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On the band functions and Bloch functions

Oktay VELİEV*
Department of Mechanical Engineering, Doğuş University, İstanbul, Turkey

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Abstract: In this paper, we consider the continuity of the band functions and Bloch functions of the differential operators generated by the differential expressions with periodic matrix coefficients.

Key words: Differential operator, band functions, Bloch functions

1. Introduction

In this paper, we consider the continuity of the Bloch eigenvalues, band functions and Bloch functions with respect to the quasimomentum of the differential operator T, generated in the space $L_2^m(\mathbb{R}^d)$ by formally self-adjoint differential expression

$$Tu = \sum_{|\alpha|=2s} Q_\alpha D_\alpha u + \sum_{|\alpha|\leq 2s-1} Q_\alpha(x) D_\alpha u + Bu, \tag{1.1}$$

where $d \geq 1$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d)$ is a multi-indeks, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$,

$$D_\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{1}{i}\frac{\partial}{\partial x_2}\right)^{\alpha_2} ... \left(\frac{1}{i}\frac{\partial}{\partial x_d}\right)^{\alpha_d},$$

Q_α, for each α, is an $m \times m$ matrix. Here the entries of $Q_\alpha(x)$, for $|\alpha| \leq 2s-1$, are bounded functions that are periodic with respect to a lattice Ω, the entries of Q_α, for $|\alpha| = 2s$, are real numbers and B is a bounded operator commuting with the shift operators $S_\omega : u(x) \to u(x+\omega)$, for $\omega \in \Omega$. Note that $L_2^m(\mathbb{R}^d)$ is the space of the vector-valued functions $f = (f_1, f_2, ..., f_m)$ with $f_k \in L_2(\mathbb{R}^d)$, for $k = 1, 2, ..., m$.

To describe briefly the scheme of this paper, let us introduce the following notations. Let Γ be the lattice dual to Ω. Denote by F and $F^\star$ the fundamental domains of the lattices Ω and Γ, respectively. Let T_t be the operator generated in $L_2^m(F)$ by (1.1) and the quasiperiodic conditions

$$u(x+\omega) = e^{i\langle t,\omega\rangle} u(x), \ \forall \omega \in \Omega, \tag{1.2}$$

where $t \in F^\star$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^d$. Denote by $SBC(H)$ the set of below-bounded self-adjoint operators, with compact resolvents acting in the Hilbert space H. If

$$T_t \in SBC(L_2^m(F)), \tag{1.3}$$

*Correspondence: oveliev@dogus.edu.tr
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then the spectrum of T_t consists of the eigenvalues. Let $\lambda_1(t), \lambda_2(t), ...$ be the eigenvalues of T_t numerated in the nondecreasing order

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots. \tag{1.4}$$

The eigenvalues and eigenfunctions of T_t, for $t \in F^\star$, are called the Bloch eigenvalues and Bloch functions of T, respectively. The function $\lambda_n : F^\star \to \mathbb{R}$ is said to be the n-th band function of T.

To investigate the continuity of the band functions and Bloch functions of T, first we study the continuity of the eigenvalues and eigenfunctions for the family $\{A_t : t \in E\}$ of the operators $A_t \in SBC(H)$, where E is a metric space and A_t continuously depends on $t \in E$ in the generalized sense defined in [2, Chap. 4, page 202]. For the simplicity of the notations, the parameter of the family $\{A_t : t \in E\}$ is also denoted by t, the eigenvalues of A_t also are denoted by $\lambda_1(t), \lambda_2(t), ...$ and are numerated in the nondecreasing order as in the case T_t (see (1.4)). In Theorems 2.4- 2.6, we prove that the eigenvalue $\lambda_n(t)$ and corresponding eigenspace of A_t continuously depend on $t \in E$. Moreover, we prove that if $\lambda_n(t_0)$ is a simple eigenvalue, then the eigenvalues $\lambda_n(t)$ are simple in some neighborhood of t_0 and the corresponding normalized eigenfunctions $\Psi_{n,t}$ of A_t can be chosen so that $\|\Psi_{n,t} - \Psi_{n,t_0}\| \to 0$ as $t \to t_0$, where $\|\cdot\|$ is the norm of H. Then, we use these results for the family $\{T_t : t \in F^\star\}$ of the operators T_t. Namely, in Theorem 2.7 we prove that, if (1.3) and an additional condition (2.6) hold, then T_t continuously depends on $t \in F^\star$ in the generalized sense and hence the results obtained for A_t continue to hold for T_t. In order not to deviate from the purpose of this paper, we do not discuss the conditions on (1.1) under which (1.3) and (2.6) hold, since the consideration of these conditions is long technical. Nevertheless, in the end of this paper (see Corollary 2.8), we give an example which shows that the obtained results include the continuity of the band functions and Bloch functions of a large class of the differential operators generated by (1.1).

