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Bound for the cocharacters of the identities of irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$

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Dedicated to Vesselin Drensky on his 70th birthday

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Abstract: For each irreducible finite dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of 2×2 traceless matrices, an explicit uniform upper bound is given for the multiplicities in the cocharacter sequence of the polynomial identities satisfied by the given representation.

Key words: Weak polynomial identities, simple Lie algebra, irreducible representation, cocharacter sequence

1. Introduction

Let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a finite dimensional representation of the Lie algebra $\mathfrak{g}$ over a field K of characteristic zero; that is, $\mathfrak{gl}(V)=\mathrm{End}_K(V)$, the space of all K-linear transformations of the finite dimensional K-vector space V, viewed as a Lie algebra with Lie product $[A,B]:=A\circ B-B\circ A$ for $A,B\in\mathrm{End}_K(V)$, and ρ is a homomorphism of Lie algebras. Denote by $F_m:=K\langle x_1,\dots,x_m\rangle$ the free associative K-algebra with m generators. Consider F_m as a subalgebra of F_{m+1} in the obvious way, and write $F:=\bigcup_{m=1}^{\infty}F_m$ for the free associative algebra of countable rank. We say that $f=0$ *is an identity of the representation* ρ *of* $\mathfrak{g}$ (or briefly, of the pair $(\mathfrak{g},\rho)$) for some $f\in F_m$ if for any elements $A_1,\dots,A_m\in\mathfrak{g}$ we have the following equality in the associative K-algebra $\mathrm{End}_K(V)$:

$$f(\rho(A_1),\dots,\rho(A_n))=0\in\mathrm{End}_K(V).$$

Note that an identity of the representation ρ of the Lie algebra $\mathfrak{g}$ is also called in the literature a *weak polynomial identity* for the pair $(\mathrm{End}_K(V),\rho(\mathfrak{g}))$. This notion was introduced and powerfully applied first by Razmyslov [13–16] (see Drensky [8] for a recent survey on weak polynomial identities). Set

$$I(\mathfrak{g},\rho):=\{f\in F\mid f=0\text{ is an identity of }(\mathfrak{g},\rho)\}.$$

Clearly $I(\mathfrak{g},\rho)$ is an ideal in F stable with respect to all K-algebra endomorphisms of F of the form $x_i\mapsto u_i$, where u_i for $i=1,2,\dots$ is an element of the Lie subalgebra of F generated by $x_1,x_2,\dots$. In particular, the general linear group $\mathrm{GL}_m(K)$ acts on F_m via K-algebra automorphisms: for $g=(g_{ij})_{i,j=1}^m$ we have $g\cdot x_j=\sum_{i=1}^m g_{ij}x_i$, and $I(\mathfrak{g},\rho)\cap F_m$ is a $\mathrm{GL}_m(K)$-invariant subspace of F_m. The *multilinear component* of

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F_m is

$$P_m := \mathrm{Span}_K\{x_{\pi(1)} \cdots x_{\pi(m)} \mid \pi \in S_m\},$$

where S_m is the symmetric group of degree m. It is well known that when $\mathrm{char}(K) = 0$, the ideal $I(\mathfrak{g}, \rho)$ is determined by the multilinear components $I(\mathfrak{g}, \rho) \cap P_m$, $m = 1, 2, \dots$. Identifying S_m with the subgroup of permutation matrices in $\mathrm{GL}_m(K)$ we get its action on F_m via K-algebra automorphisms (more explicitly, $\pi \in S_m$ is the automorphism of F_m given by $x_i \mapsto x_{\pi(i)}$), and the subspaces P_m and $I(\mathfrak{g}, \rho) \cap P_m$ are S_m-invariant. Define the *mth cocharacter of* $(\mathfrak{g}, \rho)$ as

$$\chi_m(\mathfrak{g}, \rho) := \text{ the character of the } S_m\text{-module } P_m/(I(\mathfrak{g}, \rho) \cap P_m).$$

We call

$$\chi(\mathfrak{g}, \rho) := (\chi_m(\mathfrak{g}, \rho) \mid m = 1, 2, \dots)$$

the *cocharacter sequence of* $(\mathfrak{g}, \rho)$. The irreducible S_m-modules are labeled by partitions of m; let χ^λ denote the character of the irreducible S_m-module associated to the partition $\lambda = (\lambda_1, \dots, \lambda_m) \vdash m$. We have

$$\chi_m(\mathfrak{g}, \rho) = \sum_{\lambda \vdash m} \mathrm{mult}_\lambda(\mathfrak{g}, \rho)\chi^\lambda,$$

and we are interested in the multiplicities $\mathrm{mult}_\lambda(\mathfrak{g}, \rho)$ of the irreducible S_m-characters in the cocharacter sequence. Note that the value of $\chi_m(\mathfrak{g}, \rho)$ on the identity element of S_m is

