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Generalized elliptical quaternions with some applications

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Abstract: In this article, quaternions, which is a preferred and elegant method for expressing spherical rotations, are generalized with the help of generalized scalar product spaces, and elliptical rotations on any given ellipsoid are examined by them. To this end, firstly, we define the generalized elliptical scalar product space which accepts the given ellipsoid as a sphere and determines skew symmetric matrices, and the generalized vector product related to this scalar product space. Then we define the generalized elliptical quaternions by using these notions. Finally, elliptical rotations on any ellipsoid in the space are examined by using the unit generalized elliptical quaternions. The formulas and results obtained are supported with numerical examples.

Key words: Generalized elliptical scalar product, generalized elliptical vector product, generalized elliptical quaternion, rotation matrix, elliptical rotation

1. Introduction

As rotations in the Euclidean plane can be expressed with complex numbers, 3D rotations in the Euclidean space can be expressed with the help of quaternions. Quaternions, which were defined by Hamilton in 1843 with the idea of generalizing complex numbers, attracted many researchers who studied in fields of geometry, mechanics, kinematics, robotics, and physics [5, 14, 20, 48, 56]. It is part of a wide application area as it is used to express rotational and reflectional movements in 3D space. One of the prominent applications of quaternions in recent years is spherical linear interpolations, called SLERP shortly [21, 27, 52]. In addition to Hamilton's quaternion for the Euclidean space, many types of quaternions have been defined and their applications in different spaces have been studied. For example, rotational motions on the standard ellipsoids and cones are examined by the elliptical quaternions [36]. The rotational movements on hyperboloids are studied with hyperbolic split quaternions [26, 35, 37, 53]. On the other hand, with the help of dual quaternions defined by dual numbers, 3D rigid motions and screw motions can be examined [2–4, 43, 50, 51], and they are also used in theoretical kinematics and 3D computer graphics, robotics and computer vision [7, 15, 17, 18, 55, 57]. Especially screw theory is an important tool in robot mechanics, multibody dynamics, and mechanical design [7]. Dual quaternion algebra was produced by Alexander McAulay in 1898 [33] and developed by Kotelnikov's using them in the field of mechanics [24].

Furthermore, the idea of generalizing quaternions has been one of the aims of mathematicians. As products of this idea, quaternions such as octonions, hyper dual quaternions [9–11], biquaternions [6, 41, 47],

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2010 *AMS Mathematics Subject Classification:* 15A63, 15A66, 70B05, 70B10, 70E17, 53A17 hyperbolic quaternions [25, 29], generalized quaternions [1, 22, 23, 28, 32, 42, 44, 54], elliptical quaternions [36, 40], hyperbolic split quaternions [53], and hybrid numbers [38] are available in the literature [36]. The elliptical and hyperbolic quaternion concepts are defined with the help of generalized scalar products. For the generalized scalar product spaces, and for skew symmetric and orthogonal matrices defined by the generalized scalar product, see [16, 30, 31, 34, 45, 46]. In Table 1, we give a summary of different types of quaternions briefly. Detailed information about these quaternions and their geometry can be found in Özdemir's book [39].

Table 1. Some types of real quaternions.

Type of reel quaternions	**Properties of the generators** $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$
Hamiltonian quaternions $\mathbb{H}$ [19]	$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1,$
Split quaternions $\widehat{\mathbb{H}}$ [8]	$\mathbf{i}^2=-1,\ \mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=1,$
Hyperbolic quaternions $\mathbb{H}_{\mathbb{H}}$ [29] (non-associative)	$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=1,$
Segre quaternions $\mathbb{H}_{\mathbb{S}}$ [49](commutative)	$\mathbf{i}^2=\mathbf{k}^2=\mathbf{ijk}=-1,\quad \mathbf{j}^2=1,$
Elliptical quaternions $\mathbb{H}_{\mathbb{E}}$ [36]	$\mathbf{i}^2=-a_1,\ \mathbf{j}^2=-a_2,\ \mathbf{k}^2=-a_3,\ \mathbf{ijk}=\sqrt{a_1a_2a_3}$
Hyperbolic split quaternions $\mathbb{H}_{\mathbb{HS}}$ [53]	$\mathbf{i}^2=-a_1,\ \mathbf{j}^2=a_2,\ \mathbf{k}^2=a_3,\ \mathbf{ijk}=\sqrt{\lvert a_1a_2a_3\rvert}$
Hybrid numbers $\mathbb{K}$ [38]	$\mathbf{i}^2=-1,\ \mathbf{j}^2=0,\ \mathbf{k}^2=1,\ \mathbf{ik}=-\mathbf{ki}=\mathbf{j}+\mathbf{i}$
Generalized quaternions $\mathbb{GH}$ [22]	$\mathbf{i}^2=\alpha,\ \mathbf{j}^2=\beta,\ \mathbf{k}^2=\mathbf{ijk}=-\alpha\beta$

In addition to these quaternions, the quaternions obtained by taking q_i numbers as generalized complex numbers (complex $\mathbb{C}$, dual $\mathbb{D}$ and double numbers $\mathbb{P}$) are given in Table 2 below.

Table 2. Some types of generalized complex number quaternions.

Type of complex quaternions	**Properties of the generators** $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$
Biquaternions $\mathbb{H}(\mathbb{C})$ [33]	$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ $q_i\in\mathbb{C}$ (complex numbers)
Dual quaternions $\mathbb{H}(\mathbb{D})$ [24]	$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ $q_i\in\mathbb{D}$ (dual numbers)
Perplex quaternions $\mathbb{H}(\mathbb{P})$	$\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$ $q_i\in\mathbb{P}$ (double numbers)

Our aim in this study is to examine the elliptical rotational motions not only on the standard ellipsoids with the equation $ax^2+by^2+cz^2=r^2$ as in [36] but also on the rotated ellipsoids with the equation in the most general form

$$Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz=r^2, \tag{1.1}$$

with the help of the generalized elliptic quaternions without using any affine transformations. Thus, the conclusions related to quaternions derived in [36] will be special cases (for $A,B,C\in\mathbb{R}^+$ and $D=E=F=0$) of the conclusions of this study.

To achieve this aim, we will define the generalized elliptical scalar product which accepts the (1.1) equation as a sphere, and define the generalized elliptical vector product by using the skew symmetric matrix of this scalar product. Then we will use this information to define the generalized elliptical quaternion.

Since this study will focus on the generalization of real quaternions, let us briefly recall some information about Hamiltonian quaternions

$$\mathbb{H}=\left\{q_0+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}:\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1\right\}.$$

A quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be written as $q = s_q + \mathbf{v}_q$ where the symbols $s_q = q_0$ and $\mathbf{v}_q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ denote the scalar and vector parts of q, respectively. If $s_q = 0$ then q is called a pure quaternion. The conjugate of q is denoted by $\overline{q}$, and defined as $\overline{q} = s_q - \mathbf{v}_q$. The norm of q is defined by $\sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and is denoted by N_q. We say that $q_0 = q/N_q$ is a unit quaternion if $q \neq 0$. Every unit quaternion can be written in the form $q_0 = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta$ where $\boldsymbol{\varepsilon}_0$ is a unit vector satisfying the equality $\boldsymbol{\varepsilon}_0^2 = -1$. With the help of quaternion multiplication, rotation matrices can be easily obtained. Let p and q be two quaternions. Then, the linear transformation $R_q : \mathbb{H} \to \mathbb{H}$ defined by $R_q(p) = qpq^{-1}$ is an orthogonal transformation. If q is a unit quaternion, it can be found the matrix

$$R_q = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & -2q_0q_3 + 2q_1q_2 & 2q_0q_2 + 2q_1q_3 \\ 2q_1q_2 + 2q_3q_0 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_1q_0 \\ 2q_1q_3 - 2q_2q_0 & 2q_1q_0 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

corresponding to the linear map R_q. This matrix is a rotation matrix in the 3-dimensional Euclidean space and represents a rotation through angle 2θ about the axis $\boldsymbol{\varepsilon}_0 = (q_1, q_2, q_3)$. For other types of quaternions, a rotation matrix in different spaces can be obtained by using the quaternion product. For example, with the help of unit timelike split quaternions, the rotation matrix $\widehat{R}_q$ is obtained as

$$\widehat{R}_q = \begin{bmatrix} q_1^2 + q_2^2 + q_3^2 + q_4^2 & 2q_1q_4 - 2q_2q_3 & -2q_1q_3 - 2q_2q_4 \\ 2q_2q_3 + 2q_4q_1 & q_1^2 - q_2^2 - q_3^2 + q_4^2 & -2q_3q_4 - 2q_2q_1 \\ 2q_2q_4 - 2q_3q_1 & 2q_2q_1 - 2q_3q_4 & q_1^2 - q_2^2 + q_3^2 - q_4^2 \end{bmatrix}$$

in 3-dimensional Lorentzian space. It can be seen that $R_q \in \mathrm{SO}(3)$ and $\widehat{R}_q \in \mathrm{SO}(1,2)$, since $\det R_q = \det \widehat{R}_q = 1$ and $RR^t = I$, $\widehat{R}I^*R = I^*$ [37].

