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Existence and multiplicity for positive solutions of a system of first order differential equations with multipoint and integral boundary conditions

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Abstract: In this paper, we state and prove theorems related to the existence and multiplicity for positive solutions of a system of first order differential equations with multipoint and integral boundary conditions. The main tool is the fixed point theory. In order to illustrate the main results, we present some examples.

Key words: Initial boundary value problems, nonlinear differential system, multipoint and integral boundary conditions, positive solutions, the fixed point theory

1. Introduction

In this paper, we consider the following nonlinear system

$$\begin{cases} u'(t) = f(t, u(t), v(t)), & t \in (0, T), \\ v'(t) = g(t, u(t), v(t)), & t \in (0, T), \end{cases} \tag{1.1}$$

asscociated with multipoint and integral boundary conditions as follows:

$$\begin{cases} u(0) = u_0, \\ v(0) = \sum_{j=1}^{N} B_j v(T_j) + \int_0^T H(t)v(t)dt, \end{cases} \tag{1.2}$$

where $f,\ g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,\ H : [0, T] \to \mathfrak{M}_n$ are given continuous functions, in which $\mathfrak{M}_n$ is the set of square matrices of order n, and $u_0 \in \mathbb{R}^n,\ B_j \in \mathfrak{M}_n\ (j = \overline{1, N}),\ 0 < T_1 < T_2 < \cdots < T_N = T$ are given constants.

Multipoint boundary value problems for ordinary differential equations play an important role in several branches of physics and applied mathematics, see [1] - [6], [8] - [20], [22] and the references given therein. Many authors have studied various aspects of boundary value problems, by using different methods and various techniques, such as the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, the fixed point theory (the fixed point theorems of Banach or Krasnoselskii, or Schaefer, the fixed point theorem in cones, etc.), the coincidence degree theory, monotone iterative techniques.

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In [3], Bolojan et al. proved the existence results of solutions to the following problem for a nonlinear first order differential system subject to nonlinear nonlocal initial conditions of the form

$$\begin{cases} x'(t) = f_1(t, x(t), y(t)), \\ y'(t) = f_2(t, x(t), y(t)), \text{ a.e. on } [0,1], \\ x(0) = \alpha[x, y], \\ y(0) = \beta[x, y], \end{cases} \tag{1.3}$$

where $f_1,\ f_2 : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ were L^1- Carathéodory functions, $\alpha,\ \beta : C([0,1])\times C([0,1]) \to \mathbb{R}$ were nonlinear continuous functionals, and the solution (x, y) was sought in $W^{1,1}(0, 1;\ \mathbb{R}^2)$. That problem was studied by using the fixed point principles by Perov, Schauder and Leray-Schauder, together with the technique that used convergent matrices and vector norms.

In [15], by applying the Banach fixed point theorem and the Schaefer fixed point theorem, Mardanov et al. proved the existence and uniqueness theorems for the system of ordinary differential equations with three-point boundary conditions as follows:

$$\begin{cases} y' = f(t, y),\ t \in (0, T), \\ Ay(0) + By(t_1) + Cy(T) = d, \end{cases} \tag{1.4}$$

where $A,\ B,\ C$ were constant square matrices of order n such that $\det(A + B + C) \neq 0,\ f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ was a given function, $d \in \mathbb{R}^n$ was a given vector, t_1 satisfied the condition of $0 < t_1 < T$, and $y : [0, T] \to \mathbb{R}^n$ was unknown.

In [16], Mardanov et al. considered the following nonlinear differential system with multipoint and integral boundary conditions

$$\begin{cases} x' = f(t, x(t)),\ t \in [0, T], \\ \sum\limits_{i=0}^{m} l_i x(t_i) + \int_0^T h(t)x(t)dt = \alpha, \end{cases} \tag{1.5}$$

where $l_i,\ i = \overline{1, m}$, are n-order constant matrices with $\det N \neq 0,\ N = \sum\limits_{i=0}^{m} l_i + \int_0^T h(t)dt;\ f : [0, T]\times\mathbb{R}^n \to \mathbb{R}^n$, $h : [0, T] \to \mathbb{R}^{n\times n}$ were given functions; the points $t_0,\ t_1, \cdots,\ t_m$ were arbitrarily chosen in the finite interval $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$. At first, a suitable Green function was constructed in order to reduce the problem into a corresponding integral equation. Next, by using the Banach contraction mapping principle and Schaefer fixed point theorem on the integral equation, the authors proved that the solution of the multipoint problem exists and it is unique.

In [10], Han considered the second-order three-point boundary value problem in the form

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in (0, 1), \\ x'(0) = 0,\ x(\eta) = x(1), \end{cases} \tag{1.6}$$

with $\eta \in (0, 1)$. By means of the fixed point theorem in cones, the existence and multiplicity of positive solutions were proved.

In [5], Boucherif applied the fixed point theorem in a cone to study the existence of positive solutions for the problem given by

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in (0, 1), \\ x(0) - cx'(0) = \int_0^1 g_0(s)x(s)ds, \\ x(1) - dx'(1) = \int_0^1 g_1(s)x(s)ds, \end{cases} \tag{1.7}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ was continuous, $g_0,\ g_1 : [0,1] \to [0,+\infty)$ were continuous and positive, c and d were nonnegative real parameters.

In [20], Truong et al. studied the following m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t)),\ t \in (0,1), \\ x'(0) = 0,\ x(1) = \sum_{j=1}^{m-2} \alpha_j x(\eta_j), \end{cases} \tag{1.8}$$

where $m \geq 3,\ \eta_j \in (0,1)$ and $\alpha_j \geq 0$, for all $j = \overline{1, m-2}$ such that $\sum_{j=1}^{m-2} \alpha_j < 1$. By applying well-known Guo-Krasnoselskii fixed point theorem and applying the monotone iterative technique, the results obtained in [20] were the existence and multiplicity of positive solutions. Furthermore, the compactness of the set of positive solutions was proved.

In [1], Agarwal et al. formulated existence results for solutions to discrete equations which approximate three-point boundary value problems for second-order ordinary differential equations. The proofs of these results were finished based on extending the notion of discrete compatibility, which was a degree-based relationship between the given boundary conditions and the lower or upper solutions chosen, to three-point boundary conditions. On the other hand, the invariance of the degree under the homotopy of the degree theory was also applied in the above proofs.

In [12], Henderson and Luca investigated the following multipoint boundary value problem for the system of nonlinear higher-order ordinary differential equations of the type

$$\begin{cases} u^{(n)}(t) = f(t, v(t)), \quad t \in (0,T),\ n \in \mathbb{N},\ n \geq 2, \\ v^{(m)}(t) = g(t, u(t)), \quad t \in (0,T),\ m \in \mathbb{N},\ m \geq 2, \end{cases} \tag{1.9}$$

with the multipoint boundary conditions

$$\begin{cases} u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(T) = \sum_{i=1}^{p-2} a_i u(\xi_i),\ p \in \mathbb{N},\ p \geq 3, \\ v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v(T) = \sum_{i=1}^{q-2} b_i v(\eta_i),\ q \in \mathbb{N},\ q \geq 3. \end{cases} \tag{1.10}$$

Under sufficient assumptions on f and g, the authors proved the existence and multiplicity of positive solutions of the above problem by applying the fixed point index theory.

Inspired and motivated by the idea of the above mentioned works, we continue to investigate the more general boundary problem of the form (1.1) - (1.2) with multipoint and integral boundary conditions. This paper consists of six sections. Section 1 is the introduction. In Section 2, we present some preliminaries. Here, the Green function is established for Problems (1.1)–(1.2) such that this problem is reduced to the equivalent integral system. Section 3 is devoted to the existence and uniqueness of solutions based on the fixed point theorems of Banach and Krasnoselskii. In Sections 4 and 5, by using the Guo-Krasnoselskii's fixed point theorem in a cone, we prove sufficient conditions for the existence and multiplicity of positive solutions. Finally, a remark is given in Section 6 for a system of multiple differential equations. In order to demonstrate the validity of the main results, three examples (Examples 3.1, 3.2, 4.1) are given.

2. Preliminaries

Let us start this section with some definitions and remarks which are used in next sections.

First, let $C([0,T];\mathbb{R}^n)$ and $C^1([0,T];\mathbb{R}^n)$ be the Banach spaces with normal norms, respectively, as follows:

$$\|u\|_{C([0,T];\mathbb{R}^n)} = \max_{0\le t\le T} |u(t)|_1 ,$$

$$\|u\|_{C^1([0,T];\mathbb{R}^n)} = \|u\|_{C([0,T];\mathbb{R}^n)} + \|u'\|_{C([0,T];\mathbb{R}^n)} ,$$

where $|x|_1 = |x_1| + \cdots + |x_n|$, with $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$.

Next, we define the norm of square matrices of order n, for all $A = (a_{ij}) \in \mathfrak{M}_n$, by

$$\|A\|_1 = \sup_{0\ne x\in\mathbb{R}^n} \frac{|Ax|_1}{|x|_1} = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| , \text{ for all } A = (a_{ij}) \in \mathfrak{M}_n.$$

We also define a cone in $\mathbb{R}^n$ and a cone in $\mathfrak{M}_n$, respectively, as follows:

$$\begin{aligned}\mathbb{R}^n_+ &= \{x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n : x_i \ge 0,\ \forall i = \overline{1,n}\},\\ \mathfrak{M}^+_n &= \{A = (a_{ij}) \in \mathfrak{M}_n : a_{ij} \ge 0,\ \forall i,j = \overline{1,n}\}.\end{aligned}$$

We recall that, let X be a Banach space, a cone $K \subset X$ is a closed convex set such that $\lambda K \subset K$, for all $\lambda \ge 0$ and $K \cap (-K) = \{0\}$. Of course, we shall always assume implicitly that $K \ne \{0\}$. Given a cone $K \subset X$, we can define a partial ordering $\le$ (or $\ge$) with respect to K by $x \le y$ (or $y \ge x$) iff $y - x \in K$, and we can check which properties of the usual $\le$ for the reals, i.e. $\le$ with respect to $\mathbb{R}_+$, remain valid for $\le$ with respect to any K due to the properties of a cone, (see [7]).

Therefore, we can define here that, $\forall x, y \in \mathbb{R}^n$, $x \le y$ (or $y \ge x$) iff $y - x \in \mathbb{R}^n_+$; and $\forall A, B \in \mathfrak{M}_n$, $A \le B$ (or $B \ge A$) iff $B - A \in \mathfrak{M}^+_n$. For each $x \in \mathbb{R}^n$, we can write $x > 0$ to indicate that $x \ge 0$ and $x \ne 0$; and for each $A \in \mathfrak{M}_n$, we also write $A > 0$ iff $A \ge 0$ and $A \ne 0$. It is clear to see that many properties of the usual $\le$ for the reals remain valid for $\le$ with respect to the cones $\mathbb{R}^n_+$, $\mathfrak{M}^+_n$.

We also recall here the well-known fixed point theorems in order to use in next sections as follows.

