Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 1 – 36
© TÜBİTAK
doi:10.55730/1300-0098.3343

Bipolar soft ideal rough set with applications in COVID-19

Heba I. MUSTAFA*
Department of Mathematics, Faculty of Science, Zagazig University, Egypt

Received: 04.08.2021 • **Accepted/Published Online:** 07.07.2022 • **Final Version:** 13.01.2023

Abstract: Bipolar soft rough set represents an important mathematical model to deal with uncertainty. This theory represents a link between bipolar soft set and rough set theories. This study introduced the concept of topological bipolar soft set by combining a bipolar soft set with topologies. Also, the topological structure of bipolar soft rough set has been discussed by defining the bipolar soft rough topology. The main objective of this paper is to present some solutions to develop and modify the approach of the bipolar soft rough sets. Two kinds of bipolar soft ideal approximation operators which represent extensions of bipolar soft rough approximation operator have been presented. Moreover, a new kind of bipolar approximation space via two ideals, called bipolar soft biideal approximation space, was introduced and studied by two different methods. Their properties are discussed and the relationships between these methods and the previous ones are proposed. The importance of these methods is reducing the vagueness of uncertainty areas by increasing the bipolar lower approximations and decreasing the bipolar upper approximations. Also, the bipolar soft biideal rough sets represent two opinions instead of one opinion. Finally, an application in multicriteria group decision making (MCGDM) in COVID-19 by using bipolar soft ideal rough sets is suggested by using two methods.

Key words: Bipolar soft sets, topological bipolar soft sets, bipolar soft rough topology, bipolar soft ideal rough sets, multicriteria group decision making

1. Introduction

There are many real-world problems for uncertainty and impreciseness, such as social science, engineering, economics, environmental science, artificial intelligence and medical science. This impreciseness could also be caused by the coarseness within the illustration of given knowledge. Different methods of mathematical modelling were introduced to illuminate these issues such as theory of probability, theory of fuzzy sets, theory of rough sets [1–3], interval mathematics, vague set theory, graph theory, and decision making theory. These theories reduced the space between the classical mathematical styles and the imprecise real-world information.

Anyhow all of these theories have their immanent difficulties [4], and a disadvantage that actuated Molodtsov [5, 6] to introduce the idea of soft sets as a new mathematical tool to solve some of their difficulties. Soft set theory has a significant use in game theory, smoothness of functions, medicine, operational research and probability theory [7]. Their algebraic analysis and applications developed rapidly. Maji et al. [8] presented some basic algebraic operations on soft sets, and Ali et al. [9] recommended some new operations on soft sets. Cagman et al. [10] presented soft topology and soft topological spaces. By generalizing the structure of soft set and soft topology, many concepts such as soft group [11], N-soft sets [12, 13], sum of soft topological spaces

*Correspondence: dr_heba_ibrahim@yahoo.com
2010 *AMS Mathematics Subject Classification:* 23584

[14], multipolar neutrosophic soft set [15] were introduced. Several researchers [16, 17] discussed the properties and the applications of soft sets in decision making.

The concept of fuzzy set was introduced by Zadeh [18] and many of algebraic properties in classical set theory were convenient with these sets. The connections between hesitancy, intuitive and bipolarity with the fuzzy set were studied in the types of intuitionistic fuzzy sets [19], intuitionistic fuzzy relations [20], intuitionistic hesitant fuzzy set [21], hesitant fuzzy set [22]. As a sophisticated version of the fuzzy set, the notion of the fuzzy soft set was depicted and studied in various directions [23–25]. Many authors studied the notion of fuzzy soft set in several directions in the form of intuitionistic fuzzy soft sets [26], interval-valued fuzzy soft sets [27], interval-valued intuitionistic fuzzy soft sets [28], hesitant fuzzy soft sets [29], interval-valued hesitant fuzzy soft sets [30], and generalized hesitant fuzzy soft sets [31].

The concept of bipolar valued fuzzy set [32] was introduced by extending the grade of membership of fuzzy set from [0,1] to [−1,1]. Bipolar soft sets were defined and investigated by [33]. Some scholars generalized bipolar-valued fuzzy sets and bipolar soft sets by presenting bipolar fuzzy relations [34], bipolar complex fuzzy sets [35], bipolar fuzzy soft set [36, 37], fuzzy bipolar soft sets [38], hesitant bipolar-valued fuzzy soft sets [39], bipolar neutrosophic soft sets [40], bipolar multifuzzy soft sets [41], bipolar fuzzy soft expert sets [42], bipolar fuzzy soft graphs [43], rough fuzzy bipolar soft sets [44]. Karaaslan and Karatas [45] redefined bipolar soft sets and studied its topological structure.

Pawlak [1] introduced the classical rough set models in the early of the eighties as a modern role for modeling the vagueness of data that collected from real-life problems. Soft set represents a different mathematical model to deal with the uncertainty in data collected from real-life situations. By combining soft set and rough set, Feng [46] introduced soft rough sets and soft rough approximations to solve many problems that acquired of intelligent systems identified by inadequate information. By generalizing soft rough sets, the concepts of soft rough fuzzy sets [47], intiuitionstic fuzzy soft rough sets [48], modified soft rough sets [49, 50], modified soft rough sets on a complete atomic Boolean lattice [51], soft fuzzy rough sets [52], soft prerough sets [53], soft β-rough sets [54], and the others were defined. Karaaslan and Cagman [55] defined the concept of bipolar soft rough sets which is a fusion of rough set and bipolar soft set and proposed a decision making method to select the best alternatives. In [56], Shaber et al. studied roughness through bipolar soft set by introducing a new hybrid model called modified rough bipolar soft set.

Ideal is an important notion in topological spaces and has been studied by Kuratowski [57] and Jankovic et al. [58]. It plays an effective role in solving many topological problems. It can be applied in rough set theory in [59–62]. In [63], Mustafa generalized the soft rough set theory by using an ideal by defining the concept of soft approximation space via ideal to reduce the soft boundary region. This paper is devoted to the further generalization of the bipolar soft rough set theory by using the ideal notion to reduce the bipolar soft boundary region and its properties are derived.

1.1. Motivations and contributions of the paper

The main aims of this study are divided into three goals. Firstly, the concept of topological bipolar soft set is defined for the first time by combining bipolar soft set with topologies. This new concept is clarified with real examples. Also, the topological structure of bipolar soft rough set is studied by defining the bipolar soft rough topology. Some new results of bipolar soft rough topology related to bipolar soft rough closure and bipolar soft rough interior are presented. The second goal of this paper is to present a new approach to modify and generalize bipolar soft rough set by using an ideal which is an improvement of [55]. Two kinds of bipolar soft ideal approximation operators which represent extensions of bipolar soft rough approximation operator have been introduced. The main properties of the current method are studied and comparisons between our methods and the previous ones are established. The bipolar soft rough approximations [55] are a special case of the current approximations. The proposed approximations are more accurate than [55]. Therefore, these methods are very useful in real life applications and can be used for discovering the vagueness of the data. Moreover, new bipolar soft approximation spaces by using two ideals, called bipolar soft biideal approximation spaces, are presented. These approximations are discussed by two different methods.

The importance of these approximations was its dependent on ideals which were topological tools and the two ideals represent two opinions instead of one opinion. Finally the third purpose of this contribution is to illustrate the importance of these methods in medical applications. In fact, an application in multicriteria group decision making (MCGDM) in COVID-19 by using bipolar soft ideal rough set is suggested by using two methods. These approaches are the best tool in decision making about the infection of COVID-19 by using bipolar information (positive and negative) and an ideal. Bipolar soft ideal rough sets are used to find the patients which will be prone to COVID-19. This helps the doctors to make the best decision.

2. Preliminaries

In this section, some basic notions that are useful for discussion in the next sections are recalled.

Definition 2.1 *[6] Let U be the initial universe, $\xi \neq \varphi$ be the collection of parameters, attributes or decision variables and $\eta \subseteq \xi$. A pair (Ω, η) is called a soft set over U if Ω is a mapping from the set η to the set of all subsets of U, $\Omega : \eta \longrightarrow 2^U$.*

Thus, a soft set is a parametrized family of subsets of U. For each $e \in \eta$, $\Omega(e)$ can be interpreted as a subset of U, which is usually called the set of $e-$ approximate elements of (Ω, η). But $\Omega(e)$ can be also regarded as a mapping $\Omega(e) : U \longrightarrow \{0, 1\}$, and then $\Omega(e)(u) = 1$ is equivalent to $u \in \Omega(e)$.

Definition 2.2 *[33] Let $\xi = \{e_1, e_2, e_3, ..., e_n\}$ be the set of parameters. The not set of ξ is defined by $\neg\xi = \{\neg e_1, \neg e_2, \neg e_3, ..., \neg e_n\}$, where for all i, $\neg e_i =$ not e_i.*

Definition 2.3 *[33] Let U be a universal set and ξ be a set of parameters, $\eta \subseteq \xi$. Then (Ω, Ψ, η) is said to be a bipolar soft set on U, If $\Omega : \eta \longrightarrow 2^U$ and $\Psi : \neg\eta \longrightarrow 2^U$ with the property that for each $a \in \eta$ $\Omega(a) \cap \psi(\neg a) = \varphi$.*

The set of all bipolar soft set over U will be denoted by BSS_U.

Definition 2.4 *[33] Let (Ω_1, ψ_1, η), $(\Omega_2, \psi_2, \eta) \in BSS_U$. Then (Ω, Ψ, η) is said to be a subset of (Ω_2, ψ_2, η), denoted by $(\Omega_1, \psi_1, \eta) \sqsubseteq (\Omega_2, \psi_2, \eta)$ if*
(1) $\Omega_1(a) \subseteq \Omega_2(a)$ and
(2) $\psi_1(\neg a) \supseteq \psi_2(\neg a)$ for each $a \in \eta$.

Definition 2.5 *[33] Let $(\Omega, \Psi, \eta) \in BSS_U$. The complement of (Ω, Ψ, η) is denoted by $(\Omega, \Psi, \eta)^c$ and is defined by $(\Omega, \Psi, \eta)^c = (\Omega^c, \Psi^c, A)$, where $\Omega^c(a) = \Psi(\neg a)$ and $\Psi^c(\neg a) = \Omega(a)$ for all $a \in \eta$.*

Definition 2.6 *[33] Let $(\Omega, \Psi, \eta) \in BSS_U$. Then (Ω, Ψ, η) is said to be a universal bipolar soft set, denoted by $U_A = (\amalg, \Phi, \eta)$ with the property that for each $a \in \eta$, $\amalg(a) = U$ and $\Phi(\neg a) = \varphi$.*

Definition 2.7 *[33] Let $(\Omega, \Psi, \eta) \in BSS_U$. Then (Ω, Ψ, η) is said to be an empty bipolar soft set, denoted by $\Phi_A = (\Phi, \amalg, \eta)$ with the property that for each $a \in \eta$, $\Phi(a) = \varphi$ and $\amalg(\neg a) = U$.*

Definition 2.8 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$. Then $P = (U,\ (\Omega, \Psi, \eta))$ is said to be a bipolar soft approximations space (BSA-space for short).*

Definition 2.9 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$. Then the mappings $\Omega : \eta \longrightarrow 2^U$ and $\Psi : \neg\eta \longrightarrow 2^U$ are said to be positive and negative soft sets of bipolar soft set (Ω, Ψ, η), respectively.*

From now onwards, the complement of X will be denoted by $\grave{X}$.

Definition 2.10 *[55] Let $P = (U,\ (\Omega, \Psi, \eta))$ be a BSA-space. Then the soft approximation spaces represented by $P^+ = (U, \Omega)$ and $P^- = (U, \Psi)$ are said to be positive and negative soft approximations space of bipolar soft set, respectively.*

Definition 2.11 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$ and $P = (U,\ (\Omega, \Psi, \eta))$ be a BSA-space. Then (Ω, Ψ, η) is said to be a semiintersection bipolar soft set if $\Omega(a_i) \cap \Psi(\neg a_j) = \varphi$ for all $a_i \in \eta$, $\neg a_j \in \neg\eta$.*

Definition 2.12 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$, $P = (U,\ (\Omega, \Psi, \eta))$ be a BSA-space and $X \subseteq U$. Then,*

$$\underline{S}_{P^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \subseteq X\}$$

$$\underline{S}_{P^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \cap \acute{X} \neq 0\}$$

$$\overline{S}_{P^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap X \neq 0\}$$

$$\overline{S}_{P^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \subseteq \acute{X}\}$$

are called soft $P-$lower positive approximation (SPL^+-approximation), soft $P-$lower negative approximation (SPL^--approximation), soft P-upper positive approximation (SPU^+-approximation) and soft $P-$upper negative approximation (SPU approximation) of X, respectively.

Definition 2.13 *[55] Let $\underline{S}_{P^+}(X)$, $\underline{S}_{P^-}(X), \overline{S}_{P^+}(X)$ and $\overline{S}_{P^-}(X)$ be SPL^+-approximation, SPL^-- approximation, SPU^+-approximation and SPU^--approximation of X, respectively. Then,*

$$\underline{BS}_P(X) = (\underline{S}_{P^+}(X),\ \underline{S}_{P^-}(X)),$$

$$\overline{BS}_P(X) = (\overline{S}_{P^+}(X),\ \overline{S}_{P^-}(X))$$

are said to be bipolar soft rough approximation of X with respect to BSA-space $P = (U,\ (\Omega, \Psi, \eta))$. Moreover, $\underline{BS}_P(X)$ and $\overline{BS}_P(X)$ are said to be bipolar soft $P-$ lower approximation and bipolar soft $P-$upper approximation of X, respectively.

Definition 2.14 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$ and $P = (U, (\Omega, \Psi, \eta))$ be the corresponding BSA-space. Let $X \subseteq U$. Then,*

$$BPOS_P(X) = (\underline{S}_{P^+}(X),\ \overline{S}_{P^-}(X))$$

$$BNEG_P(X) = ((\overline{S}_{P^+}(X))\acute{},\ (\underline{S}_{P^-}(X))\acute{})$$

$BBND_P(X) = (\overline{S}_{P^+}(X) \backslash \underline{S}_{P^+}(X),\ \underline{S}_{P^-}(X) \backslash \overline{S}_{P^-}(X))$

$B\mu_P(X) = (\mu_P^+(X), \mu_P^-(X))$ *where* $\mu_P^+(X) = \frac{|\underline{S}_{P^+}(X)|}{|\overline{S}_{P^+}(X)|}$ *and* $\mu_P^-(X) = \frac{|\underline{S}_{P^-}(X)|}{|\overline{S}_{P^-}(X)|}$

are said to be bipolar soft P positive region (BSP^+-region), bipolar soft P-negative region ($BSP^- -$region) and bipolar soft P-boundary region (BSB-region)of X, respectively. If $\underline{BS}_{IP}(X) = \overline{BS}_{IP}(X)$, X is said to be bipolar soft P-definable; otherwise X is called bipolar soft P-rough set.

Definition 2.15 *[55] Let $(\Omega, \Psi, \eta) \in BS_U$ and $P = (U, (\Omega, \Psi, \eta))$ be the corresponding BSA-space. Let $X \subseteq U$. Then, X is said to be bipolar soft P-definable If*
$\underline{BS}_P(X) = \overline{BS}_P(X)$; otherwise X is called bipolar soft P-rough set.

Definition 2.16 *[55] Let $(\Omega, \Psi, \eta) \in BSS_U$ then (Ω, Ψ, η) is called full if $\bigcup\limits_{a \in \eta} \Omega(a) = U$ and $\bigcup\limits_{\neg a \in \neg \eta} \Psi(\neg a) = U$.*

Definition 2.17 *Let (Ω, Ψ, η) be a full bipolar soft set over U, $P = (U, (\Omega, \Psi, \eta))$ be the corresponding BSA-space and $X \subseteq U$. Then,*

(1) X is roughly bipolar soft P-definable if $\underline{BS}_P(X) \neq (\varphi, U)$ and $\overline{BS}_P(X) \neq (U, \varphi)$.

(2) X is internally bipolar soft P-indefinable if $\underline{BS}_P(X) = (\varphi, U)$ and $\overline{BS}_P(X) \neq (U, \varphi)$

(3) X is externally bipolar soft P-indefinable if $\underline{BS}_P(X) \neq (\varphi, U)$ and $\overline{BS}_P(X) = (U, \varphi)$.

(4) X is totally bipolar soft P-indefinable if $\underline{BS}_P(X) = (\varphi, U)$ and $\overline{BS}_P(X) = (U, \varphi)$.

Definition 2.18 *[58] A nonempty collection I of subsets of a set X is said to be an ideal on X, if it satisfies the following conditions*

(1) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$;

(2) $A \in I$ and $B \subseteq A \Rightarrow B \in I$.

3. Topological bipolar soft sets and related concepts

In this section, the concept of topological bipolar soft sets is studied by combining a bipolar soft set with topologies and their properties are investigated.

Definition 3.1 *Let $(\Omega, \Psi, \eta) \in BSS_U$, then (Ω, Ψ, η) is called*

(1) Topological if $\{(\Omega(a), \Psi(\neg a)) : a \in \eta,\ \neg a \in \neg\eta\}$ is a topology on $U \times U$.

(2) Keeping intersection if for each $a, b \in \eta$ there exists $c \in \eta$ such that
$\Omega(a) \cap \Omega(b) = \Omega(c)$ and $\Psi(\neg a) \cup \Psi(\neg b) = G(\neg c)$.

(3) Keeping union if for each $a, b \in \eta$ there exists $c \in \eta$ such that $\Omega(a) \cup \Omega(b) = \Omega(c)$ and $\Psi(\neg a) \cap \Psi(\neg b) = \Psi(\neg c)$.

Define two operations $\sqcup$ and $\sqcap$ on $U \times U$ by
$(\Omega(a), \Psi(\neg a)) \sqcap (\Omega(b), \Psi(\neg b)) = (\Omega(a) \cap \Omega(b), \Psi(\neg a) \cup \Psi(\neg b))$ and
$(\Omega(a), \Psi(\neg a)) \sqcup (\Omega(b), \Psi(\neg b)) = (\Omega(a) \cup \Omega(b), \Psi(\neg a) \cap \Psi(\neg b))$.

Proposition 3.2 *Let $(\Omega, \Psi, \eta) \in BSS_U$. Then if (Ω, Ψ, η) is topological, then (Ω, Ψ, η) is full, keeping intersection and keeping union.*

Proof *Obvious.*

Example 3.3 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$, $\eta = \{a_1, a_2, a_3, a_4\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega, \Psi, \eta) \in BSS(U)$ *given by Table 1.*

Table 1.

(Ω, Ψ, η)	a_1	a_2	a_3	a_4
h_1	1	-1	1	-1
h_2	1	1	1	-1
h_3	-1	-1	1	-1
h_4	1	1	1	-1
h_5	0	0	1	-1

So (Ω, Ψ, η) *is topological.*

Example 3.4 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$, $\eta = \{a_1, a_2, a_3, a_4\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega, \Psi, \eta) \in BSS(U)$ *given by Table 2.*

Table 2.