2. Generalized elliptical inner and vector products

In this section, let us explain briefly the generalized scalar product spaces that we will use to generalize quaternions and the concept of skew and orthogonal matrix in these spaces. If the map

$$\mathcal{B} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \to \mathcal{B}(\mathbf{u}, \mathbf{v})$$

is linear in each argument, that is,

$$\mathcal{B}(a\mathbf{u} + b\mathbf{v}, \mathbf{w}) = a\mathcal{B}(\mathbf{u}, \mathbf{w}) + b\mathcal{B}(\mathbf{v}, \mathbf{w}),$$
$$\mathcal{B}(\mathbf{u}, c\mathbf{v} + d\mathbf{w}) = c\mathcal{B}(\mathbf{u}, \mathbf{v}) + d\mathcal{B}(\mathbf{u}, \mathbf{w}),$$

where, $a, b, c, d \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, then it is called a bilinear form. It is possible to express a bilinear form $\mathcal{B}$ in the space $\mathbb{R}^n$ with matrices as $\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t\Omega\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, where the matrix Ω is called "*the matrix associated with the form*" with respect to the standard basis and it is unique. Also the number $\sqrt{|\det\Omega|}$ will be called "*constant of the scalar product*" and denoted by Δ. A bilinear form is said to be symmetric or skew symmetric if

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) = \mathcal{B}(\mathbf{v}, \mathbf{u}) \quad \text{or} \quad \mathcal{B}(\mathbf{u}, \mathbf{v}) = -\mathcal{B}(\mathbf{v}, \mathbf{u}),$$

respectively. The norm of a vector associated with the scalar product $\mathcal{B}$ is defined as

$$\|\mathbf{u}\|_{\mathcal{B}} = \sqrt{|\mathcal{B}(\mathbf{u}, \mathbf{u})|}.$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are called $\mathcal{B}$-orthogonal if $\mathcal{B}(\mathbf{u},\mathbf{v}) = 0$. In addition, if their norms are 1, then they are called $\mathcal{B}$-orthonormal vectors. If $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is B-orthonormal base, that is $\|\mathbf{u}\|_{\mathcal{B}} = \|\mathbf{v}\|_{\mathcal{B}} = \|\mathbf{w}\|_{\mathcal{B}} = 1$ and $\mathbf{u},\mathbf{v},\mathbf{w}$ are $\mathcal{B}$-orthogonal to each others, then $|\det(\mathbf{u},\mathbf{v},\mathbf{w})| = \Delta$.

Now, we will define a positive definite scalar product in the real vector space $\mathbb{R}^3$, which is the 3D version of the one we gave for $\mathbb{R}^2$ in [13]:

For any vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ and a positive definite real matrix

$$\Omega = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$$

the map

$$\begin{aligned}\mathcal{B}_\Omega(\mathbf{u},\mathbf{v}) &= \mathbf{u}^T\Omega\mathbf{v} \\ &= Au_1v_1 + Bu_2v_2 + Cu_3v_3 + D(u_1v_2 + u_2v_1) + E(u_1v_3 + u_3v_1) + F(u_2v_3 + u_3v_2)\end{aligned}$$

is called *generalized (Euclidean) elliptical inner product* or $\mathcal{B}_\Omega$-inner product, and the real vector space $\mathbb{R}^3$ equipped with the $\mathcal{B}_\Omega$-inner product is denoted by $\mathbb{R}^3_{\mathcal{B}_\Omega}$. Here, Ω is the associated matrix of $\mathcal{B}_\Omega$-scalar product, and the constant of the scalar product is

$$\Delta = \sqrt{\det\Omega} = \sqrt{ABC + 2FDE - AF^2 - CD^2 - BE^2}.$$

According to $\mathcal{B}_\Omega$-inner product,

$$\|\mathbf{u}\|_{\mathcal{B}_\Omega} = \sqrt{\mathcal{B}_\Omega(\mathbf{u},\mathbf{u})} = \sqrt{Au_1^2 + Bu_2^2 + Cu_3^2 + 2Du_1u_2 + 2Eu_1u_3 + 2Fu_2u_3}$$

and $\mathcal{B}_\Omega$-sphere having center at the origin and radius r has the equation

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = r^2$$

which is an ellipsoid since $\det\Omega > 0$ and $r^2 > 0$. In addition, as usual, $\mathcal{B}_\Omega$-measure of an angle between the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\theta = \cos^{-1}\left(\frac{\mathcal{B}_\Omega(\mathbf{u},\mathbf{v})}{\|\mathbf{u}\|_{\mathcal{B}_\Omega}\|\mathbf{v}\|_{\mathcal{B}_\Omega}}\right).$$

Remark 2.1 *The generalized elliptical inner product is determined by the generalized l_p-metric given in [12], for $p = 2$ and $\lambda_i = 1$. Using the same notation of the generalized l_p-metric, the associated matrix of the $\mathcal{B}_\Omega$-inner product is*

$$\Omega = \begin{bmatrix} v_{11}^2 + v_{21}^2 + v_{31}^2 & v_{11}v_{12} + v_{21}v_{22} + v_{31}v_{32} & v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33} \\ v_{11}v_{12} + v_{21}v_{22} + v_{31}v_{32} & v_{12}^2 + v_{22}^2 + v_{32}^2 & v_{12}v_{13} + v_{22}v_{23} + v_{32}v_{33} \\ v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33} & v_{12}v_{13} + v_{22}v_{23} + v_{32}v_{33} & v_{13}^2 + v_{23}^2 + v_{33}^2 \end{bmatrix}$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three linear independent vectors such that $\mathbf{v}_i = (v_{i1}, v_{i2}, v_{i3})$. In this case, Ω is a positive definite real matrix already, since

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = 1$$

is an ellipsoid, where $A = v_{11}^2+v_{21}^2+v_{31}^2$, $B = v_{12}^2+v_{22}^2+v_{32}^2$, $C = v_{13}^2+v_{23}^2+v_{33}^2$, $D = v_{11}v_{12}+v_{21}v_{22}+v_{31}v_{32}$, $E = v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33}$ *and* $F = v_{12}v_{13} + v_{22}v_{23} + v_{32}v_{33}$.

Before defining a generalized elliptical quaternion, we also need the generalized (Euclidean) elliptical vector product in $\mathbb{R}^3_{\mathcal{B}_\Omega}$. It is known that skew symmetric matrices can be used to represent vector products as matrix multiplications in three dimensional space. If $\mathcal{B}_\Omega(T\mathbf{u},\mathbf{v}) = -\mathcal{B}_\Omega(\mathbf{u},T\mathbf{v})$ for all vectors $\mathbf{u},\mathbf{v} \in \mathbb{R}^3_{\mathcal{B}_\Omega}$, then T is called $\mathcal{B}_\Omega$-skew symmetric matrix. We know that T is $\mathcal{B}_\Omega$-skew symmetric matrix if and only if $T^t\Omega = -\Omega T$. The following theorem determines the $\mathcal{B}_\Omega$-skew symmetric matrices of $\mathbb{R}^3_{\mathcal{B}_\Omega}$:

Theorem 2.2 *In* $\mathbb{R}^3_{\mathcal{B}_\Omega}$, $\mathcal{B}_\Omega$ *-skew symmetric matrices are as in the following form*

$$T = \lambda \begin{bmatrix} \frac{(EF-CD)u_3-(DF-BE)u_2}{\det\Omega} & \frac{(DF-BE)u_1-(BC-F^2)u_3}{\det\Omega} & \frac{(BC-F^2)u_2-(EF-CD)u_1}{\det\Omega} \\ \frac{(AC-E^2)u_3-(DE-AF)u_2}{\det\Omega} & \frac{(DE-AF)u_1-(EF-CD)u_3}{\det\Omega} & \frac{(EF-CD)u_2-(AC-E^2)u_1}{\det\Omega} \\ \frac{(DE-AF)u_3-(AB-D^2)u_2}{\det\Omega} & \frac{(AB-D^2)u_1-(DF-BE)u_3}{\det\Omega} & \frac{(DF-BE)u_2-(DE-AF)u_1}{\det\Omega} \end{bmatrix} \tag{2.1}$$

where $\lambda, u_1, u_2, u_3 \in \mathbb{R}$.

