Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 333 – 350
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doi:10.55730/1300-0098.3363

Approximation results for the moments of random walk with normally distributed interference of chance

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Received: 30.11.2021 • **Accepted/Published Online:** 14.12.2022 • **Final Version:** 13.01.2023

Abstract: In this study, a random walk process $(X(t))$ with normally distributed interference of chance is considered. In the literature, this process has been shown to be ergodic and the limit form of the ergodic distribution has been found. Here, unlike previous studies, the moments of the $X(t)$ process are investigated. Although studies investigating the moment problem for various stochastic processes (such as renewal-reward processes) exist in the literature, it has not been considered for random walk processes, as it requires the use of new mathematical tools. Therefore, in this study, firstly, the exact formulas for the first four moments of the ergodic distribution of the $X(t)$ process, which is a modification of the random walk process, are found. Due to the extremely complex mathematical structure of the exact formulas, in the second part of the study, three-term asymptotic expansions are attained for these moments. Based on the asymptotic expansions, simple and useful approximation formulas, for the moments of the process $X(t)$ are proposed. In order to show that the approximate formulas are close enough to the exact formulas, a special example is given at the end of the study and the accuracy of the approximate formulas is examined on this example.

Key words: Random walk, discrete interference of chance, ergodic distribution, approximation formulas, normal distribution

1. Introduction

A number of interesting problems arising in reliability, queuing, inventory, stock control theories, mathematical insurance, financial mathematics, mathematical biology, and physics can be expressed by renewal, renewal-reward, random walk processes and their various modifications. A large number of important studies on renewal and renewal-reward processes exist in the literature (Aliyev et al. [2, 5]; Bektas et al. [7]; Borovkov [8]; Brown and Solomon [9]; Feller [12]; Hanalioglu et al. [16, 17]; Kamislik et al. [19]; Khaniyev et al. [21]; Khaniyev and Mammadova [22] and etc.). There are also many valuable studies in the literature that investigated random walks (Aliyev et al. [2]; Aliyev and Khaniyev [3, 4]; Borovkov [8]; Chang [10]; Chang and Peres [11]; Feller [12]; Gihman and Skorohod [13]; Janssen and van Leeuwaarden [18]; Khaniyev [20]; Lotov [23]; Rogozin [24]; Spitzer [25] and etc.).

In this study, a special class of stochastic processes defined by A. N. Kolmogorov and known as “Stochastic

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2010 *AMS Mathematics Subject Classification:* Primary 60G50, Secondary 60K15

processes with a discrete interference of chance" is investigated. Gihman and Skorohod [13] proved the general ergodic theorem for this class. However, the simplified expressions for the ergodic distribution and its characteristics have not been achieved so far because mathematical structures of the exact expressions of the probability and numerical characteristics of the process are very complex. To overcome these difficulties, the studies using approximation methods have increased in recent years (Aliyev et al. [5]; Alsmeyer [6]; Borovkov [8]; Chang [10]; Chang and Peres [11]; Gokpinar et al. [14]; Hanalioglu et al.[15, 17]; Janssen and van Leeuwaarden [18]; Khaniyev and Mammadova [22]; Lotov [23] etc.). Particularly, Chang and Peres's [11], Janssen and Leeuwaarden's [18], and Lotov's [23] studies, which aimed to present approximation results for the boundary functionals of the random walk, are of great interest. Lotov [23] found the simple approximation formulas for the first three moments of ladder height of Gaussian random walk. Chang and Peres [11], studied ladder heights of random walk using Reimann zeta-function. In the study of Janssen and Leeuwaarden [18], the cumulants of the boundary functional of the random walk were investigated.

As is seen, in most of the studies conducted so far, only the boundary functionals of the random walk were examined. However, to investigate the asymptotic behavior of the stationary characteristics of the random walk process is also important for solving various applied problems. However, studies are rare on this topic. For this reason, in this study, a semi-Markovian random walk $(X(t))$ with normally distributed interference of chance is considered and approximation formulas are proposed for the first four moments of the ergodic distribution of the process $X(t)$.

Note that it is possible to describe many stochastic models that arise in applied fields with random walk process and its modifications. Let us take the following model as an example.

The Model. Consider the motion of a high-energy particle in one-dimensional space. Suppose that the particle in a state $z > 0$ at the initial time. Moreover, assume that the particle change its state upwards–downwards (or right and left) with random jumps $\{\eta_n, n = 1, 2, \dots\}$ at random times $T_n \equiv \sum_{i=1}^{n} \xi_i, n = 1, 2, \dots$. Also, assume that there is a special barrier at the zero level. When the particle reaches this barrier, intervention is made immediately to the particle's "natural motion"and the one is brought to a new random initial state (ζ_1). Then, starting from the new random state ζ_1, the particle continues its motion similar to first period. In this study, it is assumed that the discrete interference of chance is expressed by truncated normal distribution. The motivation of choosing normally distributed interference of chance can be explained as follows. The new initial state ζ_1 forms under the influence of many physical factors, the distribution of the random variable ζ_1 can be accepted as normal distribution according to the central limit theorem. Such an assumption is reasonable and acceptable according to main principles of probability theory. Thus, the above-described motion of high-energy particle can be expressed by a random walk process $X(t)$ under normal intervention. Examining the stochastic behavior of this particle, which continues its motion for a long time, is of interest in terms of both mathematics and physics.

Also, the random walk process $X(t)$ with normally distributed interference of chance may represent the variation of capital of an insurance company over time in reinsurance models. Another example of interfering stochastic processes is the random variation of the "resources" of a mechanical system operating under a periodic maintenance and repair policy over time.

Similar problems that can be expressed by means of random walk processes and their modifications can also be encountered in stock control, queue, stochastic finance and other areas.

Let us now give mathematical definition of the stochastic process $X(t)$, which can express the abovementioned models.

2. Mathematical construction of process X(t)

Let $\{(\xi_n,\ \eta_n, \zeta_n), n = 1,\ 2, \dots\}$ be a sequence of random triples defined on some probability space $(\Omega, \mathcal{F},\ P)$, such that triples are independent and identically distributed. $\xi_n,\ n \geq 1$, take only positive values; $\eta_n,\ \ n \geq 1$, take both positive and negative values. Suppose that the random variables $\xi_n,\ \eta_n$, and ζ_n are mutually independent from each other. Moreover, $\zeta_n = \max\{0;\ Y_n\}$, $n \geq 1$, where $\{Y_n, n = 1,\ 2, \dots\}$ is a sequence of normally distributed random variables with parameters $\left(a, \sigma^2\right)$, $a > 0, \sigma > 0$. In other words, the probability density function of Y_n can be written as follows:

$$f_Y(x) = \frac{1}{\sigma}\varphi\left(\frac{x-a}{\sigma}\right), \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), \quad x \in R. \tag{2.1}$$

Here, $\varphi(x)$ is the probability density function of standard normal distribution. Let us denote the distribution function of ζ_n by $\pi(z)$. Then, it holds that $\pi(z) = P\{\zeta_n \leq z\} = \Phi((z-a)/\sigma)$, when $z \geq 0$ and $\pi(z) = P\{\zeta_n \leq z\} = 0$ when $z < 0$. Here, $\Phi(x)$ is the cumulative distribution function of standard normal distribution. Note that the random variable ζ_n represents the discrete interference of chance.

