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On the distribution of adjacent zeros of solutions to first-order neutral differential equations

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Abstract: The purpose of this paper is to study the distribution of zeros of solutions to a first-order neutral differential equation of the form

$$[x(t)+p(t)x(t-\tau)]'+q(t)x(t-\sigma)=0, \quad t\geq t_0,$$

where $p\in C([t_0,\infty),[0,\infty))$, $q\in C([t_0,\infty),(0,\infty))$, $\tau,\sigma>0$, and $\sigma>\tau$. We obtain new upper bound estimates for the distance between consecutive zeros of solutions, which improve upon many of the previously known ones. The results are formulated so that they can be generalized without much effort to equations for which the distribution of zeros problem is related to the study of this property for a first-order delay differential inequality. The strength of our results is demonstrated via two illustrative examples.

Key words: Distribution of zeros, neutral differential equations, oscillation

1. Introduction

In this work, we study the upper bounds for the distance between adjacent zeros of solutions to a first-order neutral differential equation of the form

$$[x(t)+p(t)x(t-\tau)]'+q(t)x(t-\sigma)=0, \quad t\geq t_0, \tag{E}$$

where $p\in C([t_0,\infty),[0,\infty))$, $q\in C([t_0,\infty),(0,\infty))$, and $\sigma>\tau>0$. By a solution of (E), we understand a nontrivial real-valued function $x\in C([t_0-\sigma,\infty),\mathbb{R})$ such that $x(t)+p(t)x(t-\tau)$ is continuously differentiable and (E) is satisfied on $[t_0,\infty)$. It is straightforward to show using the method of steps that Eq. (E) has a unique solution $x\in C([t_0-\sigma,\infty),\mathbb{R})$ such that $x=\varphi$ on $[t_0-\sigma,t_0]$, where $\varphi\in C([t_0-\sigma,t_0],\mathbb{R})$ is a given initial function. As is customary, a solution x is called oscillatory if it has infinitely many zeros and nonoscillatory otherwise. Equation itself is termed oscillatory if all its solutions are oscillatory.

With regard to a large number of applications of first-order functional differential equations in many fields of natural sciences and engineering (see, for example, [6, 10, 13, 14] for more details), the oscillation theory of such equations has been developed extensively over the last few decades. The interest in this subject is evidenced by numerous published monographs [1, 3, 6, 10, 11, 13]. Most efforts, however, were dedicated toward studying

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the existence or nonexistence of oscillatory solutions whereas only a few authors were interested in determining the location of zeros of solutions of such equations. As a matter of fact, the problem of estimating the distance between consecutive zeros of solutions of first-order delay differential equations has been considered to be one of the main challenges in the oscillation theory. Recently, there has been increasing interest concerning this problem, and we mention particularly important work [7] studying the distribution of zeros for the first-order delay differential equation

$$x'(t) + p(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{1.1}$$

with $p \in C([t_0, \infty), [0, \infty))$ and $\sigma > 0$. In [2] and [8], the authors improved the techniques from [7] to study the distribution of zeros of first-order differential equations with several constant and distributed delays, respectively. Similarly, many authors [4–6, 8, 15–19] have developed and extended the methods used to study the distribution of zeros for Eq. (1.1) to study the same property for the first-order neutral differential equations of the form (E) and their generalizations.

The purpose of this work is to continue the study on the distribution of zeros for Eq. (E) and to obtain better upper bound estimates for the distance between adjacent zeros of all its solutions. Motivated by the method used in [4, 15, 16], we first study the properties of a positive solution $X(t)$ to the companion first-order delay differential inequality

$$X'(t) + P(t)X(t-\delta) \leq 0, \qquad t \geq T_0 + \zeta, \tag{1.2}$$

where $P \in C([T_0 + \zeta, \infty), [0, \infty))$, $T_0 \geq t_0$ and $\delta, \zeta > 0$ on any bounded real interval. This makes our results more general so that they can be applied without much effort to any differential equation for which the distribution of zeros problem is related to the study of the same property for a first-order delay differential inequality (1.2). The results established in our work improve and extend the ones obtained in [4, 6–8, 15, 16, 19], which are illustrated by two examples.

2. Properties of positive solutions of the companion inequality (1.2)

In order to obtain our main results, we study the properties of a positive solution $X(t)$ of the first-order delay differential inequality (1.2). In the sequel, we set

$$\int_{t-\delta}^{t} P(v)dv \geq \eta \geq 0 \quad \text{for } t \geq T_0 + \zeta + \delta \tag{2.1}$$

and also, we will make use of two constants $\zeta_1, \zeta_2 \geq 0$ such that

$$\zeta \geq \max\{\zeta_1, \zeta_2\} + \delta. \tag{2.2}$$

For η satisfying (2.1), we define the sequence $\{\Upsilon_n(\eta)\}_{n\geq 0}$ as follows:

$$\begin{aligned} \Upsilon_0(\eta) &= 1, \\ \Upsilon_1(\eta) &= \frac{1}{1-\eta}, \\ \Upsilon_{n+1}(\eta) &= \frac{1}{1-\eta-\frac{\eta^2}{2}\Upsilon_n(\eta)}, \quad n = 1, 2, \dots \end{aligned} \tag{2.3}$$

Lemma 2.1 *Assume that* $n \in \mathbb{N}_0$ *and* (2.1) *holds. If there exist* $T_1 \geq t_0 + \zeta + n\delta$ *and a function* $X(t)$ *satisfying* (1.2) *on* $[T_0 + \zeta_1, T_1]$ *with* $X'(t) \leq 0$ *on* $[T_0 + \zeta_2,\, T_0 + \zeta]$ *and* $X(t) > 0$ *on* $[T_0 + \zeta_1, T_1]$*, then*

$$\frac{X(t-\delta)}{X(t)} \geq \Upsilon_n(\eta) > 0 \quad \textit{for } t \in [T_0 + \zeta + n\delta, T_1], \tag{2.4}$$

where $\Upsilon_n(\eta)$ *is defined by* (2.3).

Proof In view of $X(t) > 0$ on $[T_0 + \zeta_1, T_1]$ and (2.2), it follows from (1.2) that

$$X'(t) \leq 0 \quad \text{for } t \in [T_0 + \zeta, T_1].$$

This together with $X'(t) \leq 0$ on $[T_0 + \zeta_2, T_0 + \zeta]$ implies that

$$X'(t) \leq 0 \quad \text{for } t \in [T_0 + \zeta_2, T_1]. \tag{2.5}$$

Therefore,

$$\frac{X(t-\delta)}{X(t)} \geq 1 = \Upsilon_0(\eta) \quad \text{for } t \in [T_0 + \zeta, T_1].$$

Integrating inequality (1.2) from $t - \delta$ to t, we obtain

$$X(t) - X(t-\delta) + \int_{t-\delta}^{t} P(v) X(v-\delta) dv \leq 0 \quad \text{for } t \in [T_0 + \zeta + \delta, T_1]. \tag{2.6}$$

Since $X'(t) \leq 0$ on $[T_0 + \zeta_2,\, T_1]$ and $\zeta \geq \zeta_2 + \delta$, then

$$X(v-\delta) \geq X(t-\delta) \qquad \text{for } t - \delta \leq v \leq t \text{ and } t \in [T_0 + \zeta + \delta, T_1].$$

Combining this together with 2.6, we get

$$X(t-\delta)\left(1 - \int_{t-\delta}^{t} P(v) dv\right) \geq X(t) > 0 \quad \text{for } t \in [T_0 + \zeta + \delta, T_1].$$

