselian fields. Journal of Algebra 1990; 128 (1): 126-179. doi: 10.1016/0021-8693(90)90047-R

[21] Jacobson N. Finite-Dimensional Division Algebras over Fields. Berlin: Springer-Verlag, X, 1996.

[22] de Jong, AJ. The period-index problem for the Brauer group of an algebraic surface. Duke Mathematical Journal 2004; 123 (1): 71-94. doi: 10.1215/S0012-7094-04-12313-9

[23] Lang S. Algebra. Third revised edition, Graduate Texts in Mathematics, 211, New York, NY: Springer, 2002.

[24] Lieblich M. Twisted sheaves and the period-index problem. Compositio Mathematica 2008; 144 (1): 1-31. doi: 10.1112/S0010437X07003144

[25] Merkur'ev AS, Suslin AA. K-cohomology of Severi-Brauer varieties and norm residue homomomorphisms. (English. Russian original) Mathematics of the USSR, Izvestiya 1983; 21: 307-340; translation from Izvestiya Akademii Nauk SSSR, Seriya Matematicheskaya 1982; 46 (5): 1011-1046.

[26] Merkur'ev AS. Brauer groups of fields. Communications in Algebra 1983; 11 (22): 2611-2624. doi: 10.1080/00927878308822983

[27] Parimala R, Suresh V. Period-index and u-invariant questions for function fields over complete discretely valued fields. Inventiones Mathematicae 2014; 197 (1): 215-235. doi: 10.1007/s00222-013-0483-y

[28] Pierce R. Associative Algebras. Graduate Texts in Mathematics, 88, New York-Heidelberg-Berlin: Springer-Verlag, 1982.

[29] Reiner I. Maximal Orders. London Mathematical Society Monographs, 5, London-New York-San Francisco: Academic Press, a subsidiary of Harcourt Brace Jovanovich, Publishers, XII, 1975.

[30] Serre J-P. Local Fields. Translated from the French original by M.J. Greenberg, Graduate Texts in Mathematics, 67, New York-Heidelberg-Berlin: Springer-Verlag, 1979.

[31] Serre J-P. Galois Cohomology. Translated from the French original by Patrick Ion, Berlin-Heidelberg-New York: Springer-Verlag, X, 1997.

[32] Tignol J-P, Wadsworth AR. Value Functions on Simple Algebras, and Associated Graded Rings. Springer Monographs in Mathematics, Cham: Springer, 2015.

[33] Tomchin IL, Yanchevskij VI. On defects of valued division algebras. (English. Russian original) St. Petersburg Mathematical Journal 1992; 3 (3): 631-647; translation from Algebra i Analiz 1991; 3 (3): 147-164.

[34] Weil A. Basic Number Theory. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 144, Berlin-Heidelberg-New York: Springer-Verlag, XVIII, 1967.

[35] Yamazaki T. Reduced norm map of division algebras over complete discrete valuation fields of certain type. Compositio Mathematica 1998; 112 (2): 127-145. doi: 10.1023/A:1016062718500

[36] Zheglov AB. Wild division algebras over fields of power series. (English. Russian original) Sbornik Mathematics 2004; 195 (6): 783-817; translation from Matematicheskij Sbornik 2004; 195 (6): 21-56. doi: 10.1070/SM2004v195n06ABEH000825

[37] Zhukov I. Higher dimensional local fields. In: Invitation to higher local fields. Extended version of talks given at the conference on higher local fields, Münster, Germany, August 29-September 5, 1999; Coventry: Geometry and Topology Publications, Geometry and Topology Monographs 2000; 3: 5-18.