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Geometric singularities and regularity of solution of the Stokes system in nonsmooth domains

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Abstract: This paper deals with the geometrical singularities of the weak solution of the mixed boundary value problem governed by the stationary Stokes system in two-dimensional nonsmooth domains with corner points and points at which the type of boundary conditions changes. The presence of these points on the boundary generally generates local singularities in the solution. We will see the impact of the geometrical singularities of the boundary or the mixed boundary conditions on the qualitative properties of the solution including its regularity. Moreover, the asymptotic singular representations for the solution which inherently depend on the zeros of certain transcendental functions are presented.

Key words: Stokes flow, corner singularities, regularity, mixed boundary conditions, nonsmooth domain

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a 2-dimensional bounded domain, whose boundary $\partial\Omega$ comprises the corner points and points at which the type of boundary conditions changes. Note that a point $P \in \partial\Omega$ is said to be a corner point if there exists a neighborhood $\eta(P)$ of P such that $\Omega \cap \eta(P)$ is diffeomorphic to a cone κ intersected with unit disc. For simplicity, we are considering a bounded plane polygonal domain (see Figure 1) with corner points $(\omega \neq \pi)$ and points $(\omega = \pi)$ at which the type of boundary conditions changes. The boundary points where the boundary conditions change are also referred to as corner points or vertices. The obtained results for a polygonal domain can be extended to a 2-dimensional bounded domain, i.e. (Lipschitz continuous) $C^{0,1}$ with corner points. We considered one point as a special case of interest of corner points with an angle $\omega = \pi$ on one side of the domain Ω, where the Neumann boundary condition, the Dirichlet boundary condition, respectively, is prescribed.

For the polygonal domain Ω with the vertices $P_1, ..., P_N$, we introduce the following notations. Let $P_{N+1} = P_1$, $\mathcal{J} = \{1, ..., N\}$, $\Gamma_i \, (i \in \mathcal{J})$ be the open edge connecting the vertices P_{i+1} and P_i, $\Gamma_0 = \Gamma_N$, and $\omega_i \, (i \in \mathcal{J})$ be the interior angle made by Γ_{i-1}, Γ_i. Let $\mathcal{J}_D = \{i \in \mathcal{J} \colon \text{ on } \Gamma_i \text{ the Dirichlet boundary conditions are prescribed}\}$ and $\mathcal{J}_N = \{i \in \mathcal{J} \colon \text{ on } \Gamma_i \text{ the Neumann boundary conditions are prescribed}\}$.

We assume that $\mathcal{J}_D$, $\mathcal{J}_N$ are nonempty disjoint sets and $\mathcal{J} = \mathcal{J}_D \cup \mathcal{J}_N$. Moreover, let Γ_D, Γ_N be given by $\Gamma_D = \bigcup_{i\in\mathcal{J}_D} \overline{\Gamma}_i$, $\Gamma_N = \bigcup_{i\in\mathcal{J}_N} \overline{\Gamma}_i$. We have $\Gamma_D \cap \Gamma_N = \emptyset$ and $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$.

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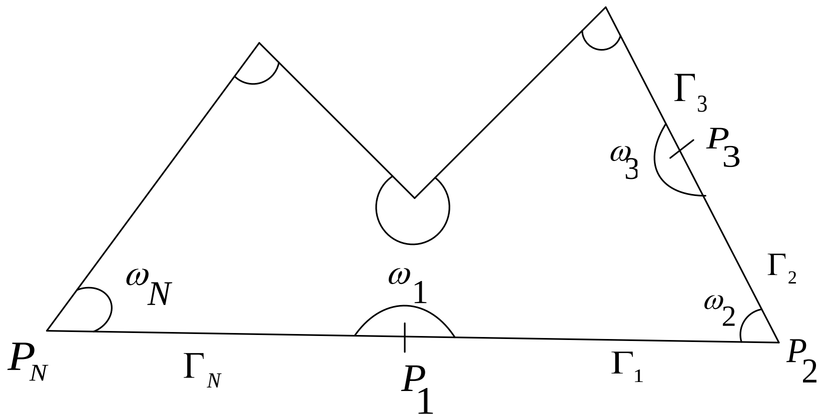

Figure 1. Schematic illustration of a polygonal domain with vertices $P_1, ..., P_N$.

The velocity and pressure formulation of the stationary Stokes system on a domain Ω is

$$\begin{cases} -\nu\Delta\mathbf{v} + \nabla q = \mathbf{f} & \text{in} \quad \Omega, \\ \operatorname{div}\mathbf{v} = 0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{v} = (v_1, v_2)$ is the velocity vector field with the cartesian components v_1, v_2, ν is the viscosity parameter of the fluid flow, i.e. $(\nu > 0)$, q is the hydrostatic pressure and $\mathbf{f}$ is a given volume force density.

The following mixed boundary conditions are considered on the boundary $\partial\Omega$:

$$\mathbf{v} = \mathbf{h}_1 \quad \text{on} \quad \Gamma_D, \tag{1.2}$$

$$\mathcal{S}\big[\mathbf{v},\, q\big]\mathbf{n} = \mathbf{h}_2 \quad \text{on} \quad \Gamma_N, \tag{1.3}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal vector to the boundary and $\mathcal{S}\big[\mathbf{v},\, q\big]$ is the hydrostatic stress tensor with the cartesian components

$$\mathcal{S}\big[\mathbf{v},\, q\big] = -q\delta_{i\,j} + \nu\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big). \tag{1.4}$$

Here, $\delta_{i\,j}$ is the Kronecker symbol. Furthermore, it is noted that if the second equation of (1.1) becoming $-\operatorname{div}\mathbf{u} = g$ for a given function g satisfying the property $\int_\Omega g\, d\mathrm{x} = 0$, then a particular regularity of g is required for proving the regularity of the pressure function or for handling the nonzero boundary data. Generally, for incompressible flows the function g is set equal to zero to satisfy the incompressibility condition. For simplicity, we are considering g equal to zero. Therefore, for a smooth boundary, smooth given data and boundary conditions, the system (1.1) has a smooth solution [41]. The system (1.1) with the boundary conditions (1.2)-(1.3) is known as the stationary Stokes system with mixed boundary conditions [27, 34].

The Navier-Stokes equations or even the Stokes equations are solved for Dirichlet boundary conditions [8, 10, 11, 15, 17] but this is not common in some situations like finite channel flow models [17, 26]. Usually, these boundary conditions are used in the upstream of the channel and on the fixed walls but not downstream of the channel, because the downstream velocity depends on the flow in a channel which is unknown. The situation becomes more intricate when the boundary of the domain has corners or edges and the Neumann boundary conditions are applied on parts of the boundary [25, 32]. Therefore, the second equation of (1.1) helps to characterize the different types of Neumann boundary conditions with the Green theorem. In numerical methods, the condition (1.3) is used on the downstream boundary [13].

The corner singularity theory has been established for compressible viscous Stokes and Navier-Stokes systems on polygonal and polyhedral domains, and the singular behavior of the solution structure near the corners and edges has been studied in [17, 25, 35]. The key point of the corner singularity theory is to split the solution into regular and singular parts. In [25], the method of special ansatzes and spherical coordinates are used to calculate the singular terms for the Dirichlet problem of the Stokes system. Analogously, in [38] the Fourier transform is used for Lame's system with various boundary conditions to obtain the singular functions. There are some results about the regularity issues for stationary incompressible Stokes and Navier-Stokes systems on bounded domains with corners. In the singularity expansion method for the Stokes problem, the spectral problems related to the corner singularities of solutions to elliptic equations were discussed in [8, 9, 19]. Serre [37] has investigated the existence of the solution of the stationary Navier-Stokes equation for an irregular boundary data for a connected and open bounded subset but has not analyzed the regularity of the considered problem in a cornered domain where the types of boundary conditions change. Kellog and Osborn showed the $H^2 \times H^1$ regularity result for the solution of the Stokes problem in a convex polygonal domain in [21]. The H^s-regularity (s being real and nonnegative) of solutions to the Stokes system on convex domains with corners was studied by [10]. Kweon [28] has considered zero Dirichlet boundary conditions to examine the regularity results of the incompressible Navier-Stokes equations in a nonconvex polygonal domain. These results have been extended for compressible Navier-Stokes equations in a nonconvex polyhedral cylinder in $\mathbb{R}^3$ with inflow boundary conditions [30, 31]. Also, Mazya [33] has considered the stokes problem with mixed boundary conditions in a polyhedral domain to analyze the existence and regularity of the problem in weighted Sobolev spaces. Moreover, the results are established on the point estimates of Green's matrix. The Helmholtz decomposition was used to obtain regularity results of the compressible Stokes system in a nonconvex polygonal domain with no-slip boundary conditions in [29]. The treatment of corner singularities and regularity results of the stationary Stokes and Navier-Stokes equations on polygonal domains with convex and nonconvex corners are comprehensively explored in [4].

However, the above-cited literature reveals that the main singularity and regularity properties are investigated by using the classical Sobolev spaces, and by employing the Fourier transform, the method of special ansatzes, and spherical coordinates.