Consequently,

$$\frac{X(t-\delta)}{X(t)} \geq \frac{1}{1 - \int_{t-\delta}^{t} P(v) dv} \geq \frac{1}{1-\eta} = \Upsilon_1(\eta) \quad \text{for } t \in [T_0 + \zeta + \delta, T_1]. \tag{2.7}$$

Integrating inequality (1.2) from $v - \delta$ to $t - \delta$ and $t - \delta \leq v \leq t$, we get

$$X(v-\delta) \geq X(t-\delta) + \int_{v-\delta}^{t-\delta} P(v_1) X(v_1 - \delta) dv_1 \quad \text{for } t \in [T_0 + \zeta + 2\delta, T_1].$$

Using the latter inequality in (2.6), we obtain

$$X(t) - X(t-\delta) + X(t-\delta) \int_{t-\delta}^{t} P(v) dv + \int_{t-\delta}^{t} P(v) \int_{v-\delta}^{t-\delta} P(v_1) X(v_1 - \delta) dv_1 dv \leq 0$$

for $t \in [T_0+\zeta+2\delta, T_1]$. Using the nonincreasing nature of $X(t)$ on $[T_0+\zeta_2, T_1]$ and rearranging, we have

$$X(t-\delta)\left(1-\int_{t-\delta}^{t} P(v)dv - \frac{X(t-2\delta)}{X(t-\delta)}\int_{t-\delta}^{t} P(v)\int_{v-\delta}^{t-\delta} P(v_1)dv_1dv\right) \geq X(t) > 0$$

for $t \in [T_0+\zeta+2\delta, T_1]$. Let $t^* \in [t-\delta, t]$, $T_0+\zeta+2\delta \leq t \leq T_1$ such that $\int_{t-\delta}^{t^*} p(v)dv = \eta$. Then

$$X(t-\delta)\left(1-\eta - \frac{X(t-2\delta)}{X(t-\delta)}\int_{t-\delta}^{t^*} P(v)\int_{v-\delta}^{t-\delta} P(v_1)dv_1dv\right) \geq X(t) > 0. \tag{2.8}$$

Clearly,

$$\begin{aligned}\int_{t-\delta}^{t^*} P(v)\int_{v-\delta}^{t-\delta} P(v_1)dv_1dv &= \int_{t-\delta}^{t^*} P(v)\int_{v-\delta}^{v} P(v_1)dv_1dv - \int_{t-\delta}^{t^*} P(v)\int_{t-\delta}^{v} P(v_1)dv_1dv \\ &\geq \eta^2 - \int_{t-\delta}^{t^*} P(v)\int_{t-\delta}^{v} P(v_1)dv_1dv.\end{aligned} \tag{2.9}$$

However,

$$\int_{t-\delta}^{t^*} dv \int_{t-\delta}^{v} P(v)P(v_1)dv_1 = \int_{t-\delta}^{t^*} dv_1 \int_{v}^{t^*} P(v_1)P(v)dv.$$

Therefore,

$$\int_{t-\delta}^{t^*} dv \int_{t-\delta}^{v} P(v)P(v_1)dv_1 = \frac{1}{2}\int_{t-\delta}^{t^*} dv_1 \int_{t-\delta}^{t^*} P(v_1)P(v)dv = \frac{1}{2}\left(\int_{t-\delta}^{t^*} P(v)dv\right)^2 = \frac{\eta^2}{2}.$$

Substituting this into (2.9), we obtain

$$\int_{t-\delta}^{t^*} P(v)\int_{v-\delta}^{t-\delta} P(v)dv \geq \frac{\eta^2}{2} \qquad \text{for } t \in [T_0+\zeta+2\delta, T_1].$$

Combining the above inequality together with (2.8), we find

$$X(t-\delta)\left(1-\eta-\frac{X(t-2\delta)}{X(t-\delta)}\frac{\eta^2}{2}\right) \geq X(t) > 0 \qquad \text{for } t \in [T_0+\zeta+2\delta, T_1]. \tag{2.10}$$

In view of (2.7), it follows that

$$\frac{X(t-2\delta)}{X(t-\delta)} \geq \Upsilon_1(\eta) \quad \text{for } t \in [T_0+\zeta+2\delta, T_1].$$

Using the above inequality in (2.10), we get

$$\frac{X(t-\delta)}{X(t)} \geq \frac{1}{1-\eta-\frac{\eta^2}{2}\frac{X(t-2\delta)}{X(t-\delta)}} \geq \frac{1}{1-\eta-\frac{\eta^2}{2}\Upsilon_1(\eta)} = \Upsilon_2(\eta) \quad \text{for } t \in [T_0+\zeta+2\delta, T_1].$$

Repeating this procedure n times we obtain (2.4). The proof is complete. □

Let

$$\Psi_1(v) = P(v),$$
$$\Psi_n(v) = P(v-(n-1)\delta)\int_{t-\delta}^{v}\Psi_{n-1}(v_1)dv_1, \quad t-\delta \le v \le t, \quad t \ge T_0+\zeta+n\delta, \quad n = 2,3,\dots. \tag{2.11}$$

Lemma 2.2 *Assume* (2.1) *holds and*

$$\int_{t-\delta}^{t}\Psi_1(v)dv + \sum_{l=2}^{n}\prod_{i=2}^{l}\Upsilon_{n+2-i}(\eta)\int_{t-\delta}^{t}\Psi_l(v)dv \ge 1 \quad \text{for } t \ge T_0+\zeta+(n+1)\delta, \tag{2.12}$$

where $n\in\mathbb{N}$ *and* $\Psi_n(t)$ *is defined by* (2.11)*. If there exists a function* $X(t)$ *satisfying* (1.2) *on* $[T_0+\zeta,\, T_1]$ *such that* $X'(t)\le 0$ *on* $[T_0+\zeta_2,\, T_0+\zeta]$ *and* $X(t)>0$ *on* $[T_0+\zeta_1, T_1]$*, then* $X(t)$ *cannot be positive on* $[T_0+\zeta,\, T_1]$*,* $T_1 \ge T_0+\zeta+(n+1)\delta$*.*

Proof Assume, for the sake of contradiction, that $X(t)$ is positive on $[T_0+\zeta,\, T_1]$, where $T_1 \ge T_0+\zeta+(n+1)\delta$. Integrating inequality (1.2) from $t-\delta$ to t, we obtain

$$X(t)-X(t-\delta)+\int_{t-\delta}^{t}P(v)X(v-\delta)dv \le 0 \text{ for } t\in[T_0+\zeta+\delta, T_1]. \tag{2.13}$$

Integrating the latter inequality by parts, it follows that

$$\begin{aligned}\int_{t-\delta}^{t}P(v)X(v-\delta)dv &= \int_{t-\delta}^{t}X(v-\delta)d\left(\int_{t-\delta}^{v}\Psi_1(v_1)dv_1\right)\\ &= X(t-\delta)\int_{t-\delta}^{t}\Psi_1(v)dv - \int_{t-\delta}^{t}X'(v-\delta)\int_{t-\delta}^{v}\Psi_1(v_1)dv_1dv\end{aligned}$$

for $t\in[T_0+\zeta+\delta, T_1]$. From this and (1.2), we get

$$\begin{aligned}\int_{t-\delta}^{t}P(v)X(v-\delta)dv &\ge X(t-\delta)\int_{t-\delta}^{t}\Psi_1(v)dv + \int_{t-\delta}^{t}X(v-2\delta)P(v-\delta)\int_{t-\delta}^{v}\Psi_1(v_1)dv_1dv\\ &= X(t-\delta)\int_{t-\delta}^{t}\Psi_1(v)dv + \int_{t-\delta}^{t}\Psi_2(v)X(v-2\delta)dv\end{aligned}$$

for $t\in[T_0+\zeta+2\delta, T_1]$. By repeating this argument, we have

$$\int_{t-\delta}^{t}P(v)X(v-\delta)dv \ge \sum_{l=1}^{n}X(t-l\delta)\int_{t-\delta}^{t}\Psi_l(v)dv + \int_{t-\delta}^{t}\Psi_{n+1}(v)X(v-(n+1)\delta)dv$$

for $t\in[T_0+\zeta+(n+1)\delta, T_1]$. In view of $X(t)>0$ for $t\in[T_0+\zeta_1, T_1]$ and $\zeta \ge \zeta_1+\delta$, it follows that

$$\int_{t-\delta}^{t}P(v)X(v-\delta)dv \ge \sum_{l=1}^{n}X(t-l\delta)\int_{t-\delta}^{t}\Psi_l(v)dv \tag{2.14}$$

for $t \in [T_0 + \zeta + (n+1)\delta, T_1]$. Clearly,

$$X(t - l\delta) = \left(\prod_{i=2}^{l} \frac{X(t - i\delta)}{X(t - (i-1)\delta)} \right) X(t - \delta), \qquad l = 1, 2, \ldots. \tag{2.15}$$

