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Separation, connectedness, and disconnectedness

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Abstract: The aim of this paper is to introduce the notions of hereditarily disconnected and totally disconnected objects in a topological category and examine the relationship as well as interrelationships between them. Moreover, we characterize each of T_2, connected, hereditarily disconnected, and totally disconnected objects in some topological categories and compare our results with the ones in the category of topological spaces.

Key words: Closure operator, T_2 spaces, connected spaces, hereditarily disconnected spaces, totally disconnected spaces

1. Introduction

For a topological space A, we have:

(1) A is connected;

(2) A and $\emptyset$ are the only subsets of A which are both open and closed;

(3) Every continuous function from A into a discrete space must be constant.

The fact (3) is used by several authors [13, 14, 24, 25] to motivate similar situations in a more general categorical setting. Baran, in [2, 3], introduced the notion of (strong) closedness in topological categories by using initial, final, discrete, and indiscrete structures which are available in a topological category and used them to generalize the fact (2) as well as each of compact, sober, $T_i, i = 1, 2, 3, 4$ objects in topological categories in [2, 5, 11]. In view of this, it will be useful to present important theorems in general topology such as the Tietze Extension Theorem, the Tychonoff Theorem, and the Urysohn Lemma among others in setting of a topological category. In 2021 and 2022, the presentation of the Tietze Extension Theorem and the Urysohn Lemma is given in [19, 21].

There is also another approach introduced by Clementino and Tholen in [14] to define the notion of connectedness in a complete category $\mathcal{E}$ [16, 25]. If the diagonal morphism $\delta_A = \langle 1_A, 1_A\rangle : A \to A \times A$ is c-dense, then an object A of $\mathcal{E}$ is called c-connected, where c is a closure operator of $\mathcal{E}$ [15].

The characterization of each of these various forms of connected objects as well as the relationships among these forms in some topological categories are studied in [5, 12].

One of the other important notions of topology to deal with is the notion of disconnectedness which is used in Boolean algebra, functional analysis, logic, and algebraic geometry [1]. The two most common notions of disconnectedness, which are equivalent in the realm of compact T_2 spaces, are: (a) hereditarily disconnected

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spaces (those A with connected subspaces consist of at most one point [17]) which were introduced by Hausdorff [20] and (b) totally disconnected spaces which go also by the name totally separated spaces (those A with the quasicomponent of any point $a \in A$ consists of the point a alone [1, 16**?**] which were introduced by Sierpinski [27].

In this paper, we introduce various forms of hereditarily disconnected and totally disconnected objects in a topological category. Moreover,

(i) we examine the interrelationships as well as the relationships among these forms.

(ii) we show that the notion of closedness induces a closure operator in the categories of **RRel** (resp. **PBorn**) of all reflexive relation (resp. prebornological) spaces and there is a partition of a reflexive space consisting of strongly closed subsets.

(iii) we give the characterization of each of T_2, connected, hereditarily disconnected, and totally disconnected objects in these categories and compare our results with those in the category of topological spaces. Moreover, in the realm of KT_2 reflexive spaces, closed and open subsets are the same and there is a partition of a space consisting of closed subsets.

2. Preliminaries

The category **PBorn** of prebornological spaces has as objects $(A_1, \mathcal{F})$, where $\mathcal{F}$ is a family of subsets of A_1 that contains all finite subsets of A_1 and is closed under finite union and as morphisms $f : (A_1, \mathcal{F}) \to (B_1, \mathcal{G})$ are functions such that $f(C) \in \mathcal{G}$ if $C \in \mathcal{F}$. It is a topological category [23].

The category **RRel** of reflexive relation spaces (spatial graphs) has as objects (A_1, R), where R is a reflexive relation on a set A_1 and as morphisms $f : (A_1, R) \to (B_1, S)$ are relation preserving functions, i.e. if sRt, then $f(s)Sf(t)$ for all $s, t \in A_1$ [16, 25].

An epimorphism $f : (A_1, R) \longrightarrow (B_1, S)$ is final in **RRel** iff for all $s, t \in B_1$, sSt holds in B_1 precisely when there exist $u, v \in A_1$ such that uRv and $f(u) = s$ and $f(v) = t$ [25].

A source $f_i : (A_1, R) \longrightarrow (B_i, R_i), i \in I$ is initial in **RRel** for all $u, v \in A_1$, uRv iff $f_i(u)R_if_i(v)$ for all $i \in I$ [16, 25]. **RRel** is a topological category.

Let B be a set, $x \in B$, and the infinity wedge $\bigvee_x^\infty B$ (resp. $B^2 \bigvee_\Delta B^2$) be taking countably many disjoint copies of B and identifying them at the point x (resp. two distinct copies of B^2 identified along the diagonal Δ) [2].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(a, b)_1 = (a, b, a)$ and $A(a, b)_2 = (a, a, b)$ and the skewed axis map $S : B^2 \vee_\Delta B^2 \to B^2$ is given by $S(a, b)_1 = (a, b, b)$ and $S(a, b)_2 = (a, a, b)$. The fold map $\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\bigtriangledown((a, b)_i) = (a, b)$ for $i = 1, 2$.

The skewed x-axis map $S_x : B \bigvee_x B \to B^2$ is given by $S_x(a_1) = (a, a)$ and $S_x(a_2) = (x, a)$. The fold map at x, ∇_x: $B \bigvee_x B \to B$ is given by $\nabla_x(a_i) = a$ for $i = 1, 2$.

$A_x^\infty : \bigvee_x^\infty B \to B^\infty$ is given by $A_x^\infty(a_i) = (x, ..., x, a, x, x, ...)$, where a_i is in the i-th component of $\bigvee_x^\infty B$ and $B^\infty = B \times B \times ...$ is the countable cartesian product of B, and $\triangledown_x^\infty : \bigvee_x^\infty B \longrightarrow B$ is given by $\triangledown_x^\infty(a_i) = a$ for all $i \in I$, where I is the index set $\{i : a_i$ is in the i-th component of $\bigvee_x^\infty B\}$ [2].