(Ω, Ψ, η)	a_1	a_2	a_3	a_4
h_1	-1	-1	-1	0
h_2	1	1	0	1
h_3	-1	-1	1	0
h_4	1	1	0	0
h_5	1	-1	-1	1

So (Ω, Ψ, η) *is keeping intersection but not topological.*

Proposition 3.5 *Let* η *be a finite set. Let* $(\Omega, \Psi, \eta) \in BSS_U$. *If* (Ω, Ψ, η) *is topological, then* $(\Omega, \Psi, \eta)^c$ *is also topological.*

Proof *Let*

$\tau = \{(\Omega(a), \Psi(\neg a)) : a \in \eta\}$ *and* $\omega = \{(\Omega^c(a), \Psi^c(\neg a)) : a \in \eta\}$.

Since (Ω, Ψ, η) *is a topological bipolar soft set over* U, *then* τ *is a topology on* $U \times U$. *Thus* $(\Omega(a), \Psi(\neg a)) = (\varphi, U)$ *for* $a \in \eta$. *So,* $(U, \varphi) = (\varphi, U)^c = (\Omega^c(a), \Psi^c(\neg a))$. *Therefore,* $(U, \varphi) \in \omega$. *Similarly,* $(\varphi, U) \in \omega$.

For any $(\Omega^c(a), \Psi^c(\neg a))$, $(\Omega^c(b), \Psi^c(\neg b)) \in \omega$,

$$
\begin{aligned}
(\Omega^c(a), \Psi^c(\neg a)) \sqcap (\Omega^c(b), \Psi^c(\neg b)) &= (\Omega^c(a) \cap \Omega^c(b), \Psi^c(\neg a) \cup \Psi^c(\neg b)) \\
&= ((\Omega(a) \cup \Omega(b))^c, (\Psi(\neg a) \cap \Psi(\neg b))^c)
\end{aligned}
$$

Since τ *is a topology on* $U \times U$, *then* $(\Omega(a) \cup \Omega(b), \Psi(\neg a) \cap \Psi(\neg b) = (\Omega(c), \Psi(\neg c))$ *for some* $c \in \eta$. *Hence,* $(\Omega^c(a) \cap \Omega^c(b), \Psi^c(\neg a) \cup \Psi^c(\neg b)) = (\Omega^c(c), \Psi^c(\neg c)) \in \omega$.

Similarly, $(\Omega^c(a), \Psi^c(\neg a)) \sqcup (\Omega^c(b), \Psi^c(\neg b)) \in \omega$.

Since η *is a finite set, then* ω *is a topology on* $U \times U$. *Thus,* $(\Omega, \Psi, \eta)^c$ *is also a topological bipolar soft set over* U.

Proposition 3.6 *Let η and κ be two finite sets. Let (Ω_1, Ψ_1, η) and $(\Omega_2, \Psi_2, \kappa)$ be two topological bipolar soft sets over U.*
If $\tau \wedge \omega = \{(\varphi, U)\}$, then $(\Omega_1, \Psi_1, \eta) \cup_\xi (\Omega_2, \Psi_2, \kappa)$ is also a topological bipolar soft set over U, where $\tau = \{(\Omega_1(a), \Psi_1(\neg a)) : a \in \eta\}, \omega = \{(\Omega_2(b), \Psi_2(\neg b)) : b \in \kappa\}, \tau \wedge \omega = \{W \cap_\xi V : W \in \tau$ and $V \in \omega\}$.

Proof Denote $(\lambda_{\eta\times\kappa},\ \mu_{\neg\eta\times\neg\kappa}) = (\Omega_1, \Psi_1, \eta) \cup_\xi (\Omega_2, \Psi_2, \kappa)$, $\Lambda = \{(\lambda(a,b), \mu(\neg a, \neg b)) :\ (a,b) \in \eta \times \kappa\}$.

Since τ and ω are two topologies on $U \times U$, then there exists $a, c \in \eta$ and $b, d \in \kappa$ such that $(\Omega_1(a), \Psi_1(\neg a)) \cup (\Omega_2(b), \Psi_2(\neg b)) = (\varphi, U)$ and $(\Omega_1(c), \Psi_1(\neg c)) \cup (\Omega_2(d), \Psi_2(\neg d)) = (U, \varphi)$.
So, $(\varphi, U) = (\Omega_1(a) \cup \Omega_2(b), \Psi_1(\neg a) \cap \Psi_2(\neg b)) = (\lambda(a,b), \mu(\neg a, \neg b))$ and
$(U, \varphi) = (\Omega_1(c) \cup \Omega_2(d), \Psi_1(\neg c) \cap \Psi_2(\neg d))$. Therefore, (φ, U), $(U, \varphi) \in \Lambda$.

For any $(a,b), (c,d) \in \eta \times \kappa$,
$(\lambda(a,b), \mu(\neg a, \neg b)) \sqcup (\lambda(c,d), \mu(\neg c, \neg d)) = (\lambda(a,b) \cup \lambda(c,d), \mu(\neg a, \neg b) \cap \mu(\neg c, \neg d))$.
$\lambda(a,b) \cup \lambda(c,d) = (\Omega_1(a) \cup \Omega_1(c)) \cup (\Omega_2(b) \cup \Omega_2(d))$ and
$\mu(\neg a, \neg b) \cap \mu(\neg c, \neg d) = (\Psi_1(\neg a) \cap \Psi_1(\neg c)) \cap (\Psi_2(\neg b) \cap \Psi_2(\neg d))$.

Since (Ω_1, Ψ_1, η) and $(\Omega_2, \Psi_2, \kappa)$ are topological bipolar soft sets over U, then
(Ω_1, Ψ_1, η) and $(\Omega_2, \Psi_2, \kappa)$ are keeping union.
Thus $\Omega_1(a) \cup \Omega_1(c) = \Omega_1(r)$ and $\Psi_1(\neg a) \cap \Psi_1(\neg c) = \Psi_1(\neg r)$ for some $r \in \eta$. Also, $\Omega_2(b) \cup \Omega_2(d) = \Omega_2(e)$ and $\Psi_2(\neg b) \cap \Psi_2(\neg d) = \Psi_2(\neg e)$ for some $e \in \kappa$. Then

$$\begin{aligned}(\lambda(a,b), \mu(\neg a, \neg b)) \sqcup (\lambda(c,d), \mu(\neg c, \neg d)) &= (\Omega_1(r) \cup \Omega_2(e), \Psi_1(\neg r) \cap \Psi_2(\neg e) \\ &= (\lambda(r,e), \mu(\neg r, \neg e)) \in \Lambda.\end{aligned}$$

Also,
$(\lambda(a,b), \mu(\neg a, \neg b)) \sqcap (\lambda(c,d), \mu(\neg c, \neg d)) = ((\lambda(a,b) \cap \lambda(c,d), \mu(\neg a, \neg b) \cup \mu(\neg c, \neg d))$,
$\lambda(a,b) \cap \lambda(c,d) = (\Omega_1(a) \cup \Omega_2(b)) \cap (\Omega_1(c) \cup \Omega_2(d)) =$
$(\Omega_1(a) \cap \Omega_1(c)) \cup (\Omega_2(b) \cap \Omega_2(d)) \cup (\Omega_1(a) \cap \Omega_2(d)) \cup (\Omega_2(b) \cap \Omega_1(c))$,
and $\mu(\neg a, \neg b) \cup \mu(\neg c, \neg d) =$
$(\Psi_1(\neg a) \cup \Psi_1(\neg c)) \cap (\Psi_2(\neg b) \cup \Psi_2(\neg d)) \cap (\Psi_1(\neg a) \cup \Psi_2(\neg d)) \cap (\Psi_2(\neg b) \cup \Psi_1(\neg c))$.

Since $\tau \wedge \omega = \{(\varphi, U)\}$,then $(\Omega_1(a) \cap \Omega_2(d)) = (\Omega_2(b) \cap \Omega_1(c)) = \varphi$ and
$(\Psi_1(\neg a) \cup \Psi_2(\neg d)) = (\Psi_2(\neg b) \cup \Psi_1(\neg c)) = U$.

Since (Ω_1, Ψ_1, η) and (Ω_2, Ψ_2, B) are keeping intersection, then $(\Omega_1(a) \cap \Omega_1(c)) = \Omega_1(s)$ and $\Psi_1(\neg a) \cup \Psi_1(\neg c) = \Psi_1(\neg s)$ for some $s \in \eta$. Also, $\Omega_2(b) \cap \Omega_2(d) = \Omega_2(t)$ and $\Psi_2(\neg b) \cup \Psi_2(\neg d) = \Psi_2(\neg t)$ for some $t \in \kappa$. Hence,

$$\begin{aligned}(\lambda(a,b), \mu(\neg a, \neg b)) \sqcap (\lambda(c,d), \mu(\neg c, \neg d)) &= (\Omega_1(s) \cup \Omega_2(t), \Psi_1(\neg s) \cap \Psi_2(\neg t)) \\ &= (\lambda(r,e), \mu(\neg r, \neg e)) \in \Lambda.\end{aligned}$$

Since $\eta \times \kappa$ is a finite set, then Λ is a topology on $U \times U$.
Hence, $(\Omega_1, \Psi_1, \eta) \cup_\xi (\Omega_2, \Psi_2, \kappa)$ is a topological bipolar soft set over U.

□

4. Bipolar soft rough topology

In this section, the topological structure of bipolar soft rough set is studied by defining the bipolar soft rough topology. Also, some new results of bipolar soft rough topology related to bipolar soft rough closure and bipolar soft rough interior are presented.

Proposition 4.1 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *and* I *be an ideal on* U *and* $P = (U, (\Omega, \Psi, \eta))$ *be the corresponding* BSA*-space. Let* $X \subseteq U$*. Then, the collection* $\tau_{BSR}(X) = \{(U, \varphi), (\varphi, U), \underline{BS}_P(X), \overline{BS}_P(X), BBND_P(X)\}$ *forms a topology on* $U \times \neg U$ *called the bipolar soft rough topology on* (U, φ) *w.r.t.* X.

Example 4.2 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$, $\eta = \{a_1, a_2, a_3\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega, \Psi, \eta) \in BSS(U)$ *given by Table 3.*

Table 3.

(Ω, Ψ, η)	a_1	a_2	a_3
h_1	1	1	0
h_2	-1	0	1
h_3	1	1	-1
h_4	0	-1	1
h_5	-1	1	0

Let $X = \{h_2, h_3, h_4, h_5\}$, *then* $\underline{BS}_P(X) = (\{h_2, h_4\}, \{h_1, h_3, h_5\})$, $\overline{BS}_P(X) = (U, \varphi)$ *and* $BBND_P(X) = \underline{BS}_P(X)$*. So,* $\tau_{BSR}(X) = \{(U, \varphi), (\varphi, U), (\{h_2, h_4\}, \{h_1, h_3, h_5\})\}$

Proposition 4.3 *Let* $\tau_{BSR}(X)$ *be the bipolar soft rough topology on* (U, φ) *w.r.t* X*. Then the collection* $\beta_{BSR} = \{(U, \varphi), (\varphi, U), \underline{BS}_P(X), BBND_P(X)\}$ *is a base for* $\tau_{BSR}(X)$.

Proof (1) $\bigsqcup_{G \in \beta_{BSR}} G = (U, \varphi)$

(2) Let (U, φ), $\underline{BS}_P(X) \in \beta_{BSR}$. Then for each $(x, y) \in (U, \varphi) \sqcap \underline{BS}_P(X) = \underline{BS}_P(X)$, we have $(x, y) \in \underline{BS}_P(X) \sqsubseteq (U, \varphi) \sqcap \underline{BS}_P(X)$. Similarly, if (U, φ), $BBND_P(X) \in \beta_{BSR}$ or $\underline{BS}_P(X)$, $BBND_P(X) \in \beta_{BSR}$ and for each $(x, y) \in (U, \varphi) \sqcap BBND_P(X)$ or $(x, y) \in \underline{BS}_P(X) \sqcap BBND_P(X)$, we have $(x, y) \in BBND_P(X) \sqsubseteq (U, \varphi) \sqcap BBND_P(X)$ or $(x, y) \in \underline{BS}_P(X) \sqsubseteq \underline{BS}_P(X) \sqcap BBND_P(X)$. Therefore, β_{BSR} is a base for $\tau_{BSR}(X)$. □

Proposition 4.4 *Let* $((U, \varphi), \tau_{BSR}(X))$, $((V, \varphi), \tau_{BSR}(Y))$ *be bipolar soft rough topologies on* (U, φ) *and* (V, φ) *w.r.t.* X Y*, respectively. If* β^1_{BSR} *and* β^2_{BSR} *are the base for* $\tau_{BSR}(X)$ *and* $\tau_{BSR}(Y)$ *such that* $\beta^1_{BSR} \sqsubseteq \beta^2_{BSR}$*. Then* $\tau_{BSR}(Y) \sqsubseteq \tau_{BSR}(X)$.
Proof *Obvious.*

Proposition 4.5 *Let* $((U, \varphi), \tau_{BSR}(X))$ *be a bipolar soft rough topological space. Then the collection* $\tau_{BSRA}(X) = \{B_i \sqcap A : B_i \in \tau_{BSR}(X)\}$ *is said to be bipolar soft rough subspace on* (A, φ)*. Then* $((A, \varphi), \tau_{BSRA})$ *is called a bipolar* SR*-subspace of* $((U, \varphi), \tau_{BSR}(X))$.

Theorem 4.6 *Let* $((A, \varphi), \tau_{BSR}(A))$ *be a bipolar* SR*-topological space If* β_{BSR} *is a bipolar* SR*-basis for* $\tau_{BSR}(X)$*, then the collection* $\beta_{BSRB} = \{A_i \sqcap B : A_i \in \beta_{BSR}\}$ *is a bipolar* SR*-basis for the bipolar* SR*-subspace topology on* (B, φ).

Proof *Consider* $C \in \beta_{BSR}$, *by definition of bipolar* SR*-subspace topology,* $C = D \sqcap B$, *where* $D \in \tau_{BSR}$*. Since* $D \in \tau_{BSR}$, *then* $D = \bigsqcup_{A_i \in \beta_{BSR}} A_i$*. Hence,*

$$C = (\bigsqcup_{A_i \in \beta_{BSR}} A_i) \sqcap B = \bigsqcup_{A_i \in \beta_{BSR}} (A_i \sqcap B).$$

Definition 4.7 *Let* $((U, \varphi), \tau_{BSR}(X))$ *be bipolar soft rough topological space w.r.t.* $X \subseteq U$*. The bipolar soft rough interior(*$SR-$*interior) of* (B, φ) *is the union of all bipolar* SR*-open subsets of* (B, φ) *and it is denoted as* $Int_{BSR}(B, \varphi)$*.*

Theorem 4.8 *Let* $((U, \varphi), \tau_{BSR}(X))$ *be a bipolar soft rough topological space w.r.t.* $X \subseteq U, A$ *and* B *are bipolar soft rough sets over* (U, φ)*. Then*

(1) $Int_{BSR}(B, \acute{B}) \sqsubseteq (B, \acute{B})$

(2)$Int_{BSR}(\varphi, U) = (\varphi, U)$ *and* $Int_{BSR}(U, \varphi) = (U, \varphi)$

(3) B *is bipolar* $SR-$ *open*$\Longleftrightarrow Int_{BSR}(B, \acute{B}) = (B, \acute{B})$

(4) $Int_{BSR}(Int_{BSR}(B, \acute{B})) = Int_{BSR}(B, \acute{B})$

(5) $A \sqsubseteq B$ *implies* $Int_{BSR}(A, \acute{A}) \sqsubseteq Int_{BSR}(B, \acute{B})$

(6)$Int_{BSR}((A, \acute{A}) \sqcup (B, \acute{B})) \sqsubseteq Int_{BSR}(A, \acute{A}) \sqcup Int_{BSR}(B, \acute{B})$

(7) $Int_{BSR}((A, \acute{A}) \sqcap (B, \acute{B})) = Int_{BSR}(A, \acute{A}) \sqcap Int_{BSR}(B, \acute{B})$.

Proof (1) and (2) are obvious.

(3) Suppose that $Int_{BSR}(B, \acute{B}) = (B, \acute{B})$. Since $Int_{BSR}(B, \acute{B})$ is bipolar SR-open, then $(B, \acute{B})$ is bipolar $SR-$open. Conversely, if $(B, \acute{B})$ is bipolar SR-open, then the largest bipolar $SR-$open set that is contained in $(B, \acute{B})$ is $(B, \acute{B})$ itself. Hence $Int_{BSR}(B, \acute{B}) = (B, \acute{B})$.

(4) Since $Int_{BSR}(B, \acute{B})$ is bipolar $SR-$open, then by part (3) we get $Int_{BSR}(Int_{BSR}(B, \acute{B})) = Int_{BSR}(B, \acute{B})$.

(5) Suppose that $A \sqsubseteq B$. By (ii) $Int_{BSR}(A, \acute{A}) \sqsubseteq (A, \acute{A})$ and so $Int_{BSR}(A, \acute{A}) \sqsubseteq (B, \acute{B})$. Since $Int_{BSR}(A, \acute{A})$ is bipolar SR-open set contained by $(B, \acute{B})$, then $Int_{BSR}(A, \acute{A}) \sqsubseteq Int_{BSR}(B, \acute{B})$.

(6) By using (2) $Int_{BSR}(A, \acute{A}) \sqsubseteq (A, \acute{A})$ and $Int_{BSR}(B, \acute{B}) \sqsubseteq (B, \acute{B})$. Then

$Int_{BSR}(A, \acute{A}) \sqcup Int_{BSR}(B, \acute{B}) \sqsubseteq (A, \acute{A}) \sqcup (B, \acute{B})$. Since, $Int_{BSR}(A, \acute{A}) \sqcup Int_{BSR}(B, \acute{B})$ is bipolar SR-open, then $Int_{BSR}((A, \acute{A}) \sqcup (B, \acute{B})) \sqsubseteq Int_{BSR}(A, \acute{A}) \sqcup Int_{BSR}(B, \acute{B})$.

(7) By using (2) $Int_{BSR}(A, \acute{A}) \sqsupseteq (A, \acute{A})$ and $Int_{BSR}(B, \acute{B}) \sqsubseteq (B, \acute{B})$. Then

$Int_{BSR}(A, \acute{A}) \sqcap Int_{BSR}(B, \acute{B}) \sqsubseteq (A, \acute{A}) \sqcap (B, \acute{B})$. Since $Int_{BSR}(A, \acute{A}) \sqcap Int_{BSR}(B, \acute{B})$ is bipolar SR-open, then $Int_{BSR}(A, \acute{A}) \sqcap Int_{BSR}(B, \acute{B}) \sqsubseteq Int_{BSR}((A, \acute{A}) \sqcap (B, \acute{B}))$. Conversely, since $(A, \acute{A}) \sqcap (B, \acute{B}) \sqsubseteq (A, \acute{A})$ and $(A, \acute{A}) \sqcap (B, \acute{B}) \sqsubseteq (B, \acute{B})$,

then $Int_{BSR}((A, \acute{A}) \sqcap (B, \acute{B})) \sqsubseteq Int_{BSR}(A, \acute{A})$ and $Int_{BSR}((A, \acute{A}) \sqcap (B, \acute{B})) \sqsubseteq Int_{BSR}(B, \acute{B})$. Therefore, $Int_{BSR}((A, \acute{A}) \sqcap (B, \acute{B})) \sqsubseteq Int_{BSR}(A, \acute{A}) \sqcap Int_{BSR}(B, \acute{B})$. □

5. Bipolar soft ideal rough sets

5.1. Generalization of bipolar soft rough sets based on ideal

In this section, a new kind of bipolar soft rough sets based on ideal is presented. The fundamental properties of the present method are studied and compared with the other ones [55].