Note that, as was noted in the physical and mathematical literature (see for example the books [1, 3, 5] and paper [4]), the continuity of the band functions of the Schrödinger operator with a periodic potential and some other periodic operators is well-known or immediately follows from the perturbation theory described in [2]. Here we do not discuss all the numerous investigations about this, and give only detailed proof of the continuity of the band functions and Bloch functions for the large class of the systems of differential operators by using the generalized convergence in the sense of [2]. Finally, we note that, using the results of [3, Chapter 13] in a standard way, one can easily verify that T can be expressed as the direct integral of the operators T_t, for $t \in F^\star$ and that the spectrum of the operator T is the union of the spectra of the operators T_t for $t \in F^\star$.

2. Main results

First, we consider the general operators by using the generalized convergence of the closed operators defined in [2, Chap. 4, page 202]. Let us state the definition of the generalized convergence in the notation which is suitable for this investigation. Let A and B be self-adjoint operators in the Hilbert space H. Since the self-adjoint operators are closed, the graphs $G(A) = \{(u, Au) : u \in D(A)\}$ and $G(B) = \{(u, Bu) : u \in D(B)\}$ of these operators are closed linear manifolds of the Hilbert space H^2. Let S_A and S_B be respectively the unit spheres of $G(A)$ and $G(B)$. The gap $g(T, S)$ between the self-adjoint operators A and B is defined as follows:

$$g(A, B) = \max\left\{\sup_{v\in S_A}\left(\inf_{w\in G(B)}\|w - v\|\right), \sup_{v\in S_B}\left(\inf_{w\in G(A)}\|w - v\|\right)\right\}. \tag{2.1}$$

We use the following definitions and theorems of [2].

Definition 2.1 *(See [2, Chap. 4, page 202]). We say that the sequence of the closed operators A_n converges to the closed operator A in the generalized sense if $g(A_n, A) \to 0$ as $n \to \infty$.*

This convergence determines the following definition of continuity.

Definition 2.2 *We say that, a family of the closed operators A_t is continuous at E in the generalized sense, if for each $t_0 \in E$ and for any sequence $\{t_n\} \subset E$ converging to t_0, $g(A_{t_n}, A_{t_0}) \to 0$ as $n \to \infty$.*

Besides these definitions, we use Theorems 3.1 and 3.16 of [2, see Chap. 4, pages 208 and 212]. Let us formulate these theorems in the suitable form by using the above notations. Theorem 3.1 of [2] states that if the closed curve γ belongs to the resolvent set $\rho(A)$ of $A \in SBC(H)$, then there exists $\delta > 0$ such that $\gamma \in \rho(B)$, for any operator $B \in SBC(H)$ with $g(A, B) < \delta$. Theorem 3.16 of [2] states that, in this case the numbers of the eigenvalues (counting the multiplicity) of A and B lying inside γ are the same. Moreover,

$$\left\| \int_\gamma (A - \lambda I)^{-1} d\lambda - \int_\gamma (B - \lambda I)^{-1} d\lambda \right\| \to 0, \tag{2.2}$$

as $g(A, B) \to 0$.

Now using these statements, we consider a continuous family $\{A_t \in SBC(H) : t \in E\}$ of the operators by using the following notations.