Definition 2.1 *(cf.[2, 5]) Let $U : \mathcal{E} \to$* **Set** *be topological, A be an object in $\mathcal{E}$ with $x \in U(A) = B$, and $Z \subset A$.*

(1) *If the initial lift of the U-source $\{A_x^\infty : \vee_x^\infty B \to B^\infty = U(A^\infty)$ and $\triangledown_x^\infty : \vee_x^\infty B \to UD(B) = B\}$ is discrete, then $\{x\}$ is said to be closed, where D is the discrete functor.*

(2) *If $\{*\}$, the image of Z, is closed in A/Z or or $Z = \emptyset$, then Z is said to be closed, where A/Z is the final lift of the epi U-sink $Q : U(A) = B \to B/Z = (B \backslash Z) \cup \{*\}$, identifying Z with a point *.*

(3) *If the initial lift of the U-source $\{S_x : B \bigvee_x B \to U(A^2) = B^2$ and $\nabla_x : B \bigvee_x B \to UD(B) = B\}$ is discrete, then A is called T_1 at x.*

(4) *If A/Z is T_1 at * or $Z = \emptyset$, then Z is said to be strongly closed.*

(5) *If Z^C, the complement of Z is strongly closed, then Z is said to be strongly open.*

(6) *If Z^C, the complement of Z is closed, then Z is said to be open.*

Remark 2.2 **(1)** *For the category* **Top** *of topological spaces and continuous functions, the notion of openness (resp. closedness) coincides with the usual openness (resp. closedness) and if a space is T_1, then the notions of openness (resp. closedness) and strong openness (resp. closedness) coincide [2, 5].*

(2) *For the category* **PBorn***, by Theorem 3.9 of [3], closedness (resp. openness) implies strong closedness (resp. strong openness).*

Theorem 2.3 *Let (A, R) be a reflexive space, $x \in A$, and $Z \subset A$.*

(a) *(A, R) is T_1 at x iff for each $s \in A$ if sRx or xRs, then $s = x$.*

(b) *$\{x\}$ is closed iff for each $s \in A$ if sRx and xRs, then $s = x$.*

(c) *Z is closed iff for each $s \in A$ if there exist $t, u \in Z$ such that sRt and uRs, then $s \in Z$.*

(d) *Z is strongly closed iff for each $s \in A$ if there exist $t \in Z$ such that sRt or tRs, then $s \in Z$.*

(e) *Z is open iff for each $s \in A$ if there exist $t, u \in Z^C$ such that sRt and uRs, then $s \in Z^C$.*

(f) *Z is strongly closed iff Z is strongly open.*

Proof The proof of Parts (a) and (d) is similar to the proof of Lemma 3.5 and Theorem 3.6 of [9] and Parts (b) and (c) are proved in [11]. The proof of Part (e) follows from (c).
(f) Suppose Z is strongly closed and for each $s \in A$ there exist $t \in Z^C$ such that sRt or tRs. If $s \notin Z^C$, then $s \in Z$. Since Z is strongly closed and sRt or tRs, by (d), $t \in Z$, a contradiction. Hence, $s \in Z^C$ and by (d),

Z^C is strongly closed and by 2.1, Z is strongly open. Similarly, if Z is strongly open, then it is strongly closed. Note that by Theorem 2.3 (f), there is a partition of a reflexive space consisting of strongly closed subsets. □

Theorem 2.4 *Let (A, R) be a reflexive space.*

(1) *If $Z \subset A$ is (strongly) closed and $M \subset Z$ is (strongly) closed, then $M \subset A$ is (strongly) closed.*

(2) *If $M_i \subset A, i \in I$ is (strongly) closed for each $i \in I$, then $\bigcap_{i\in I} M_i$ is (strongly) closed.*

(3) *If $M_i \subset A, i \in I$ is strongly closed for each $i \in I$, then $\bigcup_{i\in I} M_i$ is strongly closed.*

(4) *If Z_1 and Z_2 are closed, then $Z_1 \cup Z_2$ may not be closed.*

(5) *If $M_i \subset (A_i, R_i), i \in I$ is (strongly) closed for each $i \in I$, then $\prod_{i\in I} M_i$ is (strongly) closed in $\prod_{i\in I} A_i$.*

(6) *If $f : (A_1, S) \to (B_1, R)$ is a relation preserving function and $Z \subset B_1$ is (strongly) closed (open), so also is $f^{-1}(Z)$.*

Proof **(1)** Suppose $Z \subset A$ and $M \subset Z$ are strongly closed. Let R_Z (resp. R_M) be the initial structure on Z (resp. M) induced by the inclusion map $i : Z \to (A, R)$ (resp. $i : M \to (Z, R_Z)$). Suppose $M \subset Z$ and $Z \subset A$ are strongly closed and for each $x \in A$, there exists $a \in M$ such that xR_Ma or aR_Mx. Note that $xR_Ma = xR_Za = xRa$ and $aR_Mx = aR_Zx = aRx$. Since $a \in Z$, xRa or aRx, and $Z \subset A$ is strongly closed, by Theorem 2.3(c), $x \in Z$. Since $a \in M$, xR_Za or aR_Zx, and $M \subset Z$ is strongly closed, by Theorem 2.3(c), $x \in M$.
The proof for openness and closedness is similar.

(2) Suppose $M_i \subset A, i \in I$ is strongly closed for each $i \in I$ and for each $x \in A$ there exists $a \in M = \bigcap_{i\in I} M_i$ such that xRa or aRx . It follows that $a \in M_i$ for all $i \in I$. Since $a \in M_i$, xRa or aRx and M_i are strongly closed for all $i \in I$, by Theorem 2.3, $x \in M_i$ for all $i \in I$ and hence, $x \in M$. By Theorem 2.3, M is strongly closed. The proof for closedness is similar.

The proof for **(3)** can be done similarly.

(4) Let $A = \{x, y, z\}$ and define a reflexive relation R as follows:
$R = \{(x, x), (y, y), (z, z), (x, y), (x, z), (z, y)\}$. By Theorem 2.3, $Z_1 = \{x\}$, and $Z_2 = \{y\}$ are closed but $Z_1 \cup Z_2 = \{x, y\}$ is not closed since zRy and xRz but $z \notin Z_1 \cup Z_2$.

(5) Suppose $M_i \subset A_i$ are strongly closed for all $i \in I$ and for each $x \in A = \prod_{i\in I} A_i$ there is $a \in M = \prod_{i\in I} M_i$ such that xRa or aRx, where R is the product structure on A. It follows that for each $i \in I$, $x_iR_ia_i$ or $a_iR_ix_i$. Since each M_i is strongly closed, by Theorem 2.3, $x_i \in M_i$ for each $i \in I$ and consequently, $x \in M$. Hence, by Theorem 2.3, M is strongly closed.

(6) Suppose $Z \subset B_1$ is strongly closed. If $f^{-1}(Z) = \emptyset$, then by Definition 2.1, $f^{-1}(Z)$ is strongly closed. Suppose $f^{-1}(Z) \neq \emptyset$ and for each $x \in A_1$ there exists $a \in f^{-1}(Z)$ such that xSa or aSx. Note that $f(x) \in B_1$, $f(a) \in Z$ and $f(x)Rf(a)$ or $f(a)Rf(x)$ (f is a relation preserving map). Since $Z \subset B_1$ is strongly closed, by Theorem 2.3, $f(x) \in Z$. Hence, $x \in f^{-1}(Z)$ and by Theorem 2.3, $f^{-1}(Z)$ is strongly closed.

The proof for closedness and (strong) openness is similar. □

3. T_2 Objects

We give the characterization of each of $PreT_2', Pre\overline{T}_2, KT_2$, and NT_2 reflexive relation spaces and show that the notion of closedness induces a closure operator of **RRel** and **PBorn**.