Let us define the renewal sequence $\{\mathrm{T}_n\}$ and random walk $\{\mathrm{S}_n\}$ as follows, respectively

$$\mathrm{T}_0 = \mathrm{S}_0 = 0;\ \mathrm{T}_n \equiv \sum_{i=1}^{n}\xi_i;\ \mathrm{S}_n \equiv \sum_{i=1}^{n}\eta_i,\ n = 1,\ 2, \dots.$$

Now we can define a sequence of integer-valued random variables $\{\mathrm{N}_n\}$, $n = 0,\ 1,\ 2, \dots$ as

$$N_0 = 0, \quad N_1 \equiv N(z) = \inf\{n \geq 1 : z - S_n < 0\}, \qquad z > 0;$$

$$N_{n+1} \equiv \inf\{k \geq N_n + 1 : \quad \zeta_n - (S_k - S_{N_n}) < 0\}, \quad n = 1,\ 2,\ 3, \dots.$$

and $\inf\{\emptyset\} = +\infty$ is stipulated.

Let $\tau_0 \equiv 0, \quad \tau_1 \equiv \tau_1(z) \equiv \sum_{i=1}^{N(z)}\xi_i\ , \ \dots, \quad \tau_n \equiv \sum_{i=1}^{N_n}\xi_i\ ,\ n = 1,\ 2, \dots.$

and define $\nu(t)$ as $\nu(t) \equiv \max\{n \geq 0 :\ T_n \leq t\}\ , t > 0$.

Now we can construct the desired stochastic process $X(t)$ as follows:

$$X(t) \equiv \zeta_n - \left(S_{\nu(t)} - S_{N_n}\right), \ \tau_n \leq t < \tau_{n+1}, \quad n = 0,\ 1,\ 2,\ 3, \dots, \quad t > 0. \tag{2.2}$$

Here, $\zeta_0 \equiv z > 0$, ζ_n is a sequence of random variables which have truncated normal distribution (see Eq. (2.1)).

The process $X(t)$ defined by Eq. (2.2) is called "A semi-Markovian random walk with normally distributed interference of chance". A sample path of the process $X(t)$ is given in the figure below.

Remark. The main purpose of this study is to obtain simple and useful approximation formulas for the moments of the ergodic distribution of the process $X(t)$. To achieve this goal, it is first necessary to express the characteristic function of the ergodic distribution of the process in terms of the characteristic function of boundary functional $S_{N(z)}$. Note that, in the study [15], the same process $X(t)$ was discussed and it was shown

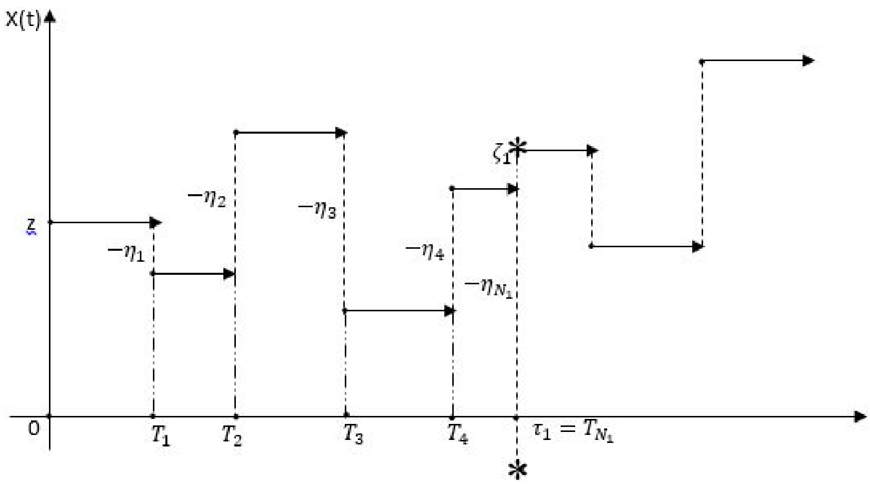

Figure. A realization of the process X(t).

that this process is ergodic. Moreover, the limiting form of ergodic distribution is found. However, unlike [15], in this study, not the ergodic distribution itself, but its moments is examined. Let us also emphasize that from the limit form of the ergodic distribution, only the one-term asymptotic expansion for moments can be obtained. Unlike distributions, one-term expansions for moments (especially higher-order moments) often give very imprecise approximations. On the other hand, three-term asymptotic expansions allow more precise and accurate approximations to be derived.

Note that the similar problem of moments has also been discussed and studied earlier in [17]. However, the stochastic process examined in [17] is a modification of the renewal–reward process. Unlike [17], the $X(t)$ process discussed in this study is a modification of the random walk process. As is known, all trajectories of the renewal–reward process are monotonous with probability 1 and are, therefore, relatively easy to analyze mathematically. On the other hand, the trajectories of the random walk processes and their modifications are not monotonous. For this reason, the moments of the $X(t)$ process are quite challenging to analyze. Therefore, in this study, the moments of a modification of the random walk process were investigated using different and more complex mathematical tools than in [17].

3. Exact expressions for moments of ergodic distribution of process X(t)

Now, let us begin our investigation by first considering the exact formula of the characteristic function of ergodic distribution, from study [15].

Lemma 3.1 *(Hanalioglu et al. [15], Lemma 3.1, p.66) Let the initial sequences of the random variables* $\{\xi_n\}$ *and* $\{\eta_n\}$ *satisfy the following supplementary conditions:* i) $E(\xi_1) < \infty$, ii) $E(\eta_1) > 0$, iii) $E(\eta_1^2) < +\infty$, *iv)* η_1 *is a non-arithmetic random variable. Then, for each* $\theta \in R/\{0\}$, *the characteristic function* $\varphi_X(\theta)$ *of the ergodic distribution of the process* $X(t)$ *can be expressed by means of characteristic functions of boundary* *functional* $S_{N(z)}$ *and random variable* η_1, *as follows:*

$$\varphi_X(\theta) \equiv \lim_{t\to\infty} E\left(\exp\left(i\theta X(t)\right)\right) = \frac{1}{E\left(N\left(\zeta_1\right)\right)} \int_0^\infty e^{i\theta z} \frac{\varphi_{S_{N(z)}}(-\theta)-1}{\varphi_\eta(-\theta)-1} d\pi(z),$$

where $\varphi_\eta(-\theta) = E\left(\exp\left(-i\theta\eta_1\right)\right)$; $\quad \varphi_{S_{N(z)}}(-\theta) = E\left(\exp\left(-i\theta S_{N(z)}\right)\right)$; $\theta \in R/\{0\}$.

Note that, Lemma 3.1 provides the basis to express the first four moments of the ergodic distribution of the process $X(t)$ by means of moments of the boundary functional $S_{N(z)}$ and some related integrals.