Notation 2.3 *In (1.4), the eigenvalues of the operator A_{t_0} are denoted by counting the multiplicity. Let us denote by $\mu_1(t_0), \mu_2(t_0), \ldots$ the eigenvalues of A_{t_0} without counting the multiplicity. In other words, $\mu_1(t_0) < \mu_2(t_0) < \cdots$ are the eigenvalues of A_{t_0} with the multiplicities $k_1, k_2, \ldots$, respectively. Since $\lambda_n(t_0) \to \infty$ as $n \to \infty$, for each n there exists p such that $n \leq k_1 + k_2 + \cdots + k_p$. This notation, together with the notation (1.4) implies that*

$$\lambda_1(t_0) = \lambda_2(t_0) = \cdots = \lambda_{s_1}(t_0) = \mu_1(t_0),$$

$$\lambda_{s_1+1}(t_0) = \lambda_{s_1+2}(t_0) = \cdots = \lambda_{s_2}(t_0) = \mu_2(t_0),$$

and

$$\lambda_{s_{p-1}+1}(t_0) = \lambda_{s_{p-1}+2}(t_0) = \cdots = \lambda_{s_p}(t_0) = \mu_p(t_0),$$

where $s_p = k_1 + k_2 + \cdots + k_p$ and $n \leq s_p$.

Now we prove the following theorems for the continuous family $\{A_t \in SBC(H) : t \in E\}$.

Theorem 2.4 *Let $\{A_t : t \in E\}$ be the continuous family of the operators $A_t \in SBC(H)$, $t_0 \in E$ and $\{t_k \in E : k \in \mathbb{N}\}$ be a sequence converging to t_0. Then, for every $r > 0$ satisfying the inequality*

$$r < \frac{1}{2} \min_{j=1,2,\ldots p} \left(\mu_{j+1}(t_0) - \mu_j(t_0)\right), \tag{2.3}$$

there exists $N > 0$ such that, each of the operators A_{t_k}, for $k > N$, has k_j eigenvalues in the interval $(\mu_j(t_0) - r, \mu_j(t_0) + r)$, where $j = 1, 2, \ldots, p$, the numbers p and $\mu_j(t_0)$ are defined in Notation 2.3. Moreover, the eigenvalues lying in $(\mu_j(t_0) - r, \mu_j(t_0) + r)$ are $\lambda_{s_{j-1}+1}(t_k), \lambda_{s_{j-1}+2}(t_k), \ldots, \lambda_{s_j}(t_k)$, where $s_0 = 0$ and s_j's, for $j \geq 1$, are defined in Notation 2.3.

Proof By (2.3), Definition 2.2 and Theorem 3.1 of [2], there exists $N > 0$ such that the circle $D_j(r) = \{z \in \mathbb{C} : |z - \mu_j(t_0)| = r\}$ belongs to the resolvent set of the operators A_{t_k}, for $k > N$. Therefore, by Theorem 3.16 of [2], A_{t_k} has k_j eigenvalues inside D_j. In the same way, we prove that A_{t_k}, for $k > N$, has no eigenvalues in the intervals $(-\infty, \mu_1(t_0) - r]$ and $[\mu_j(t_0) + r, \mu_{j+1}(t_0) - r]$, for $j = 1, 2, ..., p$, since A_{t_0} has no eigenvalues in those intervals. Therefore, the eigenvalues of A_{t_k}, for $k > N$, lying in $(\mu_j(t_0) - r, \mu_j(t_0) + r)$ are $\lambda_{s_{j-1}+1}(t), \lambda_{s_{j-1}+2}(t), ..., \lambda_{s_j}(t)$, for $j = 1, 2, ..., p$. □

Now, we are ready to prove the main results for this continuous family.

Theorem 2.5 *Let $\{A_t : t \in E\}$ be the continuous family of the operators $A_t \in SBC(H)$. Then, the eigenvalues (1.4) of A_t continuously depend on $t \in E$.*

Proof Consider the sequence $r_s \to 0$. By Theorem 2.4, there exists N_s such that, if $k > N_s$ then the eigenvalues $\lambda_{s_{j-1}+1}(t_k), \lambda_{s_{j-1}+2}(t_k), ..., \lambda_{s_j}(t_k)$ lie in $(\mu_j(t_0) - r_s, \mu_j(t_0) + r_s)$. Therefore, we have $|\lambda_n(t_k) - \lambda_n(t_0)| < r_s$, for $k > N_s$, since $n \in [s_{j-1} + 1, s_j]$ and $\lambda_n(t_0) = \mu_j(t_0)$ due to Notation 2.3. Now letting s tend to infinity, we obtain that $\lambda_n(t_k) \to \lambda_n(t_0)$ as $k \to \infty$, for any sequence $\{t_k \in E : k \in \mathbb{N}\}$ converging to t_0. Thus λ_n is continuous at t_0. □