Proof Let

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix}$$

If $T^t\Omega = -\Omega T$, then we have the following system of equations

$$\left\{ \begin{array}{l} At_{11} + Dt_{21} + Et_{31} = 0 \\ Bt_{22} + Ft_{32} + Dt_{12} = 0 \\ Ct_{33} + Ft_{23} + Et_{13} = 0 \\ At_{12} + Bt_{21} + Ft_{31} + Dt_{11} + Dt_{22} + Et_{32} = 0 \\ At_{13} + Ct_{31} + Ft_{21} + Et_{11} + Dt_{23} + Et_{33} = 0 \\ Bt_{23} + Ct_{32} + Ft_{22} + Ft_{33} + Dt_{13} + Et_{12} = 0 \end{array} \right. .$$

If one uses the following variables

$$\begin{aligned} u_1 &= Et_{12} + Ft_{22} + Ct_{32} = -Dt_{13} - Bt_{23} - Ft_{33} \\ u_2 &= At_{13} + Dt_{23} + Et_{33} = -Et_{11} - Ft_{21} - Ct_{31} \\ u_3 &= Dt_{11} + Bt_{21} + Ft_{31} = -At_{12} - Dt_{22} - Et_{32} \end{aligned}$$

then gets the following three systems of equations

$$\left\{ \begin{array}{l} At_{11} + Dt_{21} + Et_{31} = 0 \\ Dt_{11} + Bt_{21} + Ft_{31} = u_3 \\ Et_{11} + Ft_{21} + Ct_{31} = -u_2 \end{array} \right.$$

$$\left\{ \begin{array}{l} At_{12} + Dt_{22} + Et_{32} = -u_3 \\ Dt_{12} + Bt_{22} + Ft_{32} = 0 \\ Et_{12} + Ft_{22} + Ct_{32} = u_1 \end{array} \right.$$

$$\left\{ \begin{array}{l} At_{13} + Dt_{23} + Et_{33} = u_2 \\ Dt_{13} + Bt_{23} + Ft_{33} = -u_1 \\ Et_{13} + Ft_{23} + Ct_{33} = 0 \end{array} \right.$$

having the same coefficient matrix Ω. Solving each of them using the Cramer's Rule, we get the matrix (2.1). $\square$

The matrix (2.1) is $\mathcal{B}_\Omega$-skew symmetric for all $\lambda \in \mathbb{R}$; however, we have to take

$$\lambda = \Delta = \sqrt{\det \Omega} = \sqrt{ABC + 2FDE - AF^2 - BE^2 - CD^2} \tag{2.2}$$

to make it compatible with norms of $\mathcal{B}_\Omega$-vector product. Thus, we will consider the $\mathcal{B}_\Omega$-skew symmetric matrix

$$T = \begin{bmatrix} \frac{(DF-BE)u_2-(EF-CD)u_3}{\Delta} & \frac{(BC-F^2)u_3-(DF-BE)u_1}{\Delta} & \frac{(EF-CD)u_1-(BC-F^2)u_2}{\Delta} \\ \frac{(DE-AF)u_2-(AC-E^2)u_3}{\Delta} & \frac{(EF-CD)u_3-(DE-AF)u_1}{\Delta} & \frac{(AC-E^2)u_1-(EF-CD)u_2}{\Delta} \\ \frac{(AB-D^2)u_2-(DE-AF)u_3}{\Delta} & \frac{(DF-BE)u_3-(AB-D^2)u_1}{\Delta} & \frac{(DE-AF)u_1-(DF-BE)u_2}{\Delta} \end{bmatrix}$$

or shortly

$$T = \begin{bmatrix} \Delta_5 u_2 - \Delta_6 u_3 & \Delta_3 u_3 - \Delta_5 u_1 & \Delta_6 u_1 - \Delta_3 u_2 \\ \Delta_4 u_2 - \Delta_2 u_3 & \Delta_6 u_3 - \Delta_4 u_1 & \Delta_2 u_1 - \Delta_6 u_2 \\ \Delta_1 u_2 - \Delta_4 u_3 & \Delta_5 u_3 - \Delta_1 u_1 & \Delta_4 u_1 - \Delta_5 u_2 \end{bmatrix} \tag{2.3}$$

where $\Delta = \sqrt{\det \Omega}$, $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$ and $\Delta_6 = (EF - CD)/\Delta$.

Now, let us define the generalized elliptical vector product or $\mathcal{B}_\Omega$-vector product in $\mathbb{R}^3_{\mathcal{B}_\Omega}$, using the skew symmetric matrix:

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be any two vectors, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be standard unit vectors of $\mathbb{R}^3$. Then, the generalized elliptical vector product in $\mathbb{R}^3_{\mathcal{B}_\Omega}$ is the function

$$\mathcal{V}_\Omega : \mathbb{R}^3_{\mathcal{B}_\Omega} \times \mathbb{R}^3_{\mathcal{B}_\Omega} \to \mathbb{R}^3_{\mathcal{B}_\Omega}, \quad (\mathbf{u}, \mathbf{v}) \to \mathcal{V}_\Omega (\mathbf{u} \times \mathbf{v}),$$

defined by

$$\begin{aligned} \mathcal{V}_\Omega (\mathbf{u} \times \mathbf{v}) &= \begin{bmatrix} \Delta_6 u_3 - \Delta_5 u_2 & \Delta_5 u_1 - \Delta_3 u_3 & \Delta_3 u_2 - \Delta_6 u_1 \\ \Delta_2 u_3 - \Delta_4 u_2 & \Delta_4 u_1 - \Delta_6 u_3 & \Delta_6 u_2 - \Delta_2 u_1 \\ \Delta_4 u_3 - \Delta_1 u_2 & \Delta_1 u_1 - \Delta_5 u_3 & \Delta_5 u_2 - \Delta_4 u_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ &= \begin{vmatrix} \Delta_3 \mathbf{i} + \Delta_6 \mathbf{j} + \Delta_5 \mathbf{k} & \Delta_6 \mathbf{i} + \Delta_2 \mathbf{j} + \Delta_4 \mathbf{k} & \Delta_5 \mathbf{i} + \Delta_4 \mathbf{j} + \Delta_1 \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \begin{bmatrix} \Delta_3 (u_2 v_3 - u_3 v_2) + \Delta_6 (u_3 v_1 - u_1 v_3) + \Delta_5 (u_1 v_2 - u_2 v_1) \\ \Delta_6 (u_2 v_3 - u_3 v_2) + \Delta_2 (u_3 v_1 - u_1 v_3) + \Delta_4 (u_1 v_2 - u_2 v_1) \\ \Delta_5 (u_2 v_3 - u_3 v_2) + \Delta_4 (u_3 v_1 - u_1 v_3) + \Delta_1 (u_1 v_2 - u_2 v_1) \end{bmatrix} \end{aligned}$$

where $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$ and $\Delta_6 = (EF - CD)/\Delta$. Notice that if we define $\widehat{\mathbf{i}} = \Delta_3 \mathbf{i} + \Delta_6 \mathbf{j} + \Delta_5 \mathbf{k}$, $\widehat{\mathbf{j}} = \Delta_6 \mathbf{i} + \Delta_2 \mathbf{j} + \Delta_4 \mathbf{k}$

and $\widehat{\mathbf{k}} = \Delta_5\mathbf{i} + \Delta_4\mathbf{j} + \Delta_1\mathbf{k}$, then we have

$$\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right) = \left(u_2v_3 - u_3v_2\right)\widehat{\mathbf{i}} + \left(u_3v_1 - u_1v_3\right)\widehat{\mathbf{j}} + \left(u_1v_2 - u_2v_1\right)\widehat{\mathbf{k}},$$

while the well-known Euclidean vector product is

$$\mathbf{u}\times\mathbf{v} = \left(u_2v_3 - u_3v_2\right)\mathbf{i} + \left(u_3v_1 - u_1v_3\right)\mathbf{j} + \left(u_1v_2 - u_2v_1\right)\mathbf{k}.$$

The vector $\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right)$ is $\mathcal{B}_\Omega$-orthogonal to the both of the vectors $\mathbf{u}$ and $\mathbf{v}$. Consider two unit vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3_{\mathcal{B}_\Omega}$. As any unit vectors in $\mathbb{R}^3_{\mathcal{B}_\Omega}$, the end points of these vectors are on the unit $\mathcal{B}_\Omega$-sphere $S^2_{\mathcal{B}_\Omega}$

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = 1.$$

It is easy to see that $\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right)$ is also unit vector of $\mathbb{R}^3_{\mathcal{B}_\Omega}$, and it is elliptically orthogonal to the vectors $\mathbf{u}$ and $\mathbf{v}$, that is $\left\|\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right)\right\|_{\mathcal{B}_\Omega} = 1$ and $\mathcal{B}_\Omega(\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right), \mathbf{u}) = \mathcal{B}_\Omega(\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right), \mathbf{v}) = 0$. Thus, $\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right)$ is rotation axis for a $\mathcal{B}_\Omega$-rotation which transforms the vector $\mathbf{u}$ to the vector $\mathbf{v}$.

Remark 2.3 *Note that if we take $A, B, C \in \mathbb{R}^+$ and $D = E = F = 0$ for the special case considered in [36], we have the consistent following inner and vector products:*

$$\mathcal{B}_\Omega(\mathbf{u},\mathbf{v}) = \mathcal{B}_{A,B,C}(\mathbf{u},\mathbf{v}) = Au_1v_1 + Bu_2v_2 + Cu_3v_3$$

and

$$\mathcal{V}_\Omega\left(\mathbf{u}\times\mathbf{v}\right) = \mathcal{V}_{A,B,C}\left(\mathbf{u}\times\mathbf{v}\right) = \Delta\begin{vmatrix} \mathbf{i}/A & \mathbf{j}/B & \mathbf{k}/C \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{bmatrix} \Delta/A\left(u_2v_3 - u_3v_2\right) \\ \Delta/B\left(u_3v_1 - u_1v_3\right) \\ \Delta/C(u_1v_2 - u_2v_1) \end{bmatrix}$$

where

$$\Omega = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}$$

and $\Delta = \sqrt{\det\Omega} = \sqrt{ABC}$.