$$c_m(\mathfrak{g}, \rho) := \dim_K(P_m/(I(\mathfrak{g}, \rho) \cap P_m)),$$

and

$$(c_m(\mathfrak{g}, \rho) \mid m = 1, 2, \dots)$$

is called the *codimension sequence of* $(\mathfrak{g}, \rho)$. It was proved by Gordienko [10] that $\lim_{m\to\infty} \sqrt[m]{c_m(\mathfrak{g}, \rho)}$ exists and is an integer. As is observed in [10, Example 3], an obvious upper bound for $c_m(\mathfrak{g}, \rho)$ can be obtained from the fact that there is a natural K-linear embedding

$$P_m/(I(\mathfrak{g}, \rho) \cap P_m) \hookrightarrow \mathrm{Hom}_K(\rho(\mathfrak{g})^{\otimes m}, \mathrm{End}_K(V)). \tag{1.1}$$

Our starting observation is that the adjoint representation of $\mathfrak{g}$ on itself induces a natural representation of $\mathfrak{g}$ on $\rho(\mathfrak{g})^{\otimes m}$ (the mth tensor power of $\rho(\mathfrak{g})$) and on $\mathrm{End}_K(V)$, such that the image of the embedding (1.1) is contained in the subspace of $\mathfrak{g}$-module homomorphisms from $\rho(\mathfrak{g})^{\otimes m}$ to $\mathrm{End}_K(V)$. So (1.1) can be refined as

$$P_m/(I(\mathfrak{g}, \rho) \cap P_m) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(\rho(\mathfrak{g})^{\otimes m}, \mathrm{End}_K(V)). \tag{1.2}$$

This will be used to give an upper bound for the multiplicities in the cocharacter sequence $\chi(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)})$ of the d-dimensional irreducible representation

$$\rho^{(d)} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}^d) = \mathbb{C}^{d\times d}$$

of $\mathfrak{sl}_2(\mathbb{C})$ for $d = 1, 2, \dots$. Note that throughout the paper we shall identify $\mathfrak{gl}(\mathbb{C}^d)$ with the associative algebra $\mathbb{C}^{d\times d}$ of $d \times d$ complex matrices, viewed as a Lie algebra with Lie bracket $[A, B] = AB - BA$.

Theorem 1.1 *The multiplicity* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$ *in* $\chi(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$ *is nonzero only if* $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ *(i.e.* λ *has at most* 3 *nonzero parts), and in this case we have the inequality*

$$\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})\le 3^{d-2}.$$

Remark 1.2 *(i) The exact values of* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$ *are known for* $d\le 3$. *For* $d=1$ *all the multiplicities are obviously zero. It was proved in [12] (see also [7, Exercise 12.6.12]) that*

$$\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(2)})=1 \text{ for all } \lambda=(\lambda_1,\lambda_2,\lambda_3).$$

The multiplicities $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(3)})$ *are computed in [5, Theorem 3.7, Proposition 3.8]. It turns out that* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(3)})\in\{1,2,3\}$ *for each* $\lambda=(\lambda_1,\lambda_2,\lambda_3)$.

(ii) Theorem 1.1 shows in particular that for each dimension d, *there is a uniform bound (depending on* d *only) for the multiplicities* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$. *For comparison we mention that the multiplicities in the cocharacter sequence of the ordinary polynomial identities of* 2×2 *matrices are unbounded: see [6]* and *[9]. For example, for any partition* $\lambda=(\lambda_1,\lambda_2)$ *with* $\lambda_2>0$, *the multiplicity is* $(\lambda_1-\lambda_2+1)\lambda_2$. *On the other hand, the cocharacter multiplicities of any PI algebra are polynomially bounded by [2].*

(iii) There is no uniform upper bound independent of d *for the multiplicities* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$, *because by Proposition 4.1,* $\max\{\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})\mid m=1,2,\dots,\ \lambda\vdash m\}\ge d-1$ *for* $d\ge 2$.

(iv) The irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$ *are defined over* $\mathbb{Q}$. *For any field* K *of characteristic zero and any positive integer* d, *the Lie algebra* $\mathfrak{sl}_2(K)$ *has a unique (up to isomorphism)* d*-dimensional irreducible representation* $\rho_K^{(d)}$ *over* K. *By well-known general arguments, the multiplicities* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(K),\rho_K^{(d)})$ *do not depend on* K. *Therefore Theorem 1.1 implies that* $\mathrm{mult}_\lambda(\mathfrak{sl}_2(K),\rho_K^{(d)})\le 3^{d-2}$ *for any field* K *of characteristic zero.*

(v) A different interpretation and approach to the study of $\mathrm{Hom}_{\mathfrak{g}}(\rho(\mathfrak{g})^{\otimes m},\mathrm{End}_K(V))$ *for* $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$ *and* $\rho=\rho^{(d)}$ *is given in our parallel preprint [4], using classical invariant theory.*

We close the introduction by mentioning the recent paper of da Silva Macedo and Koshlukov [3, Theorem 3.7], where the codimension growth of polynomial identities of representations of Lie algebras is studied. In particular, in [3, Theorem 3.7] the identities of representations of $\mathfrak{sl}_2(\mathbb{C})$ play a decisive role.