Now, suppose that $\lambda_n(t_0)$ is a simple eigenvalue. Then, by Theorem 2.4, for every $r > 0$ satisfying (2.3), there exists $N > 0$ such that the operator A_{t_k}, for $k > N$, has a unique eigenvalue (counting the multiplicity) in the interval $(\lambda_n(t_0) - r, \lambda_n(t_0) + r)$ and this eigenvalue is $\lambda_n(t_k)$. Since $\lambda_n(t_k)$ is a simple eigenvalue, we have

$$\int_\gamma (A_{t_k} - \lambda I)^{-1} f d\lambda = (f, \Psi_{n,t_k})\Psi_{n,t_k}, \tag{2.4}$$

for $f \in H$, $k = 0$, and $k > N$, where γ is a closed curve that encloses only the eigenvalue $\lambda_n(t_k)$ and Ψ_{n,t_k} is a normalized eigenfunction corresponding to $\lambda_n(t_k)$. Using (2.2) and (2.4), we obtain the following relations:

$$(f, \Psi_{n,t_k})\Psi_{n,t_k} \to (f, \Psi_{n,t_0})\Psi_{n,t_0}$$

as $t_k \to t_0$, for each $f \in H$. Here replacing f by Ψ_{n,t_0}, we obtain

$$(\Psi_{n,t_0}, \Psi_{n,t_k})\Psi_{n,t_k} \to \Psi_{n,t_0} \tag{2.5}$$

and $|(\Psi_{n,t_0}, \Psi_{n,t_k})| \to 1$ as $t_k \to t_0$. Since a normalized eigenfunction is still normalized if it is multiplied by a factor of absolute value 1, Ψ_{k,t_k} and Ψ_{k,t_0} can be chosen so that $|(\Psi_{n,t_0}, \Psi_{n,t_k})| = (\Psi_{n,t_0}, \Psi_{n,t_k})$. Thus

$$(\Psi_{n,t_0}, \Psi_{n,t_k}) \to 1,$$

as $t_k \to t_0$. This, together with (2.5) implies that

$$\|\Psi_{n,t_k} - \Psi_{n,t_0}\| \le \|(1 - (\Psi_{n,t_0}, \Psi_{n,t_k}))\Psi_{n,t_k}\| + \|(\Psi_{n,t_0}, \Psi_{n,t_k})\Psi_{n,t_k} - \Psi_{n,t_0}\| \to 0,$$

as $t \to t_0$. It means that, $\Psi_{n,t}$ is continuous at t_0. Thus, we have:

Theorem 2.6 *Let $\{A_t : t \in E\}$ be the continuous family of the operators $A_t \in SBC(H)$ and $t_0 \in E$. If $\lambda_n(t_0)$ is a simple eigenvalue, then the eigenvalues $\lambda_n(t)$ are simple in some neighborhood of t_0 and the corresponding normalized eigenfunctions $\Psi_{n,t}$ can be chosen so that $\|\Psi_{n,t} - \Psi_{n,t_0}\| \to 0$ as $t \to t_0$.*

Now, we consider the application of Theorems 2.4- 2.6 to the differential operators T_t defined in $L_2^m(F)$ by (1.1) and (1.2). Without loss of generality, we assume that the measure of F is 1.

Theorem 2.7 *Suppose that (1.3) holds and there exist $\varepsilon > 0$ and $c > 0$ such that*

$$\|Tu + cu - Bu\| + \|u\| \geq \varepsilon \sum_{|\alpha| \leq 2s-1} \|D_\alpha u\|, \tag{2.6}$$

for all $u \in D(T_t)$ and $t \in F^*$, where $D(T_t)$ is the domain of definition of T_t, Tu and B are defined in (1.1). Then $\{T_t : t \in F^*\}$ is a continuous family of the operators in the sense of Definition 2.2 and hence Theorems 2.4- 2.6 continue to hold for this family.