It is clear for $t \in [T_0 + \zeta + (n+1)\delta, T_1]$, that $t - (i-1)\delta \in [T_0 + \zeta + (n+2-i)\delta, T_1 - (i-1)\delta]$. It follows from (2.4) that

$$\frac{X(t - i\delta)}{X(t - (i-1)\delta)} \geq \Upsilon_{n+2-i}(\eta) \qquad \text{for } t \in [T_0 + \zeta + (n+1)\delta, T_1].$$

This together with (2.14) and (2.15) implies that

$$\int_{t-\delta}^{t} P(v)X(v-\delta)dv \geq X(t-\delta) \sum_{l=1}^{n} \prod_{i=2}^{l} \Upsilon_{n+2-i}(\eta) \int_{t-\delta}^{t} \Psi_l(v)dv$$

for $t \in [T_0 + \zeta + (n+1)\delta, T_1]$. Substituting into (2.13), we obtain

$$X(t) + X(t-\delta) \left(1 - \sum_{l=1}^{n} \prod_{i=2}^{l} \Upsilon_{n+2-i}(\eta) \int_{t-\delta}^{t} \Psi_l(v)dv \right) \leq 0 \quad \text{for } t \in [T_0 + \zeta + (n+1)\delta, T_1],$$

contradicting (2.12). The proof is complete. □

Let

$$\begin{aligned} &\Phi_0(t) = P(t), \qquad t \geq T_0 + \zeta, \\ &\Phi_n(t) = \Phi_{n-1}(t) e^{\int_{t-\delta}^{t} \Phi_{n-1}(v)dv} \int_{t-\delta}^{t} \Phi_{n-1}(v)dv, \qquad t \geq T_0 + \zeta + n\delta, \quad n = 1, 2, \ldots \end{aligned} \tag{2.16}$$

Lemma 2.3 *Assume that*

$$\int_{t-\delta}^{t} \Phi_n(v)dv \geq 1 \quad \text{for } t \geq T_0 + \zeta + (n+1)\delta, \tag{2.17}$$

where $n \in \mathbb{N}$ *and* $\Phi_n(t)$ *is defined by* (2.16). *If there exists a function* $X(t)$ *satisfying* (1.2) *on* $[T_0 + \zeta,\ T_1]$ *such that* $X'(t) \leq 0$ *on* $[T_0 + \zeta_2,\ T_0 + \zeta]$ *and* $X(t) > 0$ *on* $[T_0 + \zeta_1, T_1]$, *then* $X(t)$ *cannot be positive on* $[T_0 + \zeta,\ T_1]$, $T_1 \geq T_0 + \zeta + (n+1)\delta$.

Proof Assume the contrary, i.e. $X(t) > 0$ on $[T_0 + \zeta,\ T_1]$, $T_1 \geq T_0 + \zeta + (n+1)\delta$. Integrating (1.2) from $t - \delta$ to t, we have

$$X(t) - X(t-\delta) + \int_{t-\delta}^{t} P(v)X(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + \delta, T_1].$$

Multiplying by $P(t)$ and using (1.2), we have

$$X'(t) + P(t)X(t) + P(t) \int_{t-\delta}^{t} P(v)X(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + \delta, T_1].$$

Let $Z_1(t) = e^{\int_{T_0+\zeta}^{t} P(v)dv} X(t)$, $t \geq T_0 + \zeta$. Then $Z_1(t) > 0$ on $[T_0 + \zeta, T_1]$. Since

$$(X'(t) + P(t)X(t))\, \mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} + P(t)\mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} \int_{t-\delta}^{t} P(v)X(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + \delta, T_1],$$

we see that

$$Z_1'(t) + P(t)\mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} \int_{t-\delta}^{t} P(v)X(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + \delta, T_1].$$

In view of (2.5) from the proof of Lemma 2.1, $X'(t) \leq 0$ on $[T_0 + \zeta_2, T_1]$. Hence,

$$Z_1'(t) = \mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} (X'(t) + P(t)X(t)) \leq \mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} (X'(t) + P(t)X(t-\delta)) \leq 0$$

for $t \in [T_0 + \zeta, T_1]$, and

$$Z_1'(t) + P(t)\mathrm{e}^{\int_{T_0+\zeta}^{t} P(v)dv} X(t-\delta) \int_{t-\delta}^{t} P(v)dv \leq 0 \text{ for } t \in [T_0 + \zeta + \delta, T_1].$$

Then

$$Z_1'(t) \leq 0 \quad \text{ for } t \in [T_0 + \zeta, T_1] \tag{2.18}$$

and

$$Z_1'(t) + \Phi_1(t)Z_1(t-\delta) \leq 0 \quad \text{ for } t \in [T_0 + \zeta + \delta, T_1]. \tag{2.19}$$

Integrating (2.19) from $t - \delta$ to t, we get

$$Z_1(t) - Z_1(t-\delta) + \int_{t-\delta}^{t} \Phi_1(v)Z_1(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + 2\delta, T_1].$$

Multiplying both sides of the above inequality by $\Phi_1(t)$ and using (2.19), we have

$$Z_1'(t) + \Phi_1(t)Z_1(t) + \Phi_1(t) \int_{t-\delta}^{t} \Phi_1(v)Z_1(v-\delta)dv \leq 0 \text{ for } t \in [T_0 + \zeta + 2\delta, T_1].$$

Let

$$Z_2(t) = e^{\int_{T_0+\zeta+\delta}^{t} \Phi_1(v)dv} Z_1(t) \quad \text{for } t \geq T_0 + \zeta + \delta.$$

In view of (2.18) and (2.19), it follows that

$$Z_2'(t) = \mathrm{e}^{\int_{T_0+\zeta+\delta}^{t} \Phi_1(v)dv} (Z_1'(t) + \Phi_1(t)Z_1(t)) \leq \mathrm{e}^{\int_{T_0+\zeta+\delta}^{t} \Phi_1(v)dv} (Z_1'(t) + \Phi_1(t)Z_1(t-\delta)) \leq 0$$

for $t \in [T_0 + \zeta + \delta, T_1]$, and

$$Z_2'(t) + \Phi_1(t)e^{\int_{T_0+\zeta+\delta}^{t} \Phi_1(v)dv} \int_{t-\delta}^{t} \Phi_1(v)Z_1(v-\delta)dv \text{ for } t \in [T_0 + \zeta + 2\delta, T_1].$$

Then

$$Z_2'(t) \leq 0 \qquad \text{for } t \in [T_0 + \zeta + \delta, T_1]$$

and

$$Z_2'(t)+\Phi_2(t)Z_2(t-\delta)\le 0 \text{ for } t\in[T_0+\zeta+2\delta,T_1].$$