The main focus of the present study is to analyze the existence and regularity of the weak solution of the mixed boundary value problem for the stationary Stokes system in a two-dimensional bounded domain with corner points or points at which the type of boundary conditions change. The aims are to analyze the qualitative properties of the solutions including their regularity near corner points where the types of boundary conditions change. A parametric boundary value problem for the Stokes system is obtained by employing the localization technique and the Mellin transform, which depends polynomially on the spectral parameter. Furthermore, we derive the transcendental equations for the parameter problem for various combinations of Dirichlet, Neumann, and mixed boundary conditions, which, in turn, depends on the abovementioned parameter. Analytically, it is much more difficult to determine the values of a parameter; therefore, a MATLAB program is developed with the aid of the Newton method to compute the distributions of the parameter. The existence of the generalized eigenvalues is discussed in a strip $\mathcal{R}_e\,\lambda_\mu \in [0, 1)$ with the aforementioned combinations of the boundary conditions that depend on the apex angle ω_0 of the considered domain. Moreover, it is shown that the obtained eigenvalues and the corresponding eigenfunctions generate the singular terms, which allows us to determine the optimal regularity of the weak solution of the Stokes system.

The organization of this paper is as follows: In Section 2, we introduce some function spaces and present the weak formulation of the stationary Stokes problem. In Section 3, we determine a parametric boundary eigenvalue problem with a complex parameter λ, the stationary Stokes system is being considered for various combinations of Dirichlet, Neumann, and mixed boundary conditions. Furthermore, transcendental equations for different conditions whose zeros are the eigenvalues of the operator pencil $\hat{\mathcal{U}}(\lambda)$ are derived. In Section 4, the distribution of the eigenvalues and the eigenfunctions are discussed. The obtained eigenvalues and eigensolutions yield singular terms. Additionally, some regularity results are presented. Section 5 is devoted to conclusions.

2. Analytical preliminaries

2.1. Some function spaces

Let us consider the following function spaces from [1, 14]. For $v = v(\mathrm{x})$ with $\mathrm{x} = (x_1, x_2) \in \Omega$. We denote by $D^\alpha v$ the multiindex notation for higher-order derivatives and in cartesian coordinates is described as

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \qquad \alpha = (\alpha_1,\, \alpha_2), \qquad |\alpha| = \alpha_1 + \alpha_2. \tag{2.1}$$

For $1 \le p \le \infty$, the space of all Lebesgue-measurable functions v describe on Ω and p-integrable on Ω is denoted by $L^p(\Omega)$ and is equipped with the norm

$$\|v\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |v(\mathrm{x})|^p d\mathrm{x}\right)^{\frac{1}{p}} < \infty, & \text{for} \quad 1 \le p < \infty, \\ ess\,sup\{|v(\mathrm{x})| : \mathrm{x} \in \Omega\}, & \text{for} \quad p = \infty. \end{cases}$$

We can write $\|v\|_{L^p(\Omega)} = \|v\|_{p,\,\Omega}$. For $p = 2$, then $\|v\|_{2,\,\Omega}$ is the norm in $L^2(\Omega)$. It is recognized that $L^p(\Omega)$ is a Banach space. Let $[L^p(\Omega)]^* = L^q(\Omega)$ is the corresponding dual space, where q is the dual exponent given by $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the space $L^2(\Omega)$ is a Hilbert space endowed with the inner product

$$(v, u)_0 = (v, u)_{L^2(\Omega)} = \int_\Omega v(\mathrm{x}) u(\mathrm{x}) d\mathrm{x}.$$

Now, the proper function spaces to define the weak derivatives of functions on domain Ω are introduced. The usual definition of differentiability is too strong for our intentions, and we introduce the concept of weak differentiability. The function $v \in L^p_{loc}(\Omega)$ (this means that $v \in L^p_{loc}(\Omega_0)$ for all $\Omega_0 \subset\subset \Omega$) possesses an αth weak derivative ($\alpha \in \mathbb{N}_0$) if there exists some $u \in L^p_{loc}(\Omega)$ satisfying

$$\int_\Omega v(\mathrm{x}) \frac{\partial^\alpha}{\partial \mathrm{x}^\alpha} \psi(\mathrm{x}) d\mathrm{x} = (-1)^{|\alpha|} \int_\Omega u(\mathrm{x}) \psi(\mathrm{x}) d\mathrm{x} \quad \forall \psi \in C_0^\infty.$$

If this is the case, we write $D^\alpha v = u$ as the weak derivative is unique. Now, it remains to describe the suitable function spaces. For certain given nonnegative integer $m \in \mathbb{N}_0$ and $1 \le p \le \infty$, we denote by $W^{m,p}(\Omega)$ the Sobolev spaces are defined as

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega), \quad \forall |\alpha| \le m\}.$$

This is a Banach space which equipped with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha| \le m} \int_\Omega |D^\alpha v(\mathrm{x})|^p d\mathrm{x} \Big)^{\frac{1}{p}}, \quad p \in [1, \infty),$$

$$\|v\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha| \le m} \|D^\alpha v(\mathrm{x})\|_{L^\infty(\Omega)}, \quad p = \infty.$$

We also need the seminorm

$$|v|_{W^{m,p}(\Omega)} = \Big(\sum_{|\alpha| = m} \int_\Omega |D^\alpha v(\mathrm{x})|^p d\mathrm{x} \Big)^{\frac{1}{p}}, \quad p \in [1, \infty),$$

$$|v|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha| = m} \|D^\alpha v(\mathrm{x})\|_{L^\infty(\Omega)}, \quad p = \infty.$$

The Sobolev spaces of nonintegral order are introduced below in Definition 2.1. Particularly, the space $W^{m,p}(\Omega)$ for the case $p = 2$ is a Hilbert space with the inner product $(.,.)$. For simplicity, denoted by $W^{m,2}(\Omega) = H^m(\Omega) = H^m$ with the norm $\|v\|_{W^{m,2}(\Omega)} = \|v\|_{H^m(\Omega)}$. Furthermore, we use the notation $L_0^2 = \big\{ q \in L^2(\Omega) : \int_\Omega q \, d\mathrm{x} = 0 \big\}$ and let H_0^1 denote the functions in H^1 with zero boundary values. For $m \in (0, 1)$, H_0^m denotes the closure of C_0^∞ in the topology of H^m, where C_0^∞ is the space of all C^∞ functions with compact support in Ω. When $m \ge 1$, $H_0^m = H^m \cap H_0^1$. The dual space of H_0^m is denoted by H^{-m} and is endowed with the norm $\|f\|_{-m} = \sup_{0 \ne v \in H_0^m} \frac{\langle f, v \rangle}{\|v\|_m}$, where the notation $\langle , \rangle$ stands for the duality pairing.

For vector spaces, we can write $\mathbf{H}^m = H^m \times H^m$, $\mathbf{L}^m = L^m \times L^m$, etc. Let C represent a generic constant which can have different values in different places and may depend on certain quantities as parameters.

Definition 2.1 *For a real $m \ge 0$, represented as $m = n + \sigma$ with $n \in \mathbb{N}_0$ and $0 < \sigma < 1$, the space*

$$H^m = W^{m,2}(\Omega) = \Big\{ v : \Omega \to \mathbb{R} : \|v\|_{m,2} < \infty \Big\}, \tag{2.2}$$

where

$$\|v\|_{m,2}^2 = \|v\|_{n,2}^2 + \sum_{|\alpha| = n} |\mathrm{D}^\alpha v|_{\sigma,2}^2,$$

and

$$\|v\|_{n,2} = \Big(\sum_{|\alpha| \le n} \|\mathrm{D}^\alpha v\|_{0,2}^2 \Big)^{\frac{1}{2}},$$

$$|\mathrm{D}^\alpha v|_{\sigma,2} = \Big(\int_\Omega \int_\Omega \frac{|\mathrm{D}^\alpha v(x_1) - \mathrm{D}^\alpha v(x_2)|^2}{|x_1 - x_2|^{2+2\sigma}} dx_1 dx_2 \Big)^{\frac{1}{2}},$$

is known as the Sobolev-Slobodeskij space. It is endowed with the norm $\|v\|_{m,2}$.