Theorem 2.1 (Krasnosellskii) [21]. *Let M be a nonempty bounded closed convex subset of a Banach space X. Suppose that $U : M \to X$ is a contraction and $C : M \to X$ is a compact operator such that*

$$U(x) + C(y) \in M,\ \forall x, y \in M.$$

Then, $U + C$ has a fixed point in M.

Theorem 2.2 (Guo-Krasnoselskii) [9]. *Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a cone. Assume that Ω_1, Ω_2 are two open bounded subsets of X with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $P : K \cap (\overline{\Omega_2} \setminus \Omega_1) \to K$ be a completely continuous operator satisfying one of the following conditions*

(i) $\|Pu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Pu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Pu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Pu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, the operator P has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Now, we construct an equivalent integral system for Problems (1.1)–(1.2), with $f,\ g \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, $H \in C([0,T]; \mathfrak{M}_n)$ and $B_1, \cdots, B_N \in \mathfrak{M}_n$. Here, we note more that, in order to get the existence of solutions in Section 3, furthermore, the solutions are positive with the conditions $(\tilde{H}_2)$, $(\tilde{H}_3)$ as in Sections 4 and 5 below, we shall use the following functions $f_\alpha(t,u,v) = f(t,u,v) + \alpha u$, and $g_\beta(t,u,v) = g(t,u,v) + \beta v$, for $\alpha,\ \beta \geq 0$.

Put $\sigma_\beta = I - \int_0^T e^{-\beta\tau} H(\tau)d\tau - \sum_{j=1}^N e^{-\beta T_j} B_j$.

Lemma 2.3. *Assume that* $\det \sigma_\beta \neq 0$. *The pair of functions* $(u,v) \in C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$ *is a solution of Problems (1.1)–(1.2) if and only if* (u,v) *is a solution of the following integral equations system*

$$\begin{cases} u(t) = e^{-\alpha t}u_0 + \int_0^t e^{-\alpha(t-s)} f_\alpha(s,u(s),v(s))ds, \\ v(t) = \int_0^t e^{-\beta(t-s)} g_\beta(s,u(s),v(s))ds \\ \qquad +e^{-\beta t}\sigma_\beta^{-1} \int_0^T \left(\int_s^T e^{-\beta(\tau-s)} H(\tau)d\tau \right) g_\beta(s,u(s),v(s))ds \\ \qquad +e^{-\beta t}\sigma_\beta^{-1} \sum\limits_{j=1}^N B_j \int_0^{T_j} e^{-\beta(T_j-s)} g_\beta(s,u(s),v(s))ds. \end{cases} \tag{2.1}$$

Proof of Lemma 2.3. Let $(u,v) \in C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$ be a solution of Problems (1.1)–(1.2). It is obviously that $(u,v) \in C^1([0,T];\mathbb{R}^n) \times C^1([0,T];\mathbb{R}^n)$ and (u,v) satisfies Problems (1.1)–(1.2). For each α, $\beta \geq 0$, the system (1.1) can be transformed into an equivalent form as

$$\begin{cases} u' + \alpha u = f_\alpha(t,u,v), & t \in (0,T), \\ v' + \beta v = g_\beta(t,u,v), & t \in (0,T). \end{cases} \tag{2.2}$$

Multiplying the equations in (2.2) by $e^{\alpha t}$ and $e^{\beta t}$, respectively, and integrating from 0 to t, we obtain

$$u(t) = e^{-\alpha t}u_0 + \int_0^t e^{-\alpha(t-s)} f_\alpha(s,u(s),v(s))ds,\ t \in (0,T), \tag{2.3}$$

$$v(t) = e^{-\beta t}v(0) + \int_0^t e^{-\beta(t-s)} g_\beta(s,u(s),v(s))ds,\ t \in (0,T). \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned} \int_0^T H(\tau)v(\tau)d\tau &= v(0)\int_0^T H(\tau)e^{-\beta\tau}d\tau + \int_0^T H(\tau)d\tau \left(\int_0^\tau e^{-\beta(\tau-s)} g_\beta(s,u(s),v(s))ds \right) \\ &= v(0)\int_0^T H(\tau)e^{-\beta\tau}d\tau + \int_0^T \left(\int_s^T e^{-\beta(\tau-s)} H(\tau)d\tau \right) g_\beta(s,u(s),v(s))ds, \end{aligned}$$

and

$$\sum_{j=1}^N B_j v(T_j) - v(0)\sum_{j=1}^N B_j e^{-\beta T_j} = \sum_{j=1}^N B_j \int_0^{T_j} e^{-\beta(T_j-s)} g_\beta(s,u(s),v(s))ds.$$

It implies that

$$\begin{aligned} v(0) &= \sum_{j=1}^{N} B_j v(T_j) + \int_0^T H(\tau) v(\tau) d\tau \\ &= v(0) \sum_{j=1}^{N} B_j e^{-\beta T_j} + \sum_{j=1}^{N} B_j \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s, u(s), v(s)) ds \\ &\quad + v(0) \int_0^T H(\tau) e^{-\beta\tau} d\tau + \int_0^T \left(\int_s^T e^{-\beta(\tau - s)} H(\tau) d\tau \right) g_\beta(s, u(s), v(s)) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} & v(0) \left(I - \int_0^T H(\tau) e^{-\beta\tau} d\tau - \sum_{j=1}^{N} B_j e^{-\beta T_j} \right) \\ = & \sum_{j=1}^{N} B_j \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s, u(s), v(s)) ds + \int_0^T \left(\int_s^T e^{-\beta(\tau - s)} H(\tau) d\tau \right) g_\beta(s, u(s), v(s)) ds, \end{aligned}$$

and thus,

$$\begin{aligned} v(0) = \sigma_\beta^{-1} & \sum_{j=1}^{N} B_j \int_0^{T_j} e^{-\beta(T_j - s)} g_\beta(s, u(s), v(s)) ds \\ & + \sigma_\beta^{-1} \int_0^T \left(\int_s^T e^{-\beta(\tau - s)} H(\tau) d\tau \right) g_\beta(s, u(s), v(s)) ds. \end{aligned} \tag{2.5}$$

Combining (2.3), (2.4), and (2.5), we infer that $(u(t), v(t))$ satisfies the system (2.1), therefore (u, v) is a solution of the nonlinear integral system (2.1).

Otherwise, let $(u, v) \in C([0, T]; \mathbb{R}^n) \times C([0, T]; \mathbb{R}^n)$ is a solution of the nonlinear integral equations (2.1). It is not difficult to prove that $(u, v) \in C^1([0, T]; \mathbb{R}^n) \times C^1([0, T]; \mathbb{R}^n)$ and (u, v) satisfies Problems (1.1)–(1.2).

Lemma 2.3 is proved. □

We note that, the integral equation (2.4) can be written in form

$$v(t) = \int_0^T G(t, s) g_\beta(s, u(s), v(s)) ds, \tag{2.6}$$

where the Green function $G(t,s)$ is defined as follows:

$$G(t,s)=\begin{cases} e^{-\beta(t-s)}I, & 0\le s\le t\le T,\\ 0, & 0\le t\le s\le T\end{cases} + e^{-\beta(t-s)}\sigma_\beta^{-1}\int_s^T e^{-\beta\tau}H(\tau)d\tau$$
$$+\,e^{-\beta(t-s)}\sigma_\beta^{-1}\begin{cases} \sum\limits_{j=1}^{N} e^{-\beta T_j}B_j, & 0\le s\le T_1,\\ \quad\vdots & \quad\vdots\\ \sum\limits_{j=k}^{N} e^{-\beta T_j}B_j, & T_{k-1}<s\le T_k,\\ \quad\vdots & \quad\vdots\\ e^{-\beta T}B_N, & T_{N-1}<s\le T.\end{cases} \tag{2.7}$$

The next Lemma will propose a property of the Green function $G(t,s)$.

Lemma 2.4. *Suppose that* $H(t)\ge 0,\ \forall t\in[0,T]$, *and* $B_j\ge 0,\ \forall j=\overline{1,N-1},\ B_N>0$, *such that* $\det\sigma_\beta\ne 0,\ \sigma_\beta^{-1}>0$ *and* $\sigma_\beta^{-1}B_N=(c_{ij})$, with $c_{ij}>0,\ \forall i,j=\overline{1,n}$. *Then*

$$e^{-\beta T}\sigma_\beta^{-1}B_N e^{-\beta(t-s)}\le G(t,s)\le\sigma_\beta^{-1}e^{-\beta(t-s)},\ \forall s,t\in[0,T]. \tag{2.8}$$

On the other hand, there exist positive matrices $\bar{G}_0,\ \bar{G}_1$ *such that*

$$\bar{G}_0\le G(t,s)\le\bar{G}_1,\ \forall(t,s)\in[0,T]\times[0,T]. \tag{2.9}$$

Moreover, there exists a constant $\gamma\in(0,1)$ *such that*

$$\gamma\left(I+\sigma_\beta^{-1}\right)e^{\beta s}\le G(t,s)\le\left(I+\sigma_\beta^{-1}\right)e^{\beta s},\ \forall s,t\in[0,T]. \tag{2.10}$$

Remark 2.1. If $A>0$ and A is invertible then it does not imply that $A^{-1}>0$. Indeed, we can give an example as follows:

$$A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}>0,\ A^{-1}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix},$$

obviously, we do not have $A^{-1}>0$.

Remark 2.2. Let $A=(a_{ij}),\ C=(c_{ij})\in\mathfrak{M}_n$. Assume that $c_{ij}>0,\ \forall i,j=\overline{1,n}$. Then, there exists a constant $\gamma\in(0,1)$ such that $C-\gamma A>0$.

Indeed, we have $C-\gamma A=(c_{ij}-\gamma a_{ij})\in\mathfrak{M}_n$.

By choosing $0<\gamma<\min\limits_{(i,j)}\left\{\frac{c_{ij}}{1+|a_{ij}|},1\right\}$, we get $C-\gamma A=(c_{ij}-\gamma a_{ij})>0$.