Let $U:\mathcal{E}\to$ **Set** be topological and $X\in\mathcal{E}$ with $U(X)=B$.
Let A_W (resp. S_W) be the initial lift of the U-source $A(\text{resp. S}): B^2\bigvee_\Delta B^2\to U(X^3)=B^3$ and $(B^2\bigvee_\Delta B^2)'$ be the final lift of the U-sink $\{q\circ i_1, q\circ i_2: U(X^2)=B^2\to B^2\vee_\Delta B^2\}$, where $i_k: B^2\to B^2\coprod B^2, k=1,2$ are the canonical injection maps and $q: B^2\coprod B^2\to B^2\bigvee_\Delta B^2$ is the quotient map.

Definition 3.1 *(cf. [2, 4, 22])*
(1) *If X does not contain an indiscrete subspace with (at least) two points, then X is said to be a T_0 object.*

(2) *If the initial lift of the U-source $\{id: B^2\vee_\Delta B^2\to U(B^2\vee_\Delta B^2)'=B^2\vee_\Delta B^2$ and $\nabla: B^2\vee_\Delta B^2\to U(D(B^2))=B^2\}$ is discrete, then X is said to be a T_0' object.*

(3) *If $A_W=S_W$, then X is said to be a $Pre\overline{T}_2$ object.*

(4) *If $(B^2\bigvee_\Delta B^2)'=S_W$, then X is said to be a $PreT_2'$ object.*

(5) *If X is T_0' and $Pre\overline{T}_2$, then X is said to be a KT_2 object.*

(6) *If X is T_0 and $Pre\overline{T}_2$, then X is said to be a NT_2 object.*

Theorem 3.2 *Let (A,R) be a reflexive space.*

(1) *The following are equivalent:*

(i) *(A,R) is $pre\overline{T}_2$.*

(ii) *(A,R) is KT_2.*

(iii) *R is symmetric and transitive.*

(2) *The following are equivalent:*

(i) *(A,R) is $PreT_2'$.*

(ii) *(A,R) is NT_2.*

(iii) *(A,R) is discrete.*

Proof **(1)** By Theorem 3.8 of [11] and Definition 3.1, we get $(i)\Longrightarrow(ii)$.

$(ii)\Longrightarrow(iii)$ Suppose (A,R) is KT_2 and xRy for $x,y\in A$. We have

$$\pi_1 A(x,y)_1 R \pi_1 A(y,x)_2 = xRy = \pi_1 S(x,y)_1 R \pi_1 S(y,x)_2,$$

$$\pi_2 A(x,y)_1 R \pi_2 A(y,x)_2 = yRy = \pi_2 S(x,y)_1 R \pi_2 S(y,x)_2,$$

$\pi_3 A(x,y)_1 R \pi_3 A(y,x)_2 = xRx$, and $\pi_3 S(x,y)_1 R \pi_3 S(y,x)_2 = yRx$, where $\pi_i : B^3 \to B$ are the projections, $i = 1,2,3$. Since (A,R) is KT_2, by Definition 3.1, it is $pre\overline{T}_2$ and $\pi_3 S(x,y)_1 R \pi_3 S(y,x)_2 = yRx$; thus, R is symmetric.

Suppose xRy and yRz for $x,y,z \in A$.

$$\pi_1 A(x,y)_1 R \pi_1 A(y,z)_2 = xRy = \pi_1 S(x,y)_1 R \pi_1 S(y,z)_2,$$

$$\pi_2 A(x,y)_1 R \pi_2 A(y,z)_2 = yRy = \pi_2 S(x,y)_1 R \pi_2 S(y,z)_2,$$

$\pi_3 A(x,y)_1 R \pi_3 A(y,z)_2 = xRz$, and $\pi_3 S(x,y)_1 R \pi_3 S(y,z)_2 = yRz$.

Since (A,R) is KT_2, R is transitive.

$(iii) \Longrightarrow (i)$ Suppose R is symmetric and transitive, and s,t are any points in the wedge. We show (A,R) is $Pre\overline{T}_2$, i.e.

$$\pi_1 A(s) R \pi_1 A(t), \pi_2 A(s) R \pi_2 A(t),$$

and $\pi_3 A(s) R \pi_3 A(t)$ if and only if

$$\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t),$$

and $\pi_3 S(s) R \pi_3 S(t)$. We have $s = (x,y)_1, (x,y)_2$ or (x,x) and $t = (z,w)_1, (z,w)_2$ or (z,z) for $x,y,w,z \in A$. If $s = (x,y)_1$ and $t = (z,w)_1$, then

$$\pi_1 A(s) R \pi_1 A(t) = xRz = \pi_1 S(s) R \pi_1 S(t),$$

$$\pi_2 A(s) R \pi_2 A(t) = yRw = \pi_2 S(s) R \pi_2 S(t),$$

$\pi_3 A(s) R \pi_3 A(t) = xRz$, and $\pi_3 S(s) R \pi_3 S(t) = yRw$.

If $s = (x,y)_1$ and $t = (z,w)_2$, then

$$\pi_1 A(s) R \pi_1 A(t) = xRz = \pi_1 S(s) R \pi_1 S(t), \pi_2 A(s) R \pi_2 A(t) = yRz = \pi_2 S(s) R \pi_2 S(t).$$

Note that $\pi_3 A(s) R \pi_3 A(t) = xRw$ iff $\pi_3 S(s) R \pi_3 S(t) = yRw$ (because R is symmetric and transitive). If $s = (x,y)_1$ and $t = (z,z)$, then

$$\pi_1 A(s) R \pi_1 A(t) = xRz = \pi_1 S(s) R \pi_1 S(t),$$

$$\pi_2 A(s) R \pi_2 A(t) = yRz = \pi_2 S(s) R \pi_2 S(t), \pi_3 A(s) R \pi_3 A(t) = xRz,$$

and $\pi_3 S(s) R \pi_3 S(t) = yRz$.

Suppose $s = (x,y)_2$ or (x,x) and $t = (z,w)_1, (z,w)_2$ or (z,z). Since R is symmetric and transitive, we have

$$\pi_1 A(s) R \pi_1 A(t), \pi_2 A(s) R \pi_2 A(t)$$

and $\pi_3 A(s) R \pi_3 A(t)$ iff $\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t)$ and $\pi_3 S(s) R \pi_3 S(t)$. Hence, by Definition 3.1, (A,R) is $Pre\overline{T}_2$.

(2) $(i) \implies (ii)$ If (A,R) is $PreT_2'$, then by Theorem 3.1 of [6], (A,R) is $Pre\overline{T}_2$. It remains to show that (A,R) is T_0. By Theorem 3.8 of [11], we show R is antisymmetric. Suppose yRx and xRy for $x,y \in A$. Let $s = (x,y)_1$ and $t = (y,x)_2$. $\pi_1 S(s) R \pi_1 S(t) = xRy$ $\pi_2 S(s) R \pi_2 S(t) = yRy$, and $\pi_3 S(s) R \pi_3 S(t) = yRx$. Since (A,R) is $PreT_2'$, by Definition 3.1, $(x,y)R^2(y,x)$ and $q \circ i_k(x,y) = s, q \circ i_k(y,x) = t$ for some $k = 1$ or 2. Hence, we must have $x = y$ and by Definition 3.1, (A,R) is NT_2.