3. Generalized elliptical quaternions

For real value entries A, B, C, D, E, F of the positive definite matrix Ω, and $q_0, q_1, q_2, q_3 \in \mathbb{R}$, consider four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities

$$\begin{aligned}
\mathbf{i}^2 &= -A, \quad \mathbf{j}^2 = -B, \quad \mathbf{k}^2 = -C \\
\mathbf{ij} &= -D + \Delta_5\mathbf{i} + \Delta_4\mathbf{j} + \Delta_1\mathbf{k} \\
\mathbf{ji} &= -D - \Delta_5\mathbf{i} - \Delta_4\mathbf{j} - \Delta_1\mathbf{k} \\
\mathbf{jk} &= -F + \Delta_3\mathbf{i} + \Delta_6\mathbf{j} + \Delta_5\mathbf{k} \\
\mathbf{kj} &= -F - \Delta_3\mathbf{i} - \Delta_6\mathbf{j} - \Delta_5\mathbf{k} \\
\mathbf{ki} &= -E + \Delta_6\mathbf{i} + \Delta_2\mathbf{j} + \Delta_4\mathbf{k} \\
\mathbf{ik} &= -E - \Delta_6\mathbf{i} - \Delta_2\mathbf{j} - \Delta_4\mathbf{k}
\end{aligned}$$

where $\Delta = \sqrt{\det \Omega}$, $\Delta_1 = (AB - D^2)/\Delta$, $\Delta_2 = (AC - E^2)/\Delta$, $\Delta_3 = (BC - F^2)/\Delta$, $\Delta_4 = (DE - AF)/\Delta$, $\Delta_5 = (DF - BE)/\Delta$, $\Delta_6 = (EF - CD)/\Delta$. Then $q = (q_0, q_1, q_2, q_3) = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ is called a generalized elliptical quaternion and the set of all generalized quaternions is denoted by $\mathbb{H}_\Omega^E$. This set is an associative, noncommutative division ring with our basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. If we take $A = B = C = 1$ and $D = E = F = 0$, we get the usual quaternion algebra. The generalized elliptical quaternion product table is given by

.	**1**	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	$-A$	$-D + \Delta_5\mathbf{i} + \Delta_4\mathbf{j} + \Delta_1\mathbf{k}$	$-E - \Delta_6\mathbf{i} - \Delta_2\mathbf{j} - \Delta_4\mathbf{k}$
j	**j**	$-D - \Delta_5\mathbf{i} - \Delta_4\mathbf{j} - \Delta_1\mathbf{k}$	$-B$	$-F + \Delta_3\mathbf{i} + \Delta_6\mathbf{j} + \Delta_5\mathbf{k}$
k	**k**	$-E + \Delta_6\mathbf{i} + \Delta_2\mathbf{j} + \Delta_4\mathbf{k}$	$-F - \Delta_3\mathbf{i} - \Delta_6\mathbf{j} - \Delta_5\mathbf{k}$	$-C$

or

.	**1**	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	$-A$	$-D + \widehat{\mathbf{k}}$	$-E - \widehat{\mathbf{j}}$
j	**j**	$-D - \widehat{\mathbf{k}}$	$-B$	$-F + \widehat{\mathbf{i}}$
k	**k**	$-E + \widehat{\mathbf{j}}$	$-F + \widehat{\mathbf{j}}$	$-C$

where $\widehat{\mathbf{i}} = \Delta_3\mathbf{i} + \Delta_6\mathbf{j} + \Delta_5\mathbf{k}$, $\widehat{\mathbf{j}} = \Delta_6\mathbf{i} + \Delta_2\mathbf{j} + \Delta_4\mathbf{k}$ and $\widehat{\mathbf{k}} = \Delta_5\mathbf{i} + \Delta_4\mathbf{j} + \Delta_1\mathbf{k}$, as defined in $\mathcal{B}_\Omega$-vector product.

The generalized elliptic quaternion product of two quaternions $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} = s_p + \mathbf{v}_p$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = s_q + \mathbf{v}_q$ can also be defined as

$$pq = p_0q_0 - \mathcal{B}_\Omega(\mathbf{v}_p, \mathbf{v}_q) + p_0\mathbf{v}_q + q_0\mathbf{v}_p + \mathcal{V}_\Omega(\mathbf{v}_p \times \mathbf{v}_q).$$

If p and q are pure, then

$$\begin{aligned} pq &= -\mathcal{B}_\Omega(\mathbf{v}_p, \mathbf{v}_q) + \mathcal{V}_\Omega(\mathbf{v}_p \times \mathbf{v}_q) \\ &= -\left(Ap_1q_1 + Bp_2q_2 + Cp_3q_3 + D(p_1q_2 + p_2q_1) + E(p_1q_3 + p_3q_1) + F(p_2q_3 + p_3q_2)\right) + \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \end{aligned}$$

The generalized elliptic quaternion product can be expressed as

$$L_p(q) = pq = \begin{bmatrix} p_0 & -(Ap_1 + Dp_2 + Ep_3) & -(Dp_1 + Bp_2 + Fp_3) & -(Ep_1 + Fp_2 + Cp_3) \\ p_1 & p_0 + p_3\Delta_6 - p_2\Delta_5 & p_1\Delta_5 - p_3\Delta_3 & p_2\Delta_3 - p_1\Delta_6 \\ p_2 & p_3\Delta_2 - p_2\Delta_4 & p_0 + p_1\Delta_4 - p_3\Delta_6 & p_2\Delta_6 - p_1\Delta_2 \\ p_3 & p_3\Delta_4 - p_2\Delta_1 & p_1\Delta_1 - p_3\Delta_5 & p_0 + p_2\Delta_5 - p_1\Delta_4 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

or

$$R_p(q) = qp = \begin{bmatrix} p_0 & -(Ap_1 + Dp_2 + Ep_3) & -(Dp_1 + Bp_2 + Fp_3) & -(Ep_1 + Fp_2 + Cp_3) \\ p_1 & p_0 + p_2\Delta_5 - p_3\Delta_6 & p_3\Delta_3 - p_1\Delta_5 & p_1\Delta_6 - p_2\Delta_3 \\ p_2 & p_2\Delta_4 - p_3\Delta_2 & p_0 + p_3\Delta_6 - p_1\Delta_4 & p_1\Delta_2 - p_2\Delta_6 \\ p_3 & p_2\Delta_1 - p_3\Delta_4 & p_3\Delta_5 - p_1\Delta_1 & p_0 + p_1\Delta_4 - p_2\Delta_5 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

One can see that $(ap)\,q = p\,(aq) = a\,(pq)$ for $a \in \mathbb{R}$, and $(pq)r = p(qr)$ for $p, q, r \in \mathbb{H}_\Omega^E$ since $L_pR_q = R_qL_p$.

For example, let $p, q \in \mathbb{H}_{\Omega}^{E}$ where $\Omega = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$. Then the generalized elliptic quaternion product of p and q is

$$pq = \begin{bmatrix} p_0 & -p_1+p_3 & -2p_2 & p_1-3p_3 \\ p_1 & p_0-p_2 & p_1-3p_3 & 3p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1-p_3 & p_0+p_2 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

For $p = 1 - \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $q = 2 + \mathbf{i} - 2\mathbf{j} - \mathbf{k}$, we get $pq = 16 + 9\mathbf{i} + \mathbf{j} + 7\mathbf{k}$. This can also be calculated using the following product table

.	**1**	**i**	**j**	**k**
1	**1**	**i**	**j**	**k**
i	**i**	-1	$i+k$	$1-j$
j	**j**	$-i-k$	-2	$3i+k$
k	**k**	$1+j$	$-\mathbf{i}$	-3

Conjugate, norm, and inverse of a generalized elliptical quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be defined similar to usual quaternions:

$$\begin{aligned} \overline{q} &= q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k} \\ N_q &= \sqrt{q\overline{q}} = \sqrt{\overline{q}q} = \sqrt{q_0^2 + Aq_1^2 + Bq_2^2 + Cq_3^2 + 2Dq_1q_2 + 2Eq_1q_3 + 2Fq_2q_3} \\ q^{-1} &= \frac{\overline{q}}{N_q^2}. \end{aligned}$$

In addition, each generalized elliptic quaternion $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ can be written in the form

$$q = N_q \left(\cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta\right)$$

where

$$\cos\theta = \frac{q_0}{N_q} \quad \text{and} \quad \sin\theta = \frac{\sqrt{Aq_1^2 + Bq_2^2 + Cq_3^2 + 2Dq_1q_2 + 2Eq_1q_3 + 2Fq_2q_3}}{N_q}.$$

Here,

$$\boldsymbol{\varepsilon}_0 = \frac{(q_1, q_2, q_3)}{\sqrt{Aq_1^2 + Bq_2^2 + Cq_3^2 + 2Dq_1q_2 + 2Eq_1q_3 + 2Fq_2q_3}}$$

is a unit vector in the scalar product space $\mathbb{R}^3_{\mathcal{B}_\Omega}$ satisfying the equality $\boldsymbol{\varepsilon}_0^2 = -1$. For example, if $q = 1 + 2\mathbf{i} + \mathbf{j} + \mathbf{k} \in \mathbb{H}_\Omega$ where

$$\Omega = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

then $N_q = \sqrt{1+4+2+3+4(2)-2(2)} = \sqrt{14}$ and we can write

$$q = \frac{1}{\sqrt{14}} + \frac{(2,1,1)}{\sqrt{13}}\frac{\sqrt{13}}{\sqrt{14}} = \cos\theta + \frac{(2,1,1)}{\sqrt{13}}\sin\theta$$

where $\boldsymbol{\varepsilon}_0 = \frac{1}{\sqrt{13}}(2,1,1)$ is a unit vector in $\mathbb{R}^3_{\mathcal{B}_\Omega}$ with $\boldsymbol{\varepsilon}_0^2 = -1$.