Proof of Lemma 2.4. By direct computations, we have

$$G(t,s)\ge e^{-\beta T}\sigma_\beta^{-1}B_N e^{-\beta(t-s)}, \tag{2.11}$$

$$G(t,s)\le\left[I+\sigma_\beta^{-1}\left(\int_0^T e^{-\beta\tau}H(\tau)d\tau+\sum_{j=1}^{N}e^{-\beta T_j}B_j\right)\right]e^{-\beta(t-s)}$$
$$=\left[I+\sigma_\beta^{-1}\left(I-\sigma_\beta\right)\right]e^{-\beta(t-s)}=\sigma_\beta^{-1}e^{-\beta(t-s)}. \tag{2.12}$$

Because $e^{-\beta T} \leq e^{-\beta(t-s)} \leq e^{\beta T}$, for all $t,\ s \in [0,T]$, we obtain (2.9) with

$$\bar{G}_0 = e^{-2\beta T}\sigma_\beta^{-1}B_N,\ \bar{G}_1 = e^{\beta T}\sigma_\beta^{-1}.$$

On the other hand,

$$\begin{aligned} G(t,s) &\geq e^{-\beta T}\sigma_\beta^{-1}B_N e^{-\beta(t-s)} \geq e^{-2\beta T}\sigma_\beta^{-1}B_N, \\ G(t,s) &\leq \sigma_\beta^{-1}e^{-\beta(t-s)} \leq \left(I+\sigma_\beta^{-1}\right)e^{-\beta(t-s)} \leq \left(I+\sigma_\beta^{-1}\right)e^{\beta s}. \end{aligned}$$

By the fact that $e^{-2\beta T}\sigma_\beta^{-1}B_N = (e^{-2\beta T}c_{ij})$, with $e^{-2\beta T}c_{ij} > 0,\ \forall i,j = \overline{1,n}$, in a similar way as in Remark 2.2, there exists a constant $\gamma \in (0,1)$ such that

$$e^{-2\beta T}\sigma_\beta^{-1}B_N - \gamma\left(I+\sigma_\beta^{-1}\right) > 0.$$

Hence

$$G(t,s) \geq e^{-2\beta T}\sigma_\beta^{-1}B_N e^{\beta s} \geq \gamma\left(I+\sigma_\beta^{-1}\right)e^{\beta s}.$$

Lemma 2.4 is proved. □

We note more that if the sign of B_j and $H(t)$ cannot be determined, we have

$$-G_{max} \leq G(t,s) \leq G_{max},\ \forall (t,s) \in [0,T]\times[0,T], \tag{2.13}$$

with

$$G_{max} = \left[I + \sigma_\beta^{-1}\left(\int_0^T e^{-\beta\tau}\left|H(\tau)\right|d\tau + \sum\nolimits_{j=1}^{N}\left|B_j\right|e^{-\beta T_j}\right)\right]e^{\beta T}, \tag{2.14}$$

where we denote the matrix $|A| = (|a_{ij}|)$, if $A = (a_{ij}) \in \mathfrak{M}_n$.

3. Existence and uniqueness

Based on the preliminaries, in this section, we prove two existence results of solutions for Problems (1.1)–(1.2), in which $f,\ g \in C([0,T]\times\mathbb{R}^n\times\mathbb{R}^n;\mathbb{R}^n)$, $H \in C([0,T];\mathfrak{M}_n)$, $B_j \in \mathfrak{M}_n$ $(j=\overline{1,N})$. The first result (Theorem 3.1) is the unique existence of a solution by applying the Banach fixed point theorem. Under weaker conditions, we obtain the second result (Theorem 3.5) by using the Krasnoselskii fixed point theorem.

We first consider the Banach space $X = C([0,T];\mathbb{R}^n)\times C([0,T];\mathbb{R}^n)$ equipped with the norm

$$\|(u,v)\|_X = \|u\|_{C([0,T];\mathbb{R}^n)} + \|v\|_{C([0,T];\mathbb{R}^n)}\,. \tag{3.1}$$

Next, based on Lemma 2.3 with respect to $\alpha = 0,\ \beta = 0$, we define an operator $\mathcal{P}: X \longrightarrow X$ as follows:

$$\begin{array}{cccc} \mathcal{P}: & X & \longrightarrow & X \\ & (u,v) & \longmapsto & (\mathcal{P}_1(u,v), \mathcal{P}_2(u,v)), \end{array}$$

in which

$$\mathcal{P}_1(u,v)(t) = u_0 + \int_0^t f(s,u(s),v(s))ds,$$

$$\mathcal{P}_2(u,v)(t) = \int_0^T G(t,s)g(s,u(s),v(s))ds,$$

where

$$G(t,s) = \begin{cases} I, & 0 \le s \le t \le T, \\ 0, & 0 \le t \le s \le T \end{cases} + \sigma^{-1}\int_s^T H(\tau)d\tau$$
$$+ \sigma^{-1}\begin{cases} \sum\limits_{j=1}^N B_j, & 0 \le s \le T_1, \\ \vdots & \vdots \\ \sum\limits_{j=k}^N B_j, & T_{k-1} < s \le T_k, \\ \vdots & \vdots \\ B_N, & T_{N-1} < s \le T, \end{cases} \tag{3.2}$$

and

$$\sigma = I - \int_0^T H(\tau)d\tau - \sum_{j=1}^N B_j.$$

We make the following assumptions.

(H_1) $H \in C([0,T];\mathfrak{M}_n)$; $B_j \in \mathfrak{M}_n$ $(j = \overline{1,N})$ such that $0 < \int_0^T \|H(t)\|_1 dt + \sum\limits_{j=1}^N \|B_j\|_1 < 1$;

(H_2) There exists a positive function $L_f \in L^1(0,T)$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})|_1 \le L_f(t)\left(|u-\bar{u}|_1 + |v-\bar{v}|_1\right), \tag{3.3}$$

for all (t,u,v), $(t,\bar{u},\bar{v}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$;

(H_3) There exists a positive function $L_g \in L^1(0,T)$ such that

$$|g(t,u,v) - g(t,\bar{u},\bar{v})|_1 \le L_g(t)\left(|u-\bar{u}|_1 + |v-\bar{v}|_1\right), \tag{3.4}$$

for all (t,u,v), $(t,\bar{u},\bar{v}) \in [0,T]\times\mathbb{R}^n\times\mathbb{R}^n$.

Remark 3.1. The assumption (H_1) leads to

$$\left\|\int_0^T H(\tau)d\tau + \sum_{j=1}^N B_j\right\|_1 \le \int_0^T \|H(t)\|_1 dt + \sum_{j=1}^N \|B_j\|_1 < 1,$$

so $\sigma \equiv I - \int_0^T H(\tau)d\tau - \sum_{j=1}^N B_j$ is invertible and

$$\left\|\sigma^{-1}\right\|_1 \le \frac{1}{1 - \left\|\int_0^T H(\tau)d\tau + \sum_{j=1}^N B_j\right\|_1} \le \frac{1}{1 - \int_0^T \|H(t)\|_1 dt - \sum\limits_{j=1}^N \|B_j\|_1}.$$

Theorem 3.1. *Suppose that* (H_1)–(H_3) *are satisfied. Additionally, assume that*

$$L = \|L_f\|_{L^1(0,T)} + \|L_g\|_{L^1(0,T)} \left\|\sigma^{-1}\right\|_1 < 1. \tag{3.5}$$

Then, Problems (1.1)–(1.2) *has a unique solution.*

Proof of Theorem 3.1.

First, we put $f_T = \max_{0\le t\le T} |f(t,0,0)|_1$, $g_T = \max_{0\le t\le T} |g(t,0,0)|_1$ and choose $R > 0$ large enough such that

$$R > \frac{|u_0|_1 + T\left(f_T + g_T \left\|\sigma^{-1}\right\|_1\right)}{1-L}. \tag{3.6}$$

Next, we will finish the proof of this theorem through a process with two steps as follows.

Step 1. Let $B_R = \{(u,v) \in X : \|(u,v)\|_X \le R\}$. We show that $\mathcal{P}(B_R) \subset B_R$.

Indeed, for $(u,v) \in B_R$ and for all $t \in [0,T]$, we have the following estimates

$$\begin{aligned} |\mathcal{P}_1(u,v)(t)|_1 &\le |u_0|_1 + \int_0^t |f(s,u(s),v(s)) - f(s,0,0)|_1\, ds + \int_0^t |f(s,0,0)|_1\, ds \\ &\le |u_0|_1 + R\,\|L_f\|_{L^1(0,T)} + T f_T, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} |\mathcal{P}_2(u,v)(t)|_1 &\le \left\|\sigma^{-1}\right\|_1 \left[\int_0^T |g(s,u(s),v(s)) - g(s,0,0)|_1\, ds + \int_0^T |g(s,0,0)|_1\, ds\right] \\ &\le \left\|\sigma^{-1}\right\|_1 \left[R\,\|L_g\|_{L^1(0,T)} + T g_T\right]. \end{aligned} \tag{3.8}$$

Combining (3.7)–(3.8) and the choice of R as in (3.6), we infer that $\mathcal{P}(B_R) \subset B_R$, it means that the operator $\mathcal{P} : B_R \to B_R$ is defined.

Step 2. We prove that the operator $\mathcal{P}$ is a contraction mapping.

Indeed, let (u,v) and $(\bar{u},\bar{v})$ be arbitrary elements in B_R. We have

$$\begin{aligned} |\mathcal{P}_1(u,v)(t) - \mathcal{P}_1(\bar{u},\bar{v})(t)|_1 &\le \int_0^t |f(s,u(s),v(s)) - f(s,\bar{u}(s),\bar{v}(s))|_1\, ds \\ &\le \|L_f\|_{L^1(0,T)} \|(u,v) - (\bar{u},\bar{v})\|_X, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &|\mathcal{P}_2(u,v)(t) - \mathcal{P}_2(\bar{u},\bar{v})(t)|_1 \\ &\le \left\|\sigma^{-1}\right\|_1 \int_0^T |g(s,u(s),v(s)) - g(s,\bar{u}(s),\bar{v}(s))|_1\, ds \\ &\le \left\|\sigma^{-1}\right\|_1 \|L_g\|_{L^1(0,T)} \|(u,v) - (\bar{u},\bar{v})\|_X. \end{aligned} \tag{3.10}$$

It follows from (3.9)–(3.10) and the assumption in Theorem 3.1 that $\mathcal{P} : B_R \to B_R$ is a contraction mapping. Applying the Banach fixed point theorem, we verify that the problem (1.1)–(1.2) has a unique solution (u,v). Theorem 3.1 is completely proved. □

Next, under weaker conditions, the second result is given without the Lipschitzian condition on g as in (H_3). We make the assumption $(\bar{H}_3)$ as below.

$(\bar{H}_3)$ $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ is a continuous function and there exist two positive functions $g_1,\ g_2 \in L^1(0,T)$ such that

$$|g(t,u,v)|_1 \leq g_1(t)\left(|u|_1 + |v|_1\right) + g_2(t),\ \forall (t,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n. \tag{3.11}$$

We now define two operators $\mathcal{U},\ \mathcal{C} : X \to X$ as follows:

$$\begin{aligned} &\mathcal{U} : X \to X \\ &(u,v) \longmapsto \left(\mathcal{P}_1(u,v), 0\right), \end{aligned} \tag{3.12}$$

with

$$\mathcal{P}_1(u,v)(t) = u_0 + \int_0^t f(s,u(s),v(s))ds, \tag{3.13}$$

and

$$\begin{aligned} &\mathcal{C} : X \to X \\ &(u,v) \longmapsto \left(0, \mathcal{P}_2(u,v)\right), \end{aligned} \tag{3.14}$$

where

$$\mathcal{P}_2(u,v)(t) = \int_0^T G(t,s)g(s,u(s),v(s))ds. \tag{3.15}$$

Obviously, $\mathcal{P} = \mathcal{U} + \mathcal{C}$.