2. Matrix computations

Denote by $\widetilde{\rho}^{(d)}:\mathfrak{sl}_2(\mathbb{C})\to\mathfrak{gl}(\mathbb{C}^{d\times d})$ the representation given by

$$\widetilde{\rho}^{(d)}(A)(L)=\rho^{(d)}(A)L-L\rho^{(d)}(A) \text{ for } A\in\mathfrak{sl}_2(\mathbb{C}),\ L\in\mathbb{C}^{d\times d}. \tag{2.1}$$

We have $\widetilde{\rho}^{(d)}\cong\rho^{(d)}\otimes{\rho^{(d)}}^*$. The representations of $\mathfrak{sl}_2(\mathbb{C})$ are self-dual, and so by the Clebsch-Gordan rules we have

$$\widetilde{\rho}^{(d)}\cong\rho^{(d)}\otimes\rho^{(d)}\cong\bigoplus_{n=1}^{d}\rho^{(2n-1)}. \tag{2.2}$$

We shall need an explicit decomposition of $\mathbb{C}^{d\times d}$ as a direct sum of minimal $\widetilde{\rho}^{(d)}$-invariant subspaces.

Set

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so e, f, h is a $\mathbb{C}$-vector space basis of $\mathfrak{sl}_2(\mathbb{C})$, with $[h,e] = 2e$, $[h,f] = -2f$, and $[e,f] = h$.

Recall that given a representation $\psi : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$, by a *highest weight vector* we mean a nonzero element $w \in V$ such that $\psi(e)(w) = 0 \in V$ and $\psi(h)(w) = nw$ for some nonnegative integer n (the nonnegative integer n is called the *weight* of w); in this case w generates a minimal $\mathfrak{sl}_2(\mathbb{C})$-invariant subspace in V, on which the representation of $\mathfrak{sl}_2(\mathbb{C})$ is isomorphic to $\rho^{(n+1)}$. Moreover, any finite dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module contains a unique (up to nonzero scalar multiples) highest weight vector.

Lemma 2.1 *Consider the $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}^{d\times d}$ via the representation $\widetilde{\rho}^{(d)}$. To simplify notation set $\rho := \rho^{(d)}$ and $\widetilde{\rho} := \widetilde{\rho}^{(d)}$.*

(i) $\rho(e)^n$ is a highest weight vector in $\mathbb{C}^{d\times d}$ of weight $2n$ for $n = 0, 1, \dots, d-1$.

(ii) $\rho(e)^{n-1}$ generates a minimal $\widetilde{\rho}$-invariant subspace V_n on which $\mathfrak{sl}_2(\mathbb{C})$ acts via $\rho^{(2n-1)}$ for $n = 1, \dots, d$.

(iii) $\mathbb{C}^{d\times d} = \bigoplus_{n=1}^{d} V_n$.

(iv) For $L_1 \in V_{n_1}$ and $L_2 \in V_{n_2}$ with $1 \le n_1 \ne n_2 \le d$ we have $\mathrm{Tr}(L_1 L_2) = 0$.

Proof (i) We have $\widetilde{\rho}(e)(\rho(e)^n) = \rho(e)\rho(e)^n - \rho(e)^n\rho(e) = 0$ and

$$\widetilde{\rho}(h)(\rho(e)^n) = \rho([h,e])\rho(e)^{n-1} + \rho(e)\rho([h,e])\rho(e)^{n-2} + \cdots + \rho(e)^{n-1}\rho([h,e]) = 2n\rho(e)^n.$$

This shows that $\rho(e)^n$ is the highest weight vector of weight $2n$ for the representation $\widetilde{\rho}$.

(ii) Statement (i) implies that $\rho(e)^{n-1}$ generates an irreducible $\mathfrak{sl}_2(\mathbb{C})$-submodule of $\widetilde{\rho}$ isomorphic to $\rho^{(2n-1)}$ for $n = 1, \dots, d$.

(iii) follows from (ii) and (2.2).