2.2. The weak solution of the Stokes problem

In this section, the weak formulation for the stationary Stokes problem (1.1)-(1.3), the solvability, and the uniqueness of the solution are presented in a detail. Let

$$\mathcal{E}(\Omega) = \Big\{\mathbf{u} \in C^\infty(\overline{\Omega})^2;\, \operatorname{div}\mathbf{u} = 0,\, \operatorname{supp}\mathbf{u} \cap \Gamma_D = \emptyset\Big\},$$

where $\operatorname{supp}\mathbf{u} = \{\mathrm{x} \big| \, \mathbf{u}(\mathrm{x}) \neq 0\}$, and $\operatorname{supp}\mathbf{u} \subset \Omega$. Moreover, let $\mathbf{u} \in C^\infty(\Omega)$ be a test function such that $\mathbf{u} = 0$ on Γ_D. Let $V^{m,p}$ be a closure of $\mathcal{E}(\Omega)$ in the norm of $W^{m,p}(\Omega)^2$, $1 \le p < \infty$ and $m \ge 0$ (m need not be an integer). Then $V^{m,p}$ is a Banach space with the norm of $W^{m,p}(\Omega)^2$. For simplicity, we denote $V^{0,2}$ and $V^{1,2}$ as H and U, respectively. They are closed subspaces of the spaces $L^2(\Omega)^2$ and $W^{1,2}(\Omega)^2$. Note that U and H, respectively are Hilbert spaces with the scalar products

$$\big(\mathbf{u}, \mathbf{v}\big)_H = \int_\Omega \mathbf{u} \cdot \mathbf{v}\, d\mathrm{x} \quad \text{and} \quad \big(\mathbf{u}, \mathbf{v}\big)_U = \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v}\, d\mathrm{x} = \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, d\mathrm{x}. \tag{2.3}$$

To seek that $\mathbf{v} \in W^{1,2}(\Omega)^2$ and $q \in L^2(\Omega)$, define the following function space

$$W(\Omega) = \Big\{\mathbf{u} \in W^{1,2}(\Omega)^2 : \;\; \mathbf{u} = 0 \;\text{ on }\; \Gamma_D\Big\}.$$

Moreover, the second equation of (1.1) yields that $\mathbf{v} - \mathbf{h}_1$ belongs to the subsequent space

$$V(\Omega) = \Big\{\mathbf{u} \in W^{1,2}(\Omega)^2 : \, \mathbf{u} = 0 \;\text{ on }\, \Gamma_D,\; \operatorname{div}\; \mathbf{u} = 0,\; \text{ in } \Omega\Big\},$$

supposing $\mathbf{h}_1$ has a divergence-free lift. Clearly, $V(\Omega) \subset W(\Omega)$ are closed subspaces of $W^{1,2}(\Omega)^2$.

The weak solution of the problem (1.1)-(1.3) is obtained from the following variational formulation: find a pair $(\mathbf{v}, q)$ such that $\mathbf{v} - \mathbf{h}_1 \in V$, $q \in L^2(\Omega)$ and

$$a\big(\mathbf{v}, \mathbf{u}\big) + b\big(q, \mathbf{u}\big) = \big(\mathbf{f}, \mathbf{u}\big) + \big(\mathbf{h}_2, \mathbf{u}\big)_{\Gamma_N}, \qquad \forall \mathbf{u} \in W, \tag{2.4}$$

where

$$a\big(\mathbf{v}, \mathbf{u}\big) = 2\nu \int_\Omega \mathrm{D}(\mathbf{u}) : \mathrm{D}(\mathbf{v})\, d\mathrm{x} \quad \text{and} \quad b\big(q, \mathbf{u}\big) = -\int_\Omega q\, (\operatorname{div}\mathbf{u})\, d\mathrm{x},$$

where $\mathrm{D}(\mathbf{u})$ is the symmetric part of the velocity gradient $\nabla\mathbf{u}$. The coercivity of the bilinear form $a(.,.)$ is ensured by Korn's inequality. Furthermore, by De Rham's theorem [39, 40], the equation (2.4) yields the following identity

$$a\big(\mathbf{v}, \mathbf{u}\big) = \big(\mathbf{f}, \mathbf{u}\big) + \big(\mathbf{h}_2, \mathbf{u}\big)_{\Gamma_N}, \qquad \forall \mathbf{u} \in V. \tag{2.5}$$

Remark 2.2 *It is noted that when $\Gamma_D = \partial\Omega$ that the bilinear form a reduces to*

$$a\big(\mathbf{v}, \mathbf{u}\big) = \nu \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, dx,$$

then in this case one does not need Korn's inequality to prove the coerciveness.

Remark 2.3 *The weak solution of the Navier-Stokes system with homogenous boundary conditions is proved in [7, 18]. For any $\mathbf{f} \in L^2$, there exists a uniquely determined weak solution $(\mathbf{v}, q)$ of the homogenous mixed boundary value problem (1.1)-(1.3) and the following estimate holds:*

$$\|\mathbf{v}\|_V + \|q\|_{L^2(\Omega)} \leq c\|\mathbf{f}\|_{L^2(\Omega)}, \tag{2.6}$$

where $c = c(\Omega)$. Hence, we have to analyze the smoothness of the weak solution $(\mathbf{v}, q)$ and see how it depends on the sizes of the angles ω_i, $i = 1, ..., N$ of our polygonal domain.

Remark 2.4 *The weak formulation of the mixed boundary value problem for the Stokes system (1.1)-(1.3) in a bounded Lipschitz domain for an arbitrary $\mathbf{h}_1$ and $\mathbf{h}_2$ is proved in [[13], Theorem 3.1, Part (iii)]. That is, if $\left|\Gamma_D\right| > 0$ and $\left|\Gamma_N\right| > 0$, there exists a unique solution $(\mathbf{v}, q) \in W \times L^2(\Omega)$ of the variational problem (2.4) that depends continuously on the data, i.e.*

$$\|\mathbf{v}\|_W + \|q\|_{L^2(\Omega)} \leq c\Big(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{h}_1\|_{H^{\frac{1}{2}}_{(\Gamma_D)}} + \|\mathbf{h}_2\|_{H^{-\frac{1}{2}}_{(\Gamma_N)}}\Big), \tag{2.7}$$

where the constant $c(\Omega, \Gamma_D)$. The pressure is unique under these conditions, and if $\left|\Gamma_N\right| = 0$, then lose the uniqueness up to a constant.

Remark 2.5 *If the given data on the right-hand sides of (1.1)-(1.3) are smoother, for example, $\mathbf{f} \in L^2(\Omega)^2$, $\mathbf{h}_1 \in [H^{\frac{3}{2}}(\Gamma_D)]^2$ and $\mathbf{h}_2 \in [H^{\frac{1}{2}}(\Gamma_N)]^2$, further if the domain is sufficiently smooth and the boundary conditions do not change their types, then it is proved in [41] that the weak solution $(\mathbf{v}, q)$ of the Stokes system belongs to $[H^2(\Omega)]^2 \times [H^1(\Omega)]$. Instead, if the domain has corner points or points upon which the type of boundary conditions changes, in general, the regularity cannot be improved accordingly (see [16, 21]). As a matter of fact, in these cases, the regularity can be described by a decomposition of the two-dimensional solution $v(x_1, x_2) = \big(v_1, v_2, q\big)^T(x_1, x_2)$ into singular and regular parts of the form*

$$v = v_{sing} + v_{reg} \ = \sum_{j,\,k} r_k^{\lambda_{j,\,k}}\, \Phi_{j,\,k}(\lambda_{j,\,k},\, r_k,\, \theta_k) + v_{reg}. \tag{2.8}$$

Here, the regular part v_{reg} belongs to $[H^2(\Omega)]^2 \times [H^1(\Omega)]$, the corner points are indicated by k with the equivalent polar coordinates $(r_k,\, \theta_k)$, the exponents $\lambda_{j,\,k}$ are the eigenvalues of the considered problem, and $\Phi_{j,\,k}$ are the corresponding generalized eigenvector fields.

Therefore, the information about the singular terms permits us to determine the Sobolev-Slobodeskij spaces wherein the weak solution of the considered boundary value problem is contained. Thus, we can subsequently formulate the regularity problem:

Definition 2.6 *(The regularity problem for the two-dimensional Stokes problem). Determine an optimal $m \in \mathbb{R}$ with $m \geq 0$, so that the leading singularity belongs to the Sobolev-Slobodeskij space $[H^{m+1}(\Omega)]^2 \times [H^m(\Omega)]$.*

3. The Stokes problem in an infinite cone

In this section, we will see the occurrence of the singular terms of the solutions of the boundary value problem governed by the stationary Stokes problem near the corners and the structure which they have. So, to analyze these results, the following steps are followed.

1. We localize the Stokes problem (1.1)-(1.3) in the neighborhood of a corner point and then consider the problem (1.1)-(1.3) in an infinite cone.

2. The problem (1.1)-(1.3) is written in local polar coordinates $(r,\, \theta)$ and then using the change of variable $r = e^{\tau}$. Afterward, the Mellin transform concerning the variable τ is applied to obtain a boundary value problem for a system of ordinary differential equations which depend on the complex parameter λ. Moreover, the operator pencil $\hat{\mathcal{U}}(\lambda)$ is used to represent the generalized form of this parametric boundary eigenvalue problem.

3. The eigenvalues and the generalized eigensolutions of this parametric boundary eigenvalue problem with various kinds of boundary conditions are obtained. They enter the asymptotic development of the solution of the model problem near the corner points. Finally, the regularity results can be followed by the general theory of ellipticity.