By induction, we obtain

$$Z_n'(t)+\Phi_n(t)Z_n(t-\delta)\le 0 \quad \text{for } t\in[T_0+\zeta+n\delta,T_1] \tag{2.20}$$

and

$$Z_n'(t)\le 0 \quad \text{for } t\in[T_0+\zeta+n\delta,T_1],$$

where

$$Z_n(t)=e^{\int_{T_0+\zeta+(n-1)\delta}^{t}\Phi_{n-1}(v)dv}Z_{n-1}(t) \quad \text{for } t\ge T_0+\zeta+(n-1)\delta$$

and $Z_n(t)>0$ on $[T_0+\zeta+(n-1)\delta,T_1]$. Integrating (2.20) from $t-\delta$ to t, we get

$$Z_n(t)-Z_n(t-\delta)+\int_{t-\delta}^{t}\Phi_n(v)Z_n(v-\delta)dv\le 0 \quad \text{for } t\in[T_0+\zeta+(n+1)\delta,T_1].$$

Therefore,

$$Z_n(t)+\left(\int_{t-\delta}^{t}\Phi_n(v)dv-1\right)Z_n(t-\delta)\le 0 \quad \text{for } t\in[T_0+\zeta+(n+1)\delta,T_1].$$

This contradiction completes the proof. □

Let

$$\begin{aligned}
&\Omega_0(t)=P(t) && \text{for } t\ge T_0+\zeta,\\
&\Omega_n(t)=\Omega_{n-1}(t)\int_{t-\delta}^{t}\Omega_{n-1}(v)e^{\int_{v-\delta}^{t}\Omega_{n-1}(v_1)dv_1}dv && \text{for } t\ge T_0+\zeta+2n\delta,\quad n=1,2,\dots.
\end{aligned} \tag{2.21}$$

Lemma 2.4 *Assume that*

$$\begin{aligned}
&\int_{t-\delta}^{t}\Omega_1(v)e^{\int_{v-\delta}^{t-\delta}\Phi_1(v_1)dv_1}dv\ge 1 && \textit{for } n=1\\
&\int_{t-\delta}^{t}\Omega_n(v)dv\ge 1 && \textit{for } n=2,3,\dots,
\end{aligned} \tag{2.22}$$

where $n\in\mathbb{N}$, $\Phi_1(t)$ *and* $\Omega_n(t)$ *are defined by* (2.16) *and* (2.21), *respectively. If there exists a function* $X(t)$ *satisfying* (1.2) *on* $[T_0+\zeta,\,T_1]$ *such that* $X'(t)\le 0$ *on* $[T_0+\zeta_2,\,T_0+\zeta]$ *and* $X(t)>0$ *on* $[T_0+\zeta_1,T_1]$, *then* $X(t)$ *cannot be positive on* $[T_0+\zeta,\,T_1]$, $T_1\ge T_0+\zeta+3n\delta$.

Proof Assume, for the sake of contradiction, that $X(t)$ is positive on $[T_0+\zeta,\,T_1]$, $T_1\ge T_0+\zeta+3n\delta$. Then $X'(t)\le 0$ for $t\in[T_0+\zeta_2,\,T_1]$. Integrating (1.2) from $t-\delta$ to t, we have

$$X(t)-X(t-\delta)+\int_{t-\delta}^{t}P(v)X(v-\delta)dv\le 0 \qquad \text{for } t\in[T_0+\zeta+\delta,T_1].$$

Multiplying by $P(t)$ and using (1.2), we have

$$X'(t) + X(t)P(t) + P(t)\int_{t-\delta}^{t} P(v)X(v-\delta)dv \le 0 \qquad \text{for } t \in [T_0+\zeta+\delta, T_1]. \tag{2.23}$$

Let $B_1(t) = e^{\int_{T_0+\zeta}^{t} P(v)dv} X(t),\ t \ge T_0+\zeta$. Then $B_1(t) > 0$ on $[T_0+\zeta, T_1]$. In view of (1.2), $X'(t) \le 0$ on $[T_0+\zeta_2, T_1]$, it follows that

$$B_1'(t) = (X'(t) + P(t)X(t))\, e^{\int_{T_0+\zeta}^{t} P(v)dv} \le (X'(t) + P(t)X(t-\delta))\, e^{\int_{T_0+\zeta}^{t} P(v)dv} \le 0$$

for $t \in [T_0+\zeta, T_1]$. Therefore, $B_1'(t) \le 0$ for $t \in [T_0+\zeta, T_1]$. Substituting into (2.23), we have

$$B_1'(t) + P(t)\int_{t-\delta}^{t} \mathrm{e}^{\int_{v-\delta}^{t} P(v_1)dv_1} P(v)B_1(v-\delta)dv \le 0 \qquad \text{for } t \in [T_0+\zeta+2\delta, T_1].$$

Since $B_1'(t) \le 0$ for $t \in [T_0+\zeta, T_1]$. Then

$$B_1'(t) + \Omega_1(t)B_1(t-\delta) \le 0 \qquad \text{for } t \in [T_0+\zeta+2\delta, T_1]. \tag{2.24}$$

Integrating form $t-\delta$ to t, we get

$$B_1(t) - B_1(t-\delta) + \int_{t-\delta}^{t} \Omega_1(v)B_1(v-\delta)dv \le 0 \quad \text{for } t \in [T_0+\zeta+3\delta, T_1]. \tag{2.25}$$

It is clear that $B_1(t)$ is the same as $Z_1(t)$ in the proof of Lemma 2.3. By using $B_1'(t) \le 0$ for $t \in [T_0+\zeta, T_1]$ and (2.19), we have

$$B_1'(t) + \Phi_1(t)B_1(t) \le 0 \text{ for } t \in [T_0+\zeta+\delta, T_1].$$

Then

$$-\frac{B_1'(t)}{B_1(t)} \ge \Phi_1(t) \quad \text{for } t \in [T_0+\zeta+\delta, T_1].$$

Integrating the above inequality from $v-\delta$ to $t-\delta$, $t-\delta \le v \le t$ for $t \in [T_0+\zeta+3\delta, T_1]$, we obtain

$$B_1(v-\delta) \ge B_1(t-\delta)e^{\int_{v-\delta}^{t-\delta} \Phi_1(v_1)dv_1} \qquad \text{for } t \in [T_0+\zeta+3\delta, T_1].$$

Substituting into (2.25), we get

$$B_1(t) + \left[\int_{t-\delta}^{t} \Omega_1(v)e^{\int_{v-\delta}^{t-\delta} \Phi_1(v_1)dv_1}dv - 1\right] B_1(t-\delta) \le 0 \quad \text{for } t \in [T_0+\zeta+3\delta, T_1].$$

This contradiction completes the proof for $n = 1$.