Lemma 3.2. *Let (H_1), (H_2) and $(\bar{H}_3)$ hold. In addition, assume that*

$$L_1 = \|L_f\|_{L^1(0,T)} + \left\|\sigma^{-1}\right\|_1 \|g_1\|_{L^1(0,T)} < 1. \tag{3.16}$$

Then, there exists a positive constant $R > 0$ such that

$$\mathcal{U}(u,v) + \mathcal{C}(\bar{u},\bar{v}) \in B_R, \tag{3.17}$$

for all (u,v), $(\bar{u},\bar{v}) \in B_R = \{(u,v) \in X : \|(u,v)\|_X \leq R\}$.

Proof of Lemma 3.2. Let (u,v), $(\bar{u},\bar{v}) \in B_R$. We have the following estimate

$$\begin{aligned} |\mathcal{P}_1(u,v)(t)|_1 &\leq |u_0|_1 + \int_0^t |f(s,u(s),v(s))|_1\, ds \\ &\leq |u_0|_1 + Tf_T + R\,\|L_f\|_{L^1(0,T)}. \end{aligned} \tag{3.18}$$

We also have an estimate for $\mathcal{P}_2(\bar{u},\bar{v})$ as follows:

$$\begin{aligned} |\mathcal{P}_2(\bar{u},\bar{v})(t)|_1 &\leq \left\|\sigma^{-1}\right\|_1 \int_0^T |g(s,\bar{u}(s),\bar{v}(s))|_1\, ds \\ &\leq \left\|\sigma^{-1}\right\|_1 \left[R\,\|g_1\|_{L^1(0,T)} + \|g_2\|_{L^1(0,T)}\right]. \end{aligned} \tag{3.19}$$

Choosing $R > 0$ large enough such that

$$R \geq \frac{|u_0|_1 + Tf_T + \left\|\sigma^{-1}\right\|_1 \|g_2\|_{L^1(0,T)}}{1 - L_1}. \tag{3.20}$$

Combining (3.18)–(3.20) and doing some direct calculations, we obtain an estimate as in (3.17). Lemma 3.2 is proved. □

Lemma 3.3. *If the conditions in the Lemma 3.2 are satisfied, then the operator $\mathcal{U} : X \to X$ is a contraction.*

Proof of Lemma 3.3. Let (u, v) and $(\bar{u}, \bar{v})$ be arbitrary elements in X. We have

$$\begin{aligned} |\mathcal{P}_1(u,v)(t) - \mathcal{P}_1(\bar{u},\bar{v})(t)|_1 &\leq \int_0^t |f(s,u(s),v(s)) - f(s,\bar{u}(s),\bar{v}(s))|_1 \, ds \\ &\leq \|L_f\|_{L^1(0,T)} \|(u,v) - (\bar{u},\bar{v})\|_X . \end{aligned} \tag{3.21}$$

Since $\|L_f\|_{L^1(0,T)} \leq L_1 < 1$, we infer that $\mathcal{P}_1 : X \to C([0,T];\mathbb{R}^n)$ is a contraction mapping, so is the operator $\mathcal{U} = (\mathcal{P}_1, 0) : X \to X$. Lemma 3.3 is proved. □

Lemma 3.4. *If the conditions in the Lemma 3.2 are satisfied, then the operator $\mathcal{C} : B_R \to X$ is continuous and compact.*

Proof of Lemma 3.4.

Step 1: *$\mathcal{P}_2$ is continuous.* Let $\{(u_m, v_m)\} \subset B_R$ and $(u, v) \in B_R$ such that

$$\|(u_m, v_m) - (u, v)\|_X \to 0, \text{ as } m \to +\infty. \tag{3.22}$$

By the continuity of g and the Lebesgue's dominated convergence theorem, we get

$$\int_0^T |g(t, u_m(t), v_m(t)) - g(t, u(t), v(t))|_1 \, dt \to 0, \text{ as } m \to +\infty. \tag{3.23}$$

Using (3.23), we infer that

$$\begin{aligned} &\sup_{0 \leq t \leq T} |\mathcal{P}_2(u_m, v_m)(t) - \mathcal{P}_2(u, v)(t)|_1 \\ &\leq \left\|\sigma^{-1}\right\|_1 \int_0^T |g(t, u_m(t), v_m(t)) - g(t, u(t), v(t))|_1 \, dt \to 0, \text{ as } m \to +\infty. \end{aligned} \tag{3.24}$$

Step 2: *$\mathcal{P}_2(B_R)$ is relatively compact.* It follows from the continuity of g that there exists $m_R > 0$ such that $|g(t, u(t), v(t))|_1 \leq m_R$ for all $(u, v) \in B_R$, $\forall t \in [0, T]$. Hence, the set $\mathcal{P}_2(B_R)$ is bounded in $C([0,T];\mathbb{R}^n)$.

Taking arbitrary $(u, v) \in B_R$ and $t_1, \ t_2 \in [0, T], \ t_2 < t_1$, we obtain

$$|\mathcal{P}_2(u,v)(t_1) - \mathcal{P}_2(u,v)(t_2)|_1 = \left| \int_{t_2}^{t_1} g(s, u(s), v(s)) ds \right|_1 \leq m_R |t_1 - t_2| , \tag{3.25}$$

it leads to $\mathcal{P}_2(B_R)$ is equicontinuous. Therefore, the set $\mathcal{P}_2(B_R)$ is relatively compact in $C([0,T];\mathbb{R}^n)$ due to the Arzelà-Ascoli's theorem. Lemma 3.4 is proved. □

Theorem 3.5. *Suppose that the conditions in Lemma 3.2 are satisfied. Then, Problems* (1.1)–(1.2) *have a solution.*

Proof of Theorem 3.5. Combining Lemmas 3.2, 3.3, 3.4 and applying Theorem 2.1 (Krasnoselskii), it is clear to see that $\mathcal{P} = \mathcal{U} + \mathcal{C}$ has a fixed point.

Theorem 3.5 is completely proved. □

Remark 3.2. The result obtained in Theorem 3.5 leads that the set of solutions of Problems (1.1)–(1.2) is compact in X, it means that the set

$$S = \{(u, v) \in B_R : (u, v) = \mathcal{U}(u, v) + \mathcal{C}(u, v)\}$$

is compact in X. Indeed, by the fact that $\mathcal{U} : X \to X$ is a contraction, $(I - \mathcal{U}) : X \to X$ is invertible and $(I - \mathcal{U})^{-1} : X \to X$ is continuous, and therefore, S can be written as follows:

$$S = \{(u, v) \in B_R : (u, v) = (I - \mathcal{U})^{-1}\mathcal{C}(u, v)\} = (I - \mathcal{U})^{-1}\mathcal{C}(S).$$

By $\mathcal{C} : B_R \to X$ is continuous and compact, and by $(I - \mathcal{U})^{-1} : X \to X$ is continuous, it implies that $(I - \mathcal{U})^{-1}\mathcal{C} : B_R \to X$ is continuous and compact. Hence, $S = (I - \mathcal{U})^{-1}\mathcal{C}(S)$ is relatively compact in X, since S is bounded. In order to prove the compactness of S, it remains to check that S is closed in X.

Suppose that $\{(u_m, v_m)\} \subset S$, $\|(u_m, v_m) - (u, v)\|_X \to 0$. By the continuity of $(I - \mathcal{U})^{-1}\mathcal{C}$, we have

$$\begin{aligned} & \left\|(u, v) - (I - \mathcal{U})^{-1}\mathcal{C}(u, v)\right\|_X \\ \leq \; & \|(u, v) - (u_m, v_m)\|_X + \left\|(I - \mathcal{U})^{-1}\mathcal{C}(u_m, v_m) - (I - \mathcal{U})^{-1}\mathcal{C}(u, v)\right\|_X \to 0. \end{aligned}$$

Thus $(u, v) = (I - \mathcal{U})^{-1}\mathcal{C}(u, v)$, so $(u, v) \in S$. We verify that S is compact in X.

Remark 3.3. Using Lemma 2.3, with respect to $\alpha > 0, \beta > 0$, we also obtain similar results. It is proved that Theorems 3.1 and 3.5 remain valid for this case, where L and L_1, respectively, are defined as follows:

$$L = \alpha T + \|L_f\|_{L^1(0,T)} + \left(\beta T + \|L_g\|_{L^1(0,T)}\right) \left\|e^{\beta T}\sigma_\beta^{-1}\right\|_1,$$

$$L_1 = \alpha T + \|L_f\|_{L^1(0,T)} + \left(\beta T + \|g_1\|_{L^1(0,T)}\right) \left\|e^{\beta T}\sigma_\beta^{-1}\right\|_1.$$

Example 3.1. We consider the following problem

$$\begin{cases} u_1'(t) = \delta_1 e^{-t}\left[\cos(|u,(t)|_1) + \sin^2(|v(t)|_1)\right], \; t \in (0,1), \\ u_2'(t) = \delta_2 e^{-t}\left[\cos(|v(t)|_1) + \sin^2(|u(t)|_1)\right], \; t \in (0,1), \\ v_1'(t) = \dfrac{\delta_1 |u(t)|_1}{e^{2t} + |u(t)|_1 + |v(t)|_1}, \; t \in (0,1), \\ v_2'(t) = \dfrac{\delta_2 |v(t)|_1}{e^{2t} + |u(t)|_1 + |v(t)|_1}, \; t \in (0,1), \\ (u_1(0), u_2(0))^T = u_0 \in \mathbb{R}^2, \\ v_1(0) = \dfrac{1}{8}v_1(1/2) + \dfrac{1}{16}v_1(1) + \dfrac{1}{4}\int_0^1 e^{-t}v_1(t)dt, \\ v_2(0) = \dfrac{1}{8}v_1(1/2) + \dfrac{1}{16}v_2(1/2) + \dfrac{1}{8}v_1(1) + \dfrac{1}{32}v_2(1) \\ \qquad + \dfrac{1}{8}\int_0^1 e^{-t}v_1(t)dt + \dfrac{1}{8}\int_0^1 e^{-t}v_2(t)dt, \end{cases} \tag{3.26}$$

where $(|\delta_1| + |\delta_2|)\left(1 - e^{-1}\right)\left[1 + \left(1 + e^{-1}\right)\dfrac{2e(31 + 203e)}{(4 + 9e)(4 + 25e)}\right] < 1$.

Problem (3.26) has the form Problems (1.1)–(1.2) with respect to $n = 2$, $f(t, u, v) = (f_1(t, u, v), f_2(t, u, v))^T$, $g(t, u, v) = (g_1(t, u, v), g_2(t, u, v))^T$, where

$$\begin{aligned} f_1(t,u,v) &= \delta_1 e^{-t}\left[\cos(|u(t)|_1) + \sin^2(|v(t)|_1)\right], \\ f_2(t,u,v) &= \delta_2 e^{-t}\left[\cos(|v(t)|_1) + \sin^2(|u(t)|_1)\right], \\ g_1(t,u,v) &= \frac{\bar{\delta}_1 |u(t)|_1}{e^{2t} + |u(t)|_1 + |v(t)|_1}, \\ g_2(t,u,v) &= \frac{\bar{\delta}_2 |v(t)|_1}{e^{2t} + |u(t)|_1 + |v(t)|_1}, \end{aligned}$$

and $N = 2$, $T_1 = \dfrac{1}{2}$, $T_2 = T = 1$,

$$H(t) = e^{-t}\begin{bmatrix} 1/4 & 0 \\ 1/8 & 1/8 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1/8 & 0 \\ 1/8 & 1/16 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1/16 & 0 \\ 1/8 & 1/32 \end{bmatrix}.$$

It is easy to see that the assumptions (H_1) are satisfied, since

$$\int_0^1 \|H(t)\|_1\,dt + \|B_1\|_1 + \|B_2\|_1 = \frac{13}{16} - \frac{3}{8e} < 1.$$

On the other hand, the assumptions (H_2), (H_3) are satisfied with $L_f(t) = (|\delta_1| + |\delta_2|)\,e^{-t}$, $L_g(t) = 2(|\bar{\delta}_1| + |\bar{\delta}_2|)e^{-2t}$.