3.1. Localization and the model problem

Assume that Ω is a polygonal domain. To show that the weak solution $(\mathbf{v}, q)$ of the underlying boundary value problem is regular, we have to investigate its behaviour near the corner points $P_i\,(i \in \mathcal{J})$. Let us consider the corner point P_N as origin and denote $\omega_N = \omega_0 \in (0, 2\pi)$. An appropriate infinite differentiable cut-off function $\chi(|\mathrm{x}|) = \chi(r)$ depending on the distance r from the point P_N is defined as

$$\chi(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le \epsilon, \\ 0 & \text{for} \quad r \ge 2\epsilon. \end{cases}$$

The number ϵ is so small that P_N is the only corner point of the domain Ω that lies inside the circle $\{\mathrm{x} : |\mathrm{x}| \le 2\epsilon\}$. We multiply both sides of (1.1) and (1.2)-(1.3) by the smooth cut-off function χ, then substitute $(\mathbf{u},\, p) = (\chi\mathbf{v},\, \chi q)$ in (1.1) and likewise in (1.2)-(1.3). The derivatives are considered in the distribution sense. Thus, the boundary value problem is set into an infinite cone

$$S = \big\{(r,\, \theta) : 0 < r < \infty,\, 0 < \theta < \omega_0\big\},$$

and coincides with the original problem near the point P_N. The Stokes system (1.1) becomes

$$\left\{ \begin{aligned} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{F} \quad & \text{in} \quad S, \\ \operatorname{div}\mathbf{u} = G \qquad\quad & \text{in} \quad S, \end{aligned} \right. \tag{3.1}$$

where $\mathbf{F} = \chi\,\mathbf{f} - 2\nu\nabla\chi\cdot\nabla\mathbf{v} - \nu\,\mathbf{v}\,\Delta\chi + q\,\nabla\chi$ and $G = \mathbf{v}\cdot\nabla\chi$. The behavior of $(\mathbf{u}, p)$ near the corner point P_N determines the regularity of the solution $(\mathbf{v}, q)$ in the neighborhood of the point P_N. If we suppose that the right-hand side in (1.1) is $\mathbf{f} \in L^2(\Omega)^2$, then $\mathbf{F} \in L^2(S)^2$ and $G \in H^1(S)$. Besides, the following boundary conditions are prescribed on the subsequent edges $\Gamma_{S,\,0}\,(\theta = 0)$ and $\Gamma_{S,\,\omega_0}\,(\theta = \omega_0)$ of the cone (see Figure 2). Just one condition is considered per edge to differentiate between the mixed boundary conditions. Therefore, the obtained boundary conditions are:

Dirichlet boundary conditions:

$$\mathbf{u} = \mathbf{H}_1 \quad \text{on} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \quad \text{if} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \subset \Gamma_D, \tag{3.2}$$

where $\chi \mathbf{h}_1 = \mathbf{H}_1$.
Neumann boundary conditions:

$$\mathcal{S}\big[\mathbf{u}, p\big]\mathbf{n} = \mathbf{H}_2 \quad \text{on} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0}, \quad \text{if} \quad \Gamma_{S,\,0},\, \Gamma_{S,\,\omega_0} \subset \Gamma_N, \tag{3.3}$$

where $\chi\, \mathbf{h}_2 + \nu\, \mathbf{n}\big(\nabla \chi \cdot \mathbf{v} + \mathbf{v} \cdot \chi\big) = \mathbf{H}_2$, and the notation $(\cdot)$ denotes the vector direct product between two vectors.
Mixed boundary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{H}_1 & \text{on} \quad \Gamma_{S,\,0} \quad \text{if} \quad \Gamma_{S,\,0} \subset \Gamma_D, \\ \mathcal{S}\big[\mathbf{u}, p\big]\mathbf{n} = \mathbf{H}_2 & \text{on} \quad \Gamma_{S,\,\omega_0} \quad \text{if} \quad \Gamma_{S,\,\omega_0} \subset \Gamma_N. \end{cases} \tag{3.4}$$

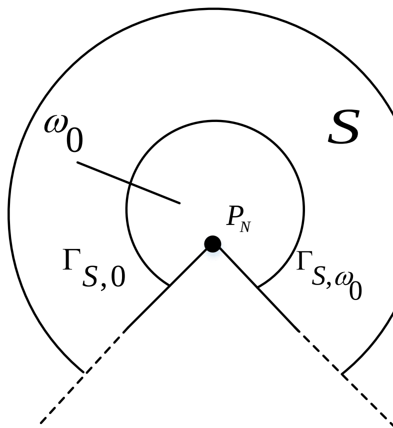

Figure 2. The infinite cone S with opening angle ω_0.

It is observed that the right-hand sides of the obtained boundary conditions have similar smoothness as the original problem in the domain Ω. To analyze the regularity results of the boundary value problem (3.1)-(3.4) , we rewrite the operators in polar coordinates. Hence, the transformed form is

$$\begin{aligned} -\nu\Big(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\Big) + \frac{\partial p}{\partial r} &= F_r, \\ -\nu\Big(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\Big) + \frac{1}{r}\frac{\partial p}{\partial \theta} &= F_\theta, \\ \frac{1}{r}\frac{\partial}{\partial r}(r\, u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}u_\theta &= G, \end{aligned} \tag{3.5}$$

where (u_r, u_θ) are the polar components of the velocity vector $\overline{\mathbf{u}}$, (F_r, F_θ) are the polar components of $\overline{\mathbf{F}}$ and are given by

$$\overline{\mathbf{u}} = \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \overline{\mathbf{F}} = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = A\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Similarly, the boundary conditions (3.2)-(3.4) emerge as

$$\overline{\mathbf{u}}\big|_{\theta=0,\,\omega_0} = (u_r, u_\theta)^T\big|_{\theta=0,\,\omega_0} = \overline{\mathbf{H}}^1, \tag{3.6}$$

$$\begin{cases} \frac{1}{r}\,\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\,u_\theta\big|_{\theta=0,\,\omega_0} = \overline{H}^2_r, \\ -p + 2\nu\big(\frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}\,u_r\big)\big|_{\theta=0,\,\omega_0} = \overline{H}^2_\theta, \end{cases} \tag{3.7}$$

$$\begin{cases} u_r\big|_{\theta=0} = \overline{H}^1_r, \\ u_\theta\big|_{\theta=0} = \overline{H}^1_\theta, \\ \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta\big|_{\theta=\omega_0} = \overline{H}^2_r, \\ -p + 2\nu\big(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r}u_r\big)\big|_{\theta=\omega_0} = \overline{H}^2_\theta, \end{cases} \tag{3.8}$$

and $\overline{\mathbf{H}}^m = (\overline{H}^m_r, \overline{H}^m_\theta)^T$, where $m = 1$ for Dirichlet and $m = 2$ for Neumann boundary conditions. They hold in the infinite cone where $\overline{\mathbf{u}}(r,\theta) = \mathbf{u}(x_1,x_2)$, $\overline{p}(r,\theta) = p(x_1,x_2)$, $\overline{\mathbf{F}}(r,\theta) = \mathbf{F}(x_1,x_2)$ and $\overline{G}(r,\theta) = G(x_1,x_2)$.

Now, the variable τ is introduced by the relation $r = e^\tau$. Accordingly, the system (3.5) is set on the infinite strip with width ω_0 as

$$\begin{aligned} -\nu\Big(\frac{\partial^2 \tilde{u}_\tau}{\partial \tau^2} + \frac{\partial^2 \tilde{u}_\tau}{\partial \theta^2} - \tilde{u}_\tau - 2\,\frac{\partial \tilde{u}_\theta}{\partial \theta}\Big) + \frac{\partial \tilde{p}}{\partial \tau} - \tilde{p} &= \tilde{F}_\tau \quad \text{in} \quad \bar{S}, \\ -\nu\Big(\frac{\partial^2 \tilde{u}_\theta}{\partial \tau^2} + \frac{\partial^2 \tilde{u}_\theta}{\partial \theta^2} - \tilde{u}_\theta + 2\,\frac{\partial \tilde{u}_\tau}{\partial \theta}\Big) + \frac{\partial \tilde{p}}{\partial \theta} &= \tilde{F}_\theta \quad \text{in} \quad \bar{S}, \\ \frac{\partial \tilde{u}_\tau}{\partial \tau} + \tilde{u}_\tau + \frac{\partial \tilde{u}_\theta}{\partial \theta} &= \tilde{G} \quad \text{in} \quad \bar{S}. \end{aligned} \tag{3.9}$$