Remark 3.1 *One can easily derive the special case considered in [36]: If we take $A, B, C \in \mathbb{R}^+$ and $D = E = F = 0$, then we get the set of elliptical quaternions $\mathbb{H}^E_{A,B,C} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ with four basic elements $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the equalities*

$$\begin{aligned} \mathbf{i}^2 &= -A, \quad \mathbf{j}^2 = -B, \quad \mathbf{k}^2 = -C \\ \mathbf{ij} &= \tfrac{\Delta}{C}\mathbf{k} = -\,\mathbf{ji} \\ \mathbf{jk} &= \tfrac{\Delta}{A}\mathbf{i} = -\mathbf{kj} \\ \mathbf{ki} &= \tfrac{\Delta}{B}\mathbf{j} = -\,\mathbf{ik} \end{aligned}$$

where $\Delta = \sqrt{ABC}$, and the elliptic quaternion product

$$pq = p_0q_0 - \mathcal{B}_{AB,C}\left(\mathbf{v}_p, \mathbf{v}_q\right) + p_0\mathbf{v}_q + q_0\mathbf{v}_p + \mathcal{V}_{A,B,C}\left(\mathbf{v}_p \times \mathbf{v}_q\right).$$

In addition, if we take $A = B = C = 1$ and $D = E = F = 0$, we get the usual quaternion algebra.

Now we generate elliptical rotation matrix by using a unit generalized elliptical quaternion:

Theorem 3.2 *If $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos\theta + \boldsymbol{\varepsilon}_0 \sin\theta \in \mathbb{H}^E_\Omega$ is a unit generalized elliptic quaternion, then for any $\mathbf{v} \in \mathbb{R}^3_{\mathcal{B}_\Omega}$, the linear map $R^q_\theta(\mathbf{v}) = q\mathbf{v}q^{-1}$ gives an elliptical rotation through the elliptical angle 2θ, about the axis $\boldsymbol{\varepsilon}_0$. The elliptical rotation matrix corresponding to the quaternion q is*

$$R^q_\theta = \begin{bmatrix} 2q_0^2 - 1 + 2(\delta'_{q_1} + q_0\Delta'_{6,5}) & 2(\delta''_{q_1} + q_0\Delta''_{5,3}) & 2(\delta'''_{q_1} + q_0\Delta'''_{3,6}) \\ 2(\delta'_{q_2} + q_0\Delta'_{2,4}) & 2q_0^2 - 1 + 2(\delta''_{q_2} + q_0\Delta''_{4,6}) & 2(\delta'''_{q_2} + q_0\Delta'''_{6,2}) \\ 2(\delta'_{q_3} + q_0\Delta'_{4,1}) & 2(\delta''_{q_3} + q_0\Delta''_{1,5}) & 2q_0^2 - 1 + 2(\delta'''_{q_3} + q_0\Delta'''_{5,4}) \end{bmatrix} \tag{3.1}$$

where $\delta'_{q_i} = q_i\left(Aq_1 + Dq_2 + Eq_3\right)$, $\delta''_{q_i} = q_i\left(Dq_1 + Bq_2 + Fq_3\right)$, $\delta'''_{q_i} = q_i\left(Eq_1 + Fq_2 + Cq_3\right)$, $\Delta'_{i,j} = \left(\Delta_i q_3 - \Delta_j q_2\right)$, $\Delta''_{i,j} = \left(\Delta_i q_1 - \Delta_j q_3\right)$, $\Delta'''_{i,j} = \left(\Delta_i q_2 - \Delta_j q_1\right)$.

Proof One can see that R_θ is a linear transformation that preserves the norm. In addition, $R^q_\theta(\mathbf{v}) = q\mathbf{v}q^{-1} = q\mathbf{v}\overline{q}$, since $N_q = 1$. Then we have $R^q_\theta(\mathbf{i}) = q\left(L_\mathbf{i}(\overline{q})\right) = L_q\left(L_\mathbf{i}(\overline{q})\right)$, $R^q_\theta(\mathbf{j}) = L_q\left(L_\mathbf{j}(\overline{q})\right)$ and $R^q_\theta(\mathbf{k}) = L_q\left(L_\mathbf{k}(\overline{q})\right)$. Then one gets

$$\begin{aligned} R^q_\theta(\mathbf{i}) &= \left(2q_0^2 - 1 + 2(\delta'_{q_1} + q_0\Delta'_{6,5})\right)\mathbf{i} + 2(\delta'_{q_2} + q_0\Delta'_{2,4})\mathbf{j} + 2(\delta'_{q_3} + q_0\Delta'_{4,1})\mathbf{k} \\ R^q_\theta(\mathbf{j}) &= 2(\delta''_{q_1} + q_0\Delta''_{5,3})\mathbf{i} + \left(2q_0^2 - 1 + 2(\delta''_{q_2} + q_0\Delta''_{4,6})\right)\mathbf{j} + 2(\delta''_{q_3} + q_0\Delta''_{1,5})\mathbf{k} \\ R^q_\theta(\mathbf{k}) &= 2(\delta'''_{q_1} + q_0\Delta'''_{3,6})\mathbf{i} + 2(\delta'''_{q_2} + q_0\Delta'''_{6,2})\mathbf{j} + \left(2q_0^2 - 1 + 2(\delta'''_{q_3} + q_0\Delta'''_{5,4})\right)\mathbf{k}. \end{aligned}$$

Thus, we have the matrix (3.1) satisfying $\det R^q_\theta = 1$ and $(R^q_\theta)^t \Omega R^q_\theta = \Omega$. Then it is an elliptical rotation matrix on the ellipsoid $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = r^2$. In addition, since $\mathbf{v}_q \parallel \boldsymbol{\varepsilon}_0$, we have $\mathcal{V}_\Omega(\mathbf{v}_q, \boldsymbol{\varepsilon}_0) = \mathcal{V}_\Omega(\boldsymbol{\varepsilon}_0, \mathbf{v}_q) = 0$ and so $q\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_0 q$. Then we have $R_q\left(\boldsymbol{\varepsilon}_0\right) = q\boldsymbol{\varepsilon}_0 q^{-1} = \boldsymbol{\varepsilon}_0 qq^{-1} = \boldsymbol{\varepsilon}_0$. Thus, $\boldsymbol{\varepsilon}_0$ is the rotation axis. Now, let us determine the rotation angle. Let $\{\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2\}$ be $\mathcal{B}_\Omega$-orthonormal set satisfying

$$\mathcal{V}_\Omega\left(\boldsymbol{\varepsilon}_0 \times \boldsymbol{\varepsilon}_1\right) = \boldsymbol{\varepsilon}_2, \quad \mathcal{V}_\Omega\left(\boldsymbol{\varepsilon}_2 \times \boldsymbol{\varepsilon}_0\right) = \boldsymbol{\varepsilon}_1 \text{ and } \mathcal{V}_\Omega\left(\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2\right) = \boldsymbol{\varepsilon}_0.$$

If $\boldsymbol{\varepsilon}$ is a vector in the plane of the $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\varepsilon}_1$, we can write it as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 \cos\alpha + \boldsymbol{\varepsilon}_1 \sin\alpha.$$

We know that $\boldsymbol{\varepsilon}_0$ is invariant under R_θ^q. Then we need to determine how $\boldsymbol{\varepsilon}_1$ changes under R_θ^q. Thus, considering $\boldsymbol{\varepsilon}_0^2 = -1$ and $\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1 = -\boldsymbol{\varepsilon}_1\,\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}_2$, we get

$$\begin{aligned}
R_\theta^q(\boldsymbol{\varepsilon}_1) &= q\boldsymbol{\varepsilon}_1\overline{q} \\
&= (\cos\theta + \boldsymbol{\varepsilon}_0\sin\theta)\,\varepsilon_1\,(\cos\theta - \boldsymbol{\varepsilon}_0\sin\theta) \\
&= \boldsymbol{\varepsilon}_1\cos^2\theta - (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0)\cos\theta\sin\theta + (\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1)\sin\theta\cos\theta - (\boldsymbol{\varepsilon}_0\boldsymbol{\varepsilon}_1)\,\boldsymbol{\varepsilon}_0\sin^2\theta \\
&= \boldsymbol{\varepsilon}_1\cos^2\theta - (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0)\cos\theta\sin\theta - (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0)\sin\theta\cos\theta + \left(\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_0^2\right)\sin^2\theta \\
&= \boldsymbol{\varepsilon}_1\cos^2\theta + \boldsymbol{\varepsilon}_2\cos\theta\sin\theta + \boldsymbol{\varepsilon}_2\sin\theta\cos\theta - \boldsymbol{\varepsilon}_1\sin^2\theta \\
&= \boldsymbol{\varepsilon}_1\cos 2\theta + \boldsymbol{\varepsilon}_2\sin 2\theta.
\end{aligned}$$