$(ii) \implies (iii)$ Suppose (A,R) is NT_2 and xRy for $x,y \in A$. Since (A,R) is $Pre\overline{T}_2$, by Part(1), we have yRx. Since (A,R) is T_0, by Theorem 3.8 of [11], R is antisymmetric. Hence, $x = y$ and (A,R) is discrete.

$(iii) \implies (i)$ Suppose (B,R) is discrete. We show (A,R) is $PreT_2'$, i.e. by Definition 3.1, (I) and (II) are equivalent: for any pair s and t in the wedge, (I) there exists a pair $(a_1,a_2),(b_1,b_2)$ in A^2 such that $(a_1,a_2)R^2(b_1,b_2)$ (R^2 is the product structure on A^2) and $q \circ i_k(a_1,a_2) = s, q \circ i_k(b_1,b_2) = t$ for some $k = 1$ or 2 iff (II) $\pi_1 S(s) R \pi_1 S(t), \pi_2 S(s) R \pi_2 S(t)$, and $\pi_3 S(s) R \pi_3 S(t)$. If (I) holds, then it follows that $a_1 R b_1$ and $a_2 R b_2$. Since (A,R) is discrete, $a_1 = b_1$, $a_2 = b_2, s = t$ and (II) holds.
Similarly, if (A,R) is discrete, one can show easily that (II) implies (I). Hence, (A,R) is $PreT_2'$. □

Theorem 3.3 *Let (A,R) be a reflexive space.*
(1) *Every strongly closed subset of A is closed.*

(2) *If (A,R) is NT_2, then all subsets of A are (strongly) open and (strongly) closed.*

(3) *If (A,R) is KT_2, then a subset of A is strongly closed iff it is closed.*

Proof **(1)** If Z is strongly closed, then Theorem 2.3, If it is closed. Let $A = \{a,b\}$ and define a reflexive relation R as follows: $R = \{(a,a),(b,b),(b,a)\}$. By Theorem 2.3, $\{b\}$ is closed but it is not strongly closed.

(2) Suppose (A,R) is NT_2 and $Z \subset A$. By Theorem 3.2, R is discrete and by Theorem 2.3, Z is (strongly) open and (strongly) closed.

(3) If Z is strongly closed, then by Part (1), Z is closed.
Suppose (A,R) is KT_2 and Z is closed. If $Z = \emptyset$, then by Definition 2.1, Z is strongly closed. Suppose $Z \neq \emptyset$ and for each $x \in A$ there exists $c \in Z$ such that xRc or cRx. If xRc, then cRx since (A,R) is KT_2. Since xRc, cRx and Z is closed, by Theorem 2.3, $x \in Z$. Similarly, if cRx, then $x \in Z$. Hence, Z is strongly closed. □

Remark 3.4 **(1)** *For the category* **Top**, *T_0' and T_0 (resp. NT_2 and KT_2) are equivalent and they reduce to the usual T_0 (resp. T_2) axiom [2, 22].*

(2) *For the category* **PBorn**, *by Theorem 3.6 of [11] and Theorem 2.6 of [4], $T_0 = NT_2 \Rightarrow KT_2 \Rightarrow T_0'$.*

(3) *For the category* **RRel**, *by Theorem 3.8 of [11] and Theorem 3.2, $T_0 \Rightarrow T_0'$ and $NT_2 \Rightarrow KT_2$. Moreover, by Theorem 2.3(f), there is a partition of a space consisting of strongly closed subsets and by Theorem 3.2(1), there is a bijection between KT_2 structures and partitions of a space. Also, in the realm of KT_2 reflexive spaces, by Theorems 2.3(f) and 3.3(3), closed and open subsets are the same and there is a partition of a space* *consisting of closed subsets.*

(4) *In any topological category, there is no implication between T_0 and T_0' [3, 4] and by Theorem 3.1 of [6], $PreT_2'$ implies $Pre\overline{T}_2$.*

Definition 3.5 *Let A be an object in a topological category $\mathcal{E}$ and $Z \subset A$. The (strong) closure $cl(Z)$ (resp. $scl(Z)$) of Z is the intersection of all (strongly) closed subsets of A containing Z.*
The (strong) quasicomponent closure $Q(Z)$ (resp. $SQ(Z)$) of Z is the intersection of all (strongly) open and (strongly) closed subsets of A containing Z.

The notion of (strong) closedness induces appropriate closure operator in some categories [8, 10, 18, 19, 26].

Theorem 3.6 **(1)** *cl, scl, and SQ are idempotent, weakly hereditary, productive, and hereditary closure operators of* **RRel** *and $scl = SQ$.*

(2) *$scl = \delta = SQ$ and $cl = \iota = Q$, where ι is the indiscrete closure operator of* **PBorn**.

(3) *Let (A, R) be a reflexive space and $Z \subset A$.*
$scl(Z) = \{x \in A : U \cap Z \neq \emptyset$ for all strongly open subsets U of A containing x $\} = SQ(Z)$.
$cl(Z) = \{x \in A : U \cap Z \neq \emptyset$ for all open subsets U of A containing x $\}$.
$Q(M) = \{x \in A : U \cap Z \neq \emptyset$ for all closed and open subsets U of A containing $x\}$.

Proof **(1)** It follows from Exercise 2.D, Propositions 2.5 and 3.6 of [16], and Theorem 2.4. By Theorem 2.3 (f), $scl = SQ$.
The proof of Part (2) follows from Theorem 3.9 of [3].

(3) The proof is the same as the proof in the case of **Top**.

Let c be a closure operator of $\mathcal{E}$.
$T_1(c) = \{$ A $\in \mathcal{E}$: $c(\{a\}) = \{a\}$ for each $a \in A\}$.
$\triangle(c) = \{$A $\in \mathcal{E}$: c$(\Delta) = \Delta$, the diagonal$\}$.
$\nabla(c) = \{$A $\in \mathcal{E}$: c$(\Delta) = A^2\}$ [16], p.250. □

Let $\mathcal{E} =$ **Top**, K be the ordinary closure and Q be the quasicomponent closure. Then $T_1(K)$, $\triangle(K)$, $\nabla(Q)$, and $T_1(Q) = \triangle(Q)$ are the class of T_1-spaces, T_2-spaces, connected spaces, and totally disconnected spaces, respectively [16].

Let $T\mathcal{E}$ be the full subcategory of a topological category $\mathcal{E}$ consisting of all T objects, where $\mathcal{E} =$ **RRel** or **PBorn** and $T = T_0, Pre\overline{T}_2, NT_2, KT_2$.