Here, $\bar{S} = \big\{(\tau,\theta) : -\infty < \tau < \infty,\, 0 < \theta < \omega_0\big\}$ and $\tilde{\mathbf{u}} = \overline{\mathbf{u}}(e^\tau,\theta)$, $\tilde{p} = e^\tau\,\overline{p}(e^\tau,\theta)$, $\tilde{\mathbf{F}} = e^{2\tau}\,\overline{\mathbf{F}}(e^\tau,\theta)$ and $\tilde{G} = e^\tau\,\overline{G}(e^\tau,\theta)$. The Dirichlet, Neumann, and mixed boundary conditions also yield the transformed form with the boundary data $\widetilde{\mathbf{H}}^{l+1} = e^{l\tau}\overline{\mathbf{H}}^{l+1}(e^\tau,\theta)$, $l = 0,1$ as

$$\tilde{\mathbf{u}}\big|_{\theta=0,\,\omega_0} = (\tilde{u}_\tau, \tilde{u}_\theta)^T\Big|_{\theta=0,\,\omega_0} = \widetilde{\mathbf{H}}^1, \tag{3.10}$$

$$\begin{cases} \pm\nu\big(\frac{\partial \tilde{u}_\tau}{\partial \theta} + \frac{\partial \tilde{u}_\theta}{\partial \tau} - \tilde{u}_\theta\big)\Big|_{\theta=0,\,\omega_0} = \widetilde{H}^2_\tau, \\ \pm\big(-\tilde{p} + 2\nu(\frac{\partial \tilde{u}_\theta}{\partial \theta} + \tilde{u}_\tau)\big)\Big|_{\theta=0,\,\omega_0} = \widetilde{H}^2_\theta, \end{cases} \tag{3.11}$$

$$\begin{cases} \tilde{u}_\tau\big|_{\theta=0} = \widetilde{H}^1_\tau, \\ \tilde{u}_\theta\big|_{\theta=0} = \widetilde{H}^1_\theta, \\ \nu\big(\frac{\partial \tilde{u}_\tau}{\partial \theta} + \frac{\partial \tilde{u}_\theta}{\partial \tau} - \tilde{u}_\theta\big)\Big|_{\theta=\omega_0} = \widetilde{H}^2_\tau, \\ -\tilde{p} + 2\nu\big(\frac{\partial \tilde{u}_\theta}{\partial \theta} + \tilde{u}_\tau\big)\Big|_{\theta=\omega_0} = \widetilde{H}^2_\theta. \end{cases} \tag{3.12}$$

To obtain the boundary eigenvalue value problem, the Mellin transform with respect to $r \in \mathbb{R}^+_0$ is introduced as

$$\mathcal{M}[\bar{\mathbf{u}}(r)](\alpha) = \hat{\mathbf{u}}(\alpha) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \bar{\mathbf{u}}(r) r^{-\alpha-1} dr, \qquad \alpha \in \mathbb{C}. \tag{3.13}$$

Letting $r = e^{\tau}$, (3.13) yields

$$\mathcal{M}[\bar{\mathbf{u}}(r)](\alpha) = \hat{\mathbf{u}}(\alpha) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\lambda\tau}\, \tilde{\mathbf{u}}(e^{\tau}) d\tau = \mathcal{F}[\tilde{\mathbf{u}}(e^{\tau})](\lambda), \tag{3.14}$$

where $\alpha = i\lambda, \lambda \in \mathbb{C}$, $i^2 = -1$ and $\mathcal{F}[\tilde{\mathbf{u}}(e^{\tau})](\lambda)$ is the complex Fourier transform with respect to the variable τ. We have

$$\operatorname{Re}\alpha = -\operatorname{Im}\lambda, \quad \operatorname{Im}\alpha = \operatorname{Re}\lambda.$$

Now, by applying (3.14) to (3.9)-(3.12) with respect to τ, the two-point boundary value problem for the unknown functions $(\hat{u}_{\tau}, \hat{u}_{\theta}, \hat{p})$ is obtained. It depends on the complex parameter λ and holds on the interval $I = (0, \omega_0)$. Let $\hat{L}(\lambda)$ denote the matrix differential operator of the transformed form of the system (3.9) and maps $W^{2,2}(I)^2 \times W^{1,2}(I) \to L^2(I)^2 \times W^{1,2}(I)$. Therefore, one has

$$\hat{L}(\lambda)(\hat{\mathbf{u}}, \hat{p}) = (\hat{\mathbf{F}}, \hat{G}) \quad \text{on} \quad I = (0, \omega_0), \tag{3.15}$$

where

$$\hat{L}(\lambda) = \begin{pmatrix} -\nu\big[\frac{\partial^2}{\partial\theta^2} - (1+\lambda^2)\big] & 2\nu\frac{\partial}{\partial\theta} & -(1-i\lambda) \\ -2\nu\frac{\partial}{\partial\theta} & -\nu\big[\frac{\partial^2}{\partial\theta^2} - (1+\lambda^2)\big] & \frac{\partial}{\partial\theta} \\ (1+i\lambda) & \frac{\partial}{\partial\theta} & 0 \end{pmatrix}. \tag{3.16}$$

Additionally, the matrix boundary operators for different kinds of boundary conditions can be written as:
For Dirichlet boundary conditions

$$\hat{B}_{DD1}(\lambda)\big|_{\theta=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{B}_{DD2}(\lambda)\big|_{\theta=\omega_0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.17}$$

For Neumann boundary conditions

$$\hat{B}_{NN1}(\lambda)\big|_{\theta=0} = \begin{pmatrix} \nu\frac{\partial}{\partial\theta} & -\nu(1-i\lambda) & 0 \\ 2\nu & 2\nu\frac{\partial}{\partial\theta} & -1 \end{pmatrix}, \quad \hat{B}_{NN2}(\lambda)\big|_{\theta=\omega_0} = \begin{pmatrix} -\nu\frac{\partial}{\partial\theta} & \nu(1-i\lambda) & 0 \\ -2\nu & -2\nu\frac{\partial}{\partial\theta} & 1 \end{pmatrix}. \tag{3.18}$$

For mixed boundary conditions

$$\hat{B}_{DN1}(\lambda)\big|_{\theta=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \hat{B}_{DN2}(\lambda)\big|_{\theta=\omega_0} = \begin{pmatrix} \nu\frac{\partial}{\partial\theta} & -\nu(1-i\lambda) & 0 \\ 2\nu & 2\nu\frac{\partial}{\partial\theta} & -1 \end{pmatrix}. \tag{3.19}$$

Therefore, the operator $\hat{B}_{[.\,.]}(\lambda)$ is used below to define the general transformed form of the matrix boundary operators for different kinds of boundary conditions

$$\Big\{\hat{B}_{[.\,.]}(\lambda)(\hat{\mathbf{u}}, \hat{p})\Big\} = \big(\hat{\mathbf{H}}^1, \hat{\mathbf{H}}^2\big) \quad \text{on} \quad \partial I = (0, \omega_0). \tag{3.20}$$

Accordingly, the generalized form of the operator pencil $\hat{\mathcal{U}}(\lambda)$ for the two-point boundary value problem can be written as

$$\hat{\mathcal{U}}(\lambda) = \Big[\hat{L}(\lambda), \, \big\{\hat{B}_{[.\,.]}(\lambda)\big\}\Big]. \tag{3.21}$$

Therefore, the operator $\hat{\mathcal{U}}(\lambda)$ maps $W^{2,2}(I)^2 \times W^{1,2}(I)$ into $L^2(I)^2 \times W^{1,2}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$. Note that $\hat{\mathcal{U}}(\lambda)$ can be defined for every boundary point in the sense of [2, 3]. Thus, $\hat{\mathcal{U}}(\lambda)(\theta,\, \lambda) = 0$ is used to describe a generalized eigenvalue problem and the solvability of these type of problems is discussed in [24]. Besides, the eigenvalues of the operator $\hat{\mathcal{U}}(\lambda)$ are obtained with the determinant method; this means that the nontrivial solution of the generalized eigenvalue problem leads to a transcendental equation whose zeros are the eigenvalues of $\hat{\mathcal{U}}(\lambda)$. To compute the eigenvalues and the corresponding eigenfunctions, we proceed as follows.

Definition 3.1 *A complex number* $\lambda = \lambda_0$ *is known as eigenvalue of* $\hat{\mathcal{U}}(\lambda)$ *if there exists a nontrivial solution, i.e.* $\hat{u}(.,\lambda_0) \neq 0$, *which is holomorphic at* λ_0, *such that* $\hat{\mathcal{U}}(\lambda_0)\, \hat{u}(\theta,\lambda_0) = 0$. $\hat{u}(\theta,\lambda_0)$ *is called an eigenfunction of* $\hat{\mathcal{U}}(\lambda_0)$ *corresponding to the eigenvalue* λ_0. *The set of fields* $\big\{\hat{u}_0(\theta,\lambda_0),\, \hat{u}_{0,1}(\theta,\lambda_0), \ldots,\, \hat{u}_{0,s}(\theta,\lambda_0)\big\}$ *with* $\hat{u}_{0,0} = \hat{u}_0$ *is said to be a Jordan chain corresponding to the eigenvalue* λ_0, *if the equation*

$$\sum_{q=0}^{m} \frac{1}{q!} \big(\frac{\partial}{\partial\lambda}\big)^q \hat{\mathcal{U}}(\lambda)\, \hat{\mathbf{v}}_{0,m-q}(\theta,\lambda)\big|_{\lambda=\lambda_0} = 0 \quad \textit{for} \quad m = 1, 2, \ldots, s,$$

is satisfied. The number $s+1$ *is called the length of the Jordan chain.*

Remark 3.2 *It is noted [22–24] that if the complex number* λ *is not an eigenvalue of the operator* $\hat{\mathcal{U}}(\lambda)$, *then* $\hat{\mathcal{U}}(\lambda)$ *is an isomorphism between the spaces* $W^{2,2}(I)^2 \times W^{1,2}(I)$ *and* $L^2(I)^2 \times W^{1,2}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$.

3.2. The calculation of the eigenvalues

To evaluate the eigenvalues and the corresponding eigenfunctions of the stationary Stokes system for various boundary conditions, the determinant method is considered (see [5]). The result is the transcendental equations whose roots are the eigenvalues, namely, λ_μ wherein (μ is used for multiple eigenvalues, i.e., $\mu = 1, \ldots, N$).