(iv) Consider the symmetric nondegenerate bilinear form

$$\beta : \mathbb{C}^{d\times d} \times \mathbb{C}^{d\times d} \to \mathbb{C}, \quad (L, M) \mapsto \mathrm{Tr}(LM).$$

Note that β is $\widetilde{\rho}$-invariant:

$$\begin{aligned} \beta([\rho(A), L], M) + \beta(L, [\rho(A), M]) &= \mathrm{Tr}([\rho(A), L]M) + \mathrm{Tr}(L[\rho(A), M]) \\ &= \mathrm{Tr}([\rho(A), LM]) = 0 \quad \text{for any } A \in \mathfrak{sl}_2(\mathbb{C}). \end{aligned}$$

The radical of the bilinear form $\beta_{V_n} : V_n \times V_n \mapsto \mathbb{C}$ (the restriction of β to $V_n \times V_n$) is a $\widetilde{\rho}$-invariant subspace in V_n, so it is either V_n or $\{0\}$. We claim that it is not V_n. Indeed, V_n contains a nonzero diagonal matrix D with real entries, since the zero weight subspace in $\mathbb{C}^{d\times d}$ (with respect to $\widetilde{\rho}(h)$) is the subspace of diagonal matrices, and V_n intersects the zero-weight space in a 1-dimensional subspace (defined over the reals). Now being a sum of squares of nonzero real numbers, $0 \ne \mathrm{Tr}(D^2) = \beta(D, D)$. Thus β_{V_n} is nondegenerate. The representation $\widetilde{\rho}$ is multiplicity free by (2.2), and by (ii) and (iii), every $\widetilde{\rho}$-invariant subspace is of the form $\sum_{j\in J} V_j$ for some subset $J \subseteq \{1, 2, \dots, d\}$. As we showed above, the orthogonal complement of V_n (with respect to β) is disjoint from V_n, so it is the sum of the other minimal invariant subspaces V_j, $j \in \{1, \dots, d\} \setminus \{n\}$. □

The representation $\rho^{(2)}$ is the defining representation of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$, and $\rho^{(d)}$ is the $(d-1)$th symmetric tensor power of $\rho^{(2)}$. Denote by x, y the standard basis vectors in $\mathbb{C}^2$, and take the basis $x^{d-1}, x^{d-1}y, \dots, y^{d-1}$ in the $(d-1)$th symmetric tensor power of $\mathbb{C}^2$. Then denoting by $E_{i,j}$ the matrix unit with entry 1 in the (i,j) position and zeros in all other positions, the representation $\rho^{(d)}$ as a matrix representation $\mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}^{d\times d}$ is given as follows:

$$\rho^{(d)}(e) = \sum_{i=1}^{d-1} iE_{i,i+1}, \quad \rho^{(d)}(f) = \sum_{i=1}^{d-1}(d-i)E_{i+1,i}, \quad \rho^{(d)}(h) = \sum_{i=1}^{d}(d+1-2i)E_{i,i}$$

Lemma 2.2 *For $d \geq 3$ the $\mathbb{C}$-vector space $\mathbb{C}^{d\times d}$ is spanned by*

$$\{\rho^{(d)}(A_1)\cdots\rho^{(d)}(A_{d-1}) \mid A_1, \dots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C})\}.$$

Proof To simplify the notation write $\rho := \rho^{(d)}$ and $\widetilde{\rho} := \widetilde{\rho}^{(d)}$. Let $\mathcal{L}$ be the subspace of $\mathbb{C}^{d\times d}$ spanned by the products $\rho(A_1)\cdots\rho(A_{d-1})$, where $A_1, \dots, A_{d-1} \in \mathfrak{sl}_2(\mathbb{C})$. Clearly $\mathcal{L}$ is a $\widetilde{\rho}$-invariant subspace of $\mathbb{C}^{d\times d}$. Since the representation $\widetilde{\rho}$ is multiplicity free by (2.2), we have $\mathcal{L} = \sum_{j\in J} V_j$ for some subset $J \subseteq \{1, 2, \dots, d\}$ by Lemma 2.1 (ii) and (iii). Therefore to prove the equality $\mathcal{L} = \mathbb{C}^{d\times d}$ it is sufficient to show that $\mathcal{L} \cap V_n \neq \{0\}$ for each $n = 1, \dots, d$, or equivalently, that $\mathcal{L}$ is not contained in $\sum_{j\in\{1,\dots,d\}\setminus\{n\}} V_j$. Since V_d is generated by $\rho(e)^{d-1} \in \mathcal{L}$, we have $V_d \subseteq \mathcal{L}$. Moreover, to prove $\mathcal{L} \nsubseteq \sum_{j\in\{1,\dots,d\}\setminus\{n+1\}} V_j$ for $n \in \{0, 1, \dots, d-2\}$, it is sufficient to present an element $L_n \in \mathcal{L}$ with $\mathrm{Tr}(\rho(e)^n L_n) \neq 0$ by Lemma 2.1 (ii) and (iv). We shall give below such elements $L_n \in \mathcal{L}$ for $n = 0, 1, \dots, d-2$.

