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A note on ”Some properties of second-order weak subdifferentials” [Turkish Journal of Mathematics (2021)45: 955-960]

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Abstract: In this note, we provide an example to illustrate that Proposition 2.4 in [Turkish Journal of Mathematics (2021)45: 955-960)] is incorrect, and give a modification of the proposition. Two examples are provided to illustrate the modified result. Meanwhile, we establish a convex function, and correct the proof of Theorem 2.3 in [Turkish Journal of Mathematics (2021)45: 955-960)] by the function.

Key words: Second-order weak subdifferentials, the subadditionability, the convexity

1. Introduction

It is well known that the concept of weak subdifferential plays a crucial role in nonsmooth analysis [1–3]. Recently, İnceoğlu [4] introduced the notion of second-order weak subdifferentials and discussed some of its properties. Unfortunately, the subadditivity of the second-order weak subdifferentials, i.e. [4, Proposition 2.4] is incorrect. At the same time, there exists a flaw in the proof process of [4, Theorem 2.3].

In this note, we first provide an example to demonstrate that [4, Proposition 2.4] is incorrect. Secondly, we propose a modified form of [4, Proposition 2.4]. Finally, we point out that there exists a flaw in the proof process of [4, Theorem 2.3], and provide a correct proof for the theorem.

2. Preliminaries

Throughout the paper, let X be a real normed space and let X^* be the topological dual space of X. Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of real numbers and the set of nonnegative real numbers, respectively. Let $A, B \subset \mathbb{R}$, the sum $A + B$ and difference $A - B$ of A and B are defined by $A + B = \{a + b : a \in A, b \in B\}$ and $A - B = \{a - b : a \in A, b \in B\}$, respectively.

Let $F, G : X \to \mathbb{R}$ be two single-valued functions, the sum $F + G$ of functions F and G is defined by $(F + G)(x) = F(x) + G(x), \forall x \in X$.

Now we recall the definition of the second-order weak subdifferential.

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Definition 2.1 (see [4, Definition 3.1]) *Let $F : X \to \mathbb{R}$ be a single-valued function and $\bar{x} \in X$ be given, where $F(\bar{x})$ is finite. A pair $(x^*, c) \in X^* \times \mathbb{R}_+$ is called the second-order weak subgradient of F at $\bar{x}$ if*

$$F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X.$$

The set

$$\partial_w^2 F(\bar{x}) = \{(x^*, c) \in X^* \times \mathbb{R}_+ : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle - c\|x - \bar{x}\|^2, \forall x \in X\}$$

of all second-order weak subgradients of F at $\bar{x}$ is called the second-order weak subdifferential of F at $\bar{x}$. If $\partial_w^2 F(\bar{x}) \neq \emptyset$, then F is called second-order weakly subdifferentiable at $\bar{x}$.

Definition 2.2 (see [5]) Let $F : X \to \mathbb{R}$ be a single-valued function. F is called a convex function on X if for any $x, y \in X$ and $\lambda \in (0, 1)$, one has

$$F(\lambda x + (1 - \lambda)y) \leq \lambda F(x) + (1 - \lambda)F(y).$$

To prove the convexity of the second-order weak subdifferential, we introduce the following lemma.

Lemma 2.3 Let $x \in X$ be given and let $f_x(\cdot)$ be a single-valued function on X^* defined by

$$f_x(x^*) = \langle x^*, x \rangle^2, \forall x^* \in X^*.$$

Then $f_x(\cdot)$ is a convex function on X^*.

Proof Let $x \in X$, $x_1^*, x_1^* \in X^*$ and $\lambda \in (0, 1)$. Then

$$\begin{aligned} f_x(\lambda x_1^* + (1 - \lambda)x_2^*) &= \langle \lambda x_1^* + (1 - \lambda)x_2^*, x \rangle^2 \\ &= \lambda^2 \langle x_1^*, x \rangle^2 + (1 - \lambda)^2 \langle x_2^*, x \rangle^2 + 2\lambda(1 - \lambda)\langle x_1^*, x \rangle \langle x_2^*, x \rangle. \end{aligned} \tag{2.1}$$