Theorem 3.7 **(A)** **(1)** $T_1(cl) = \mathbf{T_0RRel}$ *and they are hereditary and productive.*

(2) $T_1(SQ) = \triangle(cl) = \triangle(scl) = \triangle(SQ) = \triangle(Q) = \mathbf{NT_2RRel}$ *and they are hereditary and productive.*

(3) $\mathbf{T_0'RRel} = \mathbf{RRel}$, $\mathbf{KT_2RRel} = \mathbf{Pre\overline{T}_2RRel}$ *and they are topological categories.*

(B) (1) $T_1(cl) = T_1(Q) = \triangle(cl) = \triangle(Q) = \mathbf{T_0PBorn} = \mathbf{NT_2PBorn}$ *and they are hereditary and productive.*

(2) $T_1(scl) = T_1(SQ) = \triangle(scl) = \triangle(SQ) = \mathbf{PBorn}$ *and they are topological categories.*

(3) $\mathbf{KT_2PBorn} = \mathbf{Pre\overline{T}_2PBorn} = \mathbf{Born}$, *the category of bornological spaces.*

Proof **(A) (1)** $(A,R) \in T_1(cl)$ iff $cl(\{a\}) = \{a\}$ for each $a \in A$ iff $\{a\}$ is closed for each $a \in A$ iff, by Theorem 3.8 of [11], R is antisymmetric, i.e. (A,R) is T_0.

(2) Suppose $(A,R) \in T_1(scl)$ and aRb for $a,b \in A$. Since $(A,R) \in T_1(scl)$, $scl(\{a\}) = \{a\}$ for each $a \in A$, i.e. $\{a\}$ is strongly closed for each $a \in A$. Since aRb, by Theorem 2.3, $a = b$ and thus, (A,R) is discrete.
If (A,R) is discrete, then, by Theorems 3.2(2) and 3.3(2), $\{a\}$ is strongly closed for each $a \in A$, i.e. $(A,R) \in T_1(scl)$.

Suppose $(A,R) \in \triangle(scl)$ and xRy for $x,y \in A$. Then $(x,y)R^2(y,y)$ or $(x,x)R^2(x,y)$. Since $\Delta \subset A^2$ is strongly closed, by Theorem 2.3(d), $(x,y) \in \triangle(scl) = \Delta$. Hence, (A,R) is discrete. Conversely, if (A,R) is discrete, then, by Theorems 2.3(d) and 3.3(2), $\Delta \subset A^2$ is strongly closed, and thus, $(A,R) \in \triangle(scl)$.
Suppose $(A,R) \in \triangle(cl)$ and xRy. $(x,y)R^2(y,y)$ and $(x,x)R^2(x,y)$. Since $(A,R) \in \triangle(cl)$, by Theorem 2.3(c), $(x,y) \in \Delta$. Hence, (A,R) is discrete.
If (A,R) is discrete, then $\Delta \subset A^2$ is closed (by Theorem 3.3). Hence, $(A,R) \in \triangle(cl)$.
By Theorem 3.6(1), d $scl = SQ$ and consequently, $\triangle(scl) = \triangle(SQ)$ and $T_1(scl) = T_1(SQ$.

(B) Combine Remark 3.4, Theorem 3.6, and Theorem 2.6 of [4]. □

Example 3.8 **(1)** *Let* $A = \{x,y\}$ *and define a reflexive relation* R *as follows:* $R = \{(x,x),(y,y),(y,x)\}$. *By Theorem 2.3, all subsets of* A *are open and closed but* $\emptyset$ *and* A *are the only strongly open and strongly closed subsets of* A. *By Theorem 3.7,* $(A,R) \in T_1(Q)$ *and* $(A,R) \notin T_1(SQ)$.

(2) *Let* $(\mathbb{Z}, P(\mathbb{Z}))$ *be the indiscrete prebornological space, where* $\mathbb{Z}$ *is the set of integers. By Remark 3.4 and Theorem 3.7,* $(\mathbb{Z}, P(\mathbb{Z})) \in T_1(SQ)$ *and it is* KT_2 *but* $(\mathbb{Z}, P(\mathbb{Z})) \notin T_1(Q)$ *and it is neither* NT_2 *nor* T_0.

(3) *By Remark 3.4 and Theorem 3.7, every discrete prebornological space* $(X, \mathcal{F} =$ *the set of all finite subsets of* X *) [23] is* KT_2 *but it is not* NT_2 *if* $|X| > 1$.

4. Connected objects

There are various generalizations of the notion of connectedness in a topological category [5, 13, 14, 24, 25]. In this section, we characterize each of these various connected objects in **RRel** and **PBorn**.

Definition 4.1 *([5, 13, 14, 24, 25]) Let* A *be an object in a topological category* $\mathcal{E}$.
(1) *If the only subsets of* A *both (strongly) open and (strongly) closed are* A *and* $\emptyset$*, then* A *is said to be strongly connected (resp. connected).*

(2) *If any morphism from A to discrete object is constant, then A is said to be D-connected.*

(3) *If $X \in \nabla(c)$, then X is called c-connected, where c is a closure operator of $\mathcal{E}$.*

In **Top**, *D-connectedness, strong connectedness, and Q-connectedness coincides with the usual connectedness [5, 14] and if a space is T_1, then all the notions of connectedness coincide [5].*

Let T**PBorn** be the full subcategory of **PBorn** consisting of T objects, where $T = SC =$ strongly connected, $T = C =$ connected, or $T = DC = D$-connected.

Theorem 4.2 **(1)** $\nabla(scl) = C\mathbf{PBorn} = \nabla(\mathbf{SQ}) = \mathbf{DCPBorn}$ *contains only trivial spaces (spaces of cardinality at most 1).*

(2) $\nabla(cl) = \nabla(Q) = SC\mathbf{PBorn} = \mathbf{PBorn}$.

Proof By Theorem 3.6(2), $scl = \delta = SQ$ and $cl = \iota = Q$. Hence, by Theorem 3.7 and Definition 4.1, $\nabla(scl) = C\mathbf{PBorn} = \nabla(\mathbf{SQ}) = \mathbf{DCPBorn}$ and $\nabla(cl) = \nabla(Q) = SC\mathbf{PBorn} = \mathbf{PBorn}$. □

Theorem 4.3 *A reflexive space (A, R) is connected iff it is D-connected.*

Proof Suppose (A, R) is connected, (C, S) is a discrete reflexive space, and $f : (A, R) \to (C, S)$ is a relation preserving map. If $|C| = 1$, then f is constant. Suppose $|C| > 1$ and f is not constant. Then there exist $a, b \in A$ with $a \neq b$ such that $f(a) \neq f(b)$. By Theorems 3.2(2) and 3.3(2), $\{f(a)\}$ is strongly closed (open) and by Theorem 2.4(6), $Z = f^{-1}\{f(a)\}$ is also strongly closed (open). Since (A, R) is connected, by Definition 4.1, $Z = \emptyset$ or $Z = A$. If $Z = \emptyset$, then $f(x) = f(b)$ for all $x \in A$ and if $Z = A$, then $f(x)) = f(a)$ for all $x \in A$, a contradiction since $f(a) \neq f(b)$. Hence, f must be constant and by Definition 4.1, (A, R) is D-connected.
Suppose (A, R) is D-connected and there exists a nonempty proper strongly closed (open) subset Z of A.
Let (C, S) be a discrete space and $|C| > 1$. Define $f : (A, R) \to (C, S)$ by

$$f(x) = \begin{cases} u & \text{if } x \in Z \\ v & \text{if } x \notin Z \end{cases}$$

for $x \in A$. Let $x, y \in A$ and $(x, y) \in R$. If $x, y \in Z$ or $x, y \in Z^C$, then $(f(x), f(y)) = (u, u) \in S$ or $(f(x), f(y)) = (v, v) \in S$. If $x \in Z$ and $(x, y) \in R$, then by Theorem 2.3(d), $y \in Z$ (because Z is strongly closed) and $(f(x), f(y)) = (u, u) \in S$. If $x \in Z^C$ and $(x, y) \in R$, then by Theorem 2.3(d) and (f), $y \in Z^C$ and $(f(x), f(y)) = (v, v) \in S$. Hence, f is a relation preserving map but it is not constant, a contradiction. Thus, (A, R) is connected. □