Now, to express the moments of the ergodic distribution $(E(X^n);\ n = 1,\ 2,\ 3,\ 4)$ of the process $X(t)$ via moments of the boundary functional $S_{N(z)}$. For this, we should introduce the following notations:

$$E\left(X^k\right) \equiv \lim_{t\to\infty} E\left(X^k(t)\right); m_k \equiv E\left(\eta_1^k\right); m_{k1} \equiv \frac{m_k}{km_1}; M_k(z) \equiv E\left(S_{N(z)}^k\right); M_{k1}(z) \equiv \frac{M_k(z)}{kM_1(z)};\ k = 1, 2, 3, \ldots.$$

Here, $S_n \equiv \sum_{i=1}^n \eta_i, n = 1, 2, 3, ..., S_{N(z)} \equiv \sum_{i=1}^{N(z)} \eta_i, N(z) \equiv \min\{n \geq 1: \ S_n > z\}$, $z > 0$.

Theorem 3.2 *Let, in addition to the assumptions of the Lemma 3.1, the condition* $E\left(\eta_1^6\right) < \infty$ *and* $E\left(X^4\right) < \infty$ *be satisfied. Then, the first four moments of ergodic distribution of the process* $X(t)$ *can be expressed by means of the moments of boundary functional* $S_{N(z)}$ *as follows:*

$$E(X) = \frac{1}{E\left(M_1(\zeta_1)\right)} \left\{ E\left(\zeta_1 M_1(\zeta_1)\right) - \frac{1}{2} E\left(M_2(\zeta_1)\right) \right\} + A_1, \tag{3.1}$$

$$\begin{aligned} E\left(X^2\right) &= \frac{1}{E\left(M_1(\zeta_1)\right)} \left\{ E\left({\zeta_1}^2 M_1(\zeta_1)\right) - E\left(\zeta_1 M_2(\zeta_1)\right) + \frac{1}{3} E\left(M_3(\zeta_1)\right) \right. \\ &+ A_1\left[2E\left(\zeta_1 M_1(\zeta_1)\right) - E\left(M_2(\zeta_1)\right)\right]\} - A_2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} E\left(X^3\right) &= \frac{1}{E\left(M_1(\zeta_1)\right)} \left\{ E\left({\zeta_1}^3 M_1(\zeta_1)\right) - \frac{3}{2} E\left({\zeta_1}^2 M_2(\zeta_1)\right) + E\left(\zeta_1 M_3(\zeta_1)\right) - \frac{1}{4} E\left(M_4(\zeta_1)\right) \right\} \\ &+ \frac{3A_1}{E\left(M_1(\zeta_1)\right)} \left\{ E\left({\zeta_1}^2 M_1(\zeta_1)\right) - E\left(\zeta_1 M_2(\zeta_1)\right) + \frac{1}{3} E\left(M_3(\zeta_1)\right) \right\} \\ &- \frac{3A_2}{E\left(M_1(\zeta_1)\right)} \left\{ E\left(\zeta_1 M_1(\zeta_1)\right) - \frac{1}{2} E\left(M_2(\zeta_1)\right) \right\} + A_3, \end{aligned} \tag{3.3}$$

$$\begin{aligned} E\left(X^4\right) &= \frac{1}{E\left(M_1(\zeta_1)\right)} \left\{ E\left(\zeta_1^4 M_1(\zeta_1)\right) - 2E\left(\zeta_1^3 M_2(\zeta_1)\right) + 2E\left(\zeta_1^2 M_3(\zeta_1)\right) - E\left(\zeta_1 M_4(\zeta_1)\right) + \frac{1}{5} E\left(M_5(\zeta_1)\right) \right\} \\ &+ \frac{2A_1}{E\left(M_1(\zeta_1)\right)} \left\{ 2E\left(\zeta_1^3 M_1(\zeta_1)\right) - 3E\left(\zeta_1^2 M_2(\zeta_1)\right) + 2E\left(\zeta_1 M_3(\zeta_1)\right) - \frac{1}{2} E\left(M_4(\zeta_1)\right) \right\} \\ &- \frac{6A_2}{E\left(M_1(\zeta_1)\right)} \left\{ E\left(\zeta_1^2 M_1(\zeta_1)\right) - E\left(\zeta_1 M_2(\zeta_1)\right) + \frac{1}{3} E\left(M_3(\zeta_1)\right) \right\} \\ &+ \frac{2A_3}{E\left(M_1(\zeta_1)\right)} \left\{ 2E\left(\zeta_1 M_1(\zeta_1)\right) - E\left(M_2(\zeta_1)\right) \right\} - A_4. \end{aligned} \tag{3.4}$$

Here, $A_1 \equiv m_{21}$, $A_2 \equiv m_{31} - 2m_{21}^2$, $A_3 \equiv m_{41} - 6m_{31}m_{21} + 6m_{21}^3$,
and $A_4 \equiv m_{51} - 8m_{41}m_{21} + 36m_{31}m_{21}^2 - 6m_{31}^2 - 24m_{21}^4$.

Proof In order to prove this theorem, we should initially write the Taylor expansion of the characteristic function $\varphi_X(\theta)$, as $\theta \to 0$. Since the condition $E\left(\eta_1^6\right) < \infty$ is met, the following expansion can be written (Feller [12], p.514):

$$\begin{aligned} \varphi_\eta(-\theta) &\equiv E\left(\exp\left(-i\theta\eta_1\right)\right) \\ &= 1 - i\theta m_1 + \frac{(i\theta)^2}{2!}m_2 - \frac{(i\theta)^3}{3!}m_3 + \frac{(i\theta)^4}{4!}m_4 - \frac{(i\theta)^5}{5!}m_5 + R_1(i\theta\delta). \end{aligned} \tag{3.5}$$

Here, $R_1(i\theta\delta)$ is Lagrange form of remainder term of Taylor expansion for the $\varphi_\eta(-\theta)$ and $R_1(i\theta\delta)$ can be represented as follows: $R_1(i\theta\delta) = \frac{(i\theta)^6}{6!}H_1(i\theta\delta)$ where $H_1(i\theta\delta) \equiv E\left(\eta_1^6 e^{-i\theta\delta\eta_1}\right)$, $0 < \delta < 1$, $\theta \in R$.

On the other hand, it is known that $E(S_{N(z)}^6) < \infty$ when provided the condition $E\left(\eta_1^6\right) < \infty$ (Feller [12], p.514). Then, the following expansion can be written for the characteristic function of $S_{N(z)}$ (Feller [12], p.514):

$$\begin{aligned} \varphi_{S_{N(z)}}(-\theta) &\equiv E\left(\exp\left(-i\theta S_{N(z)}\right)\right) \\ &= 1 - i\theta M_1(z) + \frac{(i\theta)^2}{2!}M_2(z) - \frac{(i\theta)^3}{3!}M_3(z) + \frac{(i\theta)^4}{4!}M_4(z) \\ &- \frac{(i\theta)^5}{5!}M_5(z) + R_2(i\theta\delta; z). \end{aligned} \tag{3.6}$$

Here, $R_2(i\theta\delta; z)$ is Lagrange form of remainder term of Taylor expansion for the $\varphi_{S_{N(z)}}(-\theta)$ and it can be represented as follows:

$$R_2(i\theta\delta; z) = \frac{(i\theta)^6}{6!}H_2(i\theta\delta; z),$$

where $H_2(i\theta\delta; z) \equiv E\left(S_{N(z)}^6 e^{-i\theta\delta S_{N(z)}}\right)$, $0 < \delta < 1$, $\theta \in R$, $z \in (0, \infty)$.