Dirichlet boundary conditions (DD): It means that Dirichlet boundary conditions are given on both sides of the corner point. The solutions of the equation

$$\sin^2(\lambda\,\omega_0) = \lambda^2\, \sin^2(\omega_0), \tag{3.22}$$

are the eigenvalues of $\hat{\mathcal{U}}(\lambda)$.

Neumann boundary conditions (NN): It means that the Neumann boundary conditions are given on both sides of the corner point. The eigenvalues are the solutions of the equation

$$\lambda^2\, \sin^2(\omega_0) - \sin^2(\lambda\,\omega_0) = 0. \tag{3.23}$$

Mixed boundary conditions (DN): It means that Dirichlet or Neumann boundary condition is given on one-side of the corner point and the other condition is given on the other side. The eigenvalues are the solutions of the equation

$$\cos^2(\lambda\,\omega_0) - \lambda^2\, \sin^2(\omega_0) = 0. \tag{3.24}$$

4. Regularity results

Let $(\mathbf{v}, q) \in W^{1,2}(\Omega)^2 \times L^2(\Omega)$ be the unique weak solution of the stationary Stokes problem. The information of the singular terms permits us to evaluate the optimal regularity of the weak solution (see [4, 38]). The exponent λ_μ (generally complex) described below are the resulting eigenvalues that can be obtained from the above derived transcendental equations of the generalized boundary eigenvalue problem. For the optimal regularity of the weak solution, it holds that the solution belongs to $H^{1+\mathcal{R}_e\lambda_{\omega_0}-\epsilon}(\Omega)$, where $\epsilon > 0$ and λ_{ω_0} is that eigenvalue which has the smallest real part $\mathcal{R}_e\,\lambda_{\omega_0}$ that lies in the interval $(0,1)$. In the subsequent figures (Figures 3 and 4), the black lines reveal the real eigenvalues, while the red lines reveal the real parts of the conjugate pair of complex eigenvalues.

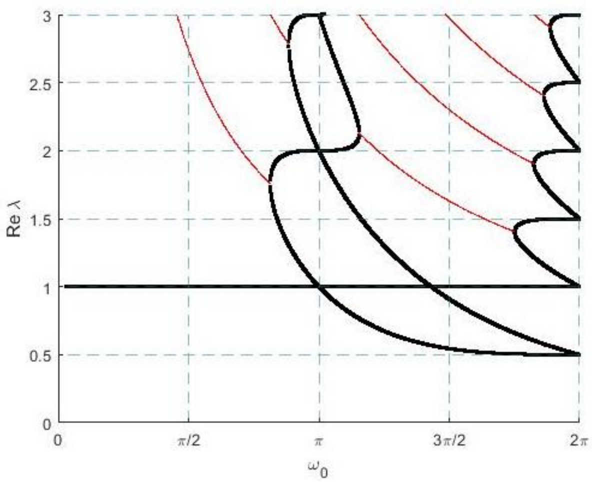

Figure 3. Distribution of the eigenvalues for DD and NN boundary conditions, where black lines $\longrightarrow$ real eigenvalues and red lines $\longrightarrow$ real parts of the conjugate pair of complex eigenvalues.

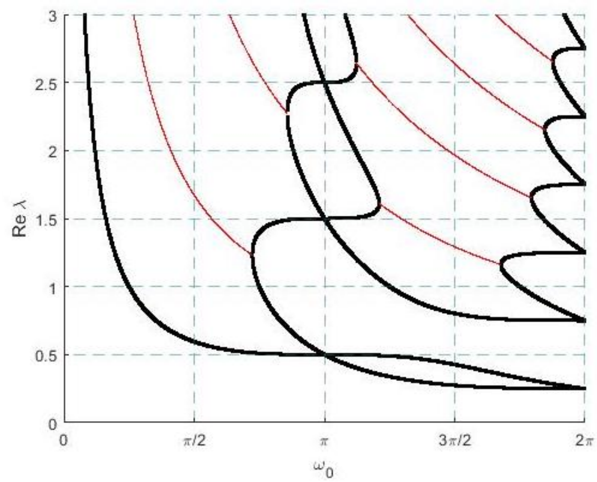

Figure 4. Distribution of the eigenvalues for Dirichlet-Neumann boundary conditions, where black lines $\longrightarrow$ real eigenvalues and red lines $\longrightarrow$ real parts of the conjugate pair of complex eigenvalues.

To estimate the singular terms in solutions, the query arises whether we have achieved all the feasible singular terms. The response is yes if all the eigenvalues are simple. The following theorem [11, 24] expresses the singular behavior of the solution of the problem (2.4) in the neighborhood of a corner point.

Theorem 4.1 *Given* $\mathbf{f} \in L^p(\Omega)^2$, $1 \le p < \infty$, $(\mathbf{v}, q)$ *is the uniquely determined weak solution of the problem (2.4) and let* P *be an isolated corner point of* Γ. *If* $\lambda_1, \lambda_2, ..., \lambda_N$ *are the eigenvalues of the operator* $\hat{\mathcal{U}}(\lambda)$, *then the solution* $(\mathbf{v}, q)$ *admits the subsequent expansion in a neighborhood* P_δ *of* P, *i.e.*

$$\big(\mathbf{v}, q\big) = \chi(r)\left[\sum_{\mu=1}^{N}\sum_{\rho=1}^{I_\mu}\sum_{\kappa=0}^{\kappa_{\mu\rho}-1} c_{\mu,\,\rho,\,\kappa}\,\mathbf{\Phi}_{\mu,\,\rho,\,\kappa}(r,\,\theta)\right] + \Big[\mathbf{v}_{reg}(r,\,\theta),\, q_{reg}(r,\,\theta)\Big], \tag{4.1}$$

with $\big(\mathbf{v}_{reg}(r,\theta), q_{reg}(r,\theta)\big) \in W^{2,p}(P_\delta)^2 \times W^{1,p}(P_\delta)$. *Here,* N *be the number of all eigenvalues of the operator* $\hat{\mathcal{U}}(\lambda)$ *in the strip* $\mathcal{R}_e\lambda_\mu \in (0, 2-\frac{2}{p})$, *the constants* $c_{\mu,\,\rho,\,\kappa}$ *depend on the data and the singular functions,* $I_\mu = \dim \operatorname{Ker} \hat{\mathcal{U}}(\lambda_\mu)$, $\kappa_{\mu\rho}$ *is the length of the Jordan chains of* $\hat{\mathcal{U}}(\lambda_\mu)$ *and the corresponding singular functions* *are given by*

$$\mathbf{\Phi}_{\mu,\,\rho,\,\kappa}(r,\,\theta) = \Big(\mathbf{v}_{\mu,\,\rho,\,\kappa}(r,\,\theta),\, q_{\mu,\,\rho,\,\kappa}(r,\,\theta)\Big), \tag{4.2}$$

with

$$\mathbf{v}_{\mu,\,\rho,\,\kappa}(r,\,\theta) = r^{\lambda_\mu} \sum_{j=0}^{\kappa} \frac{(\log r)^j}{j!}\, \Phi_\mu^{\rho,\,\kappa-j}(\theta), \tag{4.3}$$

$$q_{\mu,\,\rho,\,\kappa}(r,\,\theta) = r^{\lambda_\mu - 1} \sum_{j=0}^{\kappa} \frac{(\log r)^j}{j!}\, \varphi_\mu^{\rho,\,\kappa-j}(\theta). \tag{4.4}$$

The logarithmic terms occur only if λ_μ has the algebraic multiplicity greater than one.

For a detailed explanation of the eigenvalues and the equivalent eigenvectors, we refer to [[25] Chapter 5.1] and the results specified in [[25] Theorem 5.1.1].

Now, we briefly describe the results on the eigenvalues of the equations (3.22)-(3.24) and the corresponding eigenvectors. We consider the equation (3.22) to find the roots, and the others will be treated analogously. Ordering these solutions with the nondecreasing real part, a nondecreasing sequence of numbers $\lambda_{i,\,j} : j = 1, 2, ...$ is obtained, where $i = 1, ..., N$ is used to represent the number of vertices or corner points of Ω.

The numbers $s_{i,\,j}$ are defined by

$$s_{i,\,j} = \mathcal{R}_e\, \lambda_{i,\,j} + 1, \quad j = 1, 2, ...,$$

which is known as the order of the regularity of the solution space. Let $[\Phi_{i,\,j},\, \varphi_{i,\,j}]$ be the singular functions corresponding to the velocity and pressure with the singular exponents $\lambda_{i,\,j}$ and defined as

$$\Phi_{i,\,j} = \chi_i\, r_i^{\lambda_{i,\,j}} \tau_{i,\,j}(\theta), \qquad \varphi_{i,\,j} = \chi_i\, r_i^{\lambda_{i,\,j}-1} \xi_{i,\,j}(\theta). \tag{4.5}$$

The functions $\xi_{i,\,j}(\theta)$ and $\tau_{i,\,j}(\theta)$ are certain trigonometric pressure and velocity eigenfunctions respectively, relative to the eigenvalues $\lambda_{i,\,j},\ j = 1, 2, ...$, where χ_i is a smooth cutoff function. More information about the number $\lambda_{i,\,j}$ can be found in [22, 25]. Furthermore, the nonconvex and convex cases are discussed below separately regarding the apex angle ω_0.