Again multiplying both sides of (2.25) by $\Omega_1(t)$ and then using (2.24), we get

$$B_1'(t) + \Omega_1(t)B_1(t) + \Omega_1(t)\int_{t-\delta}^{t} \Omega_1(v)B_1(v-\delta)dv \le 0 \quad \text{for } t \in [T_0+\zeta+3\delta, T_1].$$

Put $B_2(t)=e^{\int_{T_0+\zeta+2\delta}^{t}\Omega_1(v)dv}B_1(t)$, $t\ge T_0+\zeta+2\delta$, so $B_2(t)>0$ on $[T_0+\zeta+2\delta,T_1]$. Therefore,

$$B_2'(t)+\Omega_1(t)e^{\int_{T_0+\zeta+2\delta}^{t}\Omega_1(v)dv}\int_{t-\delta}^{t}\Omega_1(v)B_1(v-\delta)dv\le 0 \quad \text{for } t\in[T_0+\zeta+4\delta,T_1]. \tag{2.26}$$

Then $B_2'(t)\le 0$ for $t\in[T_0+\zeta+3\delta,T_1]$. This together with (2.26) leads to

$$B_2'(t)+B_2(t-\delta)\Omega_1(t)\int_{t-\delta}^{t}\Omega_1(v)e^{\int_{v-\delta}^{t}\Omega_1(v_1)dv_1}dv\le 0 \quad \text{for } t\in[T_0+\zeta+5\delta,T_1],$$

that is

$$B_2'(t)+\Omega_2(t)B_2(t-\delta)\le 0 \qquad \text{for } t\in[T_0+\zeta+5\delta,T_1].$$

Repeating this arguments n times, we have

$$B_n'(t)+\Omega_n(t)B_n(t-\delta)\le 0 \qquad \text{for } t\in[T_0+\zeta+(3n-1)\delta,T_1], \tag{2.27}$$

where $B_n(t)=e^{\int_{T_0+\zeta+n\delta}^{t}\Omega_1(v)dv}B_{n-1}(t)$ for $t\ge T_0+\zeta+n\delta$ and $B_n'(t)\le 0$ for $t\in[T_0+\zeta+(3n-3)\delta,T_1]$. Integrating (2.27) from $t-\delta$ to t, we get

$$B_n(t)-B_n(t-\delta)+\int_{t-\delta}^{t}\Omega_n(v)B_n(v-\delta)dv\le 0 \quad \text{for } t\in[T_0+\zeta+3n\delta,T_1].$$

Using the nonincreasing nature of $B_n(t)$ on $[T_0+\zeta+(3n-3)\delta,T_1]$, we obtain

$$B_n(t)+\left[\int_{t-\delta}^{t}\Omega_n(v)dv-1\right]B_n(t-\delta)\le 0 \quad \text{for } t\in[T_0+\zeta+3n\delta,T_1].$$

This contradicts with (2.22). The proof is complete. □

3. Main results

In this section, we obtain new estimates for the upper bounds for the distance between zeros of all solutions of Eq. (*E*). Let $F\in C^1([t_1,\infty),[0,\infty))$, $t_1\ge t_0+\sigma$ and D_{t_1} be the least upper bound of all distances between adjacent zeros of a solution of Eq. (*E*) on $[t_1,\infty]$. We will consider the following conditions to be held:

(A1) $F(t)\ge \frac{p(t-\sigma)q(t)}{q(t-\tau)}$ for $t\ge t_1+\tau+\sigma$;

(A2) $q(t)\ge |F'(t)|$ for $t\ge t_1+\tau+\sigma$.

The proof of the following two lemmas can be found, with minor modifications, in [4, 16].

Lemma 3.1 *Assume that (A1) is satisfied and $F'(t)\le 0$ for $t\ge t_1+\tau+\sigma$. If $x(t)$ is a positive solution of Eq. (E) on $[t_1,t_2]$, $t_2\ge t_1+2\sigma$, then there exists a solution $V(t)$ of the inequality*

$$V'(t)+\frac{q(t)}{1+F(t)}V(t-(\sigma-\tau))<0 \quad \text{for } t\in[t_1+2\sigma,\,t_2],$$

such that $V(t)>0$ for $t\in[t_1+2\tau,\,t_2]$ and $V'(t)<0$ for $t\in[t_1+\sigma+\tau,\,t_2]$.

Proof Assume that

$$W(t) = x(t) + p(t)x(t-\tau) \quad \text{for } t \in [t_1+\tau,\, t_2],$$

and hence $W(t) > 0$ for $t \in [t_1+\tau,\, t_2]$. In view of (E), it follows that

$$W'(t) = [x(t) + p(t)x(t-\tau)]' = -q(t)x(t-\sigma) \quad \text{for } t \in [t_1+\tau,\, t_2].$$

Therefore, $W'(t) < 0$ for $[t_1+\sigma,\, t_2]$ and

$$W'(t) = -q(t)x(t-\sigma) = -q(t)[W(t-\sigma) - p(t-\sigma)x(t-\tau-\sigma)] \tag{3.1}$$

for $t \in [t_1+\tau+\sigma,\, t_2]$. Clearly,

$$x(t-\tau-\sigma) = \frac{-1}{q(t-\tau)}W'(t-\tau) \quad \text{for } t \in [t_1+2\tau,\, t_2+\tau].$$

Substituting into (3.1), we get

$$\begin{aligned} W'(t) &= -q(t)[W(t-\sigma) + \frac{p(t-\sigma)}{q(t-\tau)}W'(t-\tau)] \\ &= -q(t)W(t-\sigma) - \frac{q(t)p(t-\sigma)}{q(t-\tau)}W'(t-\tau) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \end{aligned}$$

Then

$$W'(t) + q(t)W(t-\sigma) + \frac{q(t)p(t-\sigma)}{q(t-\tau)}W'(t-\tau) = 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

In view (A1) and $W'(t) < 0$ for $[t_1+\sigma,\, t_2]$, it follows that

$$W'(t) + F(t)W'(t-\tau) + q(t)W(t-\sigma) \le 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \tag{3.2}$$

Let

$$G(t) = W(t) + F(t)W(t-\tau) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

Therefore,

$$G'(t) = W'(t) + F'(t)W(t-\tau) + F(t)W'(t-\tau) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \tag{3.3}$$

From (3.2), we have

$$W'(t) + F(t)W'(t-\tau) \le -q(t)W(t-\sigma) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

This together with (3.3) and $F'(t) \le 0$ for $t \ge [t_1+\tau+\sigma,\, t_2]$ leads to

$$G'(t) \le F'(t)W(t-\tau) - q(t)W(t-\sigma) \le 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2], \tag{3.4}$$

that is

$$G'(t) - F'(t)W(t-\tau) + q(t)W(t-\sigma) \le 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \tag{3.5}$$

In view of $W'(t) < 0$ on $[t_1+\sigma,\, t_2]$, it follows that

$$G(t) = W(t) + F(t)W(t-\tau) < W(t-\tau) + F(t)W(t-\tau) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

Consequently,

$$W(t-\tau) > \frac{G(t)}{1+F(t)} \qquad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \tag{3.6}$$

Therefore,

$$W(t) > \frac{G(t+\tau)}{1+F(t+\tau)} \qquad \text{for } t \in [t_1+\sigma,\, t_2-\tau].$$

Then

$$W(t-\sigma) > \frac{G(t+\tau-\sigma)}{1+F(t+\tau-\sigma)} \qquad \text{for } t \in [t_1+2\sigma,\, t_2-\tau+\sigma]. \tag{3.7}$$

Substituting from (3.6) and (3.7) into (3.5), we have

$$G'(t) - \frac{F'(t)}{1+F(t)}G(t) + \frac{q(t)}{1+F(t+\tau-\sigma)}G(t+\tau-\sigma) < 0 \quad \text{for } t \in [t_1+2\sigma,\, t_2]. \tag{3.8}$$

Let $V(t) = e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} G(t)$ for $t \geq t_1$. Then $V(t) > 0$ for $t \in [t_1+2\tau,\, t_2]$ and

$$\begin{aligned} V'(t) &= G'(t)e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} - \frac{F'(t)}{1+F(t)}e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)dv}}G(t) \\ &= e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} \frac{G'(t)\left(1+F(t)\right) - F'(t)G(t)}{1+F(t)} \end{aligned}$$

for $t \geq t_1+\tau+\sigma$. From this, (3.4) and (3.6), we have

$$\begin{aligned} V'(t) &< e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} \frac{\left[F'(t)W(t-\tau) - q(t)W(t-\sigma)\right]\left(1+F(t)\right) - \left(1+F(t)\right)F'(t)W(t-\tau)}{1+F(t)} \\ &= -e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} q(t)W(t-\sigma) < 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2]. \end{aligned}$$