From Eqs. (3.5) and (3.6), the following expansion is derived, when $\theta \to 0$:

$$\begin{aligned} \frac{\varphi_{S_{N(z)}}(-\theta) - 1}{\varphi_\eta(-\theta) - 1} &= \frac{M_1(z)}{m_1} + \frac{i\theta}{1!}.\frac{M_1(z)}{m_1}\left[m_{21} - M_{21}(z)\right] \\ &+ \frac{(i\theta)^2}{2!}.\frac{M_1(z)}{m_1}\left[2m_{21}^2 m_{31} - 2m_{21}M_{21}(z) + M_{31}(z)\right] \\ &+ \frac{(i\theta)^3}{3!}.\frac{M_1(z)}{m_1}\left[m_{41} - 6m_{31}m_{21} + 6m_{21}^3 - \left(6m_{21}^2 - 3m_{31}\right)M_{21}(z) + 3m_{21}M_{31}(z) - M_{41}(z)\right] \\ &+ \frac{(i\theta)^4}{4!}.\frac{M_1(z)}{m_1}\left[-m_{51} + 8m_{41}m_{21} + 6m_{31}^2 - 36m_{31}m_{21}^2 + 24m_{21}^4\right. \\ &- \left(4m_{41} - 24m_{31}m_{21} + 24m_{21}^3\right)M_{21}(z) + \left(12m_{21}^2 - 6m_{31}\right)M_{31}(z) \\ &- \left. 4m_{21}M_{41}(z) + M_{51}(z) + R_3(i\theta\delta; z)\right. . \end{aligned} \tag{3.7}$$

Here, $R_3(i\theta\delta; z) = \frac{(i\theta)^5}{5!} H_3(i\theta\delta; z)$; $|H_3(i\theta\delta; z)| \le c_1 E(\eta_1^6) + c_2 E\left(S_{N(z)}^6\right)$;

Also the coefficients c_1 and c_2 are finite positive numbers.

In this case, the conditions $E(\eta_1^6) < \infty$ and $E\left(S_{N(z)}^6\right) < \infty$ are satisfied because $|H_3(i\theta\delta; z)| < \infty$. Therefore, asymptotic relation $R_3(i\theta\delta; z) = o\left((i\theta)^5\right)$ is true.

On the other hand, under the condition $E(X^4) < \infty$, the following expansion can be written, as $\theta \to 0$: (Feller [12], p. 514) :

$$\begin{aligned} \varphi_X(\theta) &\equiv E(\exp(i\theta X)) = \lim_{t\to\infty} E(\exp(i\theta X(t))) \\ &= 1 + i\theta E(X) + \frac{(i\theta)^2}{2!} E(X^2) + \frac{(i\theta)^3}{3!} E(X^3) + \frac{(i\theta)^4}{4!} E(X^4) + o(\theta^4). \end{aligned} \tag{3.8}$$

Using Eq. (3.7) and Eq. (3.8), the exact expressions for the first four moments of the ergodic distribution of the process $X(t)$ can be derived as Eqs. (3.1)–(3.4). □

Remark. As it is seen even in the basic cases, it is not easy to calculate exact expressions from Theorem 3.2. To overcome this difficulty, it became necessary to find simpler and more convenient approximate formulas for the moments of the ergodic distribution of the process $X(t)$. For this reason, we will first try to derive asymptotic expansions for the moments of the boundary functional $S_{N(z)}$.

4. Asymptotic expansions for moments and related characteristics of boundary functional $S_{N(z)}$

For investigation of the boundary functional $S_{N(z)}$, we should define the first ladder epoch (ν_1^+) and the first ladder height (χ_1^+) of the random walk $\{S_n\}$, $n \ge 0$ as:

$$\nu_1^+ \equiv \min\{n \ge 1 : S_n > 0\}, \quad \chi_1^+ \equiv S_{\nu_1^+} \equiv \sum_{i=1}^{\nu_1^+} \eta_1.$$

Let (ν_n^+, χ_n^+), $n = 2, 3, \ldots$ be independent pairs with the same distribution as (ν_1^+, χ_1^+) (see Feller [12], p.392).

Now, a stochastic process $H(z)$ generated by the sequence $\{\chi_n^+\}$, $n = 1, 2, \ldots$ can be defined as follows:

$$H(z) \equiv \min\left\{n \ge 1 : \sum_{i=1}^{n} \chi_i^+ > z\right\}, \ z \ge 0.$$

The stochastic process $H(z)$ is a renewal process (see Feller [12], p.184). According to Dynkin principle, the boundary functionals $N(z)$ and $S_{N(z)}$ can be represented as follows (Rogozin [24]):

$$N(z) \equiv \sum_{i=1}^{H(z)} \nu_i^+ \ and \ S_{N(z)} \equiv \sum_{i=1}^{H(z)} \chi_i^+.$$

The main goal of this section is to study the asymptotic behavior of the following auxiliary integrals:

$$E\left(\zeta_1^n M_k\left(\zeta_1\right)\right) \equiv \int_{z=0}^{\infty} z^n M_k\left(z\right) d\pi\left(z\right); \quad n=0,\ 1,\ 2,\ldots; k=1,\ 2,\ 3,\ \ldots. \tag{4.1}$$

Here, $\pi\left(z\right)$ is a distribution function of the random variable ζ_1 that expresses a discrete interference of chance and $\zeta_1 = \max\left(0,\ Y_1\right); Y_1 \sim N\left(a,\ \ \sigma^2\right), \quad a>0, \quad \sigma>0.$

Let us give the following important properties related to standard normal distribution, before obtaining the asymptotic results for the integrals in Eq. (4.1).

Proposition 4.1 *The following equalities are true for all* $n=1,\ 2,\ 3\ldots$ and $a>0$*:*

$$\int_a^{\infty} z^n \varphi_\sigma\left(z-a\right) dz = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}\sigma^k b_k.$$

Here, $b_k \equiv \int_0^{\infty} x^k \varphi\left(x\right) dx < \infty, \quad k=0,\ 1,\ dots,\ n; \sigma^2 = \mathrm{Var}\left(Y_1\right), a\text{= }\mathrm{E}\left(Y_1\right);$

$$\varphi_\sigma\left(z\right) \equiv \frac{1}{\sigma}\varphi\left(\frac{z}{\sigma}\right),\ \varphi\left(x\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), \quad x \in R.$$

The following corollaries can be easily derived from Proposition 4.1.

Corollary 4.2 *The following exact expressions can be written:*

1) $\int_a^{\infty} \varphi_\sigma\left(z-a\right) dz = \frac{1}{2}$,

2) $\int_a^{\infty} z\varphi_\sigma\left(z-a\right) dz = \frac{a}{2} + \frac{\sigma}{\sqrt{2\pi}}$,

3) $\int_a^{\infty} z^2\varphi_\sigma\left(z-a\right) dz = \frac{a^2}{2} + \frac{2\sigma a}{\sqrt{2\pi}} + \frac{\sigma^2}{2}$.