Case 1. For the nonconvex case, that is $\omega_0 \in (\pi,\, 2\pi)$, the first 3 leading eigenvalues $\lambda_{i,\,j},\ j = 1, 2, 3$ are real and the properties

$$\frac{1}{2} < \lambda_{i,\,1} < \frac{\pi}{\omega_0} < \lambda_{i,\,2} = 1 < \lambda_{i,\,3} < \frac{2\pi}{\omega_0}, \quad \omega_0 \in (\pi, \omega^*], \tag{4.6}$$

$$\frac{1}{2} < \lambda_{i,\,1} < \frac{\pi}{\omega_0} < \lambda_{i,\,2} < \lambda_{i,\,3} = 1 < \frac{2\pi}{\omega_0}, \quad \omega_0 \in (\omega^*, 2\pi). \tag{4.7}$$

hold. In particular, $\omega^* \approx 1.4303\pi$ is the unique solution of the equation $\tan\omega - \omega = 0$ in the interval $\omega \in [0, 2\pi)$. It can be seen that for an angle $\omega_0 \in (\omega^*, 2\pi)$, there are two eigenvalues $\lambda_{i,\,1}, \lambda_{i,\,2}$ less than 1.

Case 2. For the convex case, that is $\omega_0 \in (0,\, \pi)$, $\lambda_{i,\,1}$ is a simple and unique eigenvalue that lie in the strip $0 < \mathcal{R}_e \lambda_{i,\,1} < \frac{\pi}{\omega_0}$. For this, the relative pressure eigenfunction $\xi_{i,\,j}(\theta)$ has a constant value and the velocity eigenfunction $\tau_{i,\,j}(\theta)$ is zero.

Similarly, the dual singular functions for the velocity vector and the pressure function are $\Big\{\big(\Phi^{-}_{i,\,j},\,\varphi^{-}_{i,\,j}\big)$, for $j \geq 1\Big\}$ and defined by

$$\begin{aligned}\Phi^{-}_{i,\,j} &= \chi_i\, r_i^{-\lambda_{i,\,j}}\, \tau^{-}_{i,\,j}(\theta),\\ \varphi^{-}_{i,\,j} &= \chi_i\, r_i^{-\lambda_{i,\,j}-1}\, \xi^{-}_{i,\,j}(\theta).\end{aligned} \tag{4.8}$$

Hence, the functions $\{\tau^{-}_{i,\,j}(\theta), \xi^{-}_{i,\,j}(\theta)\}$ are obtained by replacing $\lambda = -\lambda_{i,\,j}$ into the eigenfunctions $\{\tau_{i,\,j}(\theta), \xi_{i,\,j}(\theta)\}$.

Remark 4.2 *It is noted from the abovementioned results that the qualitative properties of the solution including regularity of the underlying boundary value problem depends on the properties of the eigenvalues* λ_μ. *It is observed that if* $\mathcal{R}_e\,\lambda_\mu \geq 1$, *then the general solution defined in (4.1) is regular and belong to* $[H^2(\Omega)]^2 \times [H^1(\Omega)]$. *If* $\mathcal{R}_e\,\lambda_\mu \leq 0$, *then the solution does not belong to* $[H^1(\Omega)]^2 \times [L^2(\Omega)]$. *The case* $\mathcal{R}_e\,\lambda_\mu = 0$ *represents the translation which is regular. Hence, we consider only those eigenvalues of the generalized boundary eigenvalue problem that lies in the strip* $0 \leq \mathcal{R}_e\,\lambda_\mu < 1$. *Furthermore, the generalized eigenvalues depend on the values of the apex angle* ω_0.

It is stated in Section 2 that the presence of the corner points on the boundary of the domain Ω does not affect the behavior of the regular part of the solution of the underlying boundary value problem. Thus, the following theorem describes the regularity of the singular terms of the solution of the corresponding problem in Ω.

Theorem 4.3 (**Regularity**) *Let* λ_{ω_0} *be a simple eigenvalue with the real part* $\mathcal{R}_e\,\lambda_{\omega_0}$ *lies in the interval* $(0,1)$, *and presume that it comprehends an eigenvalue with the smallest real part. The equivalent leading singular solutions of the considered boundary value problem in* S *is defined in (4.1), where the functions* $\big(\Phi^{\rho,\,\kappa-j}_{\mu}(\theta), \varphi^{\rho,\,\kappa-j}_{\mu}(\theta)\big)$ *given in (4.3) and (4.4) are the angular dependent part of the solution. Then for an arbitrary small but fixed* $\epsilon > 0$, *we have*

$$\mathbf{v}_{sing} = (\mathbf{v}_s, q_s) \in [H^{\mathcal{R}_e\,\lambda_{\omega_0}+1-\epsilon}(\Omega)]^2 \times [H^{\mathcal{R}_e\,\lambda_{\omega_0}-\epsilon}(\Omega)]. \tag{4.9}$$

Proof For the proof, we use the idea of [[17] Section 1.4.5] and [22, 25]. Let $\Omega \subset \mathbb{R}^2$ be a bounded plane polygonal domain, whose boundary comprises the corner points $P_i : 1 \leq i \leq N$. Let $\eta(P_i)$ be the neighborhood of P_i such that

$$\eta(P_i) \cap \bar{\Omega} \subseteq \big\{(r,\,\theta) : 0 \leq r < \infty,\, 0 < \theta < \omega_0\big\}, \tag{4.10}$$

with $\omega_0 < 2\pi$. Let $\mathbf{v}_{sing}$ be a function which is smooth on $\bar{\Omega} \setminus \{P_i\}$ and coincides with $r^{\lambda_\mu}\Psi(\theta)$ on $\eta(P_i) \cap \Omega$, where $\Psi(\theta) = (\Phi^{\rho,\,\kappa-j}_{\mu}(\theta), \varphi^{\rho,\,\kappa-j}_{\mu}(\theta)) \in C^\infty(0,\omega_0)$. Thus, for $p = 2$, we get

$$\mathbf{v}_s \in [H^m(\Omega)]^2 \quad \text{for any} \quad m < \mathcal{R}_e\,\lambda_{\omega_0} + 1,$$

and

$$q_s \in [H^m(\Omega)] \quad \text{for any} \quad m < \mathcal{R}_e\,\lambda_{\omega_0},$$

and hence the assertion is shown. □

Remark 4.4 *The method of proof is first developed by [6] and consists of proving that* $\mathbf{v}_{sing} \in W_r^l(\Omega)$ *for an integer* $l > m$ *and* $r < p$ *and then using the Sobolev imbeddings. So far, we have considered the case for* $p \geq 2$*. The general proof for* $p < 2$ *makes use of the weighted Sobolev spaces.*

Similar results hold for functions of the form $r^{\lambda_\mu}(\ln r)\Psi(\theta)$.

5. Conclusion

In this article, we have studied the boundary singularities and regularity of the weak solution of the mixed boundary value problem for the stationary Stokes system in a nonsmooth domain with corner points and points at which the type of boundary conditions changes. It is noted that near these points, the Stokes flow can generate infinite pressures and infinite velocity gradients. However, physical implications of these results are worth exploring to understand whether the singularities in a natural flow are discrepancies in the mathematical tools while modeling the complex phenomena. This is considered a topic of separate research efforts. Moreover, to obtain the singular terms, the transcendental equations of the generalized boundary eigenvalue problem for the Stokes system are derived for different boundary conditions. The roots of these equations are the eigenvalues of the operator $\hat{\mathcal{U}}(\lambda)$. These eigenvalues and corresponding eigensolutions produce singular terms.

To get the maximal regularity of the underlying problem, we have accounted for only those eigenvalues that lie in the strip $0 \leq \mathcal{R}_e\,\lambda_\mu < 1$. The generalized eigenvalues λ_μ depend on the values of the apex angle ω_0. It is noted from the above achieved results that if $\mathcal{R}_e\,\lambda_\mu \geq 1$ then the solution defined in (4.1) is regular and belongs to $[H^2(\Omega)]^2 \times [H^1(\Omega)]$. The case $\mathcal{R}_e\,\lambda_\mu = 0$ represents the translation which is regular. It is seen for the case of Dirichlet and Neumann boundary conditions that for an apex angle $\omega_0 \in (\omega^*, 2\pi)$, there are two eigenvalues $\lambda_{i,1}, \lambda_{i,2}$ which are less than 1. For these cases, the weak solution $(\mathbf{v}, q)$ of the considered problem has singularities, if the domain Ω has reentrant corners $(\omega_i > \pi : i = 1, 2, ...N)$. On the other hand, for the case of mixed conditions, the singularities appear at corners with $(\omega_i > \frac{\pi}{4} : i = 1, 2, ...N)$. Moreover, it is observed that if singularities exist, then splitting the solution into a singular part which defines a linear combination of explicit model singularity functions s_m for the Stokes operator with corresponding unknown coefficients C_m, and a regular part that belongs to $H^2 \times H^1$. The results to be achieved here can be extended to general three-dimensional domains (not necessarily axisymmetric or prismatic) with straight edges to analyze the edge singularities and regularity expansion of the solution.