Moreover, we have

$$\begin{aligned} \sigma &\equiv I - \int_0^1 H(t)dt - B_1 - B_2 \\ &= I - \left(1 - e^{-1}\right)\begin{bmatrix} 1/4 & 0 \\ 1/8 & 1/8 \end{bmatrix} - \begin{bmatrix} 1/8 & 0 \\ 1/8 & 1/16 \end{bmatrix} - \begin{bmatrix} 1/16 & 0 \\ 1/8 & 1/32 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} \sigma_{11} &= 1 - \frac{1 - e^{-1}}{4} - \frac{1}{8} - \frac{1}{16} = \frac{1}{4e} + \frac{9}{16} > 0, \\ \sigma_{22} &= 1 - \frac{1 - e^{-1}}{8} - \frac{1}{16} - \frac{1}{32} = \frac{1}{8e} + \frac{25}{32} > 0, \\ \sigma_{21} &= -\frac{1 - e^{-1}}{8} - \frac{1}{8} - \frac{1}{8} = \frac{1}{8e} - \frac{3}{8} < 0, \\ \sigma^{-1} &= \begin{bmatrix} \frac{1}{\sigma_{11}} & 0 \\ \frac{-\sigma_{21}}{\sigma_{11}\sigma_{22}} & \frac{1}{\sigma_{22}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_{11}} & 0 \\ \frac{-\sigma_{21}}{\sigma_{11}\sigma_{22}} & \frac{1}{\sigma_{22}} \end{bmatrix} = \begin{bmatrix} \frac{16e}{4+9e} & 0 \\ \frac{2e(3e-1)}{(4+9e)(4+25e)} & \frac{32e}{4+25e} \end{bmatrix} > 0. \end{aligned}$$

Hence

$$\begin{aligned}\left\|\sigma^{-1}\right\|_1 &= \max\left\{\frac{16e}{4+9e}+\frac{2e(3e-1)}{(4+9e)(4+25e)},\frac{32e}{4+25e}\right\}\\ &= \frac{2e(31+203e)}{(4+9e)(4+25e)}.\end{aligned}$$

Moreover

$$\begin{aligned}\|L_f\|_{L^1(0,1)} &= (|\delta_1|+|\delta_2|)\left(1-e^{-1}\right),\\ \|L_g\|_{L^1(0,1)} &= (|\delta_1|+|\delta_2|)\left(1-e^{-2}\right),\end{aligned}$$

we have

$$\begin{aligned}L &= \|L_f\|_{L^1(0,1)}+\|L_g\|_{L^1(0,1)}\left\|\sigma^{-1}\right\|_1\\ &= (|\delta_1|+|\delta_2|)\left(1-e^{-1}\right)+(|\delta_1|+|\delta_2|)\left(1-e^{-2}\right)\frac{2e(31+203e)}{(4+9e)(4+25e)}\\ &= (|\delta_1|+|\delta_2|)\left(1-e^{-1}\right)\left[1+\left(1+e^{-1}\right)\frac{2e(31+203e)}{(4+9e)(4+25e)}\right]<1.\end{aligned}$$

Then, the conditions of Theorem 3.1 are satisfied. Thus, we deduce that Problem (3.26) has a unique solution.

Example 3.2. Let us consider the following system

$$\begin{cases} u_1'(t)=\dfrac{\delta_1\left|u(t)\right|_1}{e^t+\left|u(t)\right|_1+\left|v(t)\right|_1},\ t\in(0,1),\\ u_2'(t)=\dfrac{\delta_2\left|v(t)\right|_1}{e^t+\left|u(t)\right|_1+\left|v(t)\right|_1},\ t\in(0,1),\\ v_1'(t)=\bar{\delta}_1e^{-t}\left(v_1(t)+u_2(t)\sin(\sqrt[3]{v_2(t)})\right),\ t\in(0,1),\\ v_2'(t)=\bar{\delta}_2e^{-t}\left(v_2(t)+u_1(t)\cos(\sqrt[5]{v_1(t)})\right),\ t\in(0,1),\\ (u_1(0),u_2(0))^T=u_0\in\mathbb{R}^2,\\ v_1(0)=\frac{1}{8}v_1(1/2)+\frac{1}{16}v_1(1)+\frac{1}{4}\int_0^1 e^{-t}v_1(t)dt,\\ v_2(0)=\frac{1}{8}v_1(1/2)+\frac{1}{16}v_2(1/2)+\frac{1}{8}v_1(1)+\frac{1}{32}v_2(1)\\ \qquad+\frac{1}{8}\int_0^1 e^{-t}v_1(t)dt+\frac{1}{8}\int_0^1 e^{-t}v_2(t)dt,\end{cases}\tag{3.27}$$

where $2\left(1-e^{-1}\right)\left[(|\delta_1|+|\delta_2|)+\frac{e(31+203e)}{(4+9e)(4+25e)}\max\{\left|\bar{\delta}_1\right|,\ \left|\bar{\delta}_2\right|\}\right]<1.$

Problem (3.27) has the form Problems (1.1)–(1.2) with $n=2$, $f(t,u,v)=(f_1(t,u,v),f_2(t,u,v))^T$,

$g(t,u,v) = (g_1(t,u,v), g_2(t,u,v))^T$, in which

$$\begin{aligned}
f_1(t,u,v) &= \frac{\delta_1 \left|u(t)\right|_1}{e^t + \left|u(t)\right|_1 + \left|v(t)\right|_1}, \\
f_2(t,u,v) &= \frac{\delta_2 \left|v(t)\right|_1}{e^t + \left|u(t)\right|_1 + \left|v(t)\right|_1}, \\
g_1(t,u,v) &= \bar{\delta}_1 e^{-t}\left(v_1(t) + u_2(t)\sin(\sqrt[3]{v_2(t)})\right), \\
g_2(t,u,v) &= \bar{\delta}_2 e^{-t}\left(v_2(t) + u_1(t)\cos(\sqrt[5]{v_1(t)})\right),
\end{aligned}$$

and $N = 2,\ T_1 = \dfrac{1}{2},\ T_2 = T = 1,$

$$H(t) = e^{-t}\begin{bmatrix} 1/4 & 0 \\ 1/8 & 1/8 \end{bmatrix},\ B_1 = \begin{bmatrix} 1/8 & 0 \\ 1/8 & 1/16 \end{bmatrix},\ B_2 = \begin{bmatrix} 1/16 & 0 \\ 1/8 & 1/32 \end{bmatrix}.$$

Obviously, (H_2) and $(\bar{H}_3)$ are satisfied with $L_f(t) = 2\left(|\delta_1| + |\delta_2|\right)e^{-t},\ g_1(t) = \max\{\left|\bar{\delta}_1\right|,\ \left|\bar{\delta}_2\right|\}e^{-t}$, $g_2(t) = 0$.

We note that, with $\left(1 - e^{-1}\right)\left(2\left(|\delta_1| + |\delta_2|\right) + \dfrac{592e^2 \max\{\left|\bar{\delta}_1\right|,\ \left|\bar{\delta}_2\right|\}}{(9e+4)(25e+4)}\right) < 1,$ we have

$$\begin{aligned}
L_1 &= \left\|L_f\right\|_{L^1(0,1)} + \left\|\sigma^{-1}\right\|_1 \left\|g_1\right\|_{L^1(0,1)} \\
&= 2\left(1 - e^{-1}\right)\left(|\delta_1| + |\delta_2|\right) + \frac{2e(31+203e)}{(4+9e)(4+25e)}\left(1 - e^{-1}\right)\max\{\left|\bar{\delta}_1\right|,\ \left|\bar{\delta}_2\right|\} \\
&= 2\left(1 - e^{-1}\right)\left[\left(|\delta_1| + |\delta_2|\right) + \frac{e(31+203e)}{(4+9e)(4+25e)}\max\{\left|\bar{\delta}_1\right|,\ \left|\bar{\delta}_2\right|\}\right] < 1.
\end{aligned}$$

Applying Theorem 3.5, we verify that Problem (3.27) has a solution.

4. Positive solutions

The main purpose of this section is to prove the existence of positive solutions for Problems (1.1)–(1.2), in which $f, g \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, and $H \in C([0,T]; \mathfrak{M}_n)$, $B_j \in \mathfrak{M}_n$ $(j = \overline{1,N})$. The main tool is the Guo-Krasnoselskii's fixed point theorem in a cone and applying Lemmas 2.3 and 2.4 with $\alpha > 0, \beta > 0$. For the sake of simplicity, we consider the case $u_0 = 0$.

First, based on Lemma 2.3 with $\alpha > 0, \beta > 0$, the integral system (2.1) can be written as follows:

$$\begin{cases} u(t) = \int_0^t e^{-\alpha(t-s)} f_\alpha(s, u(s), v(s))ds, \\ v(t) = \int_0^T G(t,s) g_\beta(s, u(s), v(s))ds. \end{cases} \tag{4.1}$$

Next, based on Lemma 2.4, we make here the assumptions as follows.

There exist positive constants α, β such that the following conditions are fulfilled:

$(\tilde{H}_1)$ $H \in C([0,T]; \mathfrak{M}_n)$, $B_j \in \mathfrak{M}_n$ $(j = \overline{1,N})$ such that the assumptions of Lemma 2.4 are satisfied;

$(\tilde{H}_2)$ $f : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$f(t,u,v) \geq -\alpha u, \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}^n_+ \times \mathbb{R}^n_+;$$

$(\tilde{H}_3)$ $g : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function such that

$$g(t,u,v) \geq -\beta v, \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}^n_+ \times \mathbb{R}^n_+.$$

We have the following simple lemma.

Lemma 4.1. *Suppose that* $(\tilde{H}_1)$–$(\tilde{H}_3)$ *are satisfied. Then, for each* $(u,v) \in X$ *such that* $u(t),\ v(t) \geq 0$, $\forall t \in [0,T]$, *we have* $\mathcal{P}_1(u,v)(t),\ \mathcal{P}_2(u,v)(t) \geq 0$, *for all* $t \in [0,T]$. □

We next define the cone K in X as follows:

$$K = \{(u,v) \in X : u(t) \geq 0,\ v(t) \geq 0,\ |v(t)|_1 \geq \gamma \|(u,v)\|_X,\ \forall t \in [0,T]\}, \tag{4.2}$$

where γ is defined as in (2.10) of Lemma 2.4.