For $n = 1, \dots, d-1$ we have

$$\rho(e)^n = \sum_{j=1}^{d-n} j\cdot(j+1)\cdots(j+n-1)E_{j,j+n}$$

$$\rho(f)^n = \sum_{j=1}^{d-n}(d-j)\cdot(d-j-1)\cdots(d-j-n+1)E_{j+n.j}$$

and $\rho(e)^0 = I_d = \rho(f)^0$, where I_d is the $d\times d$ identity matrix. It follows that for $n = 1, \dots, d-1$,

$$\rho(e)^n\rho(f)^n = \sum_{j=1}^{d-n} j(j+1)\cdots(j+n-1)\cdot(d-j)(d-j-1)\cdots(d-j-n+1)E_{j,j}$$

is a diagonal matrix with nonnegative integer entries, and the $(1,1)$-entry is positive. The same holds for $\rho(e)^0\rho(f)^0 = I_d$. For n with $d-1-n$ even, $\rho(h)^{d-1-n}$ is the square of a diagonal matrix with integer entries, and its $(1,1)$-entry is positive. Hence $\mathrm{Tr}(\rho(e)^n\rho(f)^n\rho(h)^{d-1-n}) \neq 0$, being a positive integer. So in this case we may take $L_n := \rho(f)^n\rho(h)^{d-1-n}$. For $n < d-2$ with $d-1-n$ odd, note that $\rho(e)\rho(f) - \rho(f)\rho(e) = \rho([e,f]) = \rho(h)$, and thus

$$\rho(f)^n\rho(h)^{d-2-n} = \rho(f)^n\rho(h)^{d-3-n}(\rho(e)\rho(f) - \rho(f)\rho(e))$$

also belongs to $\mathcal{L}$. Since $\rho(h)^{d-2-n}$ is a diagonal matrix with nonnegative integer entries, and with a positive $(1,1)$-entry, we may take $L_n := \rho(f)^n \rho(h)^{d-2-n}$ in this case. It remains to deal with the case $n = d-2$. Then

$$\rho(e)^{d-2}\rho(f)^{d-2} = (d-1)((d-2)!)^2 \cdot (E_{1,1} + E_{2,2}),$$

hence taking $L_{d-2} := \rho(f)^{d-2}\rho(h)$ we get

$$\begin{aligned}\mathrm{Tr}(\rho(e)^{d-2}L_{d-2}) &= \mathrm{Tr}((d-1)((d-2)!)^2 \cdot ((d-1)E_{11} + (d-3)E_{22}) \\ &= (2d-4)(d-1)((d-2)!)^2,\end{aligned}$$

which is nonzero for $d \geq 3$. This finishes the proof of the equality $\mathcal{L} = \mathbb{C}^{d\times d}$ for $d \geq 3$. □

3. Adjoint invariants

Denote by $\mathrm{ad} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C}))$ the *adjoint representation* of $\mathfrak{sl}_2(\mathbb{C})$ on itself, so $\mathrm{ad}(A)(B) = [A,B]$ for $A, B \in \mathfrak{sl}_2(\mathbb{C})$. Take the n-fold direct sum $\mathrm{ad}^{\oplus n} : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})^{\oplus n})$ of the adjoint representation, and write $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ for the algebra of $\mathrm{ad}^{\oplus n}$-invariant polynomial functions on $\mathfrak{sl}_2(\mathbb{C})^{\oplus n}$. There is a right action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})^n$ that commutes with $\mathrm{ad}^{\oplus n}$: for $g = (g_{ij})_{i,1=1}^n$ and $(A_1, \dots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have

$$(A_1, \dots, A_n) \cdot g := (\sum_{i=1}^n g_{i1}A_i, \dots, \sum_{i=1}^n g_{in}A_i).$$

This induces a left $\mathrm{GL}_n(\mathbb{C})$-action on the coordinate ring $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$: for $g \in \mathrm{GL}_n(\mathbb{C})$, $f \in \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ and $(A_1, \dots, A_n) \in \mathfrak{sl}_2(\mathbb{C})^n$ we have $(g \cdot f)(A_1, \dots, A_n) = f((A_1, \dots, A_n) \cdot g)$.

Lemma 3.1 *Consider the linear map* $\iota : F_m = \mathbb{C}\langle x_1, \dots, x_m\rangle \to \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ *given by*

$$\iota(f)(A_1, \dots, A_{m+d-1}) = \mathrm{Tr}(f(\rho^{(d)}(A_1), \dots, \rho^{(d)}(A_m)) \cdot \rho^{(d)}(A_{m+1}) \cdots \rho^{(d)}(A_{m+d-1}))$$

for $f \in F_m$ *and* $(A_1, \dots, A_{m+d-1}) \in \mathfrak{sl}_2(\mathbb{C})^{m+d-1}$. *It has the following properties:*

(i) The image of ι *is contained in the subalgebra* $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]^{\mathfrak{sl}_2(\mathbb{C})}$ *of* $\mathfrak{sl}_2(\mathbb{C})$*-invariants.*

(ii) For $d \geq 3$ *the kernel of* ι *is the ideal* $I(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \cap F_m$.