Set $a = f_x(x_1^*) = \langle x_1^*, x \rangle$ and $b = f_x(x_2^*) = \langle x_2^*, x \rangle$. Then it follows from (2.1) that

$$\begin{aligned} f_x(\lambda x_1^* + (1 - \lambda)x_2^*) &= \lambda^2 a^2 + (1 - \lambda)^2 b^2 + 2\lambda(1 - \lambda)ab \\ &\leq \lambda^2 a^2 + (1 - \lambda)^2 b^2 + \lambda(1 - \lambda)(a^2 + b^2) \\ &= \lambda a^2 + (1 - \lambda)b^2 = \lambda f_x(x_1^*) + (1 - \lambda)f_x(x_2^*). \end{aligned}$$

Thus $f_x(\cdot)$ is a convex function on X^*, this completes the proof. □

3. Properties of the second-order weak subdifferential

In this section, we present an important property of the second-order weak subdifferential and provide a correct proof for [4, Theorem 2.3]. We first recall a result in [4].

Proposition 3.1 (see [4, Proposition 2.4]) Let $F, G : X \to \mathbb{R}$ and $F + G : X \to \mathbb{R}$ be single-valued functions and second-order weakly subdifferentiable at $\bar{x}$. Then

$$\partial_w^2 F(\bar{x}) + \partial_w^2 G(\bar{x}) \subseteq \partial_w^2 (F + G)(\bar{x}).$$

Remark 3.2 Since the conditions do not imply the conclusion in [4, Proposition 2.4], [4, Proposition 2.4] is incorrect.

Now, we provide an example to illustrate Remark 3.2.

Example 3.3 *Let* $F, G : \mathbb{R} \to \mathbb{R}$ *be two single-valued functions with* $F(x) = 2x^2$ *and* $G(x) = 3x^2$. *Take* $\bar{x} = 0$. *Then*

$$\partial_w^2 F(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 2\}, \partial_w^2 G(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 3\},$$

and

$$\partial_w^2 (F + G)(\bar{x}) = \{(a, c) \in \mathbb{R} \times \mathbb{R}_+ : a^2 \leq c + 5\}.$$

Take $(a, c) = (\sqrt{3}, 1) \in \partial_w^2 F(\bar{x})$ *and* $(b, d) = (\sqrt{3}, 0) \in \partial_w^2 G(\bar{x})$. *Then*

$$(a, c) + (b, d) = (2\sqrt{3}, 1) \notin \partial_w^2 (F + G)(\bar{x}).$$

It is obvious that [4, Proposition 2.4] does not hold here.

Next, we give an appropriate modification to rectify Proposition 2.4 in [4].

Proposition 3.4 Let $F, G : X \to \mathbb{R}$ and $F + G : X \to \mathbb{R}$ be single-valued functions and second-order weakly subdifferentiable at $\bar{x} \in X$ and let $(a, c) \in \partial_w^2 F(\bar{x})$ and $(b, d) \in \partial_w^2 G(\bar{x})$. If

$$\langle a, x - \bar{x} \rangle \langle b, x - \bar{x} \rangle \leq 0, \forall x \in X, \tag{3.1}$$

then

$$(a, c) + (b, d) \in \partial_w^2 (F + G)(\bar{x}).$$

Proof Since $(a, c) \in \partial_w^2 F(\bar{x})$ and $(b, d) \in \partial_w^2 G(\bar{x})$,

$$F(x) - F(\bar{x}) \geq \langle a, x - \bar{x} \rangle^2 - c\|x - \bar{x}\|^2, \forall x \in X,$$

and

$$G(x) - G(\bar{x}) \geq \langle b, x - \bar{x} \rangle^2 - d\|x - \bar{x}\|^2, \forall x \in X.$$

Therefore,

$$(F + G)(x) - (F + G)(\bar{x}) \geq \langle a + b, x - \bar{x} \rangle^2 - (c + d)\|x - \bar{x}\|^2 - 2\langle a, x - \bar{x} \rangle \langle b, x - \bar{x} \rangle, \forall x \in X.$$

Then it follows from (3.1) that

$$(F(x) + G(x)) - (F(\bar{x}) + G(\bar{x})) \geq \langle a + b, x - \bar{x} \rangle^2 - (c + d)\|x - \bar{x}\|^2, \forall x \in X.$$

Thus, by virtue of the definition of the second-order weak subgradient, we have

$$(a, c) + (b, d) \in \partial_w^2 (F + G)(\bar{x}),$$

this completes the proof. □

Next, we provide an example to illustrate Proposition 3.4.