Definition 5.1 *Let $(\Omega, \Psi, \eta) \in BSS_U$, I be an ideal on U. The triple $P_I = (U, (\Omega, \Psi, \eta), I)$ is called bipolar soft ideal approximation space ($BSIA$-space for short). For any $X \subseteq U$, the bipolar soft ideal rough approximations ($BSIR-$ approximations for short) of X with respect to P_I are defined respectively as follows:*

$$\underline{BS}_{IP}(X) = (\underline{S}_{IP^+}(X),\ \underline{S}_{IP^-}(X)),$$

$$\overline{BS}_{IP}(X) = (\overline{S}_{IP^+}(X),\ \overline{S}_{IP^-}(X)),$$

where

$$\underline{S}_{IP^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap \acute{X} \in I\},$$

$$\underline{S}_{IP^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \cap \acute{X} \notin I\},$$

$$\overline{S}_{IP^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap X \notin I\},$$

$$\overline{S}_{IP^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and} \Psi(\neg a) \cap X \in I\}.$$

which are called the soft $IP-$ lower positive approximation ($SIPL^+-$ approximation), soft $IP-$ lower negative approximation ($SIPL^--$ approximation), soft $IP-$ upper positive approximations ($SIPU^+-$ approximation) and soft $IP-$ upper negative approximation ($SIPU^--$ approximation) of X, respectively.

Proposition 5.2 *Let $(\Omega, \Psi, \eta) \in BSS_U$, I be an ideal on U and $P_I = (U, (\Omega, \Psi, \eta), I)$ be the corresponding $BSIA$-space. Then,*

$$\underline{BS}_{IP}(X) = (\bigcup_{a\in\eta}\{\Omega(a) : \Omega(a) \cap \acute{X} \in I\},\ \bigcup_{\neg a\in\neg\eta}\{\Psi(\neg a) : \Psi(\neg a) \cap \acute{X} \notin I\})$$

$$\overline{BS}_{IP}(X) = (\bigcup_{a\in\eta}\{\Omega(a) : \Omega(a) \cap X \notin I\},\ \bigcup_{\neg a\in\neg\eta}\{\Psi(\neg a) : \Psi(\neg a) \cap X \in I\}).$$

Proof Obvious. □

Definition 5.3 *Let $(\Omega, \Psi, \eta) \in BSS_U$, I be an ideal on U and $P_I = (U, (\Omega, \Psi, \eta), I)$ be the corresponding $BSIA$-space. Let $X \subseteq U$. Then X is said to be bipolar soft I-definable if $\underline{BS}_{IP}(X) = \overline{BS}_{IP}(X)$; otherwise X is called bipolar soft I-rough set.*

Theorem 5.4 *Let $(\Omega, \Psi, \eta) \in BSS_U$, I be an ideal on U and $P_I = (U, (\Omega, \Psi, \eta), I)$ be the corresponding $BSIA$-space. Let $X \subseteq U$, $\Omega(a) \notin I$ and $\Psi(\neg a) \notin I$ for every $a \in \eta$, then X is bipolar soft I-definable iff $\overline{S}_{IP^+}(X) \cap \acute{X} \in I$ and $\underline{S}_{IP^-}(X) \cap X \in I$.*

Proof If X is bipolar soft I-definable. Then $\underline{S}_{IP^+}(X) = \overline{S}_{IP^+}(X)$. So $\underline{S}_{IP^+}(X) \cap \acute{X} \in I$. In fact, if $\underline{S}_{IP^+}(X) = \varphi$, then $\underline{S}_{IP^+}(X) \cap \acute{X} = \varphi \in I$. Assume that $\underline{S}_{IP^+}(X) \neq \varphi$ and $u \in \underline{S}_{IP^+}(X)$. If $u \in X$, then $\underline{S}_{IP^+}(X) \subseteq X$ and hence $\underline{S}_{IP^+}(X) \cap \acute{X} = \varphi \in I$. If $u \notin X$, then $u \in \acute{X}$ and $\exists\ a \in \eta$ s.t. $u \in \Omega(a)$ and $\Omega(a) \cap \acute{X} \in I$. Therefore $u \in \Omega(a) \cap \acute{X} \in I$ and hence $\{u\} \in I$. So, we prove that $\forall\ u \in \underline{S}_{IP^+}(X), \{u\} \in I$. Hence, $\underline{S}_{IP^+}(X) = \cup_{u\in \underline{S}_{IP^+}(X)}\{u\} \in I$ and therefore, $\underline{S}_{IP^+}(X) \cap \acute{X} \in I$. Since $\underline{S}_{IP^+}(X) = \overline{S}_{IP^+}(X)$, then $\overline{S}_{IP^+}(X) \cap \acute{X} \in I$. The other part can be proved similarly.

Conversely, assume that $\overline{S}_{IP^+}(X) \cap \acute{X} \in I$ and $\underline{S}_{IP^-}(X) \cap X \in I$. Since $\Omega(a) \notin I$ and $\Psi(\neg a) \notin I$ for every $a \in \eta$, then it is obvious that $\underline{S}_{IP^+}(X) \subseteq \overline{S}_{IP^+}(X)$ and $\underline{S}_{IP^-}(X) \supseteq \overline{S}_{IP^-}(X)$. So, it sufficient to show that $\underline{S}_{IP^+}(X) \supseteq \overline{S}_{IP^+}(X)$ and $\underline{S}_{IP^-}(X) \subseteq \overline{S}_{IP^-}(X)$. Let $u \in \overline{S}_{IP^+}(X)$, then $u \in \Omega(a)$ and $\Omega(a) \cap X \notin I$ for some $a \in \eta$. From Proposition 5.2 and hypothesis, $u \in \Omega(a) \subseteq \overline{S}_{IP^+}(X)$.

Since $\overline{S}_{IP^+}(X) \cap \acute{X} \in I$ and $\Omega(a) \cap \acute{X} \subseteq \overline{S}_{IP^+}(X) \cap \acute{X}$, then $\Omega(a) \cap \acute{X} \in I$ and hence $u \in \underline{S}_{IP^+}(X)$. Also, let $u \in \underline{S}_{IP^-}(X)$, then $u \in \Psi(\neg a)$ and $\Psi(\neg a) \cap \acute{X} \notin I$ for some $\neg a \in \neg\eta$. From Proposition 5.2 and hypothesis, $u \in \Psi(\neg a) \subseteq \underline{S}_{IP^-}(X)$.

Since $\underline{S}_{IP^-}(X) \cap X \in I$ and $\Psi(\neg a) \cap X \subseteq \underline{S}_{IP^-}(X) \cap X$, then $\Psi(\neg a) \cap X \in I$ and hence $u \in \overline{S}_{IP^-}(X)$. Consequently, $\underline{S}_{IP^-}(X) \subseteq \overline{S}_{IP^-}(X)$.

□

Proposition 5.5 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$, $\Omega(a) \notin I$ *and* $\Psi(\neg a) \notin I$ *for every* $a \in \eta$. *If* $X \cap (\cup_{a\in\eta}\Omega(a) \cup \cup_{\neg a\in\neg\eta}\Psi(\neg a)) \in I$, *then* X *is bipolar soft* I*-definable set.*

Proof Since $\Omega(a) \notin I$ and $G(\neg a) \notin I$ for every $a \in \eta$, then $\underline{S}_{IP^+}(X) \subseteq \overline{S}_{IP^+}(X)$ and $\underline{S}_{IP^-}(X) \supseteq \overline{S}_{IP^-}(X)$. Let $X \subseteq U$ s.t. $X \cap (\cup_{a\in\eta}\Omega(a) \cup \cup_{\neg a\in\neg\eta}\Psi(\neg a)) \in I$, then $\overline{S}_{IP^+}(X) = \varphi$. In fact, assume that $u \in \overline{S}_{IP^+}(X)$, then $\exists\ a \in \eta$ s.t. $u \in \Omega(a)$ and $\Omega(a) \cap X \notin I$.

Since $\Omega(a) \cap X \subseteq X \cap (\cup_{a\in\eta}\Omega(a) \cup \cup_{\neg a\in\neg\eta}\Psi(\neg a))$, then $X \cap (\cup_{a\in\eta}\Omega(a) \cup \cup_{\neg a\in\neg\eta}\Psi(\neg a)) \notin I$ a contradiction. So, $\overline{S}_{IP^+}(X) = \varphi$ and hence $\overline{S}_{IP^+}(X) \subseteq \underline{S}_{IP^+}(X)$. Let $u \in \underline{S}_{IP^-}(X)$, then $\exists\ \neg a \in \neg\eta$ s.t. $u \in \Psi(\neg a)$ and $\Psi(\neg a) \cap \acute{X} \notin I$.

Since $\Psi(\neg a) \cap X \subseteq X \cap (\cup_{a\in\eta}\Omega(a) \cup \cup_{\neg a\in\neg\eta}\Psi(\neg a)) \in I$, then $\Psi(\neg a) \cap X \in I$. Hence $u \in \overline{S}_{IP^-}(X)$ and therefore, $\underline{S}_{IP^-}(X) \subseteq \overline{S}_{IP^-}(X)$.

□

Definition 5.6 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I, J *be two ideals on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X, Y \subseteq U$. *Then,*

(1) $\underline{BS}_{IP}(X) \sqsubseteq \underline{BS}_{IP}(Y) \iff \underline{S}_{IP^+}(X) \subseteq \underline{S}_{IP^+}(Y)$ *and* $\underline{S}_{IP^-}(X) \supseteq \underline{S}_{IP^-}(Y)$

(2) $\overline{BS}_{IP}(X) \sqsubseteq \overline{BS}_{IP}(Y) \iff \overline{S}_{IP^+}(X) \subseteq \overline{S}_{IP^+}(Y)$ *and* $\overline{S}_{IP^-}(X) \supseteq \overline{S}_{IP^-}(Y)$.

The following theorems study the main properties of the current approximations.

Theorem 5.7 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I, J *be two ideals on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X, Y \subseteq U$. *Then,*

(1) $\overline{S}_{IP^+}(\varphi) = \varphi \neq \underline{S}_{IP^+}(\varphi)$

(2) $\overline{S}_{IP^+}(U) \subseteq \underline{S}_{IP^+}(U) = \bigcup_{a\in\eta}\Omega(a) \subseteq U$

(3) If $X \subseteq Y$, *then* $\underline{S}_{IP^+}(X) \subseteq \underline{S}_{IP^+}(Y)$ *and* $\overline{S}_{IP^+}(X) \subseteq \overline{S}_{IP^+}(Y)$

(4) $\underline{S}_{IP^+}(X \cup Y) \supseteq \underline{S}_{IP^+}(X) \cup \underline{S}_{IP^+}(Y)$

(5) $\underline{S}_{IP^+}(X \cap Y) \subseteq \underline{S}_{IP^+}(X) \cap \underline{S}_{IP^+}(Y)$

(6) $\overline{S}_{IP^+}(X \cup Y) = \overline{S}_{IP^+}(X) \cup \overline{S}_{IP^+}(Y)$

(7) $\overline{S}_{IP^+}(X \cap Y) \subseteq \overline{S}_{IP^+}(X) \cap \overline{S}_{IP^+}(Y)$

(8) $I \subseteq J \Longrightarrow \underline{S}_{IP^+}(X) \subseteq \underline{S}_{JP^+}(X)$

(9) $I \subseteq J \Longrightarrow \overline{S}_{IP^+}(X) \supseteq \overline{S}_{JP^+}(X)$.

Proof It is the same as the proof of Propositions 3.7 and 3.8 in [63]. □

Theorem 5.8 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I, J *be two ideals on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X, Y \subseteq U$. *Then,*

(1) $\underline{S}_{IP^-}(\varphi) \subseteq \overline{S}_{IP^-}(\varphi) = \bigcup_{\neg a \in \neg A} \Psi(\neg a)$

(2) $\underline{S}_{IP^-}(U) = \varphi \neq \overline{S}_{IP^-}(U)$

(3) If $X \subseteq Y$, *then* $\underline{S}_{IP^-}(X) \subseteq \underline{S}_{IP^-}(Y)$ *and* $\overline{S}_{IP^-}(X) \subseteq \overline{S}_{IP^-}(Y)$

(4) $\underline{S}_{IP^-}(X \cup Y) \subseteq \underline{S}_{IP^-}(X) \cap \underline{S}_{IP^-}(Y)$

(5) $\underline{S}_{IP^-}(X \cap Y) = \underline{S}_{IP^-}(X) \cup \underline{S}_{IP^+}(Y)$

(6) $\overline{S}_{IP^-}(X \cup Y) = \overline{S}_{IP^-}(X) \cap \overline{S}_{IP^-}(Y)$

(7) $\overline{S}_{IP^-}(X \cap Y) \supseteq \overline{S}_{IP^+}(X) \cup \overline{S}_{IP^+}(Y)$

(8) $I \subseteq J \Longrightarrow \underline{S}_{IP^-}(X) \subseteq \underline{S}_{JP^-}(X)$

(9) $I \subseteq J \Longrightarrow \overline{S}_{IP^-}(X) \supseteq \overline{S}_{JP^-}(X)$.

Proof (1) According to the definition of $SIPL^-$−approximation) and
$SIPU^-$−approximation) of X. $\overline{S}_{IP^-}(\varphi) = \bigcup_{\neg a \in \neg A}\{\Psi(\neg a) : \Psi(\neg a) \cap \varphi \in I\} = \bigcup_{\neg a \in \neg A} \Psi(\neg a)$

(2)
$$\begin{aligned} \underline{S}_{IP^-}(U) &= \bigcup_{\neg a \in \neg A}\{\Psi(\neg a) : \Psi(\neg a) \cap \varphi \notin I\} \\ &= \varphi \\ &\neq \bigcup_{\neg a \in \neg A}\{\Psi(\neg a) : \Psi(\neg a) \cap U \in I\}) \\ &= \overline{S}_{IP^-}(U) \end{aligned}$$

(3) Let $u \in \underline{S}_{IP^-}(Y)$. Then, $\exists\ \neg a \in \neg\eta$ such that $u \in \Psi(\neg a)$ and $\Psi(\neg a) \cap \acute{Y} \notin I$. Since $X \subseteq Y$ and I is an ideal, then $\Psi(\neg a) \cap \acute{X} \notin I$. Therefore, $u \in \underline{S}_{IP^-}(X)$. Hence, $\underline{S}_{IP^-}(Y) \subseteq \underline{S}_{IP^-}(X)$. The other part can be proved similarly.

(4) Immediately by part (3).

(5) $\underline{S}_{IP^-}(X \cap Y) \subseteq \underline{S}_{IP^-}(X) \cup \underline{S}_{IP^-}(Y)$ by part (3). Let $u \in \underline{S}_{IP^-}(X) \cup \underline{S}_{IP^-}(Y)$. Then, $u \in \underline{S}_{IP^-}(X)$ or $u \in \underline{S}_{IP^-}(Y)$. If $u \in \underline{S}_{IP^-}(X)$, then $\exists\ \neg a \in \neg\eta$ such that $u \in \Psi(\neg a)$ and $\Psi(\neg a) \cap \acute{X} \notin I$. Hence $\Psi(\neg a) \cap (X \cap Y)\acute{} = \Psi(\neg a) \cap (\acute{X} \cup \acute{Y}) \notin I$. Thus, $u \in \underline{S}_{IP^-}(X \cap Y)$. If $u \in \underline{S}_{IP^-}(Y)$,then similarly $u \in \underline{S}_{IP^-}(X \cap Y)$.

(6) and (7) Similar to 5.

(8) Let $u \in \underline{S}_{JP^-}(X)$, Then, $\exists\ \neg a \in \neg\eta$ such that $u \in \Omega(\neg a)$ and $\Omega(\neg a) \cap \acute{X} \notin J$. Since $I \subseteq J$, then $\Omega(\neg a) \cap \acute{X} \notin I$. Therefore, $u \in \underline{S}_{IP^-}(X)$.Thus, $\underline{S}_{JP^-}(X) \subseteq \underline{S}_{IP^-}(X)$. The other part can be proved similarly.

(9) Similar to (8).

□

Definition 5.9 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *,* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Then,*

(1) $(\underline{BS}_{IP}(X) \sqcup \underline{BS}_{IP}(Y)) = (\underline{S}_{IP^+}(X) \cup \underline{S}_{IP^+}(Y)\,,\ \underline{S}_{IP^-}(X) \cap \underline{S}_{IP^-}(Y))$

(2) $(\overline{BS}_{IP}(X) \sqcup \overline{BS}_{IP}(Y)) = (\overline{S}_{IP^+}(X) \cup \overline{S}_{IP^+}(Y)\,,\ \overline{S}_{IP^-}(X) \cap \overline{S}_{IP^-}(Y))$.

Definition 5.10 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *,* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Then,*

(1) $(\underline{BS}_{IP}(X) \sqcap \underline{BS}_{IP}(Y)) = (\underline{S}_{IP^+}(X) \cap \underline{S}_{IP^+}(Y)\,,\ \underline{S}_{IP^-}(X) \cup \underline{S}_{IP^-}(Y))$

(2) $(\overline{BS}_{IP}(X) \sqcap \overline{BS}_{IP}(Y)) = (\overline{S}_{IP^+}(X) \cap \overline{S}_{IP^+}(Y)\,,\ \overline{S}_{IP^-}(X) \cup \overline{S}_{IP^-}(Y))$.

Theorem 5.11 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *,* I, J *be two ideals on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X, Y \subseteq U$*. Then,*

(1) $\overline{BS}_{IP}(\varphi) \sqsubseteq \underline{BS}_{IP}(\varphi)$

(2) $\underline{BS}_{IP}(U) = \varphi \neq \overline{BS}_{IP}(U)$

(3) If $X \subseteq Y$*, then* $\underline{BS}_{IP}(Y) \sqsubseteq \underline{BS}_{IP}(X)$ *and* $\overline{BS}_{IP}(Y) \sqsubseteq\ \overline{BS}_{IP}(X)$

(4) $\underline{BS}_{IP}(X \cup Y) \sqsupseteq \underline{BS}_{IP}(X) \sqcup \underline{BS}_{IP}(Y)$

(5) $\underline{BS}_{IP}(X \cap Y) \sqsubseteq \underline{BS}_{IP}(X) \sqcap \underline{BS}_{IP}(Y)$

(6) $\overline{BS}_{IP}(X \cup Y) = \overline{BS}_{IP}(X) \sqcup \overline{BS}_{IP}(Y)$

(7) $\overline{BS}_{IP}(X \cap Y) \sqsubseteq \overline{BS}_{IP}(X) \sqcap \overline{BS}_{IP}(Y)$.