(iii) The map ι *is* $\mathrm{GL}_m(\mathbb{C})$*-equivariant, where we restrict the* $\mathrm{GL}_{m+d-1}(\mathbb{C})$*-action on* $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]$ *to the subgroup* $\mathrm{GL}_m(\mathbb{C}) \cong \{\begin{pmatrix} g & 0 \\ 0 & I_{d-1}\end{pmatrix} \mid g \in \mathrm{GL}_m(\mathbb{C})\}$ *in* $\mathrm{GL}_{m+d-1}(\mathbb{C})$.

Proof For notational simplicity we shall write ρ instead of $\rho^{(d)}$.

(i) By linearity of ι it is sufficient to show that $\iota(x_{i_1} \cdots x_{i_k})$ is an $\mathfrak{sl}_2(\mathbb{C})$-invariant for any $i_1, \dots, i_k \in \{1, \dots, m\}$. Setting $n = k+d-1$, $B_1 = A_{i_1}, \dots, B_k = A_{i_k}, B_{k+1} = A_{m+1}, \dots, B_n = A_{m+d-1}$ we have

$$\iota(x_{i_1} \cdots x_{i_k})(A_1, \dots, A_{m+d-1}) = \mathrm{Tr}(\rho(B_1) \cdots \rho(B_n)). \tag{3.1}$$

For any $X \in \mathfrak{sl}_2(\mathbb{C})$ we have

$$\begin{aligned}0 &= \mathrm{Tr}([\rho(X), \rho(B_1)\cdots\rho(B_n)]) \\ &= \mathrm{Tr}(\sum_{j=1}^{n} \rho(B_1)\cdots\rho(B_{j-1})[\rho(X), \rho(B_j)]\rho(B_{j+1})\cdots\rho(B_n)) \\ &= \mathrm{Tr}(\sum_{j=1}^{n} \rho(B_1)\cdots\rho(B_{j-1})\rho([X, B_j])\rho(B_{j+1})\cdots\rho(B_n)) \\ &= \sum_{j=1}^{n} \mathrm{Tr}(\rho(B_1)\cdots\rho(B_{j-1})\rho([X, B_j])\rho(B_{j+1})\cdots\rho(B_n)).\end{aligned}$$

The equalities (3.1) and

$$\sum_{j=1}^{n} \mathrm{Tr}(\rho(B_1)\cdots\rho(B_{j-1})\rho([X, B_j])\rho(B_{j+1})\cdots\rho(B_n)) = 0$$

mean that $\iota(x_{i_1}\cdots x_{i_k})$ is $\mathfrak{sl}_2(\mathbb{C})$-invariant, so (i) holds.

(ii) Suppose that $f \in \ker(\iota)$. Then $\mathrm{Tr}(f(\rho(A_1), \dots, \rho(A_m))B) = 0$ for all $A_1, \dots, A_m \in \mathfrak{sl}_2(\mathbb{C})$ and for all $B \in \mathbb{C}^{d\times d}$ by Lemma 2.2. By nondegeneracy of the trace we get $f(\rho(A_1), \dots, \rho(A_m)) = 0$ for all $A_1, \dots, A_m \in \mathfrak{sl}_2(\mathbb{C})$. That is, $f \in I(\mathfrak{sl}_2(\mathbb{C}), \rho)$. Thus $\ker(\iota) \subseteq I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m$. The reverse inclusion $I(\mathfrak{sl}_2(\mathbb{C}), \rho) \cap F_m \subseteq \ker(\iota)$ is obvious.

(iii) Take $g = (g_{ij})_{i,j=1}^m \in \mathrm{GL}_m(\mathbb{C})$. For $f \in F_m$ and $(A_1, \dots, A_{m+d-1}) \in \mathfrak{sl}_2(\mathbb{C})^{m+d-1}$ we have (by linearity of ρ)

$$\begin{aligned}&\iota(g\cdot f)(A_1, \dots, A_{m+d-1}) \\ &= \mathrm{Tr}(f(\sum_{i=1}^{m} g_{i1}\rho(A_i), \dots, \sum_{i=1}^{m} g_{im}\rho(A_i)) \cdot \rho(A_{m+1})\cdots\rho(A_{m+d-1})) \\ &= \mathrm{Tr}(f(\rho(\sum_{i=1}^{m} g_{i1}(A_i)), \dots, \rho(\sum_{i=1}^{m} g_{im}(A_i)) \cdot \rho(A_{m+1})\cdots\rho(A_{m+d-1})) \\ &= (g\cdot\iota(f))(A_1, \dots, A_{m+d-1}).\end{aligned}$$

This shows (iii). □

Restricting the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ to the subgroup of diagonal matrices we get an $\mathbb{N}_0^n$-grading on $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$, preserved by the action of $\mathfrak{sl}_2(\mathbb{C})$. Denote by $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ the multihomogeneous component of multidegree $(1, \dots, 1)$; this is the space of n-linear functions on $\mathfrak{sl}_2(\mathbb{C})$. The spaces $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}$ and $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]_{(1^n)}^{\mathfrak{sl}_2(\mathbb{C})}$ are S_n-invariant (where we restrict the $\mathrm{GL}_n(\mathbb{C})$-action to its subgroup S_n of permutation matrices). Lemma 3.1 has the following immediate consequence:

Corollary 3.2 *For $d \geq 3$ the restriction of ι to the multilinear component P_m of $\mathbb{C}\langle x_1, \dots, x_m\rangle$ factors through an S_m-equivariant $\mathbb{C}$-linear embedding*

$$\bar{\iota} : P_m/(I(\mathfrak{sl}_2(\mathbb{C})) \cap P_m) \to \mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]_{(1^{m+d-1})}^{\mathfrak{sl}_2(\mathbb{C})}$$

where on the right hand side we consider the restriction of the S_{m+d-1}-action to its subgroup S_m (the stabilizer in S_{m+d-1} of the elements $m+1, m+2, \dots, m+d-1$).

For a partition $\lambda \vdash m$ denote by $r(\lambda)$ the multiplicity of χ^λ in the restriction to S_m of the S_{m+d-1}-module $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^{m+d-1}]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^{m+d-1})}$. Corollary 3.2 immediately implies the following:

Corollary 3.3 *For $d \geq 3$ and any partition $\lambda \vdash m$ we have the inequality*

$$\mathrm{mult}_\lambda(\mathfrak{sl}_2(\mathbb{C}), \rho^{(d)}) \leq r(\lambda).$$

The S_n-character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^n)}$ is known:

Proposition 3.4 *For a partition $\lambda \vdash n$ denote by $\nu(\lambda)$ the multiplicity of χ^λ in the S_n-character of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}_{(1^n)}$. Then we have*

$$\nu(\lambda) = \begin{cases} 1 \text{ for } \lambda = (\lambda_1, \lambda_2, \lambda_3) \text{ with } \lambda_1 \equiv \lambda_2 \equiv \lambda_3 \text{ modulo } 2 \\ 0 \text{ otherwise.} \end{cases}$$

Proof The $\mathrm{GL}_n(\mathbb{C})$-module structure of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ is given for example in [12, Theorem 2.2]. The isomorphism types of the irreducible $\mathrm{GL}_n(\mathbb{C})$-module direct summands of the degree n homogeneous component of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]$ are labeled by partitions of n with at most 3 nonzero parts. The multiplicity $\mu(\lambda)$ of the irreducible $\mathrm{GL}_n(\mathbb{C})$-module W_λ in the degree n homogeneous component of $\mathcal{O}[\mathfrak{sl}_2(\mathbb{C})^n]^{\mathfrak{sl}_2(\mathbb{C})}$ is 1 if $\lambda_1, \lambda_2, \lambda_3$ have the same parity and is zero otherwise. Note finally that the multilinear component of W_λ is S_n-stable, and its S_n-character is χ^λ (see for example [1, Corollary 6.3.11]). □

Following [11, Section I.1] for partitions $\lambda \vdash n$ and $\mu \vdash k$ we write $\lambda \subset \mu$ is $\lambda_i \leq \mu_i$ for all i. Moreover, given $\lambda \vdash m$ and $\mu \vdash m+d-1$ with $\lambda \subset \mu$, by a *standard tableau of shape μ/λ* we mean a sequence $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(d-1)}$ of partitions $\lambda^{(i)} \vdash m+i$, where $\lambda^{(0)} = \lambda$, $\lambda^{(d-1)} = \mu$. By the well-known branching rules for the symmetric group, for $\lambda \vdash m$ the multiplicity of χ^λ in the restriction to S_m of the irreducible S_{m+d-1}-character χ^μ equals the number of standard tableaux of shape μ/λ (see for example [1, Theorem 6.4.11]). Therefore Proposition 3.4 has the following consequence.

Corollary 3.5 *We have the equality*

$$r(\lambda) = |\{T \mid T \text{ is a standard skew tableau of shape } \mu/\lambda,$$
$$\mu \vdash m+d-1,\ \mu = (\mu_1, \mu_2, \mu_3),\ \mu_1 \equiv \mu_2 \equiv \mu_3 \text{ modulo } 2\}|.$$