Using the transformation $V(t) = e^{-\int_{t_1}^{t} \frac{F'(v)}{1+F(v)}dv} G(t)$, inequality (3.8) becomes

$$V'(t) + \frac{q(t)}{1+F(t+\tau-\sigma)} e^{-\int_{t+\tau-\sigma}^{t} \frac{F'(v)}{1+F(v)}dv} V(t+\tau-\sigma) < 0 \quad \text{for } t \in [t_1+2\sigma,\, t_2],$$

that is

$$V'(t) + \frac{q(t)}{1+F(t)} V(t-(\sigma-\tau)) < 0 \quad \text{for } t \in [t_1+2\sigma,\, t_2],$$

where $V(t) > 0$ for $t \in [t_1+2\tau,\, t_2]$ and $V'(t) < 0$ for $t \in [t_1+\tau+\sigma,\, t_2]$. The proof is complete. □

Lemma 3.2 *Assume that* $(A1)-(A2)$ *are satisfied. If* $x(t)$ *is a positive solution of Eq.* (E) *on* $[t_1,\, t_2]$, $t_2 \geq t_1 + 2\sigma$, *then there exists a solution* $G(t)$ *of the inequity*

$$G'(t) + \frac{q(t) - |F'(t)|}{1+F(t+\tau-\sigma)} G(t-(\sigma-\tau)) < 0 \quad \textit{for } t \in [t_1+2\sigma,\, t_2],$$

such that $G(t) > 0$ *for all* $t \in [t_1+2\tau,\, t_2]$ *and* $G'(t) < 0$ *for all* $t \in [t_1+\tau+\sigma,\, t_2]$.

Proof Let

$$W(t) = x(t) + p(t)x(t-\tau) \qquad \text{for } t \in [t_1+\tau,\, t_2]$$

and

$$G(t) = W(t) + F(t)W(t-\tau) \qquad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

Therefore, $G(t) \geq 0$ on $[t_1+2\tau,\, t_2]$. From (3.5), we have

$$G'(t) - F'(t)W(t-\tau) + q(t)W(t-\sigma) \leq 0 \quad \text{for } t \geq [t_1+\tau+\sigma,\, t_2],$$

where $W'(t) < 0$ for $t \in [t_1+\sigma,\, t_2]$. Therefore,

$$G'(t) \leq F'(t)W(t-\tau) - q(t)W(t-\sigma) \leq |F'(t)|\, W(t-\tau) - q(t)W(t-\sigma) \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

Then

$$G'(t) < -\left(q(t) - |F'(t)|\right) W(t-\sigma) \leq 0 \quad \text{for } t \in [t_1+\tau+\sigma,\, t_2].$$

Substituting from (3.7), we have

$$G'(t) + \frac{q(t) - |F'(t)|}{1 + F(t+\tau-\sigma)} G(t-(\sigma-\tau)) < 0 \quad \text{for } t \in [t_1+2\sigma,\, t_2].$$

This completes the proof. □

Theorem 3.3 *Let $F'(t) \leq 0$ for $t \geq t_1+\tau+\sigma$. Assume that (2.1), (A1) and (2.12) are satisfied with $\delta = \sigma - \tau$ and*

$$P(t) = \frac{q(t)}{1+F(t)} \qquad t \geq t_1+\sigma+\tau.$$

Then Eq. (E) is oscillatory and $D_{t_1}(x) \leq (n+3)\sigma - (n+1)\tau$.

Proof Assume the contrary, i.e. there exists a solution $x(t)$ of Eq. (E) such that $x(t) > 0$ on $[T_0,\, T_1]$, $T_0 \geq t_1$ where $T_1 \geq T_0 + (n+3)\sigma - (n+1)\tau$. In view of Lemma 3.1, there exists a solution $V(t)$ of the inequality

$$V'(t) + \frac{q(t)}{1+F(t)} V(t-(\sigma-\tau)) < 0 \quad \text{for } t \in [T_0+2\sigma,\, T_1],$$

where $V(t) > 0$ for $t \in [T_0+2\tau,\, T_1]$ and $V'(t) < 0$ for $t \in [T_0+\sigma+\tau,\, T_1]$. Therefore, one can assume in Lemma 2.2 that $\zeta = 2\sigma$, $\zeta_1 = 2\tau$ and $\delta = \sigma - \tau$. Clearly,

$$\zeta + (n+1)\delta = (n+3)\sigma - (n+1)\tau.$$

Applying Lemma 2.2, then $V(t)$ cannot be positive on $[T_0+2\sigma,\, T_1]$, $T_1 \geq T_0 + (n+3)\sigma - (n+1)\tau$. This contradiction completes the proof. □

Theorem 3.4 *Let $F'(t) \leq 0$ for $t \geq t_1+\tau+\sigma$. Assume that (A1) and (2.17) are satisfied with $\delta = \sigma - \tau$ and*

$$P(t) = \frac{q(t)}{1+F(t)} \qquad t \geq t_1+\sigma+\tau.$$

Then Eq. (E) is oscillatory and $D_{t_1}(x) \leq (n+3)\sigma - (n+1)\tau$.

Proof Assume that there exists a solution $x(t)$ of Eq. (E) such that $x(t) > 0$ on $[T_0, T_1]$, $T_0 \geq t_1$ where $T_1 \geq T_0 + (n+3)\sigma - (n+1)\tau$. In view of Lemma 3.1, there exists a solution $V(t)$ of the inequality

$$V'(t) + \frac{q(t)}{1+F(t)} V(t - (\sigma - \tau)) < 0 \quad \text{ for } t \in [T_0 + 2\sigma, T_1],$$

where $V(t) > 0$ for $t \in [T_0 + 2\tau, T_1]$ and $V'(t) < 0$ for $t \in [T_0 + \sigma + \tau, T_1]$. If we assume in Lemma 2.3 that $\zeta = 2\sigma$, $\zeta_1 = 2\tau$, $\zeta_2 = \sigma + \tau$ and $\delta = \sigma - \tau$. It is clear that

$$\zeta + (n+1)\delta = (n+3)\sigma - (n+1)\tau.$$

Applying Lemma 2.3, then $V(t)$ can not be positive on $[T_0 + 2\sigma, T_1]$. This contradiction completes the proof. □

Theorem 3.5 *Let $F'(t) \leq 0$ for $t \geq t_1 + \tau + \sigma$. Assume that $(A1)$ and (2.22) are satisfied with $\delta = \sigma - \tau$ and*

$$P(t) = \frac{q(t)}{1+F(t)} \qquad t \geq t_1 + \sigma + \tau.$$

Then Eq. (E) is oscillatory and $D_{t_1}(x) \leq (3n+2)\sigma - 3n\tau$.

Proof Assume that there exists a solution $x(t)$ of Eq. (E) such that $x(t) > 0$ on $[T_0, T_1]$, $T_0 \geq t_1$ where $T_1 \geq T_0 + (3n+2)\sigma - 3n\tau$. In view of Lemma 3.1, there exists a solution $V(t)$ of the inequality

$$V'(t) + \frac{q(t)}{1+F(t)} V(t - (\sigma - \tau)) < 0 \quad \text{ for } t \in [T_0 + 2\sigma, T_1],$$

where $V(t) > 0$ for $t \in [T_0 + 2\tau, T_1]$ and $V'(t) < 0$ for $t \in [T_0 + \sigma + \tau, T_1]$. Therefore, one can assume in Lemma 2.4 that $\zeta = 2\sigma$, $\zeta_1 = 2\tau$, $\zeta_2 = \sigma + \tau$ and $\delta = \sigma - \tau$. Then

$$\zeta + 3n\delta = (3n+2)\sigma - 3n\tau.$$

Applying Lemma 2.4, then $V(t)$ cannot be positive on $[T_0 + 2\sigma, T_1]$. This contradiction completes the proof. □

Theorem 3.6 *Assume that (2.1), $(A1) - (A2)$ and (2.12) are satisfied with $\delta = \sigma - \tau$ and*

$$P(t) = \frac{q(t) - |F'(t)|}{1 + F(t + \tau - \sigma)}, \qquad t \geq t_1 + \sigma + \tau.$$

Then Eq. (E) is oscillatory and $D_{t_1}(x) \leq (n+3)\sigma - (n+1)\tau$.