Proof (1) and (2) are obvious. (3) Assume that $X \subseteq Y$. Since $\underline{S}_{IP^+}(X) \subseteq \underline{S}_{IP^+}(Y)$ and $\underline{S}_{IP^-}(Y) \subseteq \underline{S}_{IP^-}(X)$ by Theorems 5.7 and 5.8, then $\underline{BS}_{IP}(Y) \sqsubseteq \underline{BS}_{IP}(X)$ by Definition 5.6. The other part can be proved similarly.

(4) Since $\underline{S}_{IP^+}(X \cup Y) \supseteq \underline{S}_{IP^+}(X) \cup \underline{S}_{IP^+}(Y)$ and $\underline{S}_{IP^-}(X \cup Y) \subseteq \underline{S}_{IP^-}(X) \cap \underline{S}_{IP^-}(Y)$ by Theorems 5.7 and 5.8, then

$$\begin{aligned}\underline{BS}_{IP}(X \cup Y) &= (\underline{S}_{IP^+}(X \cup Y),\ \underline{S}_{IP^-}(X \cup Y)) \\ &\sqsupseteq (\underline{S}_{IP^+}(X) \cup \underline{S}_{IP^+}(Y),\ \underline{S}_{IP^-}(X) \cap \underline{S}_{IP^-}(Y)) \\ &= (\underline{S}_{IP^+}(X),\ \underline{S}_{IP^+}(X)) \sqcup\ (\underline{S}_{IP^+}(X),\ \underline{S}_{IP^+}(X)) \\ &= \underline{BS}_{IP}(X) \sqcup \underline{BS}_{IP}(Y).\end{aligned}$$

The other parts can be proved similarly.

The following example shows that the inclusions in part 5 in Theorem 5.11 might be strict. □

Example 5.12 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$*,* $\eta = \{a_1, a_2, a_3, a_4\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega, \Psi, \eta) \in BSS(U)$ *given by Table 4 and* $I = \{\{h_1\}\}$*.*

Let $X = \{h_1, h_3\}$ *and* $Y = \{h_2, h_4\}$*. Then* $X \cap Y = \varphi$*.*

So, $\underline{BS}_{IP}(X) = \{\{h_1, h_3\}, \{h_1, h_2, h_3, h_4\}\}$ *and* $\underline{BS}_{IP}(Y) = \{\{h_1, h_4\}, \{h_2, h_3\}\}$*. Thus,* $\underline{BS}_{IP}(X \cap Y) = \underline{BS}_{IP}(\varphi) = \varphi$ *and* $\underline{BS}_{IP}(X) \sqcap\ \underline{BS}_{IP}(Y) = \{\{h_1\}, \{h_1, h_2, h_3, h_4\}\} \neq \underline{BS}_{IP}(X \cap Y)$*.*

Proposition 5.13 *Let* (Ω, Ψ, η) *be a full bipolar soft set over* U *,* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta))$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$*. Then,*

Table 4.

(Ω,Ψ,η)	a_1	a_2	a_3	a_4
h_1	1	-1	0	1
h_2	-1	-1	1	-1
h_3	-1	0	1	1
h_4	1	-1	-1	0
h_5	0	1	0	0

(1) $X \in I \Longrightarrow \overline{BS}_{IP}(X) = (\varphi, U)$

(2) $\acute{X} \in I \Longrightarrow \underline{BS}_{IP}(X) = (U, \varphi)$.

Proof Straightforward. □

Proposition 5.14 *Let* $(\Omega,\Psi,\eta) \in BSS_U$, I, J *be two ideals on* U. *Let* $X \subseteq U$. *Then,*

(1) $I \subseteq J \Longrightarrow \underline{BS}_{IP}(X) \sqsubseteq \underline{BS}_{JP}(X)$

(2) $I \subseteq J \Longrightarrow \overline{BS}_{JP}(X) \sqsubseteq \overline{BS}_{IP}(X)$.

Proof Follows immediately by Theorems 5.7 and 5.8 and Definition 5.6.

The following example shows that the inclusions in part 2 in Proposition 5.14 might be strict.

Example 5.15 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$, $\eta = \{a_1, a_2, a_3, a_4\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega,\Psi,\eta) \in BSS(U)$ *given in Example 5.13. If* $I = \{\varphi, \{h_1\}\}$ *and* $J = \{\varphi, \{h_4\}\}$. *Let* $X = \{h_1, h_5\}$, *then* $\overline{S}_{IP^+}(X) = \{h_5\}$ *and* $\overline{S}_{IP^-}(X) = \{h_1, h_2, h_3, h_4\}$.
Therefore, $\overline{BS}_{IP}(X) = \{\{h_5\}, \{h_1, h_2, h_3, h_4\}\}$.
Also, $\overline{S}_{JP^+}(X) = \{h_1, h_3, h_{4,} h_5\}$ *and* $\overline{S}_{IP^-}(X) = \{h_1, h_2, h_3, h_4\}$.
Therefore, $\overline{BS}_{IP}(X) = \{\{h_1, h_3, h_{4,} h_5\}, \{h_1, h_2, h_3, h_4\}\}$. *So,* $\overline{BS}_{JP}(X) \sqsubseteq \overline{BS}_{IP}(X)$ *but* $I \subsetneqq J$.

Proposition 5.16 *Let* $(\Omega,\Psi,\eta) \in BSS_U$, I, J *be two ideals on* U. *Let* $X \subseteq U$. *Then the following assertions hold:*

(1) $\overline{S}_{(I\cap J)P^+}(X) = \overline{S}_{IP^+}(X) \cup \overline{S}_{JP^+}(X)$

(2) $\overline{S}_{(I\cap J)P^-}(X) = \overline{S}_{IP^-}(X) \cap \overline{S}_{JP^-}(X)$

(3) $\overline{S}_{(I\cup J)P^+}(X) = \overline{S}_{IP^+}(X) \cap \overline{S}_{JP^+}(X)$

(4) $\overline{S}_{(I\cup J)P^-}(X) = \overline{S}_{IP^-}(X) \cup \overline{S}_{JP^-}(X)$.

Proof

(1) $$\begin{aligned}\overline{S}_{(I\cap J)P^+}(X) &= \underset{a\in\eta}{\cup}\{\Omega(a) : \Omega(a) \cap X \notin (I \cap J)\}\\ &= \underset{a\in\eta}{\cup}\{\Omega(a) : \Omega(a) \cap X \notin I\} \text{ or } \underset{a\in\eta}{\cup}\{\Omega(a) : \Omega(a) \cap X \notin J\}\\ &= \underset{a\in\eta}{\cup}\{\Omega(a) : \Omega(a) \cap X \notin I\} \cup \underset{a\in\eta}{\cup}\{\Omega(a) : \Omega(a) \cap X \notin J\}\\ &= \overline{S}_{IP^+}(X) \cup \overline{S}_{JP^+}(X).\end{aligned}$$

The other parts can be proved similarly.

□

Theorem 5.17 *Let $(\Omega,\Psi,\eta)\in BSS_U$, I,J be two ideals on U Let $X\subseteq U$. Then,*
(1) $\overline{BS}_{(I\cap\ J)P}(X)=\overline{BS}_{IP}(X)\sqcup\overline{BS}_{JP}(X)$
(2) $\overline{BS}_{(I\cup\ J)P}(X)=\overline{BS}_{IP}(X)\sqcap\overline{BS}_{JP}(X)$.

Proof (1) $\overline{BS}_{(I\cap\ J)P}(X)=(\overline{S}_{(I\cap J)P^+}(X),\ \overline{S}_{(I\cap J)P^-}(X))$

$$=(\overline{S}_{IP^+}(X)\cup\overline{S}_{JP^+}(X),\ \overline{S}_{IP^-}(X)\cap\overline{S}_{JP^-}(X))$$
$$=(\overline{S}_{IP^+}(X),\ \overline{S}_{IP^-}(X))\sqcup\ (\overline{S}_{JP^+}(X),\ \overline{S}_{JP^-}(X))$$
$$=\overline{BS}_{IP}(X)\sqcup\overline{BS}_{JP}(X).$$

(2) Similar to (1).

Remark 5.18 *If $I=2^U$, then $\overline{S}_{IP^+}(X)=\underline{S}_{IP^-}(X)=\varphi$.*

□

Definition 5.19 *[64] Let (Ω,η) be a soft set over U. Then (Ω,η) is said to be an intersection complete soft set*
if for all $a_1,a_2\in\eta$, $\exists$ $a_3\in\eta$ such that $\Omega(a_3)=\Omega(a_1)\cap\Omega(a_2)$ whenever $\Omega(a_1)\cap\Omega(a_2)\neq\varphi$.

Proposition 5.20 *Let (Ω,η) be an intersection complete soft set over U, I be an ideal on U and $P=(U,(\Omega,\eta))$ be the corresponding soft approximation space. Then $\underline{S}_{IP^+}(X\cap Y)=\underline{S}_{IP^+}(X)\cap\underline{S}_{IP^+}(Y)$ for all $X,Y\subseteq U$.*

Proof By Theorem 5.7, $\underline{S}_{IP^+}(X\cap Y)\subseteq\underline{S}_{IP^+}(X)\cap\underline{S}_{IP^+}(Y)$. It is sufficient to prove the other inclusion. Let $u\in\underline{S}_{IP^+}(X)\cap\underline{S}_{IP^+}(Y)$, then $\exists$ $a_1,a_2\in\eta$ such that $u\in\Omega(a_1)$, $\Omega(a_1)\cap\acute{X}\in I$, $u\in\Omega(a_2)$ and $\Omega(a_2)\cap\acute{Y}\in I$. By properties of ideal, $(\Omega(a_1)\cap\acute{X})\cup(\Omega(a_2)\cap\acute{Y})\in I$. Since (Ω,η) is an intersection complete soft set, then $\exists$ $a_3\in\eta$ such that $u\in\Omega(a_3)=\Omega(a_1)\cap\Omega(a_2)$ and
$\Omega(a_3)\cap(X\cap Y\acute{)}=\Omega(a_3)\cap(\acute{X}\cup\acute{Y})\subseteq(\Omega(a_1)\cap\acute{X})\cup(\Omega(a_2)\cap\acute{Y})$. So $\Omega(a_3)\cap(X\cap Y\acute{)}\in I$ by properties of ideal and therefore $u\in\underline{S}_{IP^+}(X\cap Y)$.

□

Proposition 5.21 *Let $(\Omega,\Psi,\eta)\in BSS_U$, I be an ideal on U and $P_I=(U,(\Omega,\Psi,\eta),I)$ be the corresponding $BSIA$-space. If Ω is an intersection complete soft set, then $\underline{BS}_{IP}(X\cap Y)=\underline{BS}_{IP}(X)\sqcap\underline{BS}_{IP}(Y)$ for all $X,Y\subseteq U$.*

Proof From [63] $\underline{S}_{IP^+}(X\cap Y)=\underline{S}_{IP^+}(X)\cap\underline{S}_{IP^+}(Y)$. By Theorem 5.8,
$\underline{S}_{IP^-}(X\cap Y)=\underline{S}_{IP^-}(X)\cup\underline{S}_{IP^-}(Y)$. Therefore $\underline{BS}_{IP}(X\cap Y)=\underline{BS}_{IP}(X)\sqcap\underline{BS}_{IP}(Y)$ by Definition 5.10.

□

Definition 5.22 *Let $(\Omega,\Psi,\eta)\in BSS_U$, I be an ideal on U and $P_I=(U,(\Omega,\Psi,\eta),I)$ be the corresponding $BSIA$-space. Let $X\subseteq U$. Then the complement of the bipolar soft ideal lower approximation and bipolar soft ideal upper approximation of X are defined respectively by*

$\underline{BS}^c_{IP}(X) = (\underline{S}_{IP^-}(X), \underline{S}_{IP^+}(X)),$
$\overline{BS}^c_{IP}(X) = (\overline{S}_{IP^-}(X), \overline{S}_{IP^+}(X))$.

Proposition 5.23 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X, Y \subseteq U$. *Then,*
(1) $(\underline{BS}^c_{IP}(X))^c = \underline{BS}_{IP}(X)$
(2) $(\overline{BS}^c_{IP}(X))^c = \overline{BS}_{IP}(X)$
(3) $(\underline{BS}_{IP}(X) \sqcup \underline{BS}_{IP}(Y))^c = \underline{BS}^c_{IP}(X) \sqcap \underline{BS}^c_{IP}(Y)$
(4) $(\overline{BS}_{IP}(X) \sqcup \overline{BS}_{IP}(Y))^c = \overline{BS}^c_{IP}(X) \sqcap \overline{BS}^c_{IP}(Y)$
(5) $(\underline{BS}_{IP}(X) \sqcap \underline{BS}_{IP}(Y))^c = \underline{BS}^c_{IP}(X) \sqcup \underline{BS}^c_{IP}(Y)$
(6) $(\overline{BS}_{IP}(X) \sqcap \overline{BS}_{IP}(Y))^c = \overline{BS}^c_{IP}(X) \sqcup \overline{BS}^c_{IP}(Y)$
(7) $\underline{BS}_{IP}(X) \sqsubseteq \underline{BS}_{IP}(Y) \iff \underline{BS}^c_{IP}(Y) \sqsubseteq \underline{BS}^c_{IP}(X)$
(8) $\overline{BS}_{IP}(X) \sqsubseteq \overline{BS}_{IP}(Y) \iff \overline{BS}^c_{IP}(Y) \sqsubseteq \overline{BS}^c_{IP}(X)$.

Proof Immediately by using Theorem 5.11 and Definition 5.22.

Proposition 5.24 *Let* (Ω, Ψ, η) *be a semiintersection bipolar soft set over* U, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$. *Then,*
(1) $\underline{S}_{IP^+}(X) \cap \underline{S}_{IP^-}(X) = \varphi$
(2) $\overline{S}_{IP^+}(X) \cap \overline{S}_{IP^-}(X) = \varphi$.

Proof Since (Ω, Ψ, η) is a semiintersection bipolar soft set over U, then
$\Omega(a_i) \cap \Psi(\neg a_j) = \varphi \ \forall \ a_i \in \eta$ and $\neg a_j \in \neg\eta$. So it is clear that $\underline{S}_{IP^+}(X) \cap \underline{S}_{IP^-}(X) = \varphi$ and $\overline{S}_{IP^+}(X) \cap \overline{S}_{IP^-}(X) = \varphi$.

□

The following theorem presents the relationships between the current approximation in Definition 5.1 and the previous definition.

Theorem 5.25 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$. *Then,*
(1) $\overline{BS}_{IP}(X) \sqsubseteq \overline{BS}_P(X)$
(2) $\underline{BS}_P(X) \sqsubseteq \underline{BS}_{IP}(X)$
(3) $BPOS_P(X) \sqsubseteq BPOS_{IP}(X)$.

Proof Immediately from the definition.

Remark 5.26 *(1) It is noted from Theorem 5.25 that the Definition 5.1 reduces the bipolar lower approximation and increases bipolar lower approximation.*
(2) If $I = \{\varphi\}$ *in Definition 5.1, then this approximations coincide with bipolar soft rough approximations in [55]. So bipolar soft rough approximations [55] are a special case of these approximations.*

6. Another kind of bipolar soft ideal rough set

In this section, another kind of bipolar soft approximations based on ideal is introduced. Some of their properties are studied and the relationship between these approximations and the previous approximations in Definition 5.1 is discussed. These approximations are more accurate than [55].

Definition 6.1 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. For any* $X \subseteq U$, *the bipolar soft** *ideal rough approximations* (BS^*IR- *approximations for short*) *of* X *with respect to* P_I *are defined respectively as follows:*

$$\underline{BS}^*_{IP}(X) = (\underline{S}^*_{IP^+}(X),\ \underline{S}^*_{IP^-}(X)),$$
$$\overline{BS}^*_{IP}(X) = (\overline{S}^*_{IP^+}(X),\ \overline{S}^*_{IP^-}(X)),$$

where

$$\underline{S}^*_{IP^+}(X) = \{u \in X : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap \acute{X} \in I\},$$
$$\underline{S}^*_{IP^-}(X) = \underline{S}_{IP^-}(X) \cup \acute{X},$$
$$\overline{S}^*_{IP^+}(X) = \overline{S}_{IP^+}(X) \cup X,$$
$$\overline{S}^*_{IP^-}(X) = \{u \in \acute{X} : \exists \neg a \in \neg A \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \cap X \in I\},$$

which are called the soft* $IP-$*lower positive approximation* (S^*IPL^+-*approximation*), *soft** $IP-$*lower negative approximation* (S^*IPL^--*approximation*), *soft* * $IP-$*upper positive approximations* (S^*IPU^+-*approximation*) *and soft** $IP-$*upper negative approximations* (S^*IPU^--*approximation*) *of* X, *respectively.*

Definition 6.2 *Let* $(\Omega, \Psi, \eta) \in BSS_U$, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$. *Then* X *is said to be bipolar soft*$^*I$*-definable if* $\underline{BS}^*_{IP}(X) = \overline{BS}^*_{IP}(X)$; *otherwise* X *is called bipolar soft*$^*I$*-rough set.*

Moreover,

$$BPOS^*_{IP}(X) = (\underline{S}^*_{IP^+}(X),\ \overline{S}^*_{IP^-}(X))$$
$$BNEG^*_{IP}(X) = ((\overline{S}^*_{IP^+}(X))\acute{},\ (\underline{S}^*_{IP^-}(X))\acute{})$$
$$BBND^*_{IP}(X) = (\overline{S}^*_{IP^+}(X) \backslash \underline{S}^*_{IP^+}(X),\ \underline{S}^*_{IP^-}(X) \backslash \overline{S}^*_{IP^-}(X))$$
$$B\mu^*_{IP}(X) = (\mu^+_{IP}(X), \mu^-_{IP}(X)) \text{ where } \mu^+_{IP}(X) = \frac{|\underline{S}^*_{IP^+}(X)|}{|\overline{S}^*_{IP^+}(X)|} \text{ and } \mu^-_{IP}(X) = \frac{|\underline{S}^*_{IP^-}(X)|}{|\overline{S}^*_{IP^-}(X)|}$$

are called bipolar soft$^*IP-$*positive region* (BS^*IP^+*-region*), *bipolar soft** IP*-negative region* (BS^*IP^--*region*), *bipolar soft** IP*-boundary region* (BS^*IB-*region*) *of* X *and measure of accuracy of bipolar soft*$^*I$*-rough set with respect to* X, *respectively.*

The following theorem presents the relationships between the current approximations in Definitions 5.1 and 6.1.

Theorem 6.3 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *such that* (Ω, Ψ, η) *and* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$. *Then*

(1) $\underline{BS}^*_{IP}(X) \sqsubseteq \underline{BS}_{IP}(X)$.

(2) $\overline{BS}_{IP}(X) \sqsubseteq \overline{BS}^*_{IP}(X)$.

Proof Immediately.

The next theorem presents the relationships between the current approximation in Definition 6.1 and the previous definition [55].

Theorem 6.4 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *such that* (Ω, Ψ, η) *is full,* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$*. Then,*

(1) $\underline{BS}_P(X) \sqsubseteq \underline{BS}^*_{IP}(X) \sqsubseteq (X, \acute{X}) \sqsubseteq \overline{BS}^*_{IP}(X) \sqsubseteq \overline{BS}_P(X)$

(2) $BPOS_P(X) \sqsubseteq BPOS^*_{IP}(X)$

(3) $BBND_{IP}(X) \sqsubseteq BBND^*_{IP}(X)$

(4) $B\mu_P(X) \leq \ B\mu^*_{IP}(X)$.