Corollary 3.6 *For $d \geq 3$ we have the inequality $r(\lambda) \leq 3^{d-2}$.*

Proof Associate to a standard skew tableau $T = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(d-1)}$ of shape μ/λ, where $\mu = (\mu_1, \mu_2, \mu_3) \vdash m+d-1$ and $\mu_1 \equiv \mu_2 \equiv \mu_3$ modulo 2 the function $f_T : \{1, \dots, d-1\} \to \{1, 2, 3\}$, which maps $j \in \{1, \dots, d-1\}$ to the unique $i \in \{1, 2, 3\}$ such that the ith component of the partition $\lambda^{(j)}$ is 1 greater than the ith component of $\lambda^{(j-1)}$. The assignment $T \mapsto f_T$ is obviously an injective map from the set of standard skew tableaux of shape μ/λ into the set of functions $\{1,\dots,d-1\}\to\{1,2,3\}$. We claim that at most 3^{d-2} functions are contained in the image of this map. Indeed, if the three parts of $\lambda^{(d-3)}$ have the same parity, then $(f_T(d-2),f_T(d-1))\in\{(1,1),(2,2),(3,3)\}$, since the three parts of $\mu=\lambda^{(d-1)}$ must have the same parity. If the three parts of $\lambda^{(d-3)}$ do not have the same parity, say the first two components of $\lambda^{(d-3)}$ have the same parity, and the third part has the opposite parity, then $(f_T(d-2),f_T(d-1))\in\{(1,2),(2,1)\}$. Hence $r(\lambda)$ is not greater than 3-times the number of functions from a $(d-3)$-element set to a 3-element set. Thus $r(\lambda)\le 3^{d-2}$. □

3.1. Proof of Theorem 1.1

For $d\ge 3$ the statement follows from Corollary 3.3 and Corollary 3.6. For the cases $d\le 3$ see Remark 1.2 (i).

4. A lower bound

Proposition 4.1 *For $d\ge 2$ we have the equality*

$$\mathrm{mult}_{(d-1,1)}(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})=d-1.$$

Proof For $k=0,1,\dots,d-2$ consider the element

$$w_k:=x_1^k[x_1,x_2]x_1^{d-2-k}\in\mathbb{C}\langle x_1,x_2\rangle=F_2.$$

These elements are $\mathrm{GL}_2(\mathbb{C})$-highest weight vectors with weight $(d-1,1)$, hence each generates an irreducible $\mathrm{GL}_2(\mathbb{C})$-submodule isomorphic to $W_{(d-1,1)}$ (see the proof of Proposition 3.4 for the notation W_λ: it is the polynomial $\mathrm{GL}_2(\mathbb{C})$-module with highest weight $\lambda=(\lambda_1,\lambda_2)$). Moreover, the w_k $(k=0,1,\dots,d-2)$ are linearly independent modulo the ideal $I(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$: indeed, make the substitution $x_1\mapsto\rho(h)$, $x_2\mapsto\rho(e)$. Then we get

$$\begin{aligned}w_k(\rho(h),\rho(e))&=(\sum_{i=1}^{d}(d+1-2i)E_{i,i})^k\cdot(2\sum_{i=1}^{d-1}iE_{i,i+1})\cdot(\sum_{i=1}^{d}(d+1-2i)E_{i,i})^{d-2-k}\\&=2\sum_{i=1}^{d-1}i(d+1-2i)^k(d-1-2i)^{d-2-k}E_{i,i+1}.\end{aligned}$$

Denote by $Z=(Z_{i,j})_{i,j=1}^{d-1}$ the $(d-1)\times(d-1)$ matrix whose $(i,k+1)$ entry is the $(i,i+1)$-entry of $w_k(\rho(h),\rho(e))$ (i.e. the coefficient of $E_{i.i+1}$ on the right hand side of the above equality). If $i\neq\frac{d-1}{2}$, then

$$Z_{i,k+1}=2(d-1-2i)^{d-2}\cdot\left(\frac{d+1-2i}{d-1-2i}\right)^k.$$

Thus when d is even, Z is obtained from a Vandermonde matrix via multiplying each row by a nonzero integer. Since the numbers $\frac{d+1-2i}{d-1-2i}$, $i=1,\dots,d-1$ are distinct, we conclude that $\det(Z)\neq 0$. When $d=2f-1$ is odd, the $(f-1)$th row of Z is

$$(0,\dots,0,2(f-1)2^{d-2}).$$

Expand the determinant of Z along this row; the $(d-2)\times(d-2)$ minor of Z obtained by removing the $(f-1)th$ row and the last column of Z is again obtained from a Vandermonde matrix by multiplying each row by a nonzero integer. So $\det(Z)$ is nonzero also when d is odd. This shows that the elements $w_k(\rho(h),\rho(e))$, $k=0,1,\dots,d-2$ are linearly independent in $\mathbb{C}^{d\times d}$. Consequently, no nontrivial linear combination of $w_0,w_1,\dots,w_{d-2}$ belongs to $I(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})$. It follows that $F_2/(I(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})\cap F_2)$ contains the irreducible $\mathrm{GL}_2(\mathbb{C})$-module $W_{(d-1,1)}$ with multiplicity $\geq d-1$. This multiplicity is in fact equal to $d-1$, because $d-1$ is the multiplicity of $W_{(d-1,1)}$ as a summand in F_2. Recall finally that for $\lambda=(\lambda_1,\lambda_2)\vdash m$, the multiplicity of χ^λ in the cocharacter sequence coincides with the multiplicity of W_λ in $F_2/(I(\mathfrak{sl}_2(\mathbb{C}),\rho^{(d)})\cap F_2)$. □

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