Lemma 4.2. *Suppose that the following conditions are fulfilled*

(*i*) $\alpha \leq \beta$;
(*ii*) $f_\alpha(t,u,v) \leq g_\beta(t,u,v)$ *for all* $(t,u,v) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$.

Then, $\mathcal{P} : K \to K$.

Proof of Lemma 4.2. Let (u,v) be an arbitrary element in K. We have

$$\begin{aligned} \|\mathcal{P}(u,v)\|_X &= \sup_{0\leq t\leq T} |\mathcal{P}_1(u,v)(t)|_1 + \sup_{0\leq t\leq T} |\mathcal{P}_2(u,v)(t)|_1 \\ &= \sup_{0\leq t\leq T} \left| \int_0^t e^{-\alpha(t-s)} f_\alpha(s,u(s),v(s))ds \right|_1 + \sup_{0\leq t\leq T} \left| \int_0^T G(t,s) g_\beta(s,u(s),v(s))ds \right|_1 \\ &= \sup_{0\leq t\leq T} \int_0^t e^{-\alpha(t-s)} |f_\alpha(s,u(s),v(s))|_1 ds + \sup_{0\leq t\leq T} \int_0^T |G(t,s) g_\beta(s,u(s),v(s))|_1 ds \\ &\leq \int_0^T e^{\alpha s} |f_\alpha(s,u(s),v(s))|_1 ds + \int_0^T \left| \sigma_\beta^{-1} e^{\beta s} g_\beta(s,u(s),v(s)) \right|_1 ds \\ &\leq \int_0^T e^{\beta s} |g_\beta(s,u(s),v(s))|_1 ds + \int_0^T e^{\beta s} \left| \sigma_\beta^{-1} g_\beta(s,u(s),v(s)) \right|_1 ds \\ &= \int_0^T e^{\beta s} \left[|g_\beta(s,u(s),v(s))|_1 + \left| \sigma_\beta^{-1} g_\beta(s,u(s),v(s)) \right|_1 \right] ds \\ &= \int_0^T e^{\beta s} \left| \left(I + \sigma_\beta^{-1} \right) g_\beta(s,u(s),v(s)) \right|_1 ds. \end{aligned} \tag{4.3}$$

On the other hand, we also have

$$\begin{aligned}|\mathcal{P}_2(u,v)(t)|_1 &= \left|\int_0^T G(t,s)g_\beta(s,u(s),v(s))ds\right|_1 \\ &= \int_0^T |G(t,s)g_\beta(s,u(s),v(s))|_1\, ds \\ &\geq \gamma \int_0^T \left|\left(I+\sigma_\beta^{-1}\right) g_\beta(s,u(s),v(s))\right|_1 ds \geq \gamma \left\|\mathcal{P}(u,v)\right\|_X .\end{aligned} \tag{4.4}$$

It follows from (4.3), (4.4) that $|\mathcal{P}_2(u,v)(t)|_1 \geq \gamma \left\|\mathcal{P}(u,v)\right\|_X$, so $\mathcal{P}: K \to K$. Lemma 4.2 is proved. □

Theorem 4.3. *Suppose that* $(\tilde{H}_1)$ - $(\tilde{H}_3)$ *and the conditions in Lemma 4.2 are satisfied. Furthermore, the following assertions are fulfilled*

(i) *There is a constant* $\theta \in (0, 1/2T]$ *such that*

$$|f_\alpha(t,u,v)|_1 \leq \theta\left(|u|_1 + |v|_1\right),\ \forall (t,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \tag{4.5}$$

(ii) *There exist two positive constants* $r,\ R,\ r < R$, *such that*

$$\begin{aligned}&\left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \leq \frac{r}{2T},\ \forall (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n,\ \ |u|_1 \leq r,\ \gamma r \leq |v|_1 \leq r, \\ &\left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \geq \frac{R}{T},\ \forall (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n,\ \ |u|_1 \leq R,\ \gamma R \leq |v|_1 \leq R,\end{aligned} \tag{4.6}$$

or

$$\begin{aligned}&\left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \geq \frac{r}{T},\ \forall (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n,\ \ |u|_1 \leq r,\ \gamma r \leq |v|_1 \leq r, \\ &\left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \leq \frac{R}{2T},\ \forall (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n,\ \ |u|_1 \leq R,\ \gamma R \leq |v|_1 \leq R.\end{aligned} \tag{4.7}$$

Then, Problems (1.1)–(1.2) *have a solution* (u,v) *with* $u(t) \geq 0,\ v(t) \geq 0$, *for all* $t \in [0,T]$.

Proof of Theorem 4.3. Using similar calculations as in Lemma 3.4, we obtain that the operator $\mathcal{P}$ is completely continuous. Let us consider two bounded sets as follows:

$$\begin{aligned}\Omega_r &= \{(u,v) \in X : \|(u,v)\|_X < r\}, \\ \Omega_R &= \{(u,v) \in X : \|(u,v)\|_X < R\}.\end{aligned} \tag{4.8}$$

It is easy to see that Ω_r and Ω_R are open subsets of X with $0 \in \Omega_r$ and $\overline{\Omega_r} \subset \Omega_R$. We shall consider two cases.

Case 1. The (4.6) is true.

Take an arbitrary element $(u,v) \in K$ with $\|(u,v)\|_X = r$. We have the following estimates

$$|\mathcal{P}_1(u,v)(t)|_1 \leq \theta \int_0^t \left(|u(s)| + |v(s)|\right) ds \leq T\theta \left\|(u,v)\right\|_X , \tag{4.9}$$

and

$$|\mathcal{P}_2(u,v)(t)|_1 = \int_0^T |G(s,t)g_\beta(s,u(s),v(s))|_1\, ds \tag{4.10}$$
$$\leq \int_0^T \left|\bar{G}_1 g_\beta(s,u(s),v(s))\right|_1 ds \leq T\frac{r}{2T} = \frac{1}{2}\|(u,v)\|_X .$$

It follows from $(4.6)_{(1)}$, (4.9) and (4.10) that

$$\|\mathcal{P}(u,v)\|_X \leq \|(u,v)\|_X\,,\ \forall (u,v) \in K \cap \partial\Omega_r. \tag{4.11}$$

On the other hand, for each $(u,v) \in K \cap \partial\Omega_R$, we have

$$\|\mathcal{P}(u,v)\|_X \geq |\mathcal{P}_2(u,v)(t)|_1 = \left|\int_0^T G(t,s)g_\beta(s,u(s),v(s))ds\right|_1 \tag{4.12}$$
$$= \int_0^T |G(t,s)g_\beta(s,u(s),v(s))|_1\, ds$$
$$\geq \int_0^T \left|\bar{G}_0 g_\beta(s,u(s),v(s))\right|_1 ds \geq T\frac{R}{T} = \|(u,v)\|_X .$$

Combining (4.11), (4.12) and applying the first part of Theorem 2.3 (Guo-Krasnoselskii), we deduce that there exists $(u^*,v^*) \in K \cap (\overline{\Omega_R} \setminus \Omega_r)$ such that $\mathcal{P}(u^*,v^*) = (u^*,v^*)$. It means that Problems (1.1)–(1.2), with $u_0 = 0$, have positive solutions.

Case 2. The (4.7) is true.

Using the same method as in Case 1, by applying the second part of Theorem 2.3, we obtain the similar result.

Theorem 4.3 is completely proved. □

Remark 4.1. In order to show the existence of positive solutions of Problems (1.1)–(1.2) with $u_0 > 0$, we put $\bar{u}(t) = u(t) - u_0$. Then, the pair of functions $(\bar{u},v)$ is the solution of the following problem

$$\begin{cases} \bar{u}' = \bar{f}(t,\bar{u},v),\ t \in (0,T), \\ v' = \bar{g}(t,\bar{u},v),\ t \in (0,T), \\ \bar{u}(0) = 0,\ v(0) = \sum\limits_{j=1}^{N} B_j v(T_j) + \int_0^T H(t)v(t)dt, \end{cases} \tag{4.13}$$

where $\bar{f}(t,\bar{u},v) = f(t,\bar{u}+u_0,v),\ \bar{g}(t,\bar{u},v) = g(t,\bar{u}+u_0,v)$.

Applying results in Theorem 4.3 for the system (4.13), we can obtain the existence of a solution (u,v) such that $u(t) \geq u_0,\ v(t) \geq 0$ for all $t \in [0,T]$. □

We will provide an example for Theorem 4.3 below.

Example 4.1. Let us consider the nonlinear first order ordinary differential system as follows:

$$\begin{cases} u'(t) = f(u(t),v(t)),\ t \in (0,T), \\ v'(t) = g(u(t),v(t)),\ t \in (0,T), \\ u(0) = u_0,\ v(0) = B_1 v(T_1) + B_2 v(T_2) + \int_0^T H(t)v(t)dt, \end{cases} \tag{4.14}$$

where $N=2,\ T_1=\dfrac{T}{2},\ T_2=T,$

$$H(t)=e^{-t}\begin{bmatrix}1/4 & 0\\ 1/8 & 1/8\end{bmatrix},\ B_1=\begin{bmatrix}1/8 & 0\\ 1/8 & 1/16\end{bmatrix},\ B_2=\begin{bmatrix}1/16 & 1/16\\ 1/8 & 1/32\end{bmatrix},$$

and $f(u,v), g(u,v)$ are given as in (4.15) below.