Then we have that $\boldsymbol{\varepsilon}$ is rotated by the transformation R_θ^q, through the elliptical angle 2θ about the vector $\boldsymbol{\varepsilon}_0$. □

Corollary 3.3 *Every unit generalized elliptical quaternion determines an elliptical rotation on an ellipsoid, and all elliptical rotations on a given ellipsoid can be represented by unit generalized elliptical quaternions defined by the ellipsoid.*

Corollary 3.4 *If $A, B, C \in \mathbb{R}^+$ and $D = E = F = 0$, then $\Delta = \sqrt{ABC}$, $\Delta_1 = \Delta/C$, $\Delta_2 = \Delta/B$, $\Delta_3 = \Delta/A$, $\Delta_4 = \Delta_5 = \Delta_6 = 0$ and $q_0^2 + Aq_1^2 + Bq_2^2 + Cq_3^2 = 1$. Thus, we get the following elliptical rotation matrix given in [36]:*

$$R_\theta^q = \begin{bmatrix} q_0^2 + Aq_1^2 - Bq_2^2 - Cq_3^2 & 2\left(Bq_1q_2 - \frac{q_0q_3\Delta}{A}\right) & 2\left(Cq_1q_3 + \frac{q_0q_2\Delta}{A}\right) \\ 2\left(Aq_1q_2 + \frac{q_0q_3\Delta}{B}\right) & q_0^2 - Aq_1^2 + Bq_2^2 - Cq_3^2 & 2\left(Cq_2q_3 - \frac{q_0q_1\Delta}{B}\right) \\ 2\left(Aq_1q_3 - \frac{q_0q_2\Delta}{C}\right) & 2\left(Bq_2q_3 + \frac{q_0q_1\Delta}{C}\right) & q_0^2 - Aq_1^2 - Bq_2^2 + Cq_3^2 \end{bmatrix}. \tag{3.2}$$

Remark 3.5 *Let p and q be two unit generalized elliptical quaternions of $\mathbb{H}_\Omega^E$ where*

$$\Omega = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}.$$

Then $R_{\theta_1}^p$ and $R_{\theta_1}^q$ are two elliptical rotation matrices of $\mathbb{R}^3_{\mathcal{B}_\Omega}$. That is, $R_{\theta_1}^p$ and $R_{\theta_1}^q$ rotate a vector $\mathbf{v}$ elliptically on the ellipsoid $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = \|\mathbf{v}\|^2_{\mathcal{B}_\Omega}$. Then the composition of these rotation gives another elliptical rotation which can be expressed by the generalized elliptical quaternion product qp. In another words, if $R_{\theta_1}^p(\mathbf{u}) = \mathbf{v}$ and $R_{\theta_2}^q(\mathbf{v}) = \mathbf{w}$, then

$$\begin{aligned}
\mathbf{w} &= R_{\theta_2}^q(\mathbf{v}) = R_{\theta_2}^q\left(R_{\theta_1}^p(\mathbf{u})\right) = q\left(R_{\theta_1}^p(\mathbf{u})\right)q^{-1} = q\left(p\mathbf{u}p^{-1}\right)q^{-1} \\
&= (qp)\,\mathbf{u}\left(p^{-1}q^{-1}\right) = (qp)\,\mathbf{u}\,(qp)^{-1} = R_{\theta_3}^{qp}(\mathbf{u}).
\end{aligned}$$

On the other hand, if $p \in H_{\Omega_1}$ and $q \in H_{\Omega_2}$ such that $\Omega_1 \neq \Omega_2$ then the composition of corresponding rotations cannot be expressed by the generalized elliptical quaternion product, since for each generalized elliptical quaternion, the quaternion product is different.

Example 3.6 *Consider the ellipsoid with the equation*

$$5x^2 + 2y^2 + 4z^2 - 2xy - 2xz - 4yz = 1. \tag{3.3}$$

Then we have

$$\Omega = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix}$$

and $\Delta = \sqrt{10}$, $\Delta_1 = 9/\sqrt{10}$, $\Delta_2 = 19/\sqrt{10}$, $\Delta_3 = 4/\sqrt{10}$, $\Delta_4 = 11/\sqrt{10}$, $\Delta_5 = 4/\sqrt{10}$, $\Delta_6 = 6/\sqrt{10}$. *For a unit generalized elliptical quaternion* $p = (1/\sqrt{2}, 0, 1/2, 1/2)$, *one gets*

$$p = \cos(\pi/4) + \left(0, \sqrt{2}/2, \sqrt{2}/2\right) \sin(\pi/4)$$

and then by Theorem 3.2, we obtain

$$R^p_{\frac{\pi}{2}} = \begin{bmatrix} \frac{\sqrt{5}}{5} & -\frac{2}{5}\sqrt{5} & \frac{2\sqrt{5}}{5} \\ \frac{4\sqrt{5}}{5} - 1 & -\frac{3\sqrt{5}}{5} & \frac{3\sqrt{5}}{5} + 1 \\ \frac{\sqrt{5}}{5} - 1 & -\frac{2\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} + 1 \end{bmatrix}$$

which is the elliptical rotation matrix having axis of rotation $\boldsymbol{\varepsilon}_0 = \left(0, \sqrt{2}/2, \sqrt{2}/2\right)$, *and elliptical angle of rotation* $\pi/2$. *For another unit generalized elliptical quaternion* $q = (1/2, 0, \sqrt{6}/4, \sqrt{6}/4)$, *one has*

$$q = \cos(\pi/3) + \left(0, \sqrt{2}/2, \sqrt{2}/2\right) \sin(\pi/3)$$

and then we obtain

$$R^q_{\frac{2\pi}{3}} = \begin{bmatrix} \frac{\sqrt{15}-5}{10} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{15}}{5} \\ \frac{4\sqrt{15}-15}{10} & -\frac{3\sqrt{15}+5}{10} & \frac{3\sqrt{15}+15}{10} \\ \frac{\sqrt{15}-15}{10} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{15}+5}{5} \end{bmatrix}$$

which is the elliptical rotation matrix having axis of rotation $\boldsymbol{\varepsilon}_0 = \left(0, \sqrt{2}/2, \sqrt{2}/2\right)$, *and elliptical angle of rotation* $2\pi/3$.

On the other hand, since $\mathcal{B}_\Omega(\mathbf{v}_q, \mathbf{v}_p) = \sqrt{6}/4$ *and* $\mathcal{V}_\Omega(\mathbf{v}_q, \mathbf{v}_p) = \mathbf{0}$, *we have*

$$\begin{aligned} qp &= q_0p_0 - \mathcal{B}_\Omega(\mathbf{v}_q, \mathbf{v}_p) + q_0(\mathbf{v}_p) + p_0(\mathbf{v}_q) + \mathcal{V}_\Omega(\mathbf{v}_q, \mathbf{v}_p) \\ &= \left(\frac{\sqrt{2}-\sqrt{6}}{4}, 0, \frac{\sqrt{3}+1}{4}, \frac{\sqrt{3}+1}{4}\right) \\ &= \cos(7\pi/12) + \left(0, \sqrt{2}/2, \sqrt{2}/2\right) \sin(7\pi/12). \end{aligned}$$

Notice that $N_{qp}=1$ *and then we obtain*

$$R^{qp}_{\frac{7\pi}{6}}=\begin{bmatrix} -\frac{5\sqrt{3}+\sqrt{5}}{10} & \frac{\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ -\frac{5\sqrt{3}+4\sqrt{5}+10}{10} & \frac{3\sqrt{5}}{10}-\frac{\sqrt{3}}{2} & \frac{5\sqrt{3}-3\sqrt{5}+10}{10} \\ -\frac{5\sqrt{3}+\sqrt{5}+10}{10} & \frac{\sqrt{5}}{5} & \frac{5-\sqrt{5}}{5} \end{bmatrix}$$

which is the elliptical rotation matrix having axis of rotation $\boldsymbol{\varepsilon}_0=\left(0,\sqrt{2}/2,\sqrt{2}/2\right)$, *and elliptical angle of rotation* $7\pi/6$. *One can check that*

$$R^{q}_{\frac{2\pi}{3}}R^{p}_{\frac{\pi}{2}}=R^{qp}_{\frac{7\pi}{6}}.$$

Now consider the vector $\mathbf{u}=(1,2,3)$ *for an example. Then we get*

$$\begin{aligned} R^{p}_{\frac{\pi}{2}}(\mathbf{u}) &= \left(\tfrac{3\sqrt{5}}{5},\tfrac{7\sqrt{5}+10}{5},\tfrac{3\sqrt{5}+10}{5}\right)=\mathbf{v} \\ R^{q}_{\frac{\pi}{2}}(\mathbf{u}) &= \left(\tfrac{3\sqrt{15}-5}{10},\tfrac{7\sqrt{15}+20}{10},\tfrac{3\sqrt{15}+15}{10}\right)=\mathbf{v}' \end{aligned}$$

and

$$R^{q}_{\frac{2\pi}{3}}(\mathbf{v})=\left(-\tfrac{5\sqrt{3}+3\sqrt{5}}{10},\tfrac{20-7\sqrt{5}}{10},\tfrac{20-3\sqrt{5}-5\sqrt{3}}{10}\right)=\mathbf{w}=R^{qp}_{\frac{7\pi}{6}}(\mathbf{u}).$$