Theorem 4.4 *A reflexive space (A, R) is strongly connected iff for any nonempty proper subset Z of A either the conditions (I) or (II) holds.*
(I) *For some $x \in B$ if $(x, a) \notin R$ or $(b, x) \notin R$ for all $a, b \in Z$, then $x \notin Z$.*

(II) *For some $x \in B$ if $(x, a) \notin R$ or $(b, x) \notin R$ for all $a, b \in Z^C$, then $x \in Z$.*

Proof By Theorem 2.3 and Definition 4.1, we get the result. □

Theorem 4.5 *A reflexive space (A, R) is SQ-connected iff (B, R) is scl-connected iff for any $x, y \in A$ with $x \neq y$, there exists $z \in A$ such that either ($(x, z) \in R$ and $(y, z) \in R$) or ($(z, x) \in R$ and $(z, y) \in R$) holds.*

Proof By Theorem 3.6, $SQ = scl$.
Suppose (A, R) is scl-connected and there exist $x, y \in A$ with $x \neq y$, both ($(x, z) \notin R$ or $(y, z) \notin R$) and ($(z, x) \notin R$ or $(z, y) \notin R$) hold for all $z \in A$. Let $M = \{(z, w) : z, w \in A, z \neq x \text{ or } w \neq y\}$. Note that $\Delta \subset M$, $(x, y) \notin M$ and $((x, y), (z, z)) \notin R^2$ and $((z, z), (x, y)) \notin R^2$ for all $z \in A$, where R^2 is the product structure on A^2. By Theorem 2.3(d), M is strongly closed and by Definition 3.5, $(x, y) \notin scl(\Delta) = B^2$, a contradiction.
Suppose the condition holds and $(x, y) \in A^2$ with $x \neq y$. By assumption, there exists $z \in A$ such that either ($(x, z) \in R$ and $(y, z) \in R$) or ($(z, x) \in R$ and $(z, y) \in R$) holds. If the first case holds, then $((x, y), (z, z)) \in R^2$ and by Theorem 2.3(d), $(x, y) \in scl(\Delta)$ since by Theorem 3.6, $scl(\Delta)$ is strongly closed. If the second case holds, then $((z, z), (x, y)) \in R^2$ and by Theorem 2.3(d), $(x, y) \in scl(\Delta)$ since by Theorem 3.6, $scl(\Delta)$ is strongly closed. Hence, $scl(\Delta) = B^2$, i.e. (A, R) is scl-connected. □

Theorem 4.6 *A reflexive space (A, R) is cl-connected iff for any $x, y \in A$ with $x \neq y$ there exist $z, w \in A$ such that both ($(x, z) \in R$ and $(y, z) \in R$) and ($(w, x) \in R$ and $(w, y) \in R$) hold.*

Proof Combine Theorems 2.3(c) and 3.6(1) and Definition 4.1. □

Theorem 4.7 **(1)** *If (A, R) is strongly connected, then (A, R) is connected.*

(2) *If (A, R) is cl-connected, then (A, R) is scl-connected.*

Proof **(1)** Suppose (A, R) is strongly connected and $Z \subset A$ is strongly closed (open). By Theorems 2.3(f) and 3.3(1), Z is closed and open. Since (A, R) is strongly connected, by Definition 4.1, $Z = \emptyset$ or $Z = A$. For the converse implication, take Example 3.8(1) and by Theorems 4.3 and 4.4, (A, R) is connected but it is not strongly connected.

(2) Suppose (A, R) is cl-connected and $x, y \in A$ with $x \neq y$. Since (A, R) is cl-connected, by Theorem 4.6, there exist $a, b \in A$ such that both ($(x, a) \in R$ and $(y, a) \in R$) and ($(b, x) \in R$ and $(b, y) \in R$) hold. By Theorem 4.5, (A, R) is scl-connected.
Let (A, R) be the Example in 3.8(1). By Theorems 4.5 and 4.6, (A, R) is scl-connected but it is not cl-connected. □

Remark 4.8 **(1)** *In* **PBorn***, by Theorem 4.2, the notion of connectedness, D-connectedness, SQ-connectedness, and scl-connectedness (resp. strong connectedness, Q-connectedness, and cl-connectedness) are equivalent. Moreover, by Theorem 4.2 and Theorem 3.6 of [11], a prebornological space is strongly connected iff it is quasisober iff it is irreducible.*

(2) *For the category* **RRel***, by Theorems 4.3–4.7, strong connectedness (resp. cl-connectedness) implies connectedness = D-connectedness (resp. scl-connectedness). If a reflexive space is KT_2, then by Theorems 2.3(f), 3.3(3), and 4.3–4.7, connectedness, D-connectedness, and strong connectedness (resp. scl-connectedness and cl-connectedness) are equivalent. For the converse implication see Example 3.8. If a reflexive space is NT_2, then by Theorems 3.3(2) and 4.3–4.7, all the notions of connectedness are equivalent. Moreover, by Theorems*

2.3 and 4.3, and Definition 5.1 of [12], a reflexive space is connected iff it is strongly irreducible.

(3) *In any topological category, by Parts (2) and (3), there are no implications between the notion of strong connectedness (resp. cl-connectedness) and connectedness (resp. scl-connectedness).*

5. Hereditary disconnectedness and total disconnectedness

In this section, we define various forms of hereditarily disconnected and totally disconnected objects in a topological category and give the characterization of them in **PBorn** and **RRel**. Moreover, we compare our results with results in **Top**.

Definition 5.1 *Let $A \in \mathcal{E}$ and c be a closure operator of $\mathcal{E}$.*
(1) *If the only connected (resp. strongly connected, c-connected, or D-connected) subspaces of A are singletons and $\emptyset$, then A is said to be hereditarily disconnected (resp. strongly hereditarily disconnected, hereditarily c-disconnected, or hereditarily D-disconnected).*

(2) *If every quasicomponent of A contains only one point, then A is said to be totally disconnected.*

(3) *If every strongly quasicomponent of A contains only one point, then A is said to be strongly totally disconnected.*

In **Top**, the notions of strong hereditary disconnectedness, hereditary D-disconnectedness, and hereditary Q-disconnectedness (resp. total disconnectedness) coincide with the usual hereditary disconnectedness [5, 17] (resp. total disconnectedness [1, 5, 14, 17]) and if a space is T_1, then hereditary disconnectedness (resp. total disconnectedness) and strong hereditary disconnectedness (resp. strong total disconnectedness) coincide.