Proof Immediately.

Corollary 6.5 *Let* $(\Omega, \Psi, \eta) \in BSS_U$ *such that* (Ω, Ψ, η) *is full,* I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$ *If* X *is a bipolar soft* P*-definable set, then it is a bipolar soft*$^*I$*-definable set.*

Proof Immediately.

Remark 6.6 *The converse of the previous results is not true in general as illustrated in the following example.*

Example 6.7 *Let* $U = \{h_1, h_2, h_3, h_4, h_5\}$, $A = \{a_1, a_2, a_3, a_4\}$ *be the set of parameters and* $\neg A = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ *be the not set of parameters. Let* $(\Omega, \Psi, \eta) \in BSS(U)$ *given in Example 5.12 and* $I = \{\{h_1\}\}$.

If $X = \{h_1, h_5\}$, *then* $\underline{S}_{P^+}(X) = \{h_5\}$ *and* $\underline{S}_{P^-}(X) = \{h_1, h_2, h_3, h_4\}$.

Thus $\underline{BS}_P(X) = \{\{h_5\}, \{h_1, h_2, h_3, h_4\}\}$. *Also we have* $\overline{S}_{P^+}(X) = \{h_1, h_3, h_4, h_5\}$ *and* $\overline{S}_{P^-}(X) = \{h_1, h_2, h_3, h_4\}$. *Therefore,* $\overline{BS}_P(X) = \{\{h_1, h_3, h_4, h_5\}, \{h_1, h_2, h_3, h_4\}\}$. *On the other hand* $\underline{S}_{IP^+}(X) = \{h_5\}$ *and* $\underline{S}_{IP^-}(X) = \{h_5\}$.

Thus $\underline{BS}^*_{IP}(X) = \{\{h_5\}, \{h_1, h_2, h_3, h_4\}\}$. *Also we have* $\overline{S}_{IP^+}(X) = \{h_5\}$ *and* $\overline{S}_{IP^-}(X) = \{h_1, h_2, h_3, h_4\}$. *Therefore,* $\overline{BS}^*_{IP}(X) = \{\{h_5\}, \{h_1, h_2, h_3, h_4\}\}$. *So* X *is a bipolar soft*$^*I$*-definable set but it is not a bipolar soft* P*-definable.*

Remark 6.8 *It is noted from Theorem 6.4 that the Definition 6.1 reduces the bipolar boundary region and increases the bipolar accuracy measure of a set* X *by increasing the bipolar lower approximation and decreasing the bipolar upper approximation with the comparison of the method in Definition 18 in [55]. So, the suggested method is more accurate than [55] in decision making.*

According to Theorem 6.4, the following important definition can be defined.

Definition 6.9 *Let* (Ω, Ψ, η) *be a full bipolar soft set over* U, I *be an ideal on* U *and* $P_I = (U, (\Omega, \Psi, \eta), I)$ *be the corresponding* $BSIA$*-space. Let* $X \subseteq U$*. Then,*

(1) X *is roughly bipolar soft*$^*I$*-definable if* $\underline{BS}^*_{IP}(X) \neq (\varphi, U)$ *and* $\overline{BS}^*_{IP}(X) \neq (U, \varphi)$.

(2) X is internally bipolar soft$^*I$-indefinable if $\underline{BS}^*_{IP}(X) = (\varphi, U)$ and $\overline{BS}^*_{IP}(X) \neq (U, \varphi)$

(3) X is externally bipolar soft$^*I$-indefinable if $\underline{BS}^*_{IP}(X) \neq (\varphi, U)$ and $\overline{BS}^*_{IP}(X) = (U, \varphi)$.

(4) X is totally bipolar soft$^*I$-indefinable if $\underline{BS}^*_{IP}(X) = (\varphi, U)$ and $\overline{BS}^*_{IP}(X) = (U, \varphi)$.

Theorem 6.10 *Let (Ω, Ψ, η) be a full bipolar soft set over U, I be an ideal on U and $P_I = (U, (\Omega, \Psi, \eta), I)$ be the corresponding $BSIA$-space Let $X \subseteq U$. Then,*
(1) If X is roughly bipolar soft P-definable then X is roughly bipolar soft$^*I$-definable.

(2) If X is totally bipolar soft$^*I$-indefinable then X is totally bipolar soft P-indefinable.

Proof Immediately.

Remark 6.11 *Theorem 6.10 identifies the difference between bipolar soft rough approximations [55] and bipolar soft* ideal rough approximation (current method) This shows the importance of the current approach in defining the sets. For example: if X is totally bipolar soft P-indefinable, then $\underline{BS}_P(X) = (\varphi, U)$ and $\overline{BS}_P(X) = (U, \varphi)$. But, by using bipolar soft* ideal approximation, $\underline{BS}^*_{IP}(X) \neq (\varphi, U)$ and $\overline{BS}^*_{IP}(X) \neq (U, \varphi)$ and then X can be roughly bipolar soft* I-definable (Example 6.12 illustrates this fact).*

Example 6.12 *Let $U = \{h_1, h_2, h_3, h_4, h_5\}$, $\eta = \{a_1, a_2, a_3, a_4\}$ be the set of parameters and $\neg\eta = \{\neg a_1, \neg a_2, \neg a_3, \neg a_4\}$ be the not set of parameters. Let $(\Omega, \Psi, \eta) \in BSS(U)$ given by Table 5 and $I = \{\varphi, \{h_2\}, \{h_4\}, \{h_2, h_4\}\}$.*

Table 5.

(Ω, Ψ, η)	a_1	a_2	a_3	a_4
h_1	1	-1	0	1
h_2	1	-1	1	-1
h_3	-1	1	-1	1
h_4	-1	0	1	-1
h_5	0	1	-1	0

If $X = \{h_1, h_4, h_5\}$, then$\underline{S}_{P+}(X) = \varphi$ and $\underline{S}_{P-}(X) = U$. Thus $\underline{BS}_P(X) = \{\varphi, U\}$. Also we have $\overline{S}_{P+}(X) = U$ and $\overline{S}_{P-}(X) = \varphi$. Therefore, $\overline{BS}_P(X) = \{U, \varphi\}$. On the other hand $\underline{S}^*_{IP+}(X) = \{h_1, h_4\}$ and $\underline{S}^*_{IP-}(X) = \{h_2, h_3, h_4\}$. Thus $\underline{BS}^*_{IP}(X) = \{\{h_1, h_4\}, \{h_2, h_3, h_4\}\}$.

Also we have $\overline{S}^*_{IP+}(X) = \{h_1, h_2, h_3, h_5\}$ and $\overline{S}^*_{IP-}(X) = \{h_1, h_2, h_4, h_5\}$.

Therefore, $\overline{BS}^*_{IP}(X) = \{\{h_1, h_2, h_3, h_5\}, \{h_1, h_2, h_4, h_5\}\}$. So, X is a totally bipolar soft P-indefinable set but it is a roughly bipolar soft* I-definable.

7. Bipolar soft biideal approximation space

In this section, a new bipolar soft approximation space, called bipolar soft biideal approximation space, is presented by using two ideals. This approximation is discussed by two different methods. Also, their properties and the relationships between them are discussed.

Definition 7.1 *[65] Let I_1, I_2 be be two ideals on a nonempty set U. The smallest collection generating by I_1, I_2 is denoted by $\langle I_1, I_2\rangle$ and defined as:*

$$\langle I_1, I_2\rangle = \{G \cup F : G \in I_1,\ F \in I_2\}.$$

Proposition 7.2 *[65] If I_1, I_2 are two ideals on a nonempty set U and $A, B \subseteq U$. Then the collection $\langle I_1, I_2\rangle$ satisfied the following conditions:*

(1) $\langle I_1, I_2\rangle \neq \varphi$

(2) $A \in \langle I_1, I_2\rangle,\ B \subseteq A \implies B \in \langle I_1, I_2\rangle$

(3) $A, B \in \langle I_1, I_2\rangle \implies A \cup B \in \langle I_1, I_2\rangle$.

Definition 7.3 *Let $(\Omega, \Psi, \eta) \in BSS_U$ and I_1, I_2 be two ideals on U. The quadruple $P_{(I_1,I_2)} = (U, (\Omega, \Psi, \eta), I_1, I_2)$ is said to be bipolar soft biideal approximation space ($BSbIA$-space for short) related to $(U, (\Omega, \Psi, \eta), I_1, I_2)$. For any $X \subseteq U$, the bipolar soft biideal soft approximations ($BSbIR-$ approximations for short) of X with respect $P_{(I_1,I_2)}$ are defined respectively as follows:*

$$\underline{BS}_{\langle I_1,I_2\rangle P}(X) = (\underline{S}_{\langle I_1,I_2\rangle P^+}(X), \underline{S}_{\langle I_1,I_2\rangle P^-}(X)),$$

$$\overline{BS}_{\langle I_1,I_2\rangle P}(X) = (\overline{S}_{\langle I_1,I_2\rangle P^+}(X),\ \overline{S}_{\langle I_1,I_2\rangle P^-}(X)),$$

where

$$\underline{S}_{\langle I_1,I_2\rangle P^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap \acute{X}\ \in \langle I_1, I_2\rangle\},$$

$$\underline{S}_{\langle I_1,I_2\rangle P^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \cap \acute{X} \notin \langle I_1, I_2\rangle\},$$

$$\overline{S}_{\langle I_1,I_2\rangle P^+}(X) = \{u \in U : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap X\ \notin \langle I_1, I_2\rangle\},$$

$$\overline{S}_{\langle I_1,I_2\rangle P^-}(X) = \{u \in U : \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and} \Psi(\neg a) \cap X \in \langle I_1, I_2\rangle\}.$$

Remark 7.4 *(1) The bipolar soft biideal lower and upper approximations in Definition 7.3 coincide with the previous approximations in Definition 5.1 if $I_1 = I_2$.*

Definition 7.5 *Let $(\Omega, \Psi, \eta) \in BSS_U$, I_1, I_2 be two ideals on U and $P_{(I_1,I_2)} = (U, (\Omega, \Psi, \eta), I_1, I_2)$ be the corresponding $BSbIA$-space. For any $X \subseteq U$, the bipolar soft* biideal rough approximations (BS^*bIR- approximations for short) of X with respect to P and I are defined respectively as follows:*

$$\underline{BS}^*_{\langle I_1,I_2\rangle P}(X) = (\underline{S}^*_{\langle I_1,I_2\rangle P^+}(X), \underline{S}^*_{\langle I_1,I_2\rangle P^-}(X)),$$

$$\overline{BS}^*_{\langle I_1,I_2\rangle P}(X) = (\overline{S}^*_{\langle I_1,I_2\rangle P^+}(X),\ \overline{S}^*_{\langle I_1,I_2\rangle P^-}(X)),$$

where

$$\underline{S}^*_{\langle I_1,I_2\rangle P^+}(X) = \{u \in X : \exists a \in \eta \text{ such that } u \in \Omega(a) \text{ and } \Omega(a) \cap \acute{X}\ \in \langle I_1, I_2\rangle\},$$

$$\underline{S}^*_{\langle I_1,I_2\rangle P^-}(X) = \underline{S}_{\langle I_1,I_2\rangle P^-}(X) \cup \acute{X},$$

$$\overline{S}^*_{\langle I_1,I_2\rangle P^+}(X) = \overline{S}_{\langle I_1,I_2\rangle P^+}(X) \cup X,$$

$$\overline{S}^*_{\langle I_1,I_2\rangle P^-}(X) = \{u \in \acute{X}\ \ \exists \neg a \in \neg\eta \text{ such that } u \in \Psi(\neg a) \text{ and } \Psi(\neg a) \cap X \in \langle I_1, I_2\rangle\}.$$

Remark 7.6 *The properties of the current approximations in Definition 7.5 are the same as the previous one in Theorem 5.11.*

Definition 7.7 *Let* $P_{(I_1,I_2)} = (U,(\Omega,\Psi,\eta),I_1,I_2)$ *be BSbIA-space and* $X \subseteq U$.

$$\underline{BS}^*_{\{I_1,I_2\}P}(X) = \underline{BS}^*_{I_1P}(X) \sqcup \underline{BS}^*_{I_2P}(X)$$
$$\overline{BS}^*_{\{I_1,I_2\}P}(X) = \overline{BS}^*_{I_1P}(X) \sqcap \overline{BS}^*_{I_2P}(X).$$

Theorem 7.8 *Let* $P_{(I_1,I_2)} = (U,(\Omega,\Psi,\eta),I_1,I_2)$ *be BSbIA-space and* $X \subseteq U$. *Then,*

(1) $\underline{BS}^*_{\{I_1,I_2\}P}(\varphi) \subseteq \overline{BS}^*_{\{I_1,I_2\}P}(\varphi)$

(2) $\underline{BS}^*_{\{I_1,I_2\}P}(U) = \varphi \neq \overline{BS}^*_{\{I_1,I_2\}P}(U)$

(3) If $X \subseteq Y$, *then* $\underline{BS}^*_{\{I_1,I_2\}P}(X) \sqsubseteq \underline{BS}^*_{\{I_1,I_2\}P}(Y)$

(4) $\underline{BS}^*_{\{I_1,I_2\}P}(X \cup Y) \sqsupseteq \underline{BS}^*_{\{I_1,I_2\}P}(X) \sqcup \underline{BS}^*_{\{I_1,I_2\}P}(Y)$

(5) $\underline{BS}^*_{\{I_1,I_2\}P}(X \cap Y) \sqsubseteq \underline{BS}^*_{\{I_1,I_2\}P}(X) \sqcap \underline{BS}^*_{\{I_1,I_2\}P}(Y)$

(6) $\overline{BS}^*_{\{I_1,I_2\}P}(X \cup Y) \sqsupseteq \overline{BS}^*_{\{I_1,I_2\}P}(X) \sqcup \overline{BS}^*_{\{I_1,I_2\}P}(Y)$

(7) $\overline{BS}^*_{\{I_1,I_2\}P}(X \cap Y) \sqsubseteq \overline{BS}^*_{\{I_1,I_2\}P}(X) \sqcap \overline{BS}^*_{\{I_1,I_2\}P}(Y)$.

Proof Immediately by using Remark 7.6 and Definition 7.7.

The following example shows that the inclusion in part 6 cannot be replaced by equality relation.

Example 7.9 *Let* $U = \{h_1,h_2,h_3,h_4,h_5\}$, $\eta = \{a_1,a_2,a_3,a_4\}$ *be the set of parameters and* $\neg\eta = \{\neg a_1,\neg a_2,\neg a_3,\neg a_4\}$ *be the not set of parameters. Let* $(\Omega,\Psi,\eta) \in BSS(U)$ *given by Table 5 and* $I_1 = \{\varphi,\{h_1\}\}$ $I_2 = \{\varphi,\{h_5\}\}$.

Let $X = \{h_1\}$ *and* $Y = \{h_5\}$, $X \cup Y = \{h_1,h_5\}$, *So,* $\overline{BS}^*_{I_1,P}(X) = (\{h_1\},\{h_2,h_3,h_4,h_5\})$ *and* $\overline{BS}^*_{I_2,P}(X) = (\{h_1,h_2,h_3\},\{h_2,h_3,h_4,h_5\})$. *Hence,* $\overline{BS}^*_{\{I_1,I_2\}P}(X) = (\{h_1\},\{h_2,h_3,h_4,h_5\})$.

Also, $\overline{BS}^*_{I_1,P}(Y) = (\{h_3,h_5\},\{h_1,h_2,h_3,h_4\})$ *and* $\overline{BS}^*_{I_2,P}(Y) = (\{h_5\},\{h_1,h_2,h_3,h_4\})$.

Hence $\overline{BS}^*_{\{I_1,I_2\}P}(Y) = (\{h_5\},\{h_1,h_2,h_3,h_4\})$. *On the other hand*

$\overline{BS}^*_{I_1P}(X \cup Y) = (\{h_1,h_3,h_5\},\{h_2,h_3,h_4\})$ *and* $\overline{BS}^*_{I_2P}(X \cup Y) = (\{h_1,h_2,h_3,h_5\},\{h_2,h_3,h_4\})$.

So, $\overline{BS}^*_{\{I_1,I_2\}P}(X \cup Y) = (\{h_1,h_3,h_5\},\{h_2,h_3,h_4\}) \neq \overline{BS}^*_{\{I_1,I_2\}P}(X) \sqcup \overline{BS}^*_{\{I_1,I_2\}P}(Y) = (\{h_1,h_5\},\{h_2,h_3,h_4\})$.

The following theorem studies the relationships between the two methods of the current approximations in Definitions 7.5 and 7.7.

Theorem 7.10 *Let* $P_{(I_1,I_2)} = (U,(\Omega,\Psi,\eta),I_1,I_2)$ *be BSbIA-space and* $X \subseteq U$. *Then,*

(1) $\overline{BS}^*_{\langle I_1,I_2\rangle P}(X) \sqsubseteq \overline{BS}^*_{\{I_1,I_2\}P}(X)$

(2) $\underline{BS}^*_{\{I_1,I_2\}P}(X) \sqsubseteq \underline{BS}^*_{\langle I_1,I_2\rangle P}(X)$

(3) $BBND_{\langle I_1,I_2\rangle P}(X) \sqsubseteq BBND_{\{I_1,I_2\}P}(X)$

(4) $B\mu_{\{I_1,I_2\}P}(X) \leqslant B\mu_{<I_1,I_2>P}(X)$.

Proof (1) Let $u \in \overline{S}^*_{\langle I_1,I_2\rangle P^+}(X)$, then either $u \in X$ or $\exists\ a \in \eta$ s.t. $u \in \Omega(a)$ and $\Omega(a) \cap X \notin \langle I_1,I_2\rangle$. So, either $u \in X$ or $\Omega(a) \cap X \notin I_1$ and $\Omega(a) \cap X \notin I_2$. Therefore $u \in \overline{S}^*_{I_1P^+}(X)$ and $u \in \overline{S}^*_{I_2P^+}(X)$. Thus $u \in \overline{S}^*_{\{I_1,I_2\}P^+}(X)$ and hence $\overline{S}^*_{\langle I_1,I_2\rangle P^+}(X) \subseteq \overline{S}^*_{\{I_1,I_2\}P^+}(X)$.

Let $u \in \overline{S}^{*}_{\{I_1,I_2\}P^-}(X) = \overline{S}^{*}_{I_1P^-}(X) \cup \overline{S}^{*}_{I_2P^-}(X)$, then $u \in \overline{S}^{*}_{I_1P^-}(X)$ or $u \in \overline{S}^{*}_{I_2P^-}(X)$,then $u \in \acute{X}$ and $\exists$ $\neg a_1 \in \neg\eta$ s.t. $u \in \Psi(\neg a_1)$ and $\Psi(\neg a_1) \cap X \in I_1$ or $\exists$ $\neg a_2 \in \neg\eta$ s.t. $u \in \Psi(\neg a_2)$ and $\Psi(\neg a_2) \cap X \in I_2$. Since $I_1, I_2 \subseteq \langle I_1, I_2\rangle$, then $u \in \overline{S}^{*}_{\langle I_1,I_2\rangle P^-}(X)$.