Since $\|\mathbf{u}\|^2_{\mathcal{B}_\Omega}=\|\mathbf{v}\|^2_{\mathcal{B}_\Omega}=\|\mathbf{v}'\|^2_{\mathcal{B}_\Omega}=\|\mathbf{w}\|^2_{\mathcal{B}_\Omega}=15$ *the vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}'$, *and* $\mathbf{w}$ *are on the ellipsoid*

$$5x^2+2y^2+4z^2-2xy-2xz-4yz=15.$$

One can find the equation of the plane in which the rotation occurs, by $\mathcal{B}_\Omega(\mathbf{x}-\mathbf{u},\boldsymbol{\varepsilon}_0)=0$, *as* $z-x=2$ *(see Figure 1).*

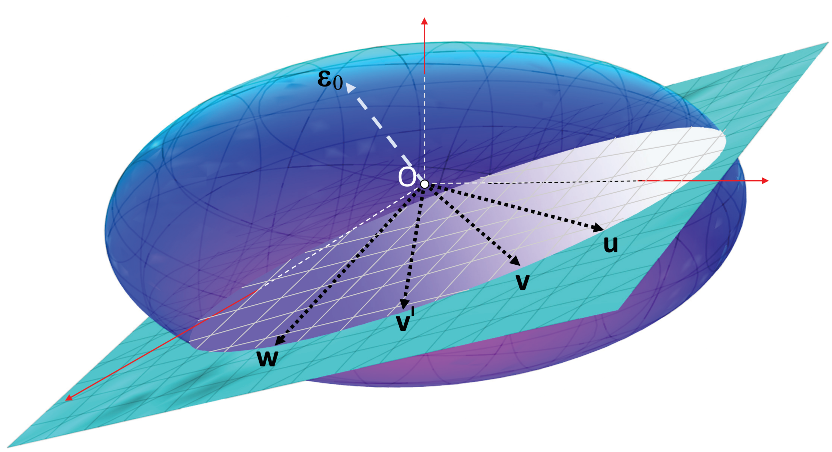

Figure. Elliptical rotations in the plane $z-x=2$.

Example 3.7 *Consider the same ellipsoid again for the generalized elliptical quaternions* $p=(1/\sqrt{2},0,1/2,1/2)$ *and* $r=(0,1/\sqrt{5},0,0)$. *Since*

$$r=\cos(\pi/2)+\left(1/\sqrt{5},0,0\right)\sin(\pi/2)$$

and $N_r = 1$, *we get*

$$R^r_\pi = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{2}{5} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

which is the elliptical rotation matrix having axis of rotation $\boldsymbol{\varepsilon}_1 = \left(1/\sqrt{5}, 0, 0\right)$, *and elliptical angle of rotation* π. *Now we have two different axes of rotation. Let us determine the quaternion product* pr : *Since* $\mathcal{B}_\Omega(\mathbf{v}_p, \mathbf{v}_r) = -\frac{1}{5}\sqrt{5}$ *and* $\mathcal{V}_\Omega(\mathbf{v}_p, \mathbf{v}_r) = (\sqrt{2}/10, 2\sqrt{2}/5, \sqrt{2}/10)$, *we get*

$$\begin{aligned} pr &= p_0 r_0 - B(\mathbf{v}_p, \mathbf{v}_r) + p_0(\mathbf{v}_r) + r_0(\mathbf{v}_p) + V(\mathbf{v}_p, \mathbf{v}_r) \\ &= \left(\tfrac{1}{\sqrt{5}}, \tfrac{\sqrt{10}+\sqrt{2}}{10}, \tfrac{2\sqrt{2}}{5}, \tfrac{\sqrt{2}}{10}\right) \\ &= \cos\theta + \left(\tfrac{\sqrt{10}+5\sqrt{2}}{20}, \tfrac{4\sqrt{10}}{20}, \tfrac{\sqrt{10}}{20}\right) + \sin\theta \end{aligned}$$

which gives the elliptical rotation having axis of rotation $\boldsymbol{\varepsilon}_1 = \left(\frac{\sqrt{10}+5\sqrt{2}}{20}, \frac{4\sqrt{10}}{20}, \frac{\sqrt{10}}{20}\right)$, *and elliptical angle of rotation* $\arccos(1/\sqrt{5})$. *Notice that* $N_{pr} = 1$. *Then we obtain*

$$R^{pr}_\theta = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{8\sqrt{5}}{25} & -\frac{12\sqrt{5}}{25} \\ \frac{4\sqrt{5}-5}{5} & \frac{7\sqrt{5}+10}{25} & -\frac{23\sqrt{5}+15}{25} \\ \frac{\sqrt{5}-5}{5} & \frac{8\sqrt{5}+10}{25} & -\frac{12\sqrt{5}+15}{25} \end{bmatrix}$$

Now one can check that

$$R^p_{\frac{\pi}{2}} R^r_\pi = R^{pr}_\theta \quad \textit{but} \quad R^r_\pi R^p_{\frac{\pi}{2}} \neq R^{pr}_\theta .$$

Theorem 3.8 *The matrix of the elliptical rotation about the unit generalized elliptical vector* $\mathbf{n} = (n_1, n_2, n_3)$ *through the angle* θ, *occurring on the ellipsoid* $Ax^2 + By^2 + Cz^2 + 2Dxy + 2Exz + 2Fyz = r^2$, *is*

$$\begin{bmatrix} \cos\theta + \delta'_a(1-\cos\theta) + \Delta'_{6,5}\sin\theta & \delta''_a(1-\cos\theta) + \Delta''_{5,3}\sin\theta & \delta'''_a(1-\cos\theta) + \Delta'''_{3,6}\sin\theta \\ \delta'_b(1-\cos\theta) + \Delta'_{2,4}\sin\theta & \cos\theta + \delta''_b(1-\cos\theta) + \Delta''_{4,6}\sin\theta & \delta'''_b(1-\cos\theta) + \Delta'''_{6,2}\sin\theta \\ \delta'_c(1-\cos\theta) + \Delta'_{4,1}\sin\theta & \delta''_c(1-\cos\theta) + \Delta''_{1,5}\sin\theta & \cos\theta + \delta'''_c(1-\cos\theta) + \Delta'''_{5,4}\sin\theta \end{bmatrix}$$

where $\delta'_{n_i} = n_i\,(An_1 + Dn_2 + En_3)$, $\delta''_{n_i} = n_i\,(Dn_1 + Bn_2 + Fn_3)$, $\delta'''_{n_i} = n_i\,(En_1 + Fn_2 + Cn_3)$, $\Delta'_{i,j} = (\Delta_i n_3 - \Delta_j n_2)$, $\Delta''_{i,j} = (\Delta_i n_1 - \Delta_j n_3)$, $\Delta'''_{i,j} = (\Delta_i n_2 - \Delta_j n_1)$.

Proof Clearly, we have

$$\Omega = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$$

and $\|\mathbf{n}\|^2_{\mathcal{B}_\Omega} = An_1^2 + Bn_2^2 + Cn_3^2 + 2Dn_1n_2 + 2En_1n_3 + 2Fn_2n_3 = 1$. We need unit generalized quaternion of

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \cos(\theta/2) + (n_1, n_2, n_3)\sin(\theta/2)$$

for elliptical rotation about the generalized elliptical unit vector $\mathbf{n}$ through the angle θ. So we have $q_0 = \cos(\theta/2)$, $q_1 = n_1 \sin(\theta/2)$, $q_2 = n_2 \sin(\theta/2)$, $q_3 = n_3 \sin(\theta/2)$. By the previous theorem, we get the following elliptical rotation matrix

$$\begin{bmatrix} \cos\theta+\delta'_a(1-\cos\theta)+\Delta'_{6,5}\sin\theta & \delta''_a(1-\cos\theta)+\Delta''_{5,3}\sin\theta & \delta'''_a(1-\cos\theta)+\Delta'''_{3,6}\sin\theta \\ \delta'_b(1-\cos\theta)+\Delta'_{2,4}\sin\theta & \cos\theta+\delta''_b(1-\cos\theta)+\Delta''_{4,6}\sin\theta & \delta'''_b(1-\cos\theta)+\Delta'''_{6,2}\sin\theta \\ \delta'_c(1-\cos\theta)+\Delta'_{4,1}\sin\theta & \delta''_c(1-\cos\theta)+\Delta''_{1,5}\sin\theta & \cos\theta+\delta'''_c(1-\cos\theta)+\Delta'''_{5,4}\sin\theta \end{bmatrix}$$

where $\delta'_{n_i} = n_i\,(An_1 + Dn_2 + En_3)$, $\delta''_{n_i} = n_i\,(Dn_1 + Bn_2 + Fn_3)$, $\delta'''_{n_i} = n_i\,(En_1 + Fn_2 + Cn_3)$, $\Delta'_{i,j} = (\Delta_i n_3 - \Delta_j n_2)$, $\Delta''_{i,j} = (\Delta_i n_1 - \Delta_j n_3)$, $\Delta'''_{i,j} = (\Delta_i n_2 - \Delta_j n_1)$. □