Theorem 5.2 *Suppose (A,R) is a reflexive space.*
(1) (A,R) *is hereditarily disconnected iff* (A,R) *is hereditarily D-disconnected.*

(2) *If (A,R) is strongly totally disconnected, then (A,R) is totally disconnected and if (A,R) is KT_2, then the converse implication also holds.*

(3) *If (A,R) is hereditarily disconnected, then (A,R) is strongly hereditarily disconnected and if (A,R) is KT_2, then the converse implication also holds.*

(4) *If (A,R) is hereditarily scl-disconnected, then (A,R) is hereditarily cl-disconnected and if (A,R) is KT_2, then the converse implication also holds.*

(5) *If (A,R) is strongly totally disconnected, then (A,R) is both strongly hereditarily disconnected and NT_2.*

Proof **(1)** It follows from Theorem 4.3.

(2) Since (A,R) is strongly totally disconnected, $SQ(x)=\{x\}$ for all $x \in A$. By Theorem 2.4 (2), $\{x\}$ is strongly closed (open) and by Theorems 2.3(f) and 3.3(3), $\{x\}$ is open and closed. Hence, $Q(x)=\{x\}$ for all $x \in A$ and by Definition 5.1, (A,R) is totally disconnected.
Suppose (A,R) is a totally disconnected KT_2. Then $Q(x)=\{x\}$ for all $x \in B$. By Theorem 2.4 (2), $\{x\}$ is closed. Since (A,R) is KT_2, by Theorem 3.3 (2), $\{x\}$ is strongly closed and by Theorems 2.3, $\{x\}$ is strongly

open. Hence, $SQ(x) = \{x\}$ for all $x \in A$ and by Definition 5.1, (A, R) is strongly totally disconnected. If (A, R) is not KT_2, then the converse implication may not hold. Take Example 3.8 and $Q(a) = \{a\}, Q(b) = \{b\}$, and $SQ(a) = A = SQ(b)$.

(3) Suppose (A, R) is hereditarily disconnected and $Z \subset A$ is strongly connected. By Theorem 4.7, Z is connected and since (A, R) is hereditarily disconnected, by Definition 5.1, $Z = \emptyset$ or $Z = \{a\}$ for some $a \in A$. For the converse implication, take the Example in (1).
Suppose (A, R) is a strongly hereditarily disconnected KT_2 space and $Z \subset A$ is connected. Since (A, R) is KT_2, by Remark 4.8, $Z \subset A$ is strongly connected and by Definition 5.1, $Z = \emptyset$ or $Z = \{a\}$ for some $a \in A$.

(4) It follows from Theorems 3.3 and 4.7 and Remark 4.8.

(5) Suppose (A, R) is strongly totally disconnected and $Z \subset A$ is strongly connected. Since (A, R) is totally strongly disconnected, $SQ(x) = \{x\}$ for all $x \in A$ and by Theorem 2.4 (2), $\{x\}$ is strongly closed for all $x \in A$. By Theorems 3.2 and 3.7, (A, R) is NT_2 and by Theorem 3.3, all subsets of Z are open and closed. Since $Z \subset A$ is strongly connected, $Z = \emptyset$ or $Z = \{a\}$ for some $a \in A$. Hence, (A, R) is hereditarily strongly disconnected. □

Let $T_1(Q)$ (resp. $T_1(SQ)$ be a class of totally disconnected (resp. strongly totally disconnected) objects and $T = HDC =$ hereditarily D-disconnected, $T = HC =$ hereditarily disconnected, or $T = HSC =$ strongly hereditarily disconnected.

Theorem 5.3 **(1)** *T**PBorn** $= \mathbf{T_1(SQ)} =$ **PBorn** for $T = HDC, HC$ and they are topological categories.*

(2) *HSC**PBorn** $= \mathbf{T_1(Q)}$ contains only trivial spaces.*

(3) *HC**RRel** $=$ **HDCRRel**, HC**RRel** $\subset$ **HSCRRel**, and $T_1(SQ) \subset T_1(Q)$.*

Proof Combine Remark 4.8 and Theorems 3.7, 4.2, and 5.2. □

Remark 5.4 **(A)** *Let (B, R) be a reflexive space.*
(1) *By Theorems 3.2 and 5.2, (B, R) is totally strongly disconnected iff it is NT_2.*

(2) *By Theorems 3.2 and 3.7, totally disconnected (resp. strongly hereditarily disconnected) space may not be NT_2 (see Example 3.8).*

(3) *By Theorems 3.3, 3.7, and 5.2, if (A, R) is KT_2, then all the notions of hereditary disconnectedness are equivalent and by Theorem 5.2, total disconnectedness implies hereditary disconnectedness.*

(4) *By Theorem 5.2, Part (1), and Theorem of 3.8 of [11], if (B, R) is strongly totally disconnected, then it is quasisober and T_0 sober.*

(B) *Let $(X, \mathcal{F})$ be a prebornological space.*

(1) *By Theorems 3.7 and 5.3 and Theorem of 3.6 of [11], $(X, \mathcal{F})$ is totally disconnected iff it is strongly hereditarily disconnected iff it is NT_2 iff it is T_0 sober.*

(2) *By Theorems 3.7(B) and 5.3 and Theorem of 3.6 of [11],* $(X, \mathcal{F})$ *is strongly totally disconnected iff it is hereditarily disconnected iff it is* T_0' *sober.*

(3) *By Remark 3.4 and Theorems 3.7 (B) and 5.3, if* $(X, \mathcal{F})$ *is totally disconnected, then it is hereditarily disconnected, strongly totally disconnected and* KT_2. *By Example 3.8, the reverse implication is not true.*

We now state some results [1, 5, 16, 17] in **Top**.

Theorem 5.5 **(1)** *Every totally disconnected space is hereditarily disconnected.*

(2) *Hereditary disconnectedness and total disconnectedness are equivalent in the realm of nonempty compact* T_2 *spaces.*

(3) *A totally disconnected space is Hausdorff.*

(4) *Every discrete space is hereditarily disconnected.*

(5) *Every hereditarily disconnected is* T_1.

(6) *A space* B *is* T_1 *iff* $\{a\}$ *is closed for each* $a \in B$.

(7) *Every strongly totally disconnected space is totally disconnected and in the realm of* T_1 *spaces they are equivalent.*

(8) *Every hereditarily disconnected space is strongly hereditarily disconnected and if a topological space is* T_1, *they coincide.*

(9) $\nabla(c) \cap \triangle(c)$ *contains only trivial spaces, where* c *is any closure operator of* **Top**.

(10) $\triangle(c) \subset T_1(c)$ *and* $T_1(c) \cap \nabla(c)$ *may contains nontrivial spaces.*

We can infer:

(1) By Theorem 5.3, a totally disconnected prebornological space is strongly totally disconnected. Moreover, By Remark 5.4 and Theorem 5.3, the prebornological space $(\mathbb{Z}, P(\mathbb{Z}))$ is both strongly totally disconnected and hereditarily disconnected but it is neither strongly hereditarily disconnected nor NT_2 nor T_0. This shows Theorem 5.5(1),(3),(5), and (7) do not hold in **PBorn**. By Theorem 5.3, every nontrivial discrete prebornological space $(X, \mathcal{F})$ is hereditarily disconnected but it is not totally disconnected. This shows Theorem 5.5 (4) holds in **PBorn**.