Corollary 4.3 *The following approximation expressions are derived for* $n=3,\ 4,\ldots$

$$\int_a^{\infty} z^n \varphi_\sigma\left(z-a\right) dz = \frac{1}{2}a^n + \frac{n\sigma}{\sqrt{2\pi}}a^{n-1} + \frac{n\left(n-1\right)\sigma^2}{4}a^{n-2} + R_n\left(a\right).$$

Here, $\frac{R_n(a)}{a^{n-2}} \to 0$*, when* $a \to \infty$*.*

Lemma 4.4 *Let the function* $g\left(z\right)$ $\left(g: R^+ \to R\right)$ *be a bounded function that converges to zero, as* $z \to \infty$*. Then, the following asymptotic relations are satisfied, as* $a \to \infty$*:*

$$\int_a^{\infty} z^n g\left(z\right) \varphi_\sigma\left(z-a\right) dz = o\left(a^n\right), \quad n=-1,\ 0,\ 1,\ 2,\ldots.$$

Proof Let us firstly prove lemma for $n = -1$. It is possible to select a finite number $b = b(\varepsilon)$, for all $\varepsilon > 0$, so that the inequality $|g(z)| \leq \varepsilon$ is satisfied, for all $z \geq b(\varepsilon)$, because of that $\lim_{z\to\infty} g(z) = 0$ }. Also, parameter a can be chosen sufficiently large so that the inequality $a \geq b(\varepsilon)$ is satisfied, for all $\varepsilon > 0$. Thus, the following inequality is provided as $a \geq b(\varepsilon)$:

$$\left|\int_a^\infty \frac{1}{z} g(z) \varphi_\sigma(z-a)\, dz\right| \leq \int_a^\infty \frac{1}{z} |g(z)| \varphi_\sigma(z-a)\, dz \leq \varepsilon \int_a^\infty \frac{1}{z} \varphi_\sigma(z-a)\, dz$$

$$\leq \frac{\varepsilon}{a} \int_a^\infty \varphi_\sigma(z-a)\, dz = \frac{\varepsilon}{2a}.$$

Then, we can obtain the following inequality, when $a \to \infty$:

$$\left|a \int_a^\infty \frac{1}{z} g(z) \varphi_\sigma(z-a)\, dz\right| \leq \frac{\varepsilon}{2}.$$

Because $\varepsilon > 0$ is arbitrarily selected, the following asymptotic relation can be written, when $a \to \infty$:

$$a \int_a^\infty \frac{1}{z} g(z) \varphi_\sigma(z-a)\, dz = o(1)$$

or

$$\int_a^\infty \frac{1}{z} g(z) \varphi_\sigma(z-a)\, dz = o\left(\frac{1}{a}\right).$$

In a similar manner, the remaining part of the Lemma 4.4 can be proved. □

Proposition 4.5 *Let the function* $\widetilde{\varphi}_\sigma(\lambda)$ *be Laplace transform of the function* $\varphi_\sigma(z)$. *Then, the following three-term asymptotic expansion holds, as* $\lambda \to 0$*:*

$$\widetilde{\varphi}_\sigma(\lambda) = \frac{1}{2} - \frac{\sigma}{\sqrt{2\pi}}\lambda + \frac{\sigma^2}{4}\lambda^2 + o\left(\lambda^2\right). \tag{4.2}$$

Proof This proposition is given in the work of Hanalioglu et al. ([15], p.67). Therefore, we do not give the proof of Proposition 4.5 here. □

Let us now restate the following proposition regarding moments of boundary functional $S_{N(z)}$, taken from the work of Khaniyev and Mammadova [22].

Proposition 4.6 *(Khaniyev and Mammadova [22]) Let the condition* $\mu_3 \equiv E\left(\chi_1^{+3}\right) < +\infty$ *be satisfied. Then, the following asymptotic expansions for the first five moments of* $S_{N(z)}$ *can be written, as* $z \to \infty$*:*

$$\begin{aligned} M_1(z) &= z + \mu_{21} + o\left(\frac{1}{z}\right), \\ M_n(z) &= z^n + \binom{n}{1}\mu_{21} z^{n-1} + \binom{n}{2}\mu_{31} z^{n-2} + o\left(z^{n-2}\right),\ n = 2,\ 3,\ 4,\ 5. \end{aligned}$$

Here, $M_k(z) \equiv E\left(S_{N(z)}^k\right)$, $k = \overline{1,\ 5}$; $\mu_k \equiv E\left(\chi_1^{+k}\right)$, $\mu_{k1} \equiv \frac{\mu_k}{k\mu_1}$, $k = 1,\ 2,\ 3$.

Using this proposition, we can derive the asymptotic results for the integrals defined in Eq. (4.1), which is the main goal of this section. The following lemma is dedicated to this aim.

Lemma 4.7 *Suppose that the condition* $\mu_3 \equiv E\left(\chi_1^{+3}\right) < +\infty$ *is satisfied. Then, the following three-term asymptotic expansions can be written for all* $n = 0,\ 1,\ 2, \ldots,$ *when* $a \to \infty$*:*

$$E\left(\zeta_1^n M_1\left(\zeta_1\right)\right) = a^{n+1} + \mu_{21} a^n + \frac{1}{2} n\left(n+1\right)\sigma^2 a^{n-1} + o\left(a^{n-1}\right). \tag{4.3}$$

Proof We can represent the auxiliary integral $E\left(\zeta_1^n M_1\left(\zeta_1\right)\right)$, defined in Eq. (4.1) as the following Lebesgue–Stiltijes integral: $E\left(\zeta_1^n M_1\left(\zeta_1\right)\right) = \int_0^\infty z^n M_1\left(z\right) d\pi\left(z\right).$

Here the distribution function $\pi\left(z\right) = \Phi\left(\frac{z-a}{\sigma}\right)\epsilon\left(z\right)$ has a positive jump at $z = 0$.

Moreover, $\epsilon\left(z\right) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$ and $\Phi(\mathrm{x})$ is the cumulative distribution function of standard normal distribution. Also, the height of this jump is $h \equiv \lim_{\theta \to 0}\left(\pi\left(+\theta\right) - \pi\left(-\theta\right)\right) = \Phi\left(\frac{-a}{\sigma}\right).$

According to the definition of the Lebesgue-Stiltijes integral for any continuous function $G\left(z\right)$, that is finite at the zero point, the following relationship can be written:

$$\begin{aligned} \int_{-0}^{\infty} G\left(z\right)) d\pi\left(z\right) &\equiv G\left(0\right) h + \int_{+0}^{\infty} G\left(z\right)) d\pi\left(z\right) \\ &= G\left(0\right)\Phi\left(\frac{-a}{\sigma}\right) + \int_{+0}^{\infty} G\left(z\right))\varphi_\sigma\left(z-a\right) d\left(z\right). \end{aligned}$$

Assuming $G\left(z\right) = z^n M_1\left(z\right)$, based on the property of the Lebesgue-Stiltijes integral, characteristics $E\left(\zeta_1^n M_1\left(\zeta_1\right)\right)$ can be written as follows, for all $n = 0,\ 1,\ 2, \ldots.$

$$E\left(\zeta_1^n M_1\left(\zeta_1\right)\right) = \int_0^\infty z^n M_1\left(z\right)\varphi_\sigma\left(z-a\right) dz + \mu_1 \Phi\left(-a/\sigma\right)\delta_n. \tag{4.4}$$