Proof For the simplicity of the notation, denote $T_t + cI - B$ by A_t. By Theorem 2.23 (c) of [2, Chap. 4, page 206], it is enough to prove that, the sequence $\{A_{t_n}\}$ converges to A_{t_0} in the generalized sense, for any sequence $\{t_n\} \subset F^*$ converging to t_0. If $u \in D(A_{t_0})$, then $e^{i\langle t_n - t_0, x\rangle} u \in D(A_{t_n})$. By the product rule of differentiation, we have

$$A_{t_n} e^{i\langle t_n - t_0, x\rangle} u = e^{i\langle t_n - t_0, x\rangle} A_{t_0} u + |t_n - t_0| \, Su, \tag{2.7}$$

where $e^{i\langle t_n - t_0, x\rangle} A_{t_0} u$ is the sum of the terms of $A_{t_n} e^{i\langle t_n - t_0, x\rangle} u$ for which no differentiation is applied to $e^{i\langle t_n - t_0, x\rangle}$ and $|t_n - t_0| \, Su$ is the sum of the other terms of $A_{t_n} e^{i\langle t_n - t_0, x\rangle} u,$ that is the sum of the terms of $A_{t_n} e^{i\langle t_n - t_0, x\rangle} u$ for which some differentiations are applied to $e^{i\langle t_n - t_0, x\rangle}$. It is clear that Su is the differential expression of order $\leq 2s - 1$, with bounded coefficients. Therefore, there exists M such that

$$\|Su\| < M \sum_{|\alpha| \leq 2s-1} \|D_\alpha u\|. \tag{2.8}$$

Now, using (2.6)-(2.8), (2.1) and Definition 2.1, we prove that $\{A_{t_n}\}$ converges to A_{t_0} in the generalized sense. Let S_t be the unit sphere of $G(A_t)$. If $v = (u, A_{t_0} u) \in S_{t_0}$, then $w = (e^{i\langle t_n - t_0, x\rangle} u, A_{t_n} e^{i\langle t_n - t_0, x\rangle} u) \in G(A_{t_n})$ and

$$\|w - v\| \leq \left\| e^{i\langle t_n - t_0, x\rangle} u - u \right\| + \left\| A_{t_n} e^{i\langle t_n - t_0, x\rangle} u - A_{t_0} u \right\|. \tag{2.9}$$

For the second term of (2.9), we have

$$\left\| A_{t_n} e^{i\langle t_n - t_0, x\rangle} u - A_{t_0} u \right\| \leq \left\| A_{t_n} e^{i\langle t_n - t_0, x\rangle} u - e^{i\langle t_n - t_0, x\rangle} A_{t_0} u \right\| + \left\| \left(e^{i\langle t_n - t_0, x\rangle} - 1 \right) A_{t_0} u \right\|. \tag{2.10}$$

On the other hand, using (2.7), (2.8) and (2.6), we obtain

$$\left\| A_{t_n} e^{i\langle t_n - t_0, x\rangle} u - e^{i\langle t_n - t_0, x\rangle} A_{t_0} u \right\| = |t_n - t_0| \, \|Su\| < |t_n - t_0| \, M \left(\sum_{|\alpha| \leq 2s-1} \|D_\alpha u\| \right) \leq \tag{2.11}$$

$$\frac{|t_n - t_0| \, M}{\varepsilon} \left(\|A_{t_0} u\| + \|u\| \right).$$

Moreover, the inclusion $(u, A_{t_0} u) \in S_{t_0}$ implies that $\|A_{t_0} u\| + \|u\| < 2$. This inequality, together with (2.9)-(2.11) implies that there exists $C > 0$ such that

$$\sup_{v \in S_{t_0}} \left(\inf_{w \in G(A_{t_n})} \|w - v\| \right) \leq C \, |t_n - t_0|.$$

In the same way, we prove that

$$\sup_{v\in S_{t_n}}\left(\inf_{w\in G(A_{t_0})}\|w-v\|\right)\leq C\left|t_n-t_0\right|.$$