Presently, the Stokes and Navier-Stokes systems with the Navier-slip boundary conditions and the free-boundary problems in bounded domains with corners have very interesting phenomena. The issues regarding their existence and regularity are considered for smooth domains, but theoretical results for the corner singularity decomposition are still not obtained. Therefore, these issues are numerically interesting. In future works, it is important to show the unique existence of the approximations for the regular parts and coefficients, and to derive their error estimates. On the other hand, it is also observed that the nonstationary compressible Stokes and Navier-Stokes equations on polygonal domains could be considered.

Conflict of interest

The author declares that there are no conflicts of interest.

References

[1] Adams RA, Fournier JJ. Sobolev spaces. Academic Press, 2003.

[2] Agmon S, Douglis A, Nirenberg L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I. Communications on Pure and Applied Mathematics 1959; 12 (4): 623-727. http://doi.org/10.1002/cpa.3160120405

[3] Agmon S, Douglis A, Nirenberg L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. Communications on Pure and Applied Mathematics 1964; 17 (1): 35-92. https://doi.org/10.1002/cpa.3160170104

[4] Anjam YN. Singularities and regularity of stationary Stokes and Navier-Stokes equations on polygonal domains and their treatments. AIMS Mathematics 2019; 5 (1): 440-466. http://dx.doi.org/10.3934/math.2020030

[5] Bab$\breve{u}$ska I, Guo BQ, Osborn JE. Regularity and numerical solution of eigenvalue problems with piecewise analytic data. SIAM Journal on Numerical Analysis 1989; 26 (6): 1534-1560. https://doi.org/10.1137/0726090

[6] Bab$\breve{u}$ska I, Kellogg RB, Pitk$\ddot{a}$ranta J. Direct and inverse error estimates for finite elements with mesh refinements. Numerische Mathematik 1979; 33 (4): 447-471. https://doi.org/10.1007/BF01399326

[7] Ben$\breve{e}$s M, K$\breve{u}$cera P. Solutions of the Navier-Stokes equations with various types of boundary conditions. Archiv Der Mathematik 2012; 98 (5): 487-497. https://doi.org/10.1007/s00013-012-0387-x

[8] Choi HJ, Kweon JR. The stationary Navier-Stokes system with no-slip boundary condition on polygons: corner singularity and regularity. Communications in Partial Differential Equations 2013; 38 (7): 1235-1255. https://doi.org/10.1080/03605302.2012.752386

[9] Chorfi N. Geometric singularities of the Stokes problem. Abstract and Applied Analysis 2014; 2014: 1-8. https://doi.org/10.1155/2014/491326

[10] Dauge M. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. part I. linearized equations. SIAM Journal on Mathematical Analysis 1989; 20 (1): 74-97. https://doi.org/10.1137/0520006

[11] Dauge M. Elliptic boundary value problems on corner domains: smoothness and asymptotics of solutions. Springer, 2006.

[12] Durand M. Singularities in elliptic problems. Singularities and constructive methods for their treatment. Springer, 1985; 104-112.

[13] Fabricius J. Stokes flow with kinematic and dynamic boundary conditions. Quarterly of Applied Mathematics 2019; 77 (3): 525-544. http://doi.org/10.1090/qam/1534

[14] Gilbarg D, Trudinger NS. Elliptic partial differential equations of second order. Springer, 2015.

[15] Girault V, Raviart PA. Finite element methods for Navier-Stokes equations: theory and algorithms. Springer Science and Business Media, 2012.

[16] Grisvard P. Behavior in a polygonal or polyhedral domain. Numerical solution of partial differential equations, III. Academic Press, 1976, 207-274.

[17] Grisvard P. Elliptic problems in nonsmooth domains. Volume 2, 2-2. Pitman Advanced Pub. Program, Boston, 1985.

[18] Hou Y, Pei S. On the weak solutions to steady Navier-Stokes equations with mixed boundary conditions. Mathematische Zeitschrift 2019; 291 (1-2): 47-54. https://doi.org/10.1007/s00209-018-2072-7

[19] Jang DK, Pyo JH. Algorithms to apply finite element dual singular function method for the Stokes equations including corner singularities. Journal of the Korean Society for Industrial and Applied Mathematics 2019; 23 (2): 115-138. https://doi.org/10.12941/jksiam.2019.23.115

[20] Kellogg B. Some simple boundary value problems with corner singularities and boundary layers. Computers and Mathematics with Applications. An International Journal 2006; 51 (5): 783-792. https://doi.org/10.1016/j.camwa.2006.03.010

[21] Kellogg RB, Osborn JE. A regularity result for the Stokes problem in a convex polygon. Journal of Functional Analysis 1976; 21 (4): 397-431. https://doi.org/10.1016/0022-1236(76)90035-5

[22] Kondratíev VA. Boundary value problems for elliptic equations in domains with conical or angular points. Trudy Moskovskogo Matematěceskogo Ob̆šcestva 1967; 16: 209-292.

[23] Kondratíev VA, Oleinik OA. Boundary value problems for partial differential equations in non-smooth domains. Russian Mathematical Surveys 1983; 38 (2): 1-86.

[24] Kozlov V, Mazya V, Rossmann J. Elliptic boundary value problems in domains with point singularities. American Mathematical Society, 1997.

[25] Kozlov V, Mazya V, Rossmann J. Spectral problems associated with corner singularities of solutions to elliptic equations. American Mathematical Society, 2001.

[26] Krăcmar S, Neustupa J. A weak solvability of a steady variational inequality of the Navier-Stokes type with mixed boundary conditions. Nonlinear Analysis: Theory, Methods and Applications 2001; 47 (6): 4169-4180. https://doi.org/10.1016/S0362-546X(01)00534-X

[27] Kŭcera P. Basic properties of solution of the non-steady Navier-Stokes equations with mixed boundary conditions in a bounded domain. Annali Dell' Universitá Di Ferrara 2009; 55 (2): 289-308. https://doi.org/10.1007/s11565-009-0082-4

[28] Kweon JR. Regularity of solutions for the Navier-Stokes system of incompressible flows on a polygon. Journal of Differential Equations 2007; 235 (1): 166-198. https://doi.org/10.1016/j.jde.2006.12.008

[29] Kweon JR. The compressible Stokes flows with no-slip boundary condition on nonconvex polygons. Journal of Mathematical Fluid Mechanics 2017; 19 (1): 47-57. https://doi.org/10.1007/s00021-016-0264-7

[30] Kweon JR. Edge singular behavior for the heat equation on polyhedral cylinders in R^3. Potential Analysis 2013; 38: 589-610. https://doi.org/10.1007/s11118-012-9288-7

[31] Kwon O, Kweon JR. Compressible Navier-Stokes equations in a polyhedral cylinder with inflow boundary condition. Journal of Mathematical Fluid Mechanics 2018; 20: 581-601. https://doi.org/10.1007/s00021-017-0336-3

[32] Lasiecka I, Szulc K, Zochowski A. Boundary control of small solutions to fluid-structure interactions arising in coupling of elasticity with Navier-Stokes equation under mixed boundary conditions. Nonlinear Analysis: Real World Applications 2018; 44: 54-85. https://doi.org/10.1016/j.nonrwa.2018.04.004

[33] Mazya V, Rossmann J. Lp estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains. Mathematische Nachrichten 2007; 280 (7): 751-793. https://doi.org/10.1002/mana.200610513

[34] Mazya V, Rossmann J. Mixed boundary value problems for the stationary Navier-Stokes system in polyhedral domains. Archive for Rational Mechanics and Analysis 2009; 194 (2): 669-712. https://doi.org/10.1007/s00205-008-0171-z

[35] Nazarov SA, Novotny A, Pileckas K. On steady compressible Navier-Stokes equations in plane domains with corners. Mathematische Annalen 1996; 304 (1): 121-150. https://doi.org/10.1007/BF01446288

[36] Orlt M, Sändig AM. Regularity of viscous Navier-Stokes flows in nonsmooth domains. Boundary value problems and integral equations in nonsmooth domains (Luminy, 1993). Lecture Notes in Pure and Applied Mathematics. Dekker, New York, 1995, 185-201.

[37] Serre D. Équations de Navier-Stokes stationnaires avec données peu réguliéres. Annali Della Scuola Normale Superiore Di Pisa, Classe Di Scienze 4e Série 1983; 10 (4): 543-559.

[38] Sändig AM, Richter U, Sändig R. The regularity of boundary value problems for the Lamé equations in a polygonal domain. Rostocker Mathematisches Kolloquium 1989; 36: 21-50. http://dx.doi.org/10.18419/opus-8314

[39] Tartar L. Topics in nonlinear analysis. Universite De Paris-Sud, Departement De Mathematique, 1978.

[40] Tartar L. An introduction to Navier-Stokes equation and oceanography. Lecture Notes of the Unione Matematica Italiana. vol. 1. Springer-Verlag, Bologna, 2006.

[41] Temam R. Navier-Stokes equations: Theory and numerical analysis. Elsevier North Holland, 1979.