Here $\delta_n = \begin{cases} 1, & if\ n = 0 \\ 0, & if\ n \neq 0; \end{cases}$; $\mu_1 \equiv E\left(\chi_1^+\right) = \lim_{z \to\ +0} M_1\left(z\right).$

Now, let us define the following notations:

$$I_{1n}\left(a\right) \equiv \int_0^a M_{1n}\left(z\right)\varphi_\sigma\left(z-a\right) dz = M_{1n}(a) * \varphi_\sigma\left(a\right); \qquad I_{2n}\left(a\right) \equiv \int_a^\infty M_{1n}\left(z\right)\varphi_\sigma\left(z-a\right) dz.$$

Here, $M_{1n}\left(z\right) \equiv z^n M_1\left(z\right),\ n = 0,\ 1,\ 2, \ldots.$

Let us study the asymptotic behavior of integrals $I_{1n}\left(a\right)$ and $I_{2n}\left(a\right)$. Firstly, we need to examine the asymptotic behavior of integrals $I_{1n}\left(a\right)$. According to the Tauber-Abel theorem, to study the asymptotic behavior of $\tilde{I}_{1n}\left(\lambda\right)$, as $\lambda \to 0$, is equivalent to investigate the asymptotic behavior of $I_{1n}\left(a\right)$, when $a \to \infty$. Here, $\tilde{I}_{1n}\left(\lambda\right)$ is Laplace transform of $I_{1n}\left(a\right)$. Then, we can write $\tilde{I}_{1n}\left(\lambda\right)$ as follows:

$$\tilde{I}_{1n}\left(\lambda\right) = \tilde{M}_{1n}\left(\lambda\right)\widetilde{\varphi}_\sigma\left(\lambda\right). \tag{4.5}$$

The Laplace transform ($\widetilde{\varphi}_\sigma(\lambda)$) of the function $\varphi_\sigma(z)$ has been found in Proposition 4.5 (see, Eq. (4.2)). Also, using Proposition 4.6, we can write the following expansion for all $n = 0,\ 1,\ 2, \ldots$

$$M_{1n}(z) \equiv z^n M_1(z) = z^{n+1} + \mu_{21} z^n + o\left(z^{n-1}\right)$$

Then, the asymptotic expansion of $\tilde{M}_{1n}(\lambda)$ can be written as follows, when $\lambda \to 0.$:

$$\tilde{M}_{1n}(\lambda) = \frac{(n+1)!}{\lambda^{n+2}} + \mu_{21}\frac{n!}{\lambda^{n+1}} + o\left(\frac{1}{\lambda^n}\right) = \frac{(n+1)!}{\lambda^{n+2}}\left\{1 + \frac{\mu_{21}}{n+1}\lambda + o\left(\lambda^2\right)\right\}. \tag{4.6}$$

Substituting Eqs. (4.2) and (4.6) into Eq. (4.5), the following expansion can be written:

$$\tilde{I}_{1n}(\lambda) = \frac{(n+1)!}{2\lambda^{n+2}} + \left[\frac{\mu_{21}}{n+1} - \frac{2\sigma}{\sqrt{2\pi}}\right]\frac{(n+1)!}{2\lambda^{n+1}} + \left[\frac{\sigma^2}{2} - \frac{2\sigma\mu_{21}}{(n+1)\sqrt{2\pi}}\right]\frac{(n+1)!}{2\lambda^n} + o\left(\frac{1}{\lambda^n}\right).$$

Using Tauber-Abel theorem for $\tilde{I}_{1n}(\lambda)$, the following asymptotic expansion for $I_{1n}(a)$ is found, as $a \to \infty$:

$$I_{1n}(a) = \frac{1}{2}a^{n+1} + \left[\frac{1}{2}\mu_{21} - \frac{\sigma(n+1)}{\sqrt{2\pi}}\right]a^n + \frac{n}{4}\left[(n+1)\sigma^2 - \frac{4\sigma\mu_{21}}{\sqrt{2\pi}}\right]a^{n-1} + o\left(a^{n-1}\right). \tag{4.7}$$

Now, we can investigate asymptotic behavior of the integral

$$I_{2n}(a) \equiv \int_a^\infty M_{1n}(z)\varphi_\sigma(z-a)\,dz,$$

as $a \to \infty$.

As stated above, the following asymptotic expansion holds for $M_{1n}(z)$, when $z \to \infty$:

$$M_{1n}(z) = z^{n+1} + \mu_{21}z^n + z^{n-1}g_1(z). \tag{4.8}$$

Here, $g_1(z) \equiv z[M_1(z) - z - \mu_{21}]$ and $\lim_{z\to\infty} g_1(z) = 0$.

Substituting Eq. (4.8) into the integral $I_{2n}(a)$, the following equation is achieved:

$$I_{2n}(a) \equiv \int_a^\infty z^{n+1}\varphi_\sigma(z-a)\,dz + \mu_{21}\int_a^\infty z^n\varphi_\sigma(z-a)\,dz + \int_a^\infty z^{n-1}g_1(z)\varphi_\sigma(z-a)\,dz. \tag{4.9}$$

If we consider Corollary 4.3 and Lemma 4.4 in Eq. (4.9), the following asymptotic expansions can be derived, when $a \to \infty$:

$$I_{2n}(a) = \frac{a^{n+1}}{2} + \left[\frac{\mu_{21}}{2} + \frac{\sigma(n+1)}{\sqrt{2\pi}}\right]a^n + \frac{n(n+1)}{4}\left[\sigma^2 + \frac{4\sigma\mu_{21}}{\sqrt{2\pi}(n+1)}\right]a^{n-1} + o\left(a^{n-1}\right). \tag{4.10}$$

In order to complete the proof of the Lemma 4.4, we just need to investigate the expression $\mu_1\Phi(-a/\sigma)\delta_n$. For this, let us denote $T \equiv a/\sigma \to \infty$, as $a \to \infty$. Using the asymptotic properties of error function (see, Abramowitz and Stegun [1], p. 298), we can write the following expansion:

$$\Phi(-T) = 1 - \Phi(T) = \frac{\varphi(T)}{T}(1 + o(1)) = o\left(\frac{1}{T}\right) = o(\frac{1}{a}).$$

Under the condition $\mu_1 \equiv E\left(\chi_1^+\right) < \infty$, we have found the following expansions, for all $n = 0, 1, 2, \ldots$, as $a \to \infty$:

$$\mu_1 \Phi\left(-T\right) \delta_n = o\left(\frac{1}{a}\right). \tag{4.11}$$

The expression in Eq. (4.4) can be rewritten as follows:

$$E\left(\zeta_1^n M_1\left(\zeta_1\right)\right) = I_{1n}\left(a\right) + I_{2n}\left(a\right) + \mu_1 \Phi\left(-a/\sigma\right) \delta_n. \tag{4.12}$$

Using the results of Eqs. (4.7), (4.10), and (4.11) in Eq. (4.12), the asymptotic expansions for $E\left(\zeta_1^n M_1\left(\zeta_1\right)\right)$ can be derived in the form of Eq. (4.3).