Therefore by (2.1), $g(A_{t_n},A_{t_o})\to 0$ as $n\to\infty$. Hence, by Definition 2.2, $\{T_t : t\in F^*\}$ is a continuous family in the generalized sense. □

By the standard estimation, one can find a large class of differential operators for which (2.6) and (1.3) hold and hence Theorems 2.4- 2.6 continue to hold. For example, we find a class of the partial differential operators satisfying (2.6) and (1.3) in the following manner. Let us write (1.1) in the form

$$Tu=Lu+Pu+Bu, \tag{2.12}$$

where

$$Lu=\sum_{|\alpha|=2s}Q_\alpha D_\alpha u,\ \ Pu=\sum_{|\alpha|\leq 2s-1}Q_\alpha(x)u.$$

First we note that, there exists $c_1>0$ for which

$$\|Pu\|\leq c_1\sum_{|\alpha|\leq 2s-1}\|D_\alpha u\|, \tag{2.13}$$

since the coefficients of the expression Pu are matrices with bounded entries. Here and in subsequent relations we denote by $c_1,c_2,...$ the positive constants. Then, we find a condition on the main term Lu of (2.12) for which, there exists $c>0$ such that

$$\|Lu+cu\|\geq(c_1+\varepsilon)\sum_{|\alpha|\leq 2s-1}\|D_\alpha u\|. \tag{2.14}$$

It is clear that (2.6) follows from (2.14) and (2.13). The condition on L, namely on the principal symbol of (1.1) is the following. Suppose that

$$L=\sum_{|\alpha|=2s}q_\alpha I_m D_\alpha, \tag{2.15}$$

where I_m is $m\times m$ unit matrix, and there exists $c_2>0$ such that

$$\sum_{|\alpha|=2s}q_\alpha\left(\xi\right)^\alpha\geq c_2\left|\xi\right|^{2s}, \tag{2.16}$$

for each $\xi\in\mathbb{R}^d$, where $q_\alpha\in\mathbb{R}$ and $(\xi)^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_d^{\alpha_d}$. Now we prove that if (2.15) and (2.16) hold, then there exists $c>0$ such that (2.14) and hence (2.6) hold. Moreover, we prove that (1.3) also holds and obtain the following consequence of Theorem 2.7.

Corollary 2.8 *Suppose that L has the form (2.15), inequality (2.16) holds, and that Pu is a formally self-adjoint differential expression. Then, the Bloch eigenvalues (1.4) of the differential operator T_t generated in $L_2^m(F)$ by (1.1) and (1.2) continuously depend on $t\in F^*$. Moreover, if $\lambda_n(t_0)$ is a simple eigenvalue, then the eigenvalues $\lambda_n(t)$ are simple in some neighborhood of t_0 and the corresponding normalized eigenfunctions $\Psi_{n,t}$ can be chosen so that $\|\Psi_{n,t}-\Psi_{n,t_0}\|\to 0$ as $t\to t_0$.*

Proof First we prove the validity of (2.14) if (2.15) and (2.16) hold. Instead of (2.14), we prove

$$\|Lu+cu\|^2 \geq (c_1+\varepsilon)^2 \left(\sum_{|\alpha|\leq 2s-1} \|D_\alpha u\|\right)^2, \tag{2.17}$$

for all $u \in L_2^m(F)$ such that $\partial^\alpha u \in L_2^m(F)$, for $|\alpha| \leq 2s$. It follows from (2.15) and (2.16) that

$$\left\| L\left(e^{i\langle\gamma,x\rangle}e_k\right) + ce_k e^{i\langle\gamma,x\rangle}\right\|^2 \geq \left(c_2 |\gamma|^{2s} + c\right)^2,$$

where $\{e_1, e_2, ..., e_m\}$ is the standard basis of $\mathbb{R}^m$ and $\gamma \in \Gamma$. Using this and the decomposition

$$u = \sum_{\gamma\in\Gamma} \left(\sum_{k=1,2,\ldots,m} u_{\gamma,k} e^{i\langle\gamma,x\rangle} e_k\right) \tag{2.18}$$

of u by the orthonormal basis $\left\{e^{i\langle\gamma,x\rangle}e_k : \gamma \in \Gamma,\ k = 1,2,...,m\right\}$, we obtain that

$$\|(Lu)+cu\|^2 \geq \sum_{\gamma\in\Gamma} \left(\sum_{k=1,2,\ldots,m} \left|u_{\gamma,k}\left(c_2|\gamma|^{2s}+c\right)\right|^2\right). \tag{2.19}$$