Proof Assume the contrary, i.e. there exists a solution $x(t)$ of Eq. (E) such that $x(t) > 0$ on $[T_0, T_1]$, $T_0 \geq t_1$ where $T_1 \geq T_0 + (n+3)\sigma - (n+1)\tau$. In view of Lemma 3.2, there exists a solution $G(t)$ of the inequality

$$G'(t) + \frac{q(t) - |F'(t)|}{1 + F(t + \tau - \sigma)} G(t - (\sigma - \tau)) < 0 \quad \text{ for } t \in [T_0 + 2\sigma, T_1],$$

where $G(t) > 0$ for $t \in [T_0 + 2\tau, T_1]$ and $G'(t) < 0$ for $t \in [T_0 + \sigma + \tau, T_1]$. By using Lemma 2.2, then $G(t)$ cannot be positive on $[T_0 + 2\sigma, T_1]$. This contradiction completes the proof. □

Theorem 3.7 *Assume that* $(A1)-(A2)$ *and* (2.17) *are satisfied with* $\delta=\sigma-\tau$ *and*

$$P(t)=\frac{q(t)-|F'(t)|}{1+F(t+\tau-\sigma)}, \qquad t\geq t_1+\sigma+\tau.$$

Then Eq. (E) *is oscillatory and* $D_{t_1}(x)\leq(3+n)\sigma-(n+1)\tau$.

Proof Assume that there exists a solution $x(t)$ of Eq. (E) such that $x(t)>0$ on $[T_0,T_1]$, $T_0\geq t_1$ where $T_1\geq T_0+(n+3)\sigma-(n+1)\tau$. In view of Lemma 3.2, there exists a solution $G(t)$ of the inequality

$$G'(t)+\frac{q(t)-|F'(t)|}{1+F(t+\tau-\sigma)}G(t-(\sigma-\tau))<0 \quad \text{for } t\in[T_0+2\sigma,T_1],$$

where $G(t)>0$ for $t\in[T_0+2\tau,T_1]$ and $G'(t)<0$ for $t\in[T_0+\sigma+\tau,T_1]$. Applying Lemma 2.3, then $G(t)$ can not be positive on $[T_0+2\sigma,T_1]$. This contradiction completes the proof. □

Theorem 3.8 *Assume that* $(A1)-(A2)$ *and* (2.22) *are satisfied with* $\delta=\sigma-\tau$ *and*

$$P(t)=\frac{q(t)-|F'(t)|}{1+F(t+\tau-\sigma)}, \qquad t\geq t_1+\sigma+\tau.$$

Then Eq. (E) *is oscillatory and* $D_{t_1}(x)\leq(3n+2)\sigma-3n\tau$.

Proof Assume that there exists a solution $x(t)$ of Eq. (E) such that $x(t)>0$ on $[T_0,T_1]$, $T_0\geq t_1$ where $T_1\geq T_0+(3n+2)\sigma-3n\tau$. In view of Lemma 3.2, there exists a solution $G(t)$ of the inequality

$$G'(t)+\frac{q(t)-|F'(t)|}{1+F(t+\tau-\sigma)}G(t-(\sigma-\tau))<0 \quad \text{for } t\in[T_0+2\sigma,T_1],$$

where $G(t)>0$ for $t\in[T_0+2\tau,T_1]$ and $G'(t)<0$ for $t\in[T_0+\sigma+\tau,T_1]$. Applying Lemma 2.4, then $G(t)$ cannot be positive on $[T_0+2\sigma,T_1]$. This contradiction completes the proof. □

Remark 3.9

It should be noted that our results improve many results from [16], [15] and [4]. For example, Lemma 2.1 improves [16, Lemma 2.1] and [4, Lemma 2.1]. Also, Lemma 2.4 improves [4, Lemma 2.4].

4. Numerical examples

Example 4.1 Consider the first-order neutral differential equation

$$[x(t)+p(t)x(t-\tau)]'+q(t)x(t-\sigma)=0 \qquad t\geq\frac{5\pi}{2}, \tag{4.1}$$

where $\tau=\frac{3\pi}{2}$, $\sigma=\frac{5\pi}{2}$, $p(t)=\frac{1.1-\sin(t)}{1.1+\cos(t)}$, $q(t)=\alpha(1.1+\sin(t))$, and $\alpha=\frac{139}{200}$. Observe that

$$F(t)=p(t-\sigma)\frac{q(t)}{q(t-\tau)} \quad = \quad \frac{1.1-\sin\left(t-\frac{5\pi}{2}\right)}{1.1+\cos\left(t-\frac{5\pi}{2}\right)}\frac{\alpha(1.1+\sin t)}{\alpha\left(1.1+\sin\left(t-\frac{3\pi}{2}\right)\right)}=1.$$

Let $\delta = \sigma - \tau = \pi$ and

$$P(t) = \frac{q(t)}{1+F(t)} = \frac{\alpha}{2}\left(1.1 + \sin(t)\right).$$

Clearly,

$$\int_{t-\delta}^{t} P(v)dv = \alpha\left(\frac{11}{20}\pi - \cos t\right) \geq \alpha\left(\frac{11}{20}\pi - 1\right) \quad \text{for all } t \geq \delta.$$

Therefore,

$$\begin{aligned}
&\int_{t-\delta}^{t} P(v)e^{\int_{v-\delta}^{v} P(v_1)dv_1} dv \int_{v-\delta}^{v} P(v_1)dv_1 \\
&\geq e\int_{t-\delta}^{t} P(v)\left(\int_{v-\delta}^{v} P(v_1)dv_1\right)^2 dv \\
&= \alpha^3 e\left(\frac{-1}{3}\cos^3(t) - \frac{121\pi^2}{400}\cos(t) - \frac{121\pi}{100}\sin(t) + \frac{1331\,\pi^3}{8000} + \frac{11\pi}{40}\right) > 1.
\end{aligned}$$

Consequently, condition (2.17) with $n=1$ is satisfied. Thus, according to Theorem (3.4), $D_{t_1}(x) \leq 4\sigma - 2\tau = 7\pi$. It is worth noting that the corresponding results from [4, 15, 16, 19] cannot give this estimation. For example, we shall show that according to [4, Theorem 3.2], the distance between adjacent zeros of all solutions of (4.1) is not greater than 8π. Let

$$\eta = \alpha\left(\frac{11}{20}\pi - 1\right)$$

and

$$f_0(\eta) = 1, \quad f_1(\eta) = \frac{1}{1-\eta}, \quad f_2(\eta) = \frac{1}{2-e^{\eta}}, \quad f_3(\eta) = \frac{f_1(\eta)}{1 - f_1(\eta) - e^{\eta f_1(\eta)}}.$$

Therefore, $\eta < 1$ and $0 < f_i(\eta) < +\infty$ for $i = 1, 2$ and $f_3(\eta) < 0$. Also,

$$\int_{t-\delta}^{t} P(v)dv + f_2(\eta)\int_{t-\delta}^{t} P(v-\delta)\int_{t-\delta}^{v} P(v_1)dv_1 dv > 1.$$

As an application of [4, Theorem 3.2] with $n=2$, we obtain $D_{t_1}(x) \leq 2\sigma + 3(\sigma - \tau) = 8\pi$.