□

Lemma 4.8 *Suppose that, the condition $\mu_3 \equiv E\left(\chi_1^{+3}\right) < +\infty$ is satisfied. Then, the following asymptotic expansions can be written for all $n = 0, 1, 2, \ldots$ and $k = 2, 3, 4, 5$ when $a \to \infty$:*

$$E\left(\zeta_1^n M_k\left(\zeta_1\right)\right) = a^{n+k} + \binom{k}{1} \mu_{21} a^{n+k-1} + \left[\binom{k}{2} \mu_{31} + \binom{n+k}{2} \sigma^2\right] a^{n+k-2} + o\left(a^{n+k-2}\right).$$

Proof This lemma can be derived by using a similar scheme for Lemma 4.7. □

5. Asymptotic expansions for first four moments of ergodic distribution of X(t)

In this section, it is aimed to derive three-term asymptotic expansions for the first four moments of the ergodic distribution of the process $X\left(t\right)$. These results are given in the following theorem.

Theorem 5.1 *Let, in addition to the conditions of Lemma 3.1, the condition $E\left(\left|\eta_1\right|^{n+2}\right) < \infty$, $(n = 1, 2, 3, 4)$ be satisfied. Then, the following three-term asymptotic expansions for the first four moments of the ergodic distribution of $X(t)$ can be written as follows, as $a \to \infty$:*

$$E\left(X^n\right) = \frac{a^n}{n+1} + c_{n1} a^{n-1} + c_{n2} a^{n-2} + o\left(a^{n-2}\right); \; n = 1, 2, 3, 4.$$

Here, $c_{11} \equiv m_{21} - \frac{1}{2}\mu_{21}$; $c_{12} \equiv \frac{1}{2}\left(\sigma^2 + \mu_{21}^2 - \mu_{31}\right)$; $c_{n1} \equiv m_{21} - \frac{1}{n+1}\mu_{21}$;

$$c_{n2} \equiv \frac{n}{2}\sigma^2 + \frac{1}{n+1}\mu_{21}^2 - \frac{n}{2} m_{31} + n m_{21}^2 - m_{21}\mu_{21}; \quad n = 2, 3, 4;$$

$$m_{k1} \equiv \frac{m_k}{k m_1}; \quad m_k \equiv E\left(\eta_1^k\right); \quad \mu_{k1} = \frac{\mu_k}{k\mu_1}; \; \mu_k = E\left(\chi_1^{+k}\right); \; k = 1, 2, 3; \quad \sigma^2 = \mathrm{Var}\left(Y_1\right); a = E\left(Y_1\right).$$

Proof The exact expressions for the first four moment of the ergodic distribution are obtained as Eqs. (3.1)–(3.4) in Section 3.

Moreover, the following asymptotic expansion can be given using Lemma 4.8, when $a \to \infty$:

$$\frac{1}{E\left(M_1\left(\zeta_1\right)\right)} = \frac{1}{a}\left\{1 - \frac{\mu_{21}}{a} + \frac{\mu_{21}^2}{a^2} + o\left(\frac{1}{a^2}\right)\right\}. \tag{5.1}$$

Taking advantage of Lemma 4.7 and Lemma 4.8, the following asymptotic expansion can be written, as $a \to \infty$:

$$E\left(\zeta_1 M_1\left(\zeta_1\right)\right) - \frac{1}{2}E\left(M_2\left(\zeta_1\right)\right) = \frac{a^2}{2}\left\{1 + \frac{\sigma^2 - \mu_{31}}{a^2} + o\left(\frac{1}{a^2}\right)\right\}. \tag{5.2}$$

Substituting Eqs. (5.1) and (5.2) into Eq. (3.1), the following asymptotic result is derived for $E\left(X\right)$:

$$E\left(X\right) = \frac{a}{2} + c_{11} + \frac{c_{12}}{a} + o\left(\frac{1}{a}\right).$$

Now, it is aimed to find the asymptotic expression for the second moment of the ergodic distribution of the process $X\left(t\right)$. The exact expression for the second moment of the ergodic distribution of the process $X\left(t\right)$ had been found in Section 3, as Eq. (3.2). Using Lemma 4.7 and Lemma 4.8, the following asymptotic result can be written,when $a \to \infty$:

$$E\left(\zeta_1^2 M_1\left(\zeta_1\right)\right) - E\left(\zeta_1 M_2\left(\zeta_1\right)\right) + \frac{1}{3}E\left(M_3\left(\zeta_1\right)\right) = \frac{a^3}{3} + \sigma^2 a + o\left(a\right). \tag{5.3}$$

Substituting Eqs. (5.1)–(5.3) into Eq. (3.2), the asymptotic expansion for $E(X^2)$ can be found as follows, when $a \to \infty$:

$$E(X^2) = \frac{a^2}{3} + c_{21}a + c_{22} + o\left(1\right)$$

The third and fourth moments of the ergodic distribution of the process can be derived similarly.

□

6. Proposed approximate formulas and an example of their closeness to exact expressions

In this section, we will propose approximate formulas for the moments of the ergodic distribution of the process $X(t)$ using the asymptotic formulas in Theorem 5.1. For this, the part consisting of the first three terms (without remainder terms) in the asymptotic expansions of ergodic moments will be denoted with the notation $\tilde{E}\left(X^n\right)$. Therefore, the suggested approximate formulas are written as follows:

$$\tilde{E}\left(X^n\right) = \frac{a^n}{n+1} + c_{n1}a^{n-1} + c_{n2}a^{n-2}, n = 1, 2, 3, 4. \tag{6.1}$$

The exact forms of the coefficients c_{n1} and c_{n2} are given in Theorem 5.1. These formulas are much simpler and easier to calculate in comparison of exact expressions. However, in addition to the simplicity of the proposed formulas, knowing that they are close enough to the exact expressions is an important feature for the approximate formulas. To illustrate this feature, a specific example is addressed below. In Example 6.1, the difference between the exact expression of the expected value of the ergodic distribution of the process $X\left(t\right)$ and its approximate formula is examined.

Example 6.1. The random variables V_{1n} and V_{2n} are assumed to be exponentially distributed with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. It is assumed that these variables are independent of each other. Let us also define the random variable $\eta_n = V_{1n} - V_{1n}$ by means of these random variables. The random sequence $S_n \equiv \sum_{i=1}^{n} \eta_i,\ n = 1,\ 2, \ldots$ forms a random walk. Here, $E\left(\eta_n\right) = E\left(V_{1n} - V_{1n}\right) = \frac{1}{2} > 0$. Also, the random variable Y_1 denoting the interference has a normal distribution with parameters $(a, \sigma^2 = 9)$. According to Theorem 3.2, the exact expression of the expected value of the ergodic distribution of the process $X(t)$ is as follows:

$$E(X) = \frac{1}{E(M_1(\zeta_1))} \left\{ E(\zeta_1 M_1(\zeta_1)) - \frac{1}{2} E(M_2(\zeta_1)) \right\} + A_1, \tag{6.2}$$

where $A_1 \equiv m_{21} = \frac{m_2}{2m_1}, m_k = E\left(\eta_1^k\right), k = 1, 2.$