On the other hand, using (2.18) we have

$$\left(\sum_{|\alpha|\leq 2s-1} \|D_\alpha u\|\right)^2 \leq c_3 \sum_{|\alpha|\leq 2s-1} \|D_\alpha u\|^2 = c_3 \sum_{\gamma\in\Gamma} \left(\sum_{k=1,2,\ldots,m} \left|u_{\gamma,k} \sum_{|\alpha|\leq 2s-1} |\gamma|^\alpha\right|^2\right). \tag{2.20}$$

Therefore, (2.17) follows from (2.19) and (2.20), if we prove that there exists $c > 0$ such that

$$c_2 |\gamma|^{2s} + c > \sqrt{c_3}\,(c_1+\varepsilon) \sum_{|\alpha|\leq 2s-1} |\gamma|^\alpha, \tag{2.21}$$

for all $\gamma \in \Gamma$. Hence, it remains to prove (2.21). Clearly, there exists c_4 such that $c_2|\gamma|^{2s}$ is greater than the right-hand side of (2.21), for $|\gamma| > c_4$. Besides, the number of $\gamma \in \Gamma$ satisfying $|\gamma| \leq c_4$ is finite. Therefore, there exists $c > 0$ such that (2.21) holds, for $|\gamma| \leq c_4$. Thus, (2.21) and hence (2.17), (2.14), and (2.6) hold.

Now, we prove that (1.3) holds, too. Let L_t and P_t be respectively the operators generated by the differential expression Lu and Pu and boundary conditions (1.2). Since the basis $\left\{e^{i\langle\gamma+t,x\rangle}e_k : \gamma \in \Gamma, k = 1,2,...,m\right\}$ of $L_2^m(F)$ is the set of the eigenfunctions of L_t and all eigenvalues of L_t are nonnegative numbers, we have $L_t \in SBC(L_2^m(F))$.

Now, we prove that $T_t \in SBC(L_2^m(F))$, that is, (1.3) holds. It readily follows from the proof of (2.21) that, there exists $c > 0$ such that

$$c_2 |\gamma|^{2s} + c > 2\sqrt{c_3}\,(c_1+\varepsilon) \sum_{|\alpha|\leq 2s-1} |\gamma|^\alpha.$$

Therefore, arguing as in the proof of (2.14), we see that, there exists c_5 such that

$$\|Pu+Bu\| \leq c_5\|u\| + \tfrac{1}{2}\|Lu+cu\|. \tag{2.22}$$

It shows that, $P_t + B$ is relatively bounded with respect to $L_t + cI$ with relative bound smaller than 1. Therefore, by Theorem 4.11 of [2, Chap.5, page 291], $T_t + cI = L_t + cI + P_t + B$ is self-adjoint and bounded from below. It remains to prove that $T_t + cI$ has a compact resolvent. For this, we use Theorem 3.17 of [2, Chap. 4, page 214] which implies that, if (2.22) holds and there exists $\mu \in \rho(L + cI)$ such that

$$c_5 \left\| (L_t + cI - \mu I)^{-1} \right\| + \frac{1}{2} \left\| (L_t + cI)\,(L_t + cI - \mu I)^{-1} \right\| < 1, \tag{2.23}$$

then $T_t + cI$ has a compact resolvent. Since $L_t + cI$ is a self-adjoint operator, it is clear and well-known that (see, for example (4.9) of [2, Chap. 5, page 290]), if $\mu = ic_6$, then

$$\left\| (L_t + cI - \mu I)^{-1} \right\| \leq \frac{1}{c_6}, \quad \left\| (L_t + cI)\,(L_t + cI - \mu I)^{-1} \right\| \leq 1.$$

Thus, if $c_6 > 2c_5$, then (2.23) holds. Hence, $T_t + cI$ is a below-bounded self-adjoint operator with a compact resolvent and (1.3) holds. Now, the proof follows from Theorem 2.7. □

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