Hence $\overline{S}^{*}_{\{I_1,I_2\}P^-}(X) \subseteq \overline{S}^{*}_{\langle I_1,I_2\rangle P^-}(X)$. Thus $\overline{BS}^{*}_{\langle I_1,I_2\rangle P}(X) \sqsubseteq \overline{BS}^{*}_{\{I_1,I_2\}P}(X)$.

(2) Similarly as (1).

(3) Immediately.

(4) Straightforward from part (1) and part (2).

□

Remark 7.11 *It is noted from Theorem 7.10 that Definition 7.5 reduces the bipolar boundary region and increases the bipolar accuracy measure of a set X by increasing the bipolar soft lower approximations and decreasing the bipolar soft upper approximations via two ideals with the comparison of the method in Definition 7.7.*

Proposition 7.12 *Let $P_{(I_1,I_2)} = (U,(\Omega,\Psi,\eta), I_1, I_2)$ be BSbIA-space and $X \subseteq U$. Then,*

(1) $\overline{BS}^{*}_{\langle I_1,I_2\rangle P}(X) \sqsubseteq \overline{BS}^{*}_{\{I_1,I_2\}P}(X) \sqsubseteq \overline{BS}^{*}_{I_iP}(X), \quad \forall i \in \{1,2\}$

(2) $\underline{BS}^{*}_{I_iP}(X) \sqsubseteq \underline{BS}^{*}_{\{I_1,I_2\}P}(X) \sqsubseteq \underline{BS}^{*}_{\langle I_1,I_2\rangle P}(X), \quad \forall i \in \{1,2\}$

(3) $BBND_{\langle I_1,I_2\rangle P}(X) \sqsubseteq BBND_{\{I_1,I_2\}P}(X) \sqsubseteq BBND_{I_iP}(X), \quad \forall i \in \{1,2\}$

(4) $B\mu_{I_iP}(X) \preceq B\mu_{\langle I_1,I_2\rangle P}(X) \preceq B\mu_{\{I_1,I_2\}P}(X), \ \forall i \in \{1,2\}$.

Proof *Sraightforward from Definition 7.7 and Theorem 7.10.*

8. Bipolar soft ideal rough sets in multicriteria group decision making

8.1. First method

In this section, the use of $BSIR$-sets in object assessment and multicriteria group decision making is presented.

Let $U = \{u_1, u_2, u_{3,} \quad u_{n\}}$ be a set of objects under observation, E be the set of parameters to evaluate the the objects in U. Let $\eta = \{a_1, a_2, \quad a_n\} \subseteq \zeta$. Let (Ω,Ψ,η) be a bipolar soft set which represent the information about objects. Consider a set of experts $S = \{D_1, D_{2,} \ldots.. D_n\}$ who evaluate the objects to identify the optimal solution and a bipolar soft set $\Theta = (\theta^{+}, \theta^{-}, S)$ based on the initial assessment derived by experts and an ideal I. In order to obtain results bipolar SIR-approximations in the form of bipolar soft sets $\Theta_{*} = (\theta^{+}_{*}, \theta^{-}_{*}, S)$ and $\Theta^{*} = (\theta^{*+}, \theta^{*-}, S)$ are computed. Following these bipolar soft sets define bipolar fuzzy soft sets $\nu_{\Theta_{*}}$, ν_{Θ} and $\nu_{\Theta^{*}}$ which describes the fuzziness of these bipolar soft sets. After then the negative and positive choice values according to each object and the choice value of each object are calculated. Finally, the optimal alternative having a maximum choice value can be selected.

Algorithm 1

(1) Start.

(2) Input the set of objects, set of criterions and the set of experts.

(3) Construct the bipolar soft set (Ω,Ψ,η) which describes the given data.

(4) Construct the bipolar soft set $\Theta = (\theta^{+}, \theta^{-}, S)$ and an ideal I which describes the initial assessment results of the group of the analysis S.

(5) Construct the $BSIR$-approximations in the form of bipolar soft sets $\Theta_* = (\theta_*^+, \theta_*^-, S)$ and $\Theta^* = (\theta^{*+}, \theta^{*-}, S)$.

(6) Define bipolar fuzzy sets ν_{Θ_*}, ν_Θ and ν_{Θ^*} corresponding to the bipolar soft sets $\Theta_* = (\theta_*^+, \theta_*^-, S)$, $\Theta = (\theta^+, \theta^-, S)$ and $\Theta^* = (\theta^{*+}, \theta^{*-}, S)$ defined by

$\nu_{\Theta_*}^+(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta_*^+ D_i}(p_k),\ \nu_{\Theta_*}^-(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta_*^- D_i}(p_k),$

$\nu_{\Theta^*}^+(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta^{*+} D_i}(p_k),\ \nu_{\Theta^*}^-(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta^{*-} D_i}(p_k),$

$\nu_{\Theta}^+(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta^+ D_i}(p_k),\ \nu_{\Theta}^-(u_k) = \frac{1}{m}\sum_{j=1}^{m} C_{\theta^- D_i}(p_k).$

(7) Compute the negative and positive choice values according to each object u_k (denoted $c_i^+(u_k)$ and $c_i^-(u_k)$), where

$c_i^+(u_k) = \nu_{\Theta_*+\Theta+\Theta^*}^+(u_k) = \nu_{\Theta_*}^+(u_k) + \nu_{\Theta}^+(u_k) + \nu_{\Theta^*}^+(u_k) - (\nu_{\Theta_*}^+(u_k) \times \nu_{\Theta}^+(u_k) \times \nu_{\Theta^*}^+(u_k))$ and

$c_i^-(u_k) = \nu_{\Theta_*+\Theta+\Theta^*}^-(u_k) = \nu_{\Theta_*}^-(u_k) + \nu_{\Theta}^-(u_k) + \nu_{\Theta^*}^-(u_k) - (\nu_{\Theta_*}^-(u_k) \times \nu_{\Theta}^-(u_k) \times \nu_{\Theta^*}^-(u_k)).$

(8) Compute the choice value of each object by $c_i(u_k) = c_i^+(u_k) + c_i^-(u_k)$.

(9) Finally the object having a maximum choice value can be selected as an optimal solution.

(10) Stop.

Example 8.1 *Suppose there is a set of experts (doctors)* $S = \{D_1, D_{2,} D_3\}$ *who want to evaluate some patients for the prone of COVID-19 infection. Let* $U = \{u_1, u_2, u_{3,}.....u_6\}$ *be the set of patients and* $\eta = \{a_{1,} a_{2,} a_3\}$ *be the set of parameters (characteristics of patients), where* $a_1 =$ *obey the protective instructions,* $e_2 =$ *vaccinated and* $e_3 =$ *comorbidity " chronic illness".*

Step 3 *Consider a semiintersection bipolar soft set* (Ω, Ψ, η) *which specify the characteristics of the patients given in Table 6.*

Table 6.

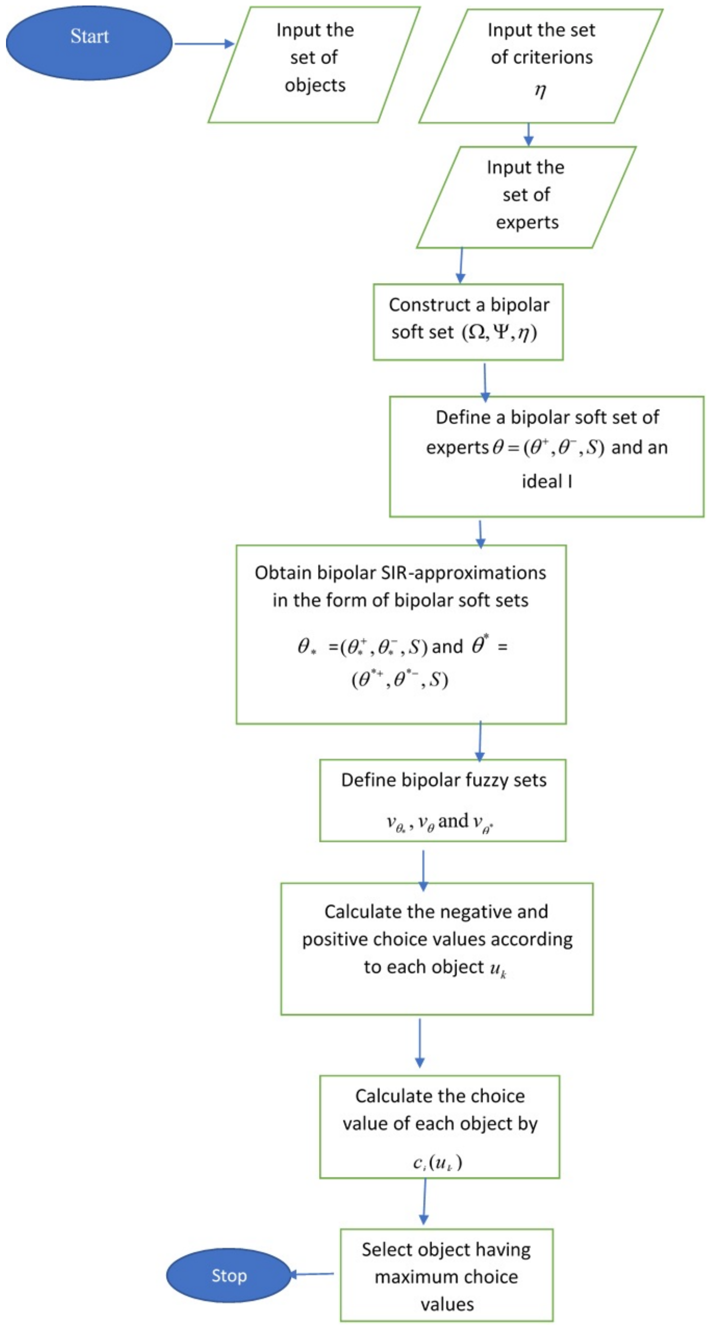
$u(\Omega,\Psi,\eta)$	a_1	a_2	a_3	a_4
u_1	1	0	1	1
u_2	1	1	0	1
u_3	0	1	1	1
u_4	-1	0	-1	0
u_5	0	-1	-1	0
u_6	0	-1	0	-1

Step 4 *Let* X_i *be the initial assessment result of the doctors. This evaluation is represented by means of bipolar soft set* $\Theta = (\theta^+, \theta^-, S)$ *whose tabular representation is given by Table 7.*

From this bipolar soft set $\theta = (\theta^+, \theta^-, S)$ *primary evaluations of experts are*

$X_1 = \theta^+(D_1) \cup \theta^-(\neg D_1) = \{u_1, u_3, u_5\}$

$X_2 = \theta^+(D_2) \cup \theta^-(\neg D_2) = \{u_2, u_4\}$



Algorithm 1.

$X_3 = \theta^+(D_3) \cup \theta^-(\neg D_3) = \{u_1, u_3, u_6\}$

Let $I = \{\varphi, \{u_4\}, \{u_5\}, \{u_4, u_5\}\}$.

Step 5 *Now, the following* $BSIR-$ *approximations are computed as*

$\theta_*^+(D_1) = \underline{S}_{IP^+}(X_1) = \{u_1, u_3\}$ $\quad$ $\theta_*^-(D_1) = \underline{S}_{IP^-}(X_1) = \{u_5, u_6\}$,

$\theta_*^+(D_2) = \underline{S}_{IP^+}(X_2) = \varphi$ $\quad$ $\theta_*^-(D_2) = \underline{S}_{IP^-}(X_2) = \{u_5, u_6\}$,

Table 7.

$(\theta^{+},\theta^{-},S)$	D_1	D_2	D_3
u_1	1	0	1
u_2	0	1	0
u_3	1	0	-1
u_4	0	-1	0
u_5	-1	0	0
u_6	0	0	-1

$\theta_{*}^{+}(D_3) = \underline{S}_{IP^{+}}(X_3) = \{u_1, u_3\}$ $\qquad$ $\theta_{*}^{-}(D_3) = \underline{S}_{IP^{-}}(X_3) = \varphi$

and

$\theta^{*+}(D_1) = \overline{S}_{IP^{+}}(X_1) = \{u_1, u_2, u_3\}$ $\qquad$ $\theta^{*-}(D_1) = \overline{S}_{IP^{-}}(X_1) = \{u_4, u_5, u_6\}$,

$\theta^{*+}(D_2) = \overline{S}_{IP^{+}}(X_2) = \{u_1, u_2, u_3\}$ $\qquad$ $\theta^{*-}(D_2) = \overline{S}_{IP^{-}}(X_2) = \{u_4, u_5, u_6\}$,

$\theta^{*+}(D_3) = \overline{S}_{IP^{+}}(X_3) = \{u_1, u_2, u_3\}$ $\qquad$ $\theta^{*-}(D_3) = \overline{S}_{IP^{-}}(X_3) = \{u_4, u_5\}$.

Step 6 *Following these $BSIR-$approximations, define bipolar soft sets $\Theta_{*} = (\theta_{*}^{+}, \theta_{*}^{-}, S)$ and $\Theta^{*} = (\theta^{*+}, \theta^{*-}, S)$, where*

$\theta_{*}^{+}(D_i) = \underline{S}_{IP^{+}}(X_i), \theta_{*}^{-}(D_i) = \underline{S}_{IP^{-}}(X_i)$, and $\theta^{*+}(D_i) = \overline{S}_{IP^{+}}(X_i), \theta^{*-}(D_i) = \overline{S}_{IP^{-}}(X_i)$. Tabular representations of these bipolar soft sets are given in Tables 8 and 9.

Table 8.

$(\theta_{*}^{+},\theta_{*}^{-},S)$	u_1	u_2	u_3	u_4	u_5	u_6
D_1	1	0	1	0	-1	-1
D_2	0	0	0	0	-1	-1
D_3	1	0	1	0	0	0

Table 9.

$(\theta^{*+},\theta^{*-},S)$	u_1	u_2	u_3	u_4	u_5	u_6
D_1	1	1	1	-1	-1	-1
D_2	1	1	1	-1	-1	-1
D_3	1	1	1	-1	-1	0

Now, define bipolar fuzzy soft set $\nu_{\Theta_{*}}(u_k) = (\nu_{\Theta_{*}}^{+}(u_k), \nu_{\Theta_{*}}^{-}(u_k))$, $\nu_{\Theta^{*}}(u_k) = (\nu_{\Theta^{*}}^{+}(u_k), \nu_{\Theta^{*}}^{-}(u_k))$ and $\nu_{\Theta}(u_k) = (\nu_{\Theta}^{+}(u_k), \nu_{\Theta}^{-}(u_k))$, where

$\nu_{\Theta_{*}}^{+}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta_{*}^{+}D_i}(p_k)$, $\nu_{\Theta_{*}}^{-}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta_{*}^{-}D_i}(p_k)$,

$\nu_{\Theta^{*}}^{+}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta^{*+}D_i}(p_k)$, $\nu_{\Theta^{*}}^{-}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta^{*-}D_i}(p_k)$, *and*

$\nu_{\Theta}^{+}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta^{+}D_i}(p_k)$ *and* $\nu_{\Theta}^{-}(u_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\theta^{-}D_i}(p_k)$.

Hence,

$\nu_{\Theta_*}(u_k) = \{(u_1, \frac{2}{3}, 0), (u_2, 0, 0), (u_3, \frac{2}{3}, 0), (u_4, 0, 0), (u_5, 0, \frac{-2}{3}), (u_6, , 0, \frac{-2}{3})\}$

$\nu_{\Theta}(u_k) = \{(u_1, \frac{2}{3}, 0), (u_2, \frac{1}{3}, 0), (u_3, \frac{1}{3}, \frac{-1}{3}), (u_4, 0, \frac{-1}{3}), (u_5, 0, \frac{-1}{3}), (u_6, 0, \frac{-1}{3})\}$

$\nu_{\Theta^*}(u_k) = \{(u_1, 1, 0), (u_2, 1, 0), (u_3, 1, 0), (u_4, 0, -1), (u_5, 0, -1), (u_6, 0, \frac{-2}{3})\}$.

Step 7 *Calculate the negative and positive choice values according to each object* u_k *(denoted* $c_i^+(u_k) = \nu^+_{\Theta_*+\Theta+\Theta^*}(u_k)$ *and* $c_i^-(u_k) = \nu^-_{\Theta_*+\Theta+\Theta^*}(u_k)$ *as given in Table 10.*

Table 10.

	u_1	u_2	u_3	u_4	u_5	u_6
c_i^+	$\frac{7}{3}$	$\frac{4}{3}$	2	0	0	0
c_i^-	0	0	$\frac{-1}{3}$	$\frac{-4}{3}$	-2	$\frac{-5}{3}$

Step 8 *Calculate the choice value of each object by* $c_i(u_k) = c_i^+(u_k) + c_i^-(u_k)$ *as given in Table 11.*

Table 11.

	u_1	u_2	u_3	u_4	u_5	u_6
c_i	$\frac{7}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{-4}{3}$	-2	$\frac{-5}{3}$

Step 9 *All the alternatives can be arranged according to their choice values: The ranking of patients with prone to COVID-19 infection from high priority to low priority is as follows:*

$u_1 \succ u_3 \succ u_2 \succ u_4 \succ u_6 \succ u_5$.

Algorithm 2

(1) *Start.*

(2) *Input the set of objects, set of criterions and the set of experts.*

(3) *Construct the bipolar soft set* (Ω, Ψ, η) *which describes the given data.*

(4) *Based on the initial assessment results of the group of the analysis* S, *construct a bipolar soft set and an ideal* I.

(5) *Construct the* $BSIR-$*approximations in the form of bipolar soft sets* $\Theta_* = (\theta_*^+, \theta_*^-, S)$ *and* $\Theta^* = (\theta^{*+}, \theta^{*-}, S)$.

(6) *Find choice values for all selected bipolar soft sets* $\Theta_* = (\theta_*^+, \theta_*^-, S)$ *and* $\Theta^* = (\theta^{*+}, \theta^{*-}, S)$.