Example 3.5 *Consider Example 3.3. Take* $\bar{x}=0$. *Then*

$$\partial_w^2 F(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+2\},\quad \partial_w^2 G(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+3\},$$

and

$$\partial_w^2 (F+G)(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+5\}.$$

Take $(a,c)=(-\sqrt{2},1)\in\partial_w^2 F(\bar{x})$ *and* $(b,d)=(\sqrt{2},1)\in\partial_w^2 G(\bar{x})$. *It is obvious that* $\langle a, x-\bar{x}\rangle\langle b, x-\bar{x}\rangle = -2x^2 \le 0, \forall x\in X$ *and*

$$(a,c)+(b,d)\in\partial_w^2(F+G)(\bar{x}).$$

Thus Proposition 3.4 holds here.

Remark 3.6 The condition (3.1) is essential in Proposition 3.4. The following example explains the case.

Example 3.7 *Consider Example 3.3. Take* $\bar{x}=0$. *Then*

$$\partial_w^2 F(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+2\}, \partial_w^2 G(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+3\},$$

and

$$\partial_w^2 (F+G)(\bar{x}) = \{(a,c)\in\mathbb{R}\times\mathbb{R}_+ : a^2 \le c+5\}.$$

Take $(a_0,c_0)=(\sqrt{2},1)\in\partial_w^2 F(\bar{x})$ *and* $(b_0,d_0)=(\sqrt{2},0)\in\partial_w^2 G(\bar{x})$. *It is obvious that*

$$\langle a_0, x-\bar{x}\rangle\langle b_0, x-\bar{x}\rangle = 2x^2 > 0, \forall x\in X \text{ and } x\neq 0$$

and

$$(a_0,c_0)+(b_0,d_0)\notin\partial_w^2(F+G)(\bar{x}),$$

that is, the condition (3.1) does not hold here and Proposition 3.4 is not applicable here. Thus the condition (3.1) is essential in Proposition 3.4.

Remark 3.8 In general, the inequality $a^2+b^2\ge(a+b)^2$ does not hold, where $a,b\in\mathbb{R}$, so there is a flaw in the proof process of [4, Theorem 2.3], that is, it follows from the equations (2.8) and (2.9) in [4] that the equation in [4, Page 957, Line -6] may not be obtained.

Next, by virtue of Lemma 2.3, we give an appropriate modification to the proof process of [4, Theorem 2.3].

Theorem 3.9 Let $\bar{x}\in X$, $F: X\to\mathbb{R}$ be a single-valued function and $\partial_w^2 F(\bar{x})\neq\emptyset$. Then the set $\partial_w^2 F(\bar{x})$ is convex.

Proof Let $(a,c)\in\partial_w^2 F(\bar{x})$, $(b,d)\in\partial_w^2 F(\bar{x})$ and $\lambda\in(0,1)$. Then

$$F(x)-F(\bar{x})\ge\langle a, x-\bar{x}\rangle^2 - c\|x-\bar{x}\|^2, \forall x\in X,$$

and

$$F(x)-F(\bar{x})\ge\langle b, x-\bar{x}\rangle^2 - d\|x-\bar{x}\|^2, \forall x\in X.$$

Therefore,

$$F(x) - F(\bar{x}) \geq \lambda\langle a, x - \bar{x}\rangle^2 + (1-\lambda)\langle b, x - \bar{x}\rangle^2 - (\lambda c + (1-\lambda)d)\|x - \bar{x}\|^2, \forall x \in X.$$

Then it follows from Lemma 2.3 that

$$F(x) - F(\bar{x}) \geq \langle \lambda a + (1-\lambda)b, x - \bar{x}\rangle^2 - (\lambda c + (1-\lambda)d)\|x - \bar{x}\|^2, \forall x \in X.$$

Thus, by virtue of the definition of the second-order weak subgradient, we have

$$\lambda(a, c) + (1-\lambda)(b, d) \in \partial_w^2 F(\bar{x}),$$

that is, the set $\partial_w^2 F(\bar{x})$ is convex. So this completes the proof. □

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