We first choose $\gamma\in(0,1)$ as in Lemma 2.4. We have

$$\begin{aligned}\sigma_\beta &\equiv I-\int_0^T e^{-\beta t}H(t)dt-e^{-\beta T/2}B_1-e^{-T\beta}B_2\\ &= I-\frac{1-e^{-(1+\beta)T}}{1+\beta}\begin{bmatrix}1/4 & 0\\ 1/8 & 1/8\end{bmatrix}-e^{-\beta T/2}\begin{bmatrix}1/8 & 0\\ 1/8 & 1/16\end{bmatrix}-e^{-T\beta}\begin{bmatrix}1/16 & 1/16\\ 1/8 & 1/32\end{bmatrix}\\ &= \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}\sigma_{11} &= 1-\frac{1-e^{-(1+\beta)T}}{4(1+\beta)}-\frac{1}{8}e^{-\beta T/2}-\frac{1}{16}e^{-T\beta}>0,\\ \sigma_{22} &= 1-\frac{1-e^{-(1+\beta)T}}{8(1+\beta)}-\frac{1}{16}e^{-\beta T/2}-\frac{1}{32}e^{-T\beta}>0,\\ \sigma_{12} &= -\frac{1}{16}e^{-T\beta}<0,\\ \sigma_{21} &= -\frac{1-e^{-(1+\beta)T}}{8(1+\beta)}-\frac{1}{8}e^{-\beta T/2}-\frac{1}{8}e^{-T\beta}<0,\ \text{with } \beta>0 \text{ small enough},\\ \det\sigma_\beta &= \sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}=\Delta>0,\\ \sigma_\beta^{-1} &= \frac{1}{\Delta}\begin{bmatrix}\sigma_{22} & -\sigma_{12}\\ -\sigma_{21} & \sigma_{11}\end{bmatrix}\equiv\begin{bmatrix}\bar\sigma_{22} & -\bar\sigma_{12}\\ -\bar\sigma_{21} & \bar\sigma_{11}\end{bmatrix}>0.\end{aligned}$$

$$\begin{aligned}I+\sigma_\beta^{-1} &= \begin{bmatrix}1+\bar\sigma_{22} & -\bar\sigma_{12}\\ -\bar\sigma_{21} & 1+\bar\sigma_{11}\end{bmatrix}\equiv\left(\bar d_{ij}\right),\\ e^{-2\beta T}\sigma_\beta^{-1}B_N &= e^{-2\beta T}\begin{bmatrix}\bar\sigma_{22} & -\bar\sigma_{12}\\ -\bar\sigma_{21} & \bar\sigma_{11}\end{bmatrix}\begin{bmatrix}1/16 & 1/16\\ 1/8 & 1/32\end{bmatrix}\\ &= e^{-2\beta T}\begin{bmatrix}\frac{\bar\sigma_{22}}{16}-\frac{\bar\sigma_{12}}{8} & \frac{\bar\sigma_{22}}{16}-\frac{\bar\sigma_{12}}{32}\\ \frac{-\bar\sigma_{21}}{16}+\frac{\bar\sigma_{11}}{8} & \frac{-\bar\sigma_{21}}{16}+\frac{\bar\sigma_{11}}{32}\end{bmatrix}\equiv\left(\bar c_{ij}\right),\\ \bar c_{ij} &> 0,\ \forall i,j=\overline{1,2}.\end{aligned}$$

By choosing $\gamma\in(0,1)$ such that

$$0<\gamma<\min_{i,j=\overline{1,2}}\left\{1,\frac{\bar c_{ij}}{1+\bar d_{ij}}\right\},$$

then

$$e^{-2\beta T}\sigma_\beta^{-1}B_N - \gamma\left(I + \sigma_\beta^{-1}\right) > 0.$$

We next choose two positive constants $r,\ R$ such that $r < R$ and $r < \gamma R$.

We note more that it is easy to compute two positive matrices $\bar{G}_0,\ \bar{G}_1$ as in Lemma 2.4.

Now, we give the functions f, g as follows:

$$f(u,v) + \alpha u = \delta\left[g(u,v) + \beta v + \alpha\left(|v_1|\sin^2(\sqrt[3]{v_2}),\ \ |v_2|\sin^2(\sqrt[5]{v_1})\right)\right], \tag{4.15}$$

$$g(u,v) - \beta v = \begin{cases} c_1\dfrac{|v|_1\,|u|_1^2}{1+|u|_1^2}\vec{d}, & u,v\in\mathbb{R}^2,\ |v|_1 \le r,\\ \dfrac{c_1(|v|_1-\gamma R)}{r-\gamma R}\dfrac{r\,|u|_1^2}{1+|u|_1^2}\vec{d} & \\ \qquad + \dfrac{c_2(|v|_1 - r)}{\gamma R - r}\left(u + \dfrac{|v|_1}{\gamma R}v\right), & u,v\in\mathbb{R}^2,\ r\le |v|_1 \le \gamma R,\\ c_2(u+v), & u,v\in\mathbb{R}^2,\ |v|_1 \ge \gamma R, \end{cases}$$

for $0 \le c_1(d_1+d_2) + 2\beta \le \dfrac{1}{2T\left\|\bar{G}_1\right\|_1},\ c_2 \ge \dfrac{1}{\gamma T}$ and $\vec{d} = (d_1, d_2) \in \mathbb{R}_+^2,\ \vec{d} > 0.$

In what follows, we verify that $(\tilde{H}_2),\ (\tilde{H}_3)$ are satisfied.

It is clear to see that $g \in C(\mathbb{R}^2\times\mathbb{R}^2;\mathbb{R}^2)$ and for all $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2$, we have

(i) $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2,\ |v|_1 \le r : g(u,v) - \beta v = c_1\dfrac{|v|_1\,|u|_1^2}{1+|u|_1^2}\vec{d} \ge 0;$

(ii) $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2,\ r\le |v|_1 \le \gamma R:$

$$\begin{aligned} g(u,v) - \beta v &= \frac{c_1(|v|_1-\gamma R)}{r-\gamma R}\frac{r\,|u|_1^2}{1+|u|_1^2}\vec{d} + \frac{c_2(|v|_1 - r)}{\gamma R - r}\left(u + \frac{|v|_1}{\gamma R}v\right)\\ &= \left(1 - \frac{|v|_1 - r}{\gamma R - r}\right)\frac{c_1 r\,|u|_1^2}{1+|u|_1^2}\vec{d} + \frac{|v|_1 - r}{\gamma R - r}c_2\left(u + \frac{|v|_1}{\gamma R}v\right)\\ &= (1-\lambda)\frac{c_1 r\,|u|_1^2}{1+|u|_1^2}\vec{d} + \lambda c_2\left(u + \frac{|v|_1}{\gamma R}v\right) \ge 0,\ \text{with } \lambda = \frac{|v|_1 - r}{\gamma R - r} \in [0,1]; \end{aligned}$$

(iii) $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2,\ |v|_1 \ge \gamma R : g(u,v) - \beta v = c_2(u+v) \ge 0.$

Thus $g(u,v) + \beta v = g(u,v) - \beta v + 2\beta v \ge 0,\ \forall (u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2$. It implies that $(\tilde{H}_3)$ holds.

On the other hand, f satisfies $(\tilde{H}_2)$. Indeed, by

$$f(t,u,v) = -\alpha u + \delta\left[g(u,v) + \beta v + \alpha\left(|v_1|\sin^2(\sqrt[3]{v_2}),\ \ |v_2|\sin^2(\sqrt[5]{v_1})\right)\right],$$

we have $f \in C(\mathbb{R}^2\times\mathbb{R}^2;\mathbb{R}^2)$ and for all $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2$,

$$\begin{aligned} f(u,v) + \alpha u &= \delta\left[g(u,v) + \beta v + \alpha\left(|v_1|\sin^2(\sqrt[3]{v_2}),\ \ |v_2|\sin^2(\sqrt[5]{v_1})\right)\right]\\ &\ge \delta\left(g(u,v) + \beta v\right) \ge 0. \end{aligned}$$

Next, the conditions in Lemma 4.2 are satisfied. We need prove that if $\alpha \le \beta$, then

$$f(u,v) + \alpha u \le g(u,v) + \beta v,\ \forall (u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2.$$

We have

$$\begin{aligned} f_\alpha(u,v) &= \delta\left[g(u,v)+\beta v+\alpha\left(|v_1|\sin^2(\sqrt[3]{v_2}),\ \ |v_2|\sin^2(\sqrt[5]{v_1})\right)\right] \le \delta\left[g(u,v)+\beta v+\alpha v\right] \\ &= \delta\left[g(u,v)+\left(1+\frac{\alpha}{\beta}\right)\beta v\right] \le \delta\left(1+\frac{\alpha}{\beta}\right) g_\beta(u,v) \\ &\le g_\beta(u,v),\ \forall (u,v)\in \mathbb{R}^2_+\times\mathbb{R}^2_+, \end{aligned}$$

with $\delta>0$ small enough, such that $0<\delta\left(1+\dfrac{\alpha}{\beta}\right)\le 1$.

It is clear that the function f satisfies the condition (4.5), i.e.

$$\exists\theta\in(0,\frac{1}{2T}]:|f_\alpha(u,v)|=|f(u,v)+\alpha u|_1\le\theta\left(|u|_1+|v|_1\right),\forall u,v\in\mathbb{R}^2.$$

Indeed, for all $(u,v)\in\mathbb{R}^2$, we have

(i) $(u,v)\in\mathbb{R}^2\times\mathbb{R}^2,\ |v|_1\le r:$

$$\begin{aligned} |g(u,v)-\beta v|_1 &= c_1\frac{|v|_1|u|_1^2}{1+|u|_1^2}(d_1+d_2) \\ &\le c_1(d_1+d_2)|v|_1\le c_1(d_1+d_2)\left(|u|_1+|v|_1\right); \end{aligned}$$

(ii) $(u,v)\in\mathbb{R}^2\times\mathbb{R}^2,\ r\le|v|_1\le\gamma R:$ with $\lambda=\dfrac{|v|_1-r}{\gamma R-r}\in[0,1]$, we get

$$\begin{aligned} |g(u,v)-\beta v|_1 &= \left|(1-\lambda)\frac{c_1 r|u|_1^2}{1+|u|_1^2}\vec{d}+\lambda c_2\left(u+\frac{|v|_1}{\gamma R}v\right)\right|_1 \\ &\le(1-\lambda)c_1\frac{r|u|_1^2}{1+|u|_1^2}(d_1+d_2)+\lambda c_2\left(|u|_1+\frac{|v|_1}{\gamma R}|v|_1\right) \\ &\le c_1(d_1+d_2)|v|_1+c_2\left(|u|_1+|v|_1\right) \\ &\le\left[c_1(d_1+d_2)+c_2\right]\left(|u|_1+|v|_1\right); \end{aligned}$$

(iii) $(u,v)\in\mathbb{R}^2\times\mathbb{R}^2,\ |v|_1\ge\gamma R:|g(u,v)-\beta v|_1=|c_2(u+v)|_1\le c_2\left(|u|_1+|v|_1\right).$

It implies from (i)–(iii) that

$$|g(u,v)-\beta v|_1\le\left[c_1(d_1+d_2)+c_2\right]\left(|u|_1+|v|_1\right),\ \forall(u,v)\in\mathbb{R}^2\times\mathbb{R}^2.$$

Hence

$$\begin{aligned} |f(u,v)+\alpha u|_1 &= \delta\left|g(u,v)+\beta v+\alpha\left(|v_1|\sin^2(\sqrt[3]{v_2}),\ \ |v_2|\sin^2(\sqrt[5]{v_1})\right)\right|_1 \\ &\le \delta\left[|g(u,v)-\beta v|_1+(2\beta+\alpha)|v|_1\ \right] \\ &\le \delta\left[\left(c_1(d_1+d_2)+c_2\right)\left(|u|_1+|v|_1\right)+(2\beta+\alpha)\left(|u|_1+|v|_1\right)\ \right] \\ &\le \delta\left[c_1(d_1+d_2)+c_2+2\beta+\alpha\ \right]\left(|u|_1+|v|_1\right) \\ &\equiv \theta\left(|u|_1+|v|_1\right),\ \forall(u,v)\in\mathbb{R}^2\times\mathbb{R}^2, \end{aligned}$$

where $\theta=\delta\left[c_1(d_1+d_2)+c_2+2\beta+\alpha\ \right]\le\dfrac{1}{2T}$, with $\delta>0$ small enough.