In this case, it is possible to write the exact expression of $E(X)$ in a simpler way. Due to the memoryless property of the exponential distribution, the boundary functional $S_{N(z)}$ can be written as: $S_{N(z)} = z + H_1$. Here the random variable H_1 is a random variable that has the same distribution as the random variables V_{1n}. Therefore, the moments of the boundary functional $S_{N(z)}$ can be calculated as follows:

$$M_1(z) = E\left(S_{N(z)}\right) = E(z + H_1) = z + 1;$$

$$M_2(z) = E\left(S_{N(z)}^2\right) = E(z + H_1)^2 = z^2 + 2z + 2.$$

Note that the random variable $\zeta_1 \equiv \max\{0;\ Y_1\}$ has a truncated normal distribution with $(a, \sigma^2 = 9)$ parameters. Considering the asymptotic expansion of the error function, the first three moments of random variable ζ_1 can be written as follows (see Abramowitz and Stegun [1], p. 298):

$$E(\zeta_1) = a + \sigma\varphi(t) - a\bar{\Phi}(t) = a + O\left(\frac{1}{a^2}\exp\left(-\frac{a^2}{2\sigma^2}\right)\right) \cong a;$$

$$E\left(\zeta_1^2\right) \cong a^2 + \sigma^2; E\left(\zeta_1^3\right) \cong a^3 + 3\sigma^2.$$

Here, $a = E(Y_1), \sigma^2 = Var(Y_1)$.

Now we can write the following expressions:

$$E(M_1(\zeta_1)) = E\left(S_{N(\zeta_1)}\right) = E(\zeta_1) + 1 \cong a + 1; E(\zeta_1 M_1(\zeta_1)) \cong a^2 + a + \sigma^2;$$

$$E(M_2(\zeta_1)) = E\left(S_{N(\zeta_1)}^2\right) = E\left(\zeta_1^2\right) + 2E(\zeta_1) + 2 \cong a^2 + 2a + \left(2 + \sigma^2\right).$$

Considering the above results, we can calculate the exact expression of $E(X)$ with the following formula:

$$E(X) \cong \frac{a^2 + 3a + 12}{2(a+1)} \tag{6.3}$$

On the other hand, in Eq. (6.1), the proposed approximate formula for the expected value of the ergodic distribution of the process $X(t)$ is as follows:

$$\tilde{E}(X) = \frac{a}{2} + c_{11} + \frac{c_{12}}{a}$$

Here, $c_{11} \equiv m_{21} - \frac{1}{2}\mu_{21}; c_{12} \equiv \frac{1}{2}\left(\sigma^2 + \mu_{21}^2 - \mu_{31}\right); \sigma^2 = \mathrm{Var}(Y_1) = 9;$

$$m_1 = E(\eta_1) = \frac{1}{2}; m_2 = E\left(\eta_1^2\right) = \frac{3}{2}; m_{21} = \frac{m_2}{2m_1} = \frac{3}{2}.$$

According to the memoryless property of the exponential distribution,

$$\mu_1 = E(H_1) = 1; \mu_2 = E\left(H_1^2\right) = 2; \mu_3 = E\left(H_1^3\right) = 6.$$

Then, $\mu_{21} = \frac{\mu_2}{2\mu_1} = 1; \mu_{31} = \frac{\mu_3}{3\mu_1} = 2; c_{11} = 1; c_{12} = \frac{5}{2}$.

Now, we can rewrite the proposed approximate formula of the expected value of the ergodic distribution of the process $X(t)$ by the following formula:

$$\tilde{E}(X) = \frac{a^2 + 2a + 5}{2a} \tag{6.4}$$

Now, the absolute and relative errors between the exact expression and approximate formula given in Eqs. (6.3) and (6.4) can be defined as follows, respectively:

$$\Delta \equiv \left| E(X) - \tilde{E}(X) \right|; \delta \equiv \frac{\Delta}{E(X)} 100\%$$

Also, let us define the parameter called accuracy percentage with $AP \equiv 100\% - \delta$ to measure how close the suggested approximate formula for the expected value is to the exact expression.

In Table below, the absolute and relative errors and accuracy percentage (AP) are calculated for expected value $E(X)$.

Table . Comparison of the exact and the approximation results for different a values.

a	30	20	15	10	9	8	7	6	5	4
Δ	0.08	0.11	0.15	0.21	0.22	0.25	0.27	0.29	0.33	0.37
δ (%)	0.50	0.98	1.65	3.17	3.67	4.50	5.26	6.16	7.62	9.25
AP (%)	99.50	99.02	98.35	96.83	96.33	95.50	94.74	93.84	92.38	90.75

Remark. As seen from Table 1, the accuracy of the expected value is more than 90% when $a \geq 4$; more than 95% when $a \geq 8$; more than 99% when $a \geq 20$. This is an indication that the approximate formulas we have proposed show a good approximation even at not very large values of the parameter a.

7. Conclusion

In this study, a random walk process $X(t)$ with normal interference of chance was discussed and the moments of the ergodic distribution of the process were examined. Note that the process $X(t)$ is also investigated in study [15]. However, in the study [15], only the ergodic distribution of the process $X(t)$ was examined and the limit form of the ergodic distribution was found.

As it is known, although the distribution is the most important probability characteristic in probability theory, moments are equally important in terms of application. In particular, characteristics such as the expected value and variance of the process are parameters that are frequently used in practice. Therefore, in this study, in Section 3, exact expressions are found for the first four moments of the ergodic distribution of the process $X(t)$. However, these expressions turned out to be very complex and difficult to calculate. In order to overcome this difficulty, in Section 5, three term asymptotic expansions are derived for the moments of the process $X(t)$. Then, based on these asymptotic expansions, in Section 6, approximation formulas are proposed for the moments of the ergodic distribution of the process $X(t)$. These proposed formulas are much simpler and more useful than the exact expressions. As is known, in addition to the simplicity of approximate formulas, one of the requested features is that these formulas must be close enough to exact expressions.

In general, it is known that approximate formulas derived from three-term asymptotic expansions are much closer to exact expressions than approximation formulas derived from one-term expansions. To illustrate this, a special example is given at the end of the study and on this example it is examined how close the proposed approximate formula for the expected value is to the exact expression.

The calculation results for the expected value of the process $X(t)$ are given in Table 1 above and it is observed that the proposed approximation formula is close enough to the exact expression even at not very large values of parameter a. Especially when $a \geq 8$, the degree of closeness (AP) of the approximation formulas to exact expressions is over 95%. This means that formulas that are both approximate and very simple can be used, without making major errors, rather than exact but very complex formulas. This provides a great computational convenience for researchers using the random walk process $(X(t))$ with normally distributed interference, especially in the field of high energy physics.

As a continuation of this work, modeling the motion of high-energy particles between two barriers and studying the random process $(X(t))$ by asymptotic methods can be of interest from both scientific and practical points of view. In particular, obtaining analytical and asymptotic results for the process $(X(t))$ describing the motion of high-energy particles when both barriers are reflecting can make an important contribution to high energy physics.

Acknowledgments

The authors would like to thank the anonymous referees for their helpful and constructive comments that greatly contributed to this paper. We would also like to thank the editor for his support during the review process.

Aynura Poladova would like to thank TOBB University of Economics and Technology for financial support of her scientific research.

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