(7) *Find the decision set by adding all the choice values of obtained bipolar soft sets.*

(8) *Input the weighting vector* $W = (\omega_L, \omega_M, \omega_H)$ *and compute the weighted evaluation value for each object.*

(9) *Find the decision set by adding all the weighted values* $\sum_i \omega_i$. *Choose the object having a maximum value.*

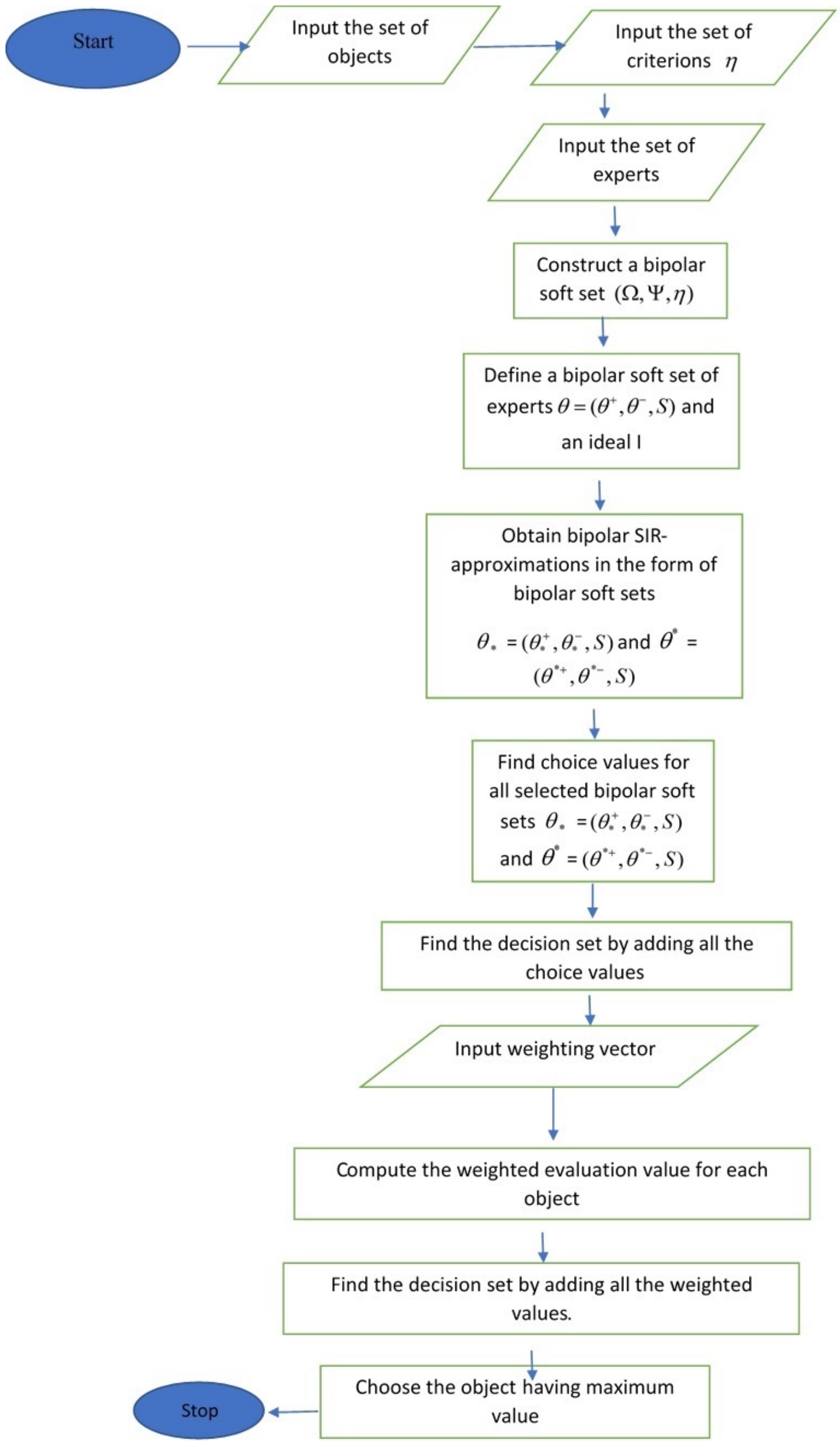
(10) *Stop.*

Example 8.2 *Consider Example 8.1. First five steps are the same as done by Algorithm 1.*

Step 6 *Find choice value for all selected bipolar soft sets* $\Theta_* = (\theta_*^+, \theta_*^-, S)$,

$\Theta = (\theta^+, \theta^-, S)$ *and* $\Theta^* = (\theta^{*+}, \theta^{*-}, S)$ *which are given in Tables 12–14.*

Step 7 *Find the (final choice value) decision set by adding all the choice values of obtained bipolar soft sets which are given in Table 15.*



Algorithm 2.

Steps 8 *and* ***9*** *Compute the weighted evaluation value for each object and find the decision set by adding all the weighted values* $\sum_i \omega_i$ *which are given in choice value table (Table 16). Choose the object having a maximum value.*

Table 12.

$(\theta^{+},\theta^{-},S)$	u_1	u_2	u_3	u_4	u_5	u_6
D_1	1	0	1	0	-1	0
D_2	0	1	0	-1	0	0
D_3	1	0	1	0	0	-1
Choice values C_1	2	1	2	-1	-1	-1

Table 13.

$(\theta^{+}_{*},\theta^{-}_{*},S)$	u_1	u_2	u_3	u_4	u_5	u_6
D_1	1	0	1	0	-1	-1
D_2	0	0	0	0	-1	-1
D_3	1	0	1	0	0	0
Choice value C_2	2	0	2	0	-2	-2

Table 14.

$(\theta^{*+},\theta^{*-},S)$	u_1	u_2	u_3	u_4	u_5	u_6
D_1	1	1	1	-1	-1	-1
D_2	1	1	1	-1	-1	-1
D_3	1	1	1	-1	-1	0
Choice value C_3	3	3	3	-3	-3	-2

Table 15.

	u_1	u_2	u_3	u_4	u_5	u_6
D_1	2	0	2	0	-2	-2
D_2	2	1	2	-1	-1	-1
D_3	3	3	3	-3	-3	-2
Final choice value	7	4	7	-4	-6	-5

Table 16.

	u_1	u_2	u_3	u_4	u_5	u_6
$\omega(o.2)$	1.4	0.8	1	-0.8	-1.2	-1
$\omega(0.3)$	2.1	1.2	1.5	-1.2	-1.8	-1.5
$\omega(0.5)$	3.5	2	2.5	-2	-3	-2.5
$Final$WC *value*	7	4	5	-4	-6	-5

The ranking of patients with prone to COVID-19 infection from high priority to low priority is as follows:
$u_1 \succ u_3 \succ u_2 \succ u_4 \succ u_6 \succ u_5$.
Thus, u_1 *is to be selected.*

8.2. Second method

Let $U = \{u_1, u_2, u_3, \quad u_n\}$ be a set of objects under observation, E be the set of parameters to evaluate the the objects in U. Let $\eta = \{a_1, a_2, \quad a_n\} \subseteq \zeta$. Consider a bipolar soft set (Ω, Ψ, η) which represents the information about objects, a set of experts $S = \{D_1, D_{2,} \ldots.. D_n\}$ who evaluate the objects to identify the optimal solution and an ideal I. Based on the initial assessment by the experts, define bipolar soft sets $\Theta = (\omega^+, \omega^-, S)$. Assume that $T_j, j = 1, 2, \ldots r$ are bipolar soft sets, $T_2, \ldots.. T_r \in BSS_U$ represented real results that are previously obtained for same or similar problems in different times or different places.

Definition 8.3 *Let* $\underline{BS}_{IT_q}(X_i) = (\underline{S}_{IT_q^+}(X_i), \underline{S}_{IT_q^-}(X_i)), \overline{BS}_{IT_q}(X_i) = (\overline{S}_{IT_q^+}(X_i), \overline{S}_{IT_q^-}(X_i))$ *be bipolar soft ideal lower and upper approximation of* $X_i, (i = 1, 2, \ldots.n)$ *related to* $T_q(q = 1, 2, \ldots.r)$. *Then,*

$$\underline{b}_I = \begin{pmatrix} \left(\underline{v}_1^{1^+}, \underline{v}_1^{1^+}\right) & \left(\underline{v}_{21}^{1^+}, \underline{v}_2^{1^+}\right) \ldots\ldots & \left(\underline{v}_n^{1^+}, v_n^{1^+}\right) \\ \left(\underline{v}_1^{2^+}, \underline{v}_1^{2^+}\right) & \left(\underline{v}_2^{2^+}, \underline{v}_2^{2^+}\right) \ldots\ldots & \left(\underline{v}_n^{2^+}, \underline{v}_n^{2^+}\right) \\ & & \\ \left(\underline{v}_1^{r^+}, \underline{v}_1^{r^+}\right) & \left(\underline{v}_2^{r^+}, \underline{v}_2^{r^+}\right) \ldots\ldots & \left(\underline{v}_n^{r^+}, \underline{v}_n^{r^+}\right) \end{pmatrix}$$

$$\overline{b}^I = \begin{pmatrix} \left(\overline{v}_1^{1^+}, \overline{v}_1^{1^+}\right) & \left(\overline{v}_2^{1^+}, \overline{v}_2^{1^+}\right) \ldots\ldots & \left(\overline{v}_n^{1^+}, \underline{v}_n^{1^+}\right) \\ \left(\overline{v}_1^{2^+}, \overline{v}_1^{2^+}\right) & \left(\overline{v}_2^{2^+}, \overline{v}_2^{2^+}\right) \ldots\ldots & \left(\overline{v}_n^{2^+}, \overline{v}_n^{2^+}\right) \\ & & \\ \left(\overline{v}_1^{r^+}, \overline{v}_1^{r^+}\right) & \left(\overline{v}_2^{r^+}, \overline{v}_2^{r^+}\right) \ldots\ldots & \left(\overline{v}_n^{r^+}, \overline{v}_n^{r^+}\right) \end{pmatrix}$$

are called bipolar soft ideal lower and upper approximations matrices, respectively, and denoted by $\underline{b}$ and $\overline{b}$. Here

$$\underline{v}_j^{q^+} = \left(\underline{v}_{1j}^{q^{j+}}, \underline{v}_{2j}^{q^+}, \ldots\ldots \underline{v}_{nj}^{q^+}\right) \quad \underline{v}_j^{q^-} = \left(\underline{v}_{1j}^{q^-}, \underline{v}_{2j}^{q^-}, \ldots\ldots \underline{v}_{nj}^{q^-}\right)$$

$$\overline{v}_i^{q^+} = \left(\overline{v}_{1i}^{q^+}, \overline{v}_{2i}^{q^+}, \ldots\ldots \overline{v}_{ni}^{q^+}\right) \quad \overline{v}_i^{q^-} = \left(\overline{v}_{1i}^{q^-}, \overline{v}_{2i}^{q^-}, \ldots\ldots \overline{v}_{ni}^{q^-}\right),$$

where

$$\underline{v}_{ij}^{q^{j+}} = \begin{cases} 1 & v_i \in \underline{S}_{IT_q^+}(X_j) \\ 0 & v_i \notin \underline{S}_{IT_q^+}(X_i) \end{cases}, \qquad \underline{v}_{ij}^{q-} = \begin{cases} 1 & v_i \in \underline{S}_{IT_q^-}(X_i) \\ 0 & v_i \notin \underline{S}_{IT_q^+}(X_i) \end{cases}$$

and

$$\overline{v}_{ij}^{q^+} = \begin{cases} 1 & v_i \in \overline{S}_{IT_q^+}(X_i) \\ 0 & v_i \notin \overline{S}_{IT_q^+}(X_i) \end{cases}, \qquad \overline{v}_{ij}^{q-} = \begin{cases} 1 & v_i \in \overline{S}_{IT_q^-}(X_i) \\ 0 & v_i \notin \overline{S}_{IT_q^-}(X_i) \end{cases}$$

Definition 8.4 *Let* $\underline{b}, \overline{b}$ *be bipolar soft ideal lower and upper approximations matrices based on* $\underline{BS}_{IT_q}(X_i) = (\underline{S}_{IT_q^+}(X_i), \underline{S}_{IT_q^-}(X_i)), \overline{BS}_{IT_q}(X_i) = (\overline{S}_{IT_q^+}(X_i), \overline{S}_{IT_q^-}(X_i))$ *for* $q = 1, , \ldots.r$ *and* $j = 1, 2, \ldots n$. *Bipolar soft ideal lower approximation vector (denoted by* $\underline{v}_I$ *) and bipolar soft ideal upper approximation vector (denoted by* $\overline{v}_I$ *) are defined by, respectively,*

$$\underline{v}_I = \sum_{j=1}^{n}\sum_{q=1}^{r}(\underline{v}_j^{q^+} \oplus v_j^{q^-})\overline{v}_I$$

$$\overline{v}_I = \sum_{j=1}^{n}\sum_{q=1}^{r}(\overline{v}_j^{q^+} + \overline{v}_j^{q^-})\overline{v}_I.$$

Definition 8.5 *Let $\underline{v}_I$, $\overline{v}_I$ be bipolar soft ideal T_q- lower approximation vector and bipolar soft ideal T_q- upper approximation vector, respectively. Vector summation $\underline{v}_I \oplus \overline{v}_I = (v_1, v_2, ...v_n)$ is called I-decision vector.*

Definition 8.6 *Let $\underline{v}_I$ $\overline{v}_I$ $=(v_1, v_2, ...v_n)$ be the decision vector. Then each v_i is called a weighted number of $u_i \in U$ and u_i is called an optimum element of U if its weighted number is maximum of v_i for all $i \in I_n$.. If there are more than one optimum elements of U, choose one of them.*

Algorithm 3

(1) Start.

(2) Input the set of objects, set of criterions and the set of experts.

(3) Take primary evaluations $X_1, X_2, ...X_n$ of experts $D_1, D_2, ...Dn$.

(4) Construct $T_1, T_2, ...T_r$ of bipolar soft sets using real results.

(5) Compute $\underline{BS}_{IT_q}(X_i)$ and $\overline{BS}_{IT_q}(X_i)$ for each $q = 1, , ...r$ and $i = 1, 2, ...n$.

(6) Construct bipolar soft ideal lower and upper approximations matrices $\underline{b}$ and $\overline{b}$.

(7) Compute $\underline{v}_I$,and $\overline{v}_I$.

(8) Compute $\underline{v}_I \oplus \overline{v}_I$.

(9) Find $max_{i \in I_n} v_i$.

(10) Stop.

Example 8.7 *Suppose there is a set of experts (doctors) $S = \{D_1, D_2, D_3\}$ who want to evaluate some patients for the prone to COVID-19 infection. Let $U = \{u_1, u_2, u_3, ...u_6\}$ be the set of patients and $\eta = \{a_1, a_2, a_3\}$ be the set of parameters (characteristics of patients), where $a_1 =$obey the protective instructions, $e_2 =$vaccinated and $e_3 =$comorbidity " chronic illness". Consider a semiintersection bipolar soft set (Ω, Ψ, η) which specifies the characteristics of the patients given in Table 6. Let $U = \{u_1, u_2, u_3, ...u_5\}$ and $A = \{a_1, a_2, a_3\}$.*

Step 3: *Primary evaluations of $D_1, D_2, ...X_n$ are $X_1 = \{u_1, u_2, , u_3\}$, $X_{2,} = \{u_1, u_3, , u_5\}$ and $X_3 = \{u_2, u_4, , u_5\}$, respectively.*

Step 4: *Real results in different three periods are expressed as bipolar soft sets as follows:*

$T_1 = \{(a_1, \{u_1\}, \{u_3, u_5\}),\ (a_2, \{u_1, u_5\}, \{u_3\}), (a_3, \{u_4, u_5\}, \{u_1, u_3\})\}$

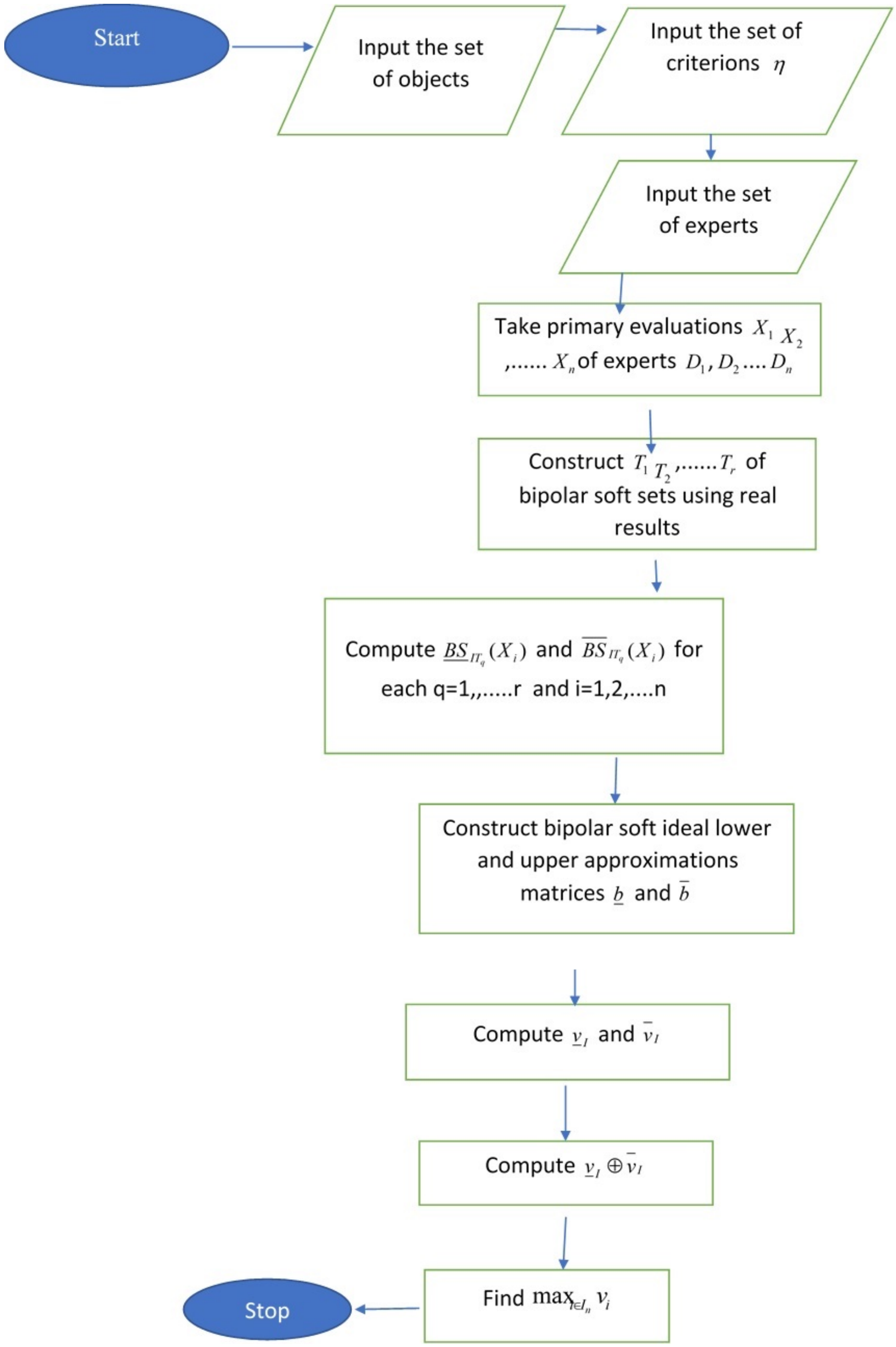
$T_2 = \{(a_1, \{u_2\}, \{u_1, u_4\}),\ (a_2, \{u_2, u_4\}, \{u_5\}), (a_3, \{u_3, u_4\}, \{u_1, u_5\})\}$

$T_3 = \{(a_1, \{u_3, u_5\}, \{u_1, u_2\}),\ (a_2, \{u_2\}, \{u_4\}), (a_3, \{u_2, u_5\}, \{u_1, u_3\})\}$.

Step 5:

$\underline{BS}_{IT_1}(X_1) = (\{u_1, u_2, u_3, u_4, u_5\}, \{\})$, $\qquad \overline{BS}_{IT_1}(X_1) = (\{u_1, u_2, u_3, u_4\}, \{u_4\})$

$\underline{BS}_{IT_1}(X_2) = (\{u_1, u_3, u_4, u_5\}, \{u_1, u_2\})$, $\qquad \overline{BS}_{IT_1}(X_2) = (\{u_1, u_3, u_4, u_5\}, \{u_2, u_4\})$



Algorithm 3.