Finally, the function g_β satisfies the condition (4.6), because

(i) $(u,v)\in\mathbb{R}^2_+\times\mathbb{R}^2_+,\ |u|_1\le r,\ \gamma r\le|v|_1\le r:$

$$|g(u,v)-\beta v|_1=c_1\frac{|v|_1|u|_1^2}{1+|u|_1^2}(d_1+d_2)\le c_1(d_1+d_2)|v|_1;$$

hence,

$$\begin{aligned}|g_\beta(u,v)|_1 &\le |g(u,v)-\beta v|_1+2\beta\,|v|_1\\ &\le [c_1(d_1+d_2)+2\beta\,]\,|v|_1\le [c_1(d_1+d_2)+2\beta\,]\,r.\end{aligned}$$

It implies that

$$\begin{aligned}\left|\bar{G}_1 g_\beta(u,v)\right|_1 &\le \left\|\bar{G}_1\right\|_1 |g_\beta(u,v)|_1\le \left\|\bar{G}_1\right\|_1 [c_1(d_1+d_2)+2\beta\,]\,r\\ &\le \left\|\bar{G}_1\right\|_1 \frac{r}{2T\left\|\bar{G}_1\right\|_1}=\frac{r}{2T}.\end{aligned}$$

(ii) $(u,v)\in\mathbb{R}_+^2\times\mathbb{R}_+^2,\ |u|_1\le R,\ \gamma R\le |v|_1\le R:$
$g_\beta(u,v)=g(u,v)-\beta v+2\beta v\ge g(u,v)-\beta v=c_2\,(u+v)\ge c_2 v;$

therefore,

$$\left|\bar{G}_0 g_\beta(u,v)\right|_1\ge c_2\,|v|_1\ge c_2\gamma R\ge\frac{R}{T}.$$

We deduce that the assumptions and the conditions in Theorem 4.3 are satisfied; hence, we verify that the system (4.14) has a positive solution. □

5. Multiplicity of positive solutions

In this section, we will show that Problems (1.1)–(1.2) can have two distinct solutions or even finitely many distinct solutions. The multiplicity of positive solutions depends strongly on the nonlinear term in (1.1). For the sake of simplicity, we just consider the case $u_0=0$.

First, in order to prove the multiplicity result, we assume that there exists $R_1<\gamma R_2<\gamma^2R_3$ such that for $j=\overline{1,2}$

$$\begin{aligned}&(\bar{G}_1)\quad \left|\bar{G}_1 g_\beta(t,u,v)\right|_1\le\frac{R_j}{2T}\ \text{for all } (t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n,\\ &\qquad\qquad |u|_1\le R_j,\ \gamma R_j\le|v|_1\le R_j,\\ &(\underline{G}_1)\quad \left|\bar{G}_0 g_\beta(t,u,v)\right|_1\ge\frac{R_{j+1}}{T}\ \text{for all } (t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n,\\ &\qquad\qquad |u|_1\le R_{j+1},\ \gamma R_{j+1}\le|v|_1\le R_{j+1},\end{aligned}$$

where γ is defined as in (2.10) of Lemma 2.4.

Theorem 5.1. *Assume that* $(\tilde{H}_1)$–$(\tilde{H}_3)$, (4.5) *and* $(\bar{G}_1)$–$(\underline{G}_1)$ *are satisfied. Then, Problems* (1.1)–(1.2) *have two solutions* (u_1,v_1) *and* (u_2,v_2) *such that*

$$\begin{aligned}&R_1<\|(u_1,v_1)\|_X\le R_2,\\ &R_2<\|(u_2,v_2)\|_X\le R_3.\end{aligned}\tag{5.1}$$

Proof of Theorem 5.1. We denote the sets

$$\Omega_j=\{(u,v)\in X:\|(u,v)\|_X<R_j\}\,,\ j=\overline{1,3}.\tag{5.2}$$

For $(u,v)\in K\cap\partial\Omega_1$, we have

$$\begin{aligned}&|u(t)|_1\le\|(u,v)\|_X=R_1,\\ &\gamma R_1=\gamma\,\|(u,v)\|_X\le|v(t)|_1\le\|(u,v)\|_X=R_1.\end{aligned}\tag{5.3}$$

It follows from (5.3) and $(\bar{G}_1)$ that

$$\left|\bar{G}_1 g_\beta(t,u(t),v(t))\right|_1 \le \frac{R_1}{2T}. \tag{5.4}$$

Combining (4.5) and (5.4), we obtain the following estimate

$$\begin{aligned}\|\mathcal{P}(u,v)\|_X &= \max_{0\le t\le T}\int_0^t |f_\alpha(s,u(s),v(s)|_1\,ds + \max_{0\le t\le T}\int_0^T |G(t,s)g_\beta(s,u(s),v(s))|_1\,ds \\ &\le \frac{1}{2}\|(u,v)\|_X + \frac{R_1}{2} = \|(u,v)\|_X\,.\end{aligned} \tag{5.5}$$

If $(u,v)\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}|u(t)|_1 &\le \|(u,v)\|_X = R_2,\\ \gamma R_2 = \gamma\|(u,v)\|_X &\le |v(t)|_1 \le \|(u,v)\| = R_2.\end{aligned} \tag{5.6}$$

It follows from (5.6) and the assumption $(\underline{G}_1)$ that

$$\|\mathcal{P}(u,v)\|_X \ge \|(u,v)\|_X\,. \tag{5.7}$$

Applying the Guo-Krasnoselskii's fixed point theorem, we verify that there exists a pair of functions $(u_1,v_1)\in K\cap(\overline{\Omega_2}\setminus\Omega_1)$ such that $\mathcal{P}(u_1,v_1)=(u_1,v_1)$.

By using the similar calculations as the previous part, we also deduce that there exists a pair of functions $(u_2,v_2)\in K\cap(\overline{\Omega_3}\setminus\Omega_2)$ which is a fixed point of the operator $\mathcal{P}$.

Theorem 5.1 is completely proved. □

Next, we shall generalize results obtained in Theorem 5.1 to have the existence of finitely many distinct solutions. For this purpose, we assume that there exists $\{R_j\}_{j=1}^p$ such that $R_{j-1}<\gamma R_j$. We make the following assumptions

$(\bar{G}_p)$ $\left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \le \dfrac{R_j}{2T}$ for all $(t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n$,
$|u|_1\le R_j,\ \gamma R_j\le|v|_1\le R_j,\ j=\overline{1,p-1}$,

$(\underline{G}_p)$ $\left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \ge \dfrac{R_{j+1}}{T}$ for all $(t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n$,
$|u|_1\le R_{j+1},\ \gamma R_{j+1}\le|v|_1\le R_{j+1},\ j=\overline{1,p-1}$;

or

$(\underline{G}_p^*)$ $\left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \ge \dfrac{R_j}{T}$ for all $(t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n$,
$|u|_1\le R_j,\ \gamma R_j\le|v|_1\le R_j,\ j=\overline{1,p-1}$,

$(\bar{G}_p^*)$ $\left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \le \dfrac{R_{j+1}}{2T}$ for all $(t,u,v)\in[0,T]\times\mathbb{R}_+^n\times\mathbb{R}_+^n$,
$|u|_1\le R_{j+1},\ \gamma R_{j+1}\le|v|_1\le R_{j+1},\ j=\overline{1,p-1}$.

Then, by the method and calculations as in the proof of Theorem 5.1, we obtain the following theorem.

Theorem 5.2. *Assume that* $(\tilde{H}_1)$–$(\tilde{H}_3)$, (4.5) *and* $(\bar{G}_p)$–$(\underline{G}_p)$ *(or* $(\underline{G}_p^*)$–$(\bar{G}_p^*)$*) are satisfied. Then, Problems* (1.1)–(1.2) *have at least* $p-1$ *solutions* (u_j,v_j), $1\le j\le p-1$ *such that*

$$R_j < \|(u,v)\|_X \le R_{j+1},\ \ j=\overline{1,p-1}. \tag{5.8}$$

Finally, assume that we have a positive sequence $\{R_j\}$ such that $\dfrac{R_j}{R_{j+1}} < \gamma < 1$ such that for each $j \in \mathbb{N}$,

$$(\bar{G}_\infty) \quad \left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \le \frac{R_j}{2T} \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \quad |u|_1 \le R_j,\ \gamma R_j \le |v|_1 \le R_j,$$

$$(\underline{G}_\infty) \quad \left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \ge \frac{R_{j+1}}{T} \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \quad |u|_1 \le R_{j+1}, \gamma R_{j+1} \le |v|_1 \le R_{j+1};$$

or

$$(\underline{G}_\infty^*) \quad \left|\bar{G}_0 g_\beta(t,u,v)\right|_1 \ge \frac{R_j}{T} \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \quad |u|_1 \le R_j, \gamma R_j \le |v|_1 \le R_j,$$

$$(\bar{G}_\infty^*) \quad \left|\bar{G}_1 g_\beta(t,u,v)\right|_1 \le \frac{R_{j+1}}{2T} \text{ for all } (t,u,v) \in [0,T] \times \mathbb{R}_+^n \times \mathbb{R}_+^n, \quad |u|_1 \le R_{j+1}, \gamma R_{j+1} \le |v|_1 \le R_{j+1}.$$

Then, we also obtain the following theorem.

Theorem 5.3. *Assume that* $(\tilde{H}_1)$–$(\tilde{H}_3)$, (4.5) *and* $(\bar{G}_\infty)$–$(\underline{G}_\infty)$ (*or* $(\underline{G}_\infty^*)$–$(\bar{G}_\infty^*)$) *are satisfied. Then, Problems* (1.1)–(1.2) *have infinitely many solutions* $\{(u_j, v_j)\}$, $j \in \mathbb{N}$ *such that*

$$R_j < \|(u,v)\|_X \le R_{j+1},\ \forall j \in \mathbb{N}. \tag{5.9}$$

6. A remark

We remark that the methods used in the above sections can be applied again to obtain the same results as above for the following problem

$$u_k'(t) = f_k(t, u_1, \cdots, u_m, u_{m+1}, \cdots, u_{m+n}),\ t \in (0,T),\ k = \overline{1, m+n}, \tag{6.1}$$

associated with the initial and multipoint conditions

$$\begin{cases} u_k(0) = u_{0k},\ k = \overline{1,m}, \\ u_k(0) = \sum\limits_{j=1}^{N_k} \mu_{kj} u_k(T_{kj}) + \int_0^T h_k(t) u_k(t) dt,\ k = \overline{m+1, m+n}, \end{cases} \tag{6.2}$$

where $f_k : [0,T] \times \mathbb{R}^{m+n} \to \mathbb{R}$, $h_k : [0,T] \to \mathbb{R}$ $(k = \overline{1, m+n})$ are given functions and u_{0k} $(k = \overline{1,m})$, $0 < T_{k1} < T_{k2} < \cdots < T_{kN_k} = T$ $(k = \overline{m+1, m+n})$, μ_{kj} $(j = \overline{1, N_k})$ are given constants, with $\max\limits_{m+1 \le k \le m+n} \left(\sum\limits_{j=1}^{N_k} |\mu_{kj}| + \int_0^T |h_k(t)|\, dt \right) \le 1$.

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