Example 4.2 Consider the first-order neutral differential equation

$$[x(t) + x(t-1)]' + \alpha x(t-3) = 0 \qquad t \geq 1, \tag{4.2}$$

where $\alpha > 0$. Equation (4.2) is a particular case of (E) with $p(t) = 1$, $q(t) = \alpha$, $\tau = 1$ and $\sigma = 3$. It is clear that

$$F(t) = p(t-\sigma)\frac{q(t)}{q(t-\tau)} = 1.$$

Let $\delta = \sigma - \tau = 2$ and define

$$P(t) = \frac{q(t)}{1+F(t)} = \frac{\alpha}{2}.$$

Then

$$\int_{t-\delta}^{t} P(v)dv = \alpha \qquad \text{for all } t \geq \delta.$$

Thus, one can choose $\eta = \alpha$ (that is defined by (2.1)). Furthermore, using the computer-algebra software (e.g., Maple), we obtain

$$\int_{t-\delta}^{t} \Psi_1(v)dv = \alpha \qquad \text{and} \qquad \Upsilon_1(\eta) = \frac{1}{1-\alpha},$$

$$\int_{t-\delta}^{t} \Psi_2(v)dv = \frac{1}{2}\alpha^2 \qquad \text{and} \qquad \Upsilon_2(\eta) = \frac{1}{1-\alpha-\frac{1}{2(1-\alpha)}\alpha^2},$$

$$\int_{t-\delta}^{t} \Psi_3(v)dv = \frac{1}{6}\alpha^3 \qquad \text{and} \qquad \Upsilon_3(\eta) = \frac{1}{1-\alpha-\frac{\alpha^2}{2\left(1-\alpha-\frac{1}{2(1-\alpha)}\alpha^2\right)}},$$

$$\int_{t-\delta}^{t} \Psi_4(v)dv = \frac{1}{24}\alpha^4 \qquad \text{and} \qquad \Upsilon_4(\eta) = \frac{1}{1-\alpha-\frac{\alpha^2}{2\left(1-\alpha-\frac{\alpha^2}{2\left(1-\alpha-\frac{1}{2(1-\alpha)}\alpha^2\right)}\right)}},$$

where $\Upsilon_n(\eta)$ and $\Psi_n(v)$ are defined by (2.3) and (2.11), respectively. Consequently,

$$\sum_{l=1}^{4}\prod_{i=2}^{l}\Upsilon_{6-i}(\eta)\int_{t-\delta}^{t}\Psi_l(v)dv = \int_{t-\delta}^{t}\Psi_1(v)dv + \Upsilon_4(\eta)\int_{t-\delta}^{t}\Psi_2(v)dv + \Upsilon_4(\eta)\Upsilon_3(\eta)\int_{t-\delta}^{t}\Psi_3(v)dv$$
$$+ \Upsilon_4(\eta)\Upsilon_3(\eta)\Upsilon_2(\eta)\int_{t-\delta}^{t}\Psi_4(v)dv > 1$$

for $\alpha = 0.4367$. By Theorem 3.3 with $n = 4$, it is easy to see that $D_{t_1}(x) \leq 16$ for all $\alpha \geq 0.4367$. It is worth noting that none of the corresponding results of [4, 15, 16, 19] can give this estimation for such α. For example, [16, Theorem 3.1] and [4, Theorem 3.2] give, respectively $D_{t_1}(x) \leq 18$ and $D_{t_1}(x) \leq 22$ for all $\alpha \geq 0.4367$.

5. Conclusion

In this work, we obtained new upper bounds for the distance between adjacent zeros of all solutions of the first-order linear neutral differential equation (E). Our results essentially improve many known results in the literature which was illustrated via examples. The generality of the obtained results, especially in Section 2, leads to study the distance between zeros for many other functional differential equations, which is left for further research.

Competing interests

The authors declare no conflicts of interest.

References

[1] Agarwal RP, Berezansky L, Braverman E, Domoshnitsky A. Theory of Functional Differential Equations with Applications. Springer, 2012.

[2] Attia ER, Chatzarakis GE. Upper bounds for the distance between adjacent zeros of first-order linear differential equations with several delays. Mathematics 2022; 10 (648): 1-15. https://doi.org/10.3390/math100406481

[3] Bainov DD, Mishev DP. Oscillation Theory for Neutral Differential Equations with Delay. CRC Press, 1991.

[4] Baker FA, El-Morshedy HA. The distribution of zeros of all solutions of first order neutral differential equations. Applied Mathematics and Computation 2015; 259 (2015): 777-789. https://doi.org/10.1016/j.amc.2015.03.004.

[5] Baker FA, El-Morshedy HA. On the distribution of zeros of solutions of a first order neutral differential equation. Scientific Journal for Damietta Faculty of Science 2015; 4 (1): 1-9.

[6] Erbe LH, Kong QK, Zhang BG. Oscillation Theory for Functional Differential Equations. Dekker, New York, 1995.

[7] El-Morshedy HA. On the distribution of zeros of solutions of first order delay differential equations. Nonlinear Analysis: Theory, Methods and Applications 2011; 74 (10): 3353-3362. https://doi.org/10.1016/j.na.2011.02.011

[8] El-Morshedy HA, Attia ER. On the distance between adjacent zeros of solutions of first order differential equations with distributed delays. Electronic Journal of Qualitative Theory of Differential Equations 2016; 2016 (8): 1-14. https://doi.org/10.14232/ejqtde.2016.1.8

[9] Gopalsamy K. Stability and Oscillation in Delay Differential Equations of Population Dynamics. Kluwer Academic Publishers, Dordrecht, 1992.

[10] Györi I, Ladas G. Oscillation Theory of Delay Differential Equations with Applications. Clarendon Press, Oxford, 1991.

[11] Kolmanovskii V, Myshkis A. Introduction to the Theory and Applications of Functional Differential Equations. Kluwer Academic Publishers, Netherlands, 1999.

[12] Kolmanovskii V, Nosov V.R. Stability of Functional Differential Equations. Academic Press, London, 1986.

[13] Kuang Y. Delay Differential Equations with Applications in Population Dynamics, in: Mathematics in Science and Engineering. Academic Press, Boston, MA, 1993.

[14] Murray JD, Mathematical Biology I: An Introduction, Interdisciplinary Applied Mathematics. Springer, New York, NY, USA, 2002.

[15] Wu HW, Cheng SS, Wang QR. Distribution of zeros of solutions of functional differential equations. Applied Mathematics and Computation 2007; 193 (1): 154-161. https://doi.org/10.1016/j.amc.2007.03.081

[16] Wu HW, Xu YT. The distribution of zeros of solutions of neutral differential equations. Applied Mathematics and Computation 2004; 156 (3): 665-677. https://doi.org/10.1016/j.amc.2003.08.026

[17] Xianhua T, Jianshe Y. Distribution of zeros of solutions of first order delay differential equations. Applied Mathematics-A Journal of Chinese Universities 1999; 14 (1999): 375-380. https://doi.org/10.1007/s11766-999-0066-2

[18] Yong Z, Zhicheng W. The distribution of zeros of solutions of neutral equations. Applied Mathematics and Mechanics 1997; 18 (1997): 1197-1204. doi: https://doi.org/10.1007/BF00713722

[19] Zhou Y. The distribution of zeros of solutions of neutral differential equations, Hiroshima Mathematical Journal 1999; 29 : 361-370.