$\underline{BS}_{IT_1}(X_3) = (\{u_2\}, \{u_1, u_2\}),$ $\qquad$ $\overline{BS}_{IT_1}(X_3) = (\{u_2\}, \{u_4\})$

$\underline{BS}_{IT_2}(X_1) = (\{u_2, u_3, u_4, u_5\}, \{\}),$ $\qquad$ $\overline{BS}_{IT_2}(X_1) = (\{u_2, u_3, u_5\}, \{\})$

$\underline{BS}_{IT_2}(X_2) = (\{u_4\}, \{\}),$ $\qquad$ $\overline{BS}_{IT_2}(X_2) = (\{u_2, u_3\}, \{\})$

$\underline{BS}_{IT_2}(X_3) = (\{u_2, u_4, u_5\}, \{u_1, u_3, u_5\}),$ $\quad \overline{BS}_{IT_2}(X_3) = (\{u_2, u_3, u_5\}, \{u_1, u_3, u_5\})$

$\underline{BS}_{IT_3}(X_1) = (\{u_1, u_2, u_4, u_5\}, \{\}),$ $\quad \overline{BS}_{IT_3}(X_1) = (\{u_1, u_2, u_4, u_5\}, \{\})$

$\underline{BS}_{IT_3}(X_2) = (\{u_1, u_5\}, \{u_2, u_3\}),$ $\quad \overline{BS}_{IT_3}(X_2) = (\{u_1, u_5\}, \{\})$

$\underline{BS}_{IT_3}(X_3) = (\{u_4, u_5\}, \{u_1, u_2, u_3\}),$ $\quad \overline{BS}_{IT_3}(X_3) = (\{u_2, u_4\}, \{u_1, u_3\})$.

Step 6: *Bipolar soft ideal lower and upper approximations matrices $\underline{b}$ and $\overline{b}$.*

For simplicity, let

$$\begin{aligned} a_0 &= (0,0,0,0,0), a_1 = (0,0,0,-1,0), a_3 = (\frac{-1}{2}, \frac{-1}{2}, 0,0,0) \\ a_4 &= (0,-1,0,-1,0), a_5 = (\frac{-1}{2}, \frac{-1}{2}, 0,0,0), a_6 = (\frac{-1}{2}, 0, \frac{-1}{2}, 0, \frac{-1}{2}) \\ a_7 &= (-1,0,-1,0,-1) \end{aligned}$$

$$\underline{b}_I = \begin{pmatrix} \{(1,1,1,1,1), a_0\} & \{(1,0,1,1,1), a_3\} & \{(0,1,0,0,0), a_5\} \\ \{(0,1,1,1,1), a_0\} & \{(0,0,0,1,0), a_0\} & \{(0,1,0,1,1), a_6\} \\ \{(0,1,1,0,1), a_0\} & \{(0,1,1,0,0), a_0\} & \{(0,1,1,0,1), a_6\} \end{pmatrix}$$

$$\overline{b}^I = \begin{pmatrix} \{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1), a_1\} & \{(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), a_4\} & \{(0, \frac{1}{2}, 0,0,0), a_1\} \\ \{(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), a_0\} & \{(0,0,0,\frac{1}{2},0), a_0\} & \{(0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}), a_7\} \\ \{(0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), a_0\} & \{(0, \frac{1}{2}, \frac{1}{2}, 0, 0), a_0\} & \{(0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}), a_7\} \end{pmatrix}$$

Step 7: *Therefore,* $\underline{v}_I = (4, 5, \frac{5}{2}, 5, \frac{7}{2})$ *and* $\overline{v}_I = (0, 0, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2})$.

Step 8: *Decision vector is* $\underline{v}_I \ \overline{v}_I = (4, 5, 3, 4.5, 4)$.

Step 9: $max_{i \in I_n v_i} = v_2 = 5$. *So, optimum element is* v_2.

9. Conclusion

Ideal is an important concept in topological spaces and plays an important role in the study of topological problems. This paper can be considered a modification and generalization of bipolar soft rough set model. New approximations called bipolar soft ideal rough approximations have been introduced and their properties have been studied. Comparisons among these approaches and previous ones have been discussed. In Section 7, two different methods of bipolar approximation spaces based on two ideals in Definitions 7.3 and 7.5 are introduced. Also, the comparisons between these methods are investigated. This method can be extended similarly by using n-ideals. Finally, an application in multicriteria group decision making by using two methods to present the importance of our approximations have been presented and three algorithms for obtaining an optimal choice by using bipolar soft ideal rough sets have been proposed.

Merits and future directions of the proposed study are listed as follows:

(1) According to this study, in ”Theorem 6.4 and its Corollary 6.5” the suggested method is more accurate than bipolar soft rough approximations in decision making by increasing the bipolar soft accuracy measure and reducing the bipolar soft boundary region of the sets. Therefore, these methods are very useful in real life applications.

(2) According to this study, in ”Theorem 6.10” the suggested method is very important in defining the sets by introducing a complete new range of bipolar soft ideal approximation spaces. For example: if X is totally bipolar soft P-indefinable, then $\underline{BS}_P(X) = (\varphi, U)$ and $\overline{BS}_P(X) = (U, \varphi)$. But, by using bipolar soft*

ideal approximation, $\underline{BS}^*_{IP}(X) \neq (\varphi, U)$ and $\overline{BS}^*_{IP}(X) \neq (U, \varphi)$ and then X can be roughly bipolar soft* I-definable.

(3) Introducing a new kind of bipolar soft rough set based on n-ideals.

(4) The bipolar soft biideal rough sets represent two opinions instead of one opinion.

(5) These approaches are the best tool in decision making about the infection of COVID-19 by using bipolar information (positive and negative) and an ideal. Bipolar soft ideal rough sets are used to find the patients which will be prone to COVID-19.

(6) In our future work we shall extend this work to some new models like bipolar neutrosophic soft rough sets, fuzzy soft rough sets and bipolar soft rough graph.

References

[1] Pawlak Z. Rough sets. International Journal of Computer and Information Sciences 1982; 11 (5): 145-172.

[2] El-Bably MK, Al-shami TM. Different kinds of generalized rough sets based on neighborhoods with a medical application. International Journal of Biomathematics 2021; 14(8): 2150086. doi: 10.1142/S1793524521500868

[3] El-Bably MK, Abo-Tabl EA, A topological reduction for predicting of a lung cancer disease based on generalized rough sets. Journal of Intelligent and Fuzzy Systems 2021; 41(2): 1861-1872. doi: 10.3233/JIFS-210167

[4] Paternain D, Jurio A, Barrenechea E, Bustince H, Bedregal B, Szmidt E. An alternative to fuzzy methods in decision-making problems. Expert Systems with Applications 2012; 39 (9): 7729-7735. doi: 10.1016/j.eswa.2012.01.081

[5] Molodtsov D. Soft set theory-first results. Computer and Mathematics with Applications 1999; 37 (4-5):19-31. doi: 10.1016/S0898-1221(99)00056-5

[6] Molodtsov D. The theory of soft sets. Moscow, URSS Publishers, (in Russian), 2004.

[7] Alcantud JCR, Santos-Garca G, Galilea EH. Glaucoma diagnosis: A soft set based decision making procedure. In J. M.Puerta, et al. (Eds.), Lecture notes in artificial intelligence, Springer 9422 (2015): 49-60. doi: 10.1007/978-3-319-24598-0_5

[8] Maji PK, Biswas R, Roy AR. Soft set theory. Computer and Mathematics with Applications 2003; 45 (4-5): 555-562. doi: 10.1016/S0898-1221(03)00016-6

[9] Ali MI, Feng F, Liu XY, Min WK, Shabir M. On some new operations in soft set theory. Computer and Mathematics with Applications 2009; 57 : 1547-1553. doi: 10.1016/j.camwa.2008.11.009

[10] Cagman N, Karatas S, Enginoglu S. Soft topology. Computer and Mathematics with Applications 2011; 62: 351-358.

[11] Aktas H, Cagman N. Soft sets and soft group. Information Sciences 2007; 1(77): 2726-2735. doi: 10.1016/j.ins.2006.12.00

[12] Fatimah F, Rosadi D, Hakim RBF, Alcantud JCR. N-soft sets and their decision-making algorithms. Soft Computing 2018; 22: 3829-3842. doi: 10.1007/s00500-017-2838-6

[13] Akram M, Ali G, Alcantud JCR, Fatima F. Parameter reductions in N-soft sets and their applications in decision-making. Expert Systems 2020; 38(8): 1-15. doi: 10.1111/exsy.12601

[14] Al-shami TM, Kočinac LDR, Asaad BA. Sum of soft topological spaces. Mathematics 2020; 8(6): 990. doi: 10.3390/math8060990

[15] Zulqarnain RM, Garg H, Siddique I, Ali R, Alsubie A et al. Algorithms for a Generalized Multipolar Neutrosophic Soft Set with Information Measures to Solve Medical Diagnoses and Decision-Making Problems, Volume 2021 Article ID 6654657. doi: 10.1155/2021/6654657

[16] Herawan T, Deris MM. A soft set approach for association rules mining. Knowledge Based Systems 2011; 24: 186-195. doi: 10.1016/j.knosys.2010.08.005

[17] Alcantud JCR, Santos-Garca G. A new criterion for soft set based decision making problems under incomplete information. International Journal of Computational Intelligence Systems 2017; 10 (1): 394-404. doi: 10.2991/ijcis.2017.10.1.27

[18] Zadeh LA. Fuzzy sets. Information and Control 1965; 8: 338-353. doi: 10.1016/S0019-9958(65)90241-X

[19] Atanassov KT, Intuitionistic fuzzy sets. Fuzzy Sets and System 1986: 20: 87-96. doi: 10.1016/S0165-0114(86)80034-3

[20] Atanassov KT, Intuitionistic fuzzy relations (IFRs). On intuitionistic fuzzy sets theory, vol 283. Studies in fuzziness and soft computing. Springer, Berlin, pp 147-193 (2012).

[21] Saikia R, Garg H, Dutta P. Fuzzy multi-criteria decision making algorithm under intuitionistic hesitant fuzzy set with novel distance measure. International Journal of Mathematical Engineering and Management Sciences 2020; 5(3): 473-487. doi: 10.33889/ijmems.2020.5.3.039

[22] Torra V. Hesitant fuzzy sets. International Journal of Intelligent Systems 2010; 25(6): 529-539. doi: 10.1002/int.20418

[23] Majumdar P, Samanta SK. Generalized fuzzy soft sets. Computer and Mathematics with Applications 2010; 59(4): 1425-1432. doi: 10.1016/j.camwa.2009.12.006

[24] Roy AR, Maji PK. A fuzzy soft set theoretic approach to decision making problems. Journal of Computational and Applied Mathematics 2007; 203: 412-418. doi: 10.1016/j.cam.2006.04.008

[25] Petchimuthu S, Garg H, Kamacı H, Atagün AO. The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM. Computational and Applied Mathematics 2020; 39:68. doi: 10.1007/s40314-020-1083-2

[26] Cağman N, Karataş S. Intuitionistic fuzzy soft set theory and its decision making. Journal of Intelligent and Fuzzy Systems 2013; 24(4): 829-836. doi: 10.3233/IFS-2012-0601

[27] S Alkhazaleh S , Salleh AR. Generalized Interval-valued fuzzy soft set. Journal of Applied Mathematics. Volume 2012, Article ID 870504, 18 pages. doi: 10.1155/2012/870504.

[28] Jiang Y, Tang Y, Chena Q, Liua H, Tang J. Interval-valued intuitionistic fuzzy soft sets and their properties. Computer and Mathematics with Applications 2010; 60: 906-918. doi: 10.1016/j.camwa.2010.05.036

[29] Wang F,.Li X, Chen X. Hesitant Fuzzy Soft Set and Its Applications in Multicriteria Decision Making. Journal of Applied Mathematics 2014, Article ID 643785, 10 pages. doi: 10.1155/2014/643785

[30] X.Peng X,.Peng Yan Y. Interval-valued Hesitant Fuzzy Soft Sets and their Application in Decision Making. Fundamenta Informaticae 2015; 141 (1): 71-93. doi: 10.3233/FI-2015-1264

[31] Chen B. Generalized hesitant fuzzy soft sets. Italian Journal of Pure and Applied Mathematics 2016; 36: 35-54.

[32] Lee KM. Bipolar-valued fuzzy sets and their applications. In: Proceedings of international conference on intelligent technologies, Bangkok, Thailand, 2000. pp. 307-312.

[33] Shabir M, Naz M. On bipolar soft sets. Retrieved from https://arxiv.org/abs/1303.1344 (2013).

[34] Ali G , Akram M, Alcantud JCR. Attributes reductions of bipolar fuzzy relation decision systems. Neural Computing with Applications 2019; 32: 10051-10071. doi: 10.1007/ s00521- 019- 04536- 8

[35] Abdallah A, Ala A, Jamil J. Bipolar complex fuzzy sets and their properties. Italian Journal of Pure and Applied Mathematics 2020; 43: 754-761.

[36] Abdullah S, Aslam M,.Ullah K. Bipolar fuzzy soft sets and its applications in decision making problem. Journal of Intelligent and Fuzzy Systems 2014; 27(2): 729-742. doi: 10.3233/IFS-131031

[37] Riaz M, Tehrim TS. Bipolar fuzzy soft mappings with application to bipolar disorders. International Journal of Biomathematics 2019; 12(7):1-31. doi.org/10.1142/S1793524519500803

[38] Shabir M, Naz M. On fuzzy bipolar soft sets, their algebraic structures and applications. Journal of Intelligent and Fuzzy Systems 2014; 26(4):1645-1656. doi: 10.3233/IFS-130844

[39] Wang JY, Wang YP, Liu L. Hesitant Bipolar-Valued Fuzzy Soft Sets and Their Application in Decision Making. Complexity 2020; 3:1-12. doi.org/10.1155/2020/6496030

[40] Ali M, Son LH, Deli I, Dien TN. Bipolar neutrosophic soft sets and applications in decision making. Journal of Intelligent and Fuzzy Systems 2017; 33(6): 4077-4087. doi: 10.3233/JIFS-17999

[41] Yang Y, Peng X , Chen H, Zeng L. A decision making approach based on bipolar multi-fuzzy soft set t heory. Journal of Intelligent and Fuzzy Systems 2014; 27(4): 1861-1872. doi: 10.3233/IFS-141152

[42] Al-Qudah Y, Hassan N. Bipolar fuzzy soft expert set and its application in decision making. International Journal of Applied Decision Sciences 2017; 10(2): 175-191. doi: 10.1504/IJADS.2017.10004211

[43] Akram M, Feng F, Saeid AB, Veoreanu-Fotea L. A new multiple criteria decision-making method based on bipolar fuzzy soft graphs. Iranian Journal of Fuzzy Systems 2018; 15(4): 73–92. doi: 10.22111/ijfs.2018.4116

[44] Malik N, Shabir M. Rough fuzzy bipolar soft sets and a pplication in decision-making problems. Soft Computing 2019; 23 (5): 1603-1614. doi:10.1007/s00500-017-2883-1

[45] Karaaslan F, Karatas S. A new approach to bipolar soft sets and its applications. Discrete Mathematics, Algorithms and Applications 2015; 7 (4): 1550054. doi.org/10.1142/S1793830915500548

[46] Feng F, Liu X, Leoreanu-Fotea V, Jun YB. Soft sets and soft rough sets. Information Sciences 2011; 181: 1125-1137. doi.org/10.1016/j.ins.2010.11.004

[47] Meng D, Zhang X, Qin K. Soft rough fuzzy sets and soft fuzzy rough sets. Computers and Mathematics with Applications 2011; 62(12): 4635-4645. doi: 10.1016/j.camwa.2011.10.049

[48] Zhang H, Shu L, Liao S. Intuitionistic Fuzzy Soft Rough Set and Its Application in Decision Making, Abstract and Applied Analysis, Volume 2014, Article ID 287314, 13 pages. doi: 10.1155/2014/28731

[49] Shabir M, Irfan Ali M, Shaheen T. Another approach to soft rough sets. Knowledge-Based Systems 2013; 40: 72-80. doi: 10.1016/j.knosys.2012.11.012

[50] Li P, Liu J, Kong Z, Liu W, Xu C. On modified soft rough sets (MSR-sets), 2017 29th Chinese Control And Decision Conference (CCDC), Chongqing 2017: 254-257. doi: 10.1109/CCDC.2017.7978101.

[51] Mustafa HI. On modified soft rough sets on a complete atomic Boolean lattice. Journal of the Egyptian Mathematical Society 2019; 15(27). doi: 10.1186/s 42787-019-0019-8.

[52] Sun B, Ma W. Soft fuzzy rough sets and its application in decision making. Artificial Intelligence Review 2014; 41: 67-80. doi: 10.1007/s10462-011-9298-7.

[53] El Sayed M, Al Qubati Abdul Gawad AQ, El-Bably MK. Soft pre-rough sets and its applications in decision making. Mathematical Biosciences and Engineering 2020; 17(5): 6045-6063. doi: 10.3934/mbe.2020321

[54] El-Bably MK, El Atik AFA, Soft beta-rough sets and their application to determine COVID-19. Turkish Journal of Mathematics 2021; 45 (3): 1133-1148

[55] Karaaslan F, Cagman N. Bipolar soft rough sets and their applications in decision making. Afrika Matematika 2018; 29 : 823-839. doi: 10.1007/s13370-018-0580-6

[56] Shabir M, Rizwan G. Modified rough bipolar soft sets. Journal of Intelligent and Fuzzy Systems 2020; 39(3): 4259-4283. doi: 10.3233/JIFS-200317

[57] Kuratowski K. Topology, Vol. I, Academic Press, New York, 1966.

[58] Jankovic D, Hamlet TR. New topologies from old via ideals. The American Mathematical Monthly 1990; 97: 295-310. doi: 10.1080/00029890.1990.11995593

[59] Kozae AM. On topology expansions by ideals and applications. Chaos, Solitons and Fractals 2002; 13: 55-60. doi: 10.1016/S0960-0779(00)00224-1

[60] Kandil A, Yakout MM, Zakaria A. Generalized rough sets via ideals. Annals of Fuzzy Mathematics and Informatics 2013; 5(3): 525-532.

[61] O.A.E.Tantawy, H.I.Mustafa, On rough approximations via ideal, Information Sciences 2013; 251: 114-125.

[62] Hosny M, M.Raafat M. On generalization of rough multiset via multiset ideals. Journal of Intelligent and Fuzzy Systems 2017; 33: 1249-1261. doi: 10.3233/JIFS-17102

[63] Mustafa HI. Soft rough approximation operators via ideal. Journal of computational analysis and applications 2019; 26(1): 18-35.

[64] Feng F, Li C, Davvaz B, Ali MI. Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Computing 2010; 14(9): 899–911. doi: 10.1007/s00500-009-0465-6

[65] Kandil A, El-Sheikh A, Hosny M, Raafat M. Bi-ideal approximation spaces and their applications. Soft Computing 2020; 24: 12989-13001. doi: 10.1007/s00500-020-04720-2