In the realm of NT_2 prebornological spaces, by Remark 5.4 and Theorem 5.3, all the notions of hereditary disconnectedness and total disconnectedness are equivalent.
By Theorems 3.6(2), 3.7 (B), and 4.2,
$\triangle(c) \cap \nabla(c)$ and $T_1(c) \cap \nabla(c)$ for $c = cl, scl, Q, SQ$ contain only trivial spaces.
$\triangle(cl) = T_1(cl) = \triangle(Q) = T_1(Q) \subset \triangle(scl) = T_1(scl) = \triangle(SQ) == T_1(SQ)$.
This shows Theorem 5.5 (9) and (10) hold in **PBorn**.

(2) In **CP**, the category of pairs (A_1, B_1), where $B_1 \subset A_1$ and functions $f : (A_1, B_1) \to (C_1, D_1)$ such that $f(B_1) \subset D_1$ [3].

All of $T_0', Pre\overline{T}_2, PreT_2'$ and KT_2 are equivalent. Indeed, for a pair space (Z,W), it is not hard to see the final structure on $Z^2 \vee_\Delta Z^2$ induced by $q\circ i_1, q\circ i_2$ and the initial structures on $Z^2 \vee_\Delta Z^2$ induced by the maps A and S are the same, namely,

$$(q\circ i_1)(W^2)\bigcup(q\circ i_2)(W^2) = W^2 \vee_\Delta W^2$$

$$= (\pi_1 S)^{-1}(W)\bigcap(\pi_2 S)^{-1}(W)\bigcap(\pi_3 S)^{-1}(W)$$

$$= (\pi_1 A)^{-1}(W)\bigcap(\pi_2 A)^{-1}(W)\bigcap(\pi_3 A)^{-1}(W).$$

The result follows from these and Theorem 3.7 of [11]. Moreover, $T_0 = NT_2 \Rightarrow KT_2$.

By Definition 3.5 and Theorem 3.8 of [3], $scl = \delta = cl = Q = SQ$, where δ is the discrete closure operator of **CP**.

$T_1(c) = \triangle(c) = \mathbf{KT_2CP} = \mathbf{CP}$ and they are topological categories, where $c = scl, cl, Q$ or $c = SQ$.

By Definition 5.1, all the notions of hereditary disconnectedness and total disconnectedness are equivalent in **CP**. A totally disconnected pair space is not NT_2, as an example, a pair space $(R,\mathbb{Z})$ is not NT_2 but it is hereditarily disconnected. This shows Theorem 5.5 (3) does not hold in **CP**.

$\triangle(c)\cap\nabla(c)$ and $T_1(c)\cap\nabla(c)$ for $c = cl, scl, Q, SQ$

contain only trivial spaces and this shows Theorem 5.5 (9) and (10) hold in **CP**.

(3) In **pqsMet**, the category of extended pseudo-quasisemi metric spaces and nonexpansive maps, by Definition 3.5 and Theorem 3.10 of [12],

$\triangle(scl) = T_1(scl) = T_1(SQ) \subset \triangle(cl) = T_1(cl) = T_1(Q)$.

Hence, every strongly totally disconnected extended pseudo-quasi-semi metric space is totally disconnected but the reverse implication is not true. Let $A = \{a,b\}$ and e be given as $e(a,a) = 0 = e(b,b), e(b,a) = \infty$ and $e(a,b) = 11$. By Theorem 3.4 of [12], $Q(a) = \{a\}, Q(b) = \{b\}$, and $SQ(a) = A = SQ(b)$ and Definition 3.5, $(A,e) \in T_1(Q)$ but $(A,e) \notin T_1(SQ)$. Also, by Theorem 3.2 of [21], (A,e) is not T_1 and by Theorem 3.13 of [21], (A,e) is not KT_2. Hence, Theorem 5.5(3) and (6) are not valid in **pqsMet**.

If a space is NT_2, then by Definitions 3.5 and 5.1, Theorem 3.10 of [12], and Theorem 3.14 of [21], $T_1(SQ) = T_1(Q)$ and $HC\mathbf{pqsMet} = \mathbf{HSCpqsMet}$. If a space is in $\triangle(scl)$, then by Theorem 4.10 of [12], all the notions of connectedness are equivalent; hence, by Definition 5.1, all the notions of hereditary disconnectedness and total disconnectedness are equivalent.

In **FCO**, the category of filter convergence spaces and continuous maps, by Definition 5.1 and Theorem 2.9 of [8], $\triangle(scl) \subset T_1(scl) = \triangle(cl) = T_1(cl)$ and thus Theorem 5.5 (6) hold in **FCO**. If a filter convergence space X is T_1, then, by Remark 4.13 of [5] and by Definition 5.1, total disconnectedness and strong hereditary disconnectedness are equivalent. If a filter convergence space is $PreT_2'$, then, by Theorem 4.10 of [7] and Definition 5.1, all the notions of hereditary disconnectedness and total disconnectedness are equivalent.

(4) By Theorems 3.2, 3.3, 3.7, and 5.2, Parts (1) and (4) of Theorem 5.5 hold in **RRel** and by Theorems 2.3 and 3.7, Part (6) of Theorem 5.5 does not hold in **RRel**. By Remark 5.4, totally disconnected (resp. strongly hereditarily disconnected) space may not be NT_2 and T_1 which shows Theorem 5.5 (3) and (5) do not hold in **RRel**. If a reflexive space is KT_2, then by Theorem 5.2, the notions of hereditary disconnectedness

(resp. total disconnectedness) and strong hereditary disconnectedness (resp. strong total disconnectedness) are equivalent and total disconnectedness implies hereditary disconnectedness. Thus, Parts (7) and (8) of Theorem 5.5 hold in **RRel**
If a reflexive space is NT_2, then, by Theorem 5.2 and Remark 5.4, all the notions of hereditary disconnectedness and total disconnectedness are equivalent.
By Theorems 3.7 and 4.2, $\triangle(C) \cap \nabla(C)$ for $C = cl, scl, Q, SQ$ contains only trivial spaces and $T_1(cl) \cap \nabla(cl)$ may contains nontrivial spaces. Let $A = \{x, y\}$ and $R = \{(x,x),(y,y),(x,y)\}$. By Theorems 2.3, 3.7 and 4.6, $(A,R) \in T_1(cl) \cap \nabla(cl)$. This shows Theorem 5.5 (9) and (10) hold in **RRel**.

(5) In any topological category, by Parts (1) and (4), there are no implications between the notion of strong total disconnectedness (resp. strong hereditary disconnectedness) and total disconnectedness (resp. hereditary disconnectedness). By (2) all the notions of hereditary disconnectedness and total disconnectedness could always be equivalent.

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