Example 3.9 *Let us consider the ellipsoid with the equation* $5x^2 + 2y^2 + 4z^2 - 2xy - 2xz - 4yz = 1$, *and find the matrix of the elliptical rotation about the axis* $\varepsilon_q = (0, \sqrt{2}/2, \sqrt{2}/2)$ *and the elliptical angle* $2\pi/3$. *We have* $\mathbf{n} = (0, \sqrt{2}/2, \sqrt{2}/2)$, $\theta = 2\pi/3$,

$$\Omega = \begin{bmatrix} 5 & -1 & -1 \\ -1 & 2 & -2 \\ -1 & -2 & 4 \end{bmatrix}$$

$\delta'_{n_1} = 0$, $\delta'_{n_2} = \delta'_{n_3} = -1$, $\delta''_{n_1} = \delta''_{n_2} = \delta''_{n_3} = 0$, $\delta'''_{n_1} = 0$, $\delta'''_{n_2} = \delta'''_{n_3} = 1$, $\Delta = \sqrt{10}$, $\Delta_1 = 9/\sqrt{10}$, $\Delta_2 = 19/\sqrt{10}$, $\Delta_3 = 4/\sqrt{10}$, $\Delta_4 = 11/\sqrt{10}$, $\Delta_5 = 4/\sqrt{10}$, $\Delta_6 = 6/\sqrt{10}$. *Then we obtain*

$$R^{\varepsilon_q}_{2\pi/3} = \begin{bmatrix} \frac{\sqrt{15}-5}{10} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{15}}{5} \\ \frac{4\sqrt{15}-15}{10} & -\frac{3\sqrt{15}+5}{10} & \frac{3\sqrt{15}+15}{10} \\ \frac{\sqrt{15}-15}{10} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{15}+5}{5} \end{bmatrix}$$

which is the elliptical rotation matrix having axis of rotation $\boldsymbol{\varepsilon}_0 = \left(0, \sqrt{2}/2, \sqrt{2}/2\right)$, *and elliptical angle of rotation* $2\pi/3$ *(see Example 3.6).*

3.1. An algorithm

Generating 3-dimensional elliptical rotation matrix of the ellipsoid $Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz = r^2$ that rotates a given vector $\mathbf{x} = (x_1, y_1, z_1)$ to given another vector $\mathbf{y} = (x_2, y_2, z_2)$ on the ellipsoid.

Step 1. Write 3×3 positive definite matrix

$$\Omega = [\omega_{ij}]_{3\times 3} = \begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}$$

where $A > 0$, $AB - D^2 > 0$ and $\det \Omega > 0$, or write three linear independent vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ such that $\mathbf{v}_i = (v_{i1}, v_{i2}, v_{i3})$, then define matrix

$$\Omega = [\omega_{ij}]_{3\times 3} = \begin{bmatrix} v_{11}^2 + v_{21}^2 + v_{31}^2 & v_{11}v_{12} + v_{21}v_{22} + v_{31}v_{32} & v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33} \\ v_{11}v_{12} + v_{21}v_{22} + v_{31}v_{32} & v_{12}^2 + v_{22}^2 + v_{32}^2 & v_{12}v_{13} + v_{22}v_{23} + v_{32}v_{33} \\ v_{11}v_{13} + v_{21}v_{23} + v_{31}v_{33} & v_{12}v_{13} + v_{22}v_{23} + v_{32}v_{33} & v_{13}^2 + v_{23}^2 + v_{33}^2 \end{bmatrix}.$$

In this case, the equation $\omega_{11}x^2+\omega_{22}y^2+\omega_{33}z^2+2\omega_{21}xy+2\omega_{31}xz+2\omega_{32}yz=1$ is an ellipsoid. If an ellipsoid is given with the equation

$$Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz=1$$

in the beginning, then write the matrix

$$\Omega=[\omega_{ij}]_{3\times 3}=\begin{bmatrix} A & D & E \\ D & B & F \\ E & F & C \end{bmatrix}.$$

In this case, Ω is positive definite already.

Step 2. Define the generalized elliptical scalar product $\mathcal{B}_\Omega$, generalized elliptical norm of a vector and some constants as follows:

$$\begin{aligned}
\mathcal{B}(\mathbf{x},\mathbf{y},\Omega) &= \omega_{11}x_1y_1+\omega_{22}x_2y_2+\omega_{33}x_3y_3+\omega_{21}(x_1y_2+u_2y_1)+\omega_{31}(x_1y_3+x_3y_1)+\omega_{32}(x_2y_3+x_3y_2)\\
\mathcal{N}(\mathbf{x},\Omega) &= \sqrt{\mathcal{B}(\mathbf{x},\mathbf{x},\Omega)}\\
\Delta &= \sqrt{\omega_{11}\omega_{22}\omega_{33}+2\omega_{32}\omega_{21}\omega_{31}-\omega_{11}\omega_{32}^2-\omega_{33}\omega_{21}^2-\omega_{22}\omega_{31}^2}\\
\Delta_1 &= (\omega_{11}\omega_{22}-\omega_{21}^2)/\Delta\\
\Delta_2 &= (\omega_{11}\omega_{33}-\omega_{31}^2)/\Delta\\
\Delta_3 &= (\omega_{22}\omega_{33}-\omega_{32}^2)/\Delta\\
\Delta_4 &= (\omega_{21}\omega_{31}-\omega_{11}\omega_{32})/\Delta\\
\Delta_5 &= (\omega_{21}\omega_{32}-\omega_{22}\omega_{31})/\Delta\\
\Delta_6 &= (\omega_{31}\omega_{32}-\omega_{33}\omega_{21})/\Delta
\end{aligned}$$

where $\mathbf{x}=(x_1,y_1,z_1)$ and $\mathbf{y}=(x_2,y_2,z_2)$.

Step 3. Define the generalized elliptical vector product $\mathcal{V}_\Omega$ as

$$\mathcal{V}(\mathbf{x},\mathbf{y},\Omega)=\begin{bmatrix} \Delta_3(x_2y_3-x_3y_2)+\Delta_6(x_3y_1-x_1y_3)+\Delta_5(x_1y_2-x_2y_1) \\ \Delta_6(x_2y_3-x_3y_2)+\Delta_2(x_3y_1-x_1y_3)+\Delta_4(x_1y_2-x_2y_1) \\ \Delta_5(x_2y_3-x_3y_2)+\Delta_4(x_3y_1-x_1y_3)+\Delta_1(x_1y_2-x_2y_1) \end{bmatrix}.$$

Step 4. Choose the vectors $\mathbf{x}=(x_1,y_1,z_1)$ and $\mathbf{y}=(x_2,y_2,z_2)$ such that $\|\mathbf{x}\|_{\mathcal{B}_\Omega}=\|\mathbf{y}\|_{\mathcal{B}_\Omega}=r$ to find the elliptical rotation matrix that rotates $\mathbf{x}$ to $\mathbf{y}$ elliptically on the ellipsoid

$$Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz=r^2.$$

Step 5. Find, $\mathcal{V}(\mathbf{x},\mathbf{y},\Omega)$ and norm of the vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathcal{V}(\mathbf{x},\mathbf{y},\Omega)$. That is, find $\mathcal{N}(\mathbf{x},\Omega)$, $\mathcal{N}(\mathbf{y},\Omega)$ and $\mathcal{N}(\mathcal{V}(\mathbf{x},\mathbf{y},\Omega),\Omega)$.

Step 6. Find the rotation axis $\mathbf{u} = (u_1, u_2, u_3)$ where

$$\begin{aligned} u_1 &= \frac{\Delta_3 (x_2 y_3 - x_3 y_2) + \Delta_6 (x_3 y_1 - x_1 y_3) + \Delta_5 (x_1 y_2 - x_2 y_1)}{\mathcal{N}(\mathcal{V}(\mathbf{x}, \mathbf{y}, \Omega), \Omega)} \\ u_2 &= \frac{\Delta_6 (x_2 y_3 - x_3 y_2) + \Delta_2 (x_3 y_1 - x_1 y_3) + \Delta_4 (x_1 y_2 - x_2 y_1)}{\mathcal{N}(\mathcal{V}(\mathbf{x}, \mathbf{y}, \Omega), \Omega)} \\ u_3 &= \frac{\Delta_5 (x_2 y_3 - x_3 y_2) + \Delta_4 (x_3 y_1 - x_1 y_3) + \Delta_1 (x_1 y_2 - x_2 y_1)}{\mathcal{N}(\mathcal{V}(\mathbf{x}, \mathbf{y}, \Omega), \Omega)} \end{aligned}$$

Step 7. Find the elliptical rotation angle using

$$\cos\theta = \frac{\mathcal{B}(\mathbf{x}, \mathbf{y}, \Omega)}{\sqrt{\mathcal{N}(\mathbf{x}, \Omega)}\sqrt{\mathcal{N}(\mathbf{y}, \Omega)}}$$

and define $K = \cos\theta$ and $S = \sin\theta$ where $S = \sqrt{1 - K^2}$.

Step 8. Define the set of generalized elliptical quaternions $\mathbb{H}_{\Omega}^{E} = \{q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},\ q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ with

$$\begin{aligned} \mathbf{i}^2 &= -\omega_{11}, \quad \mathbf{j}^2 = -\omega_{22}, \quad \mathbf{k}^2 = -\omega_{33} \\ \mathbf{ij} &= -\omega_{21} + \Delta_5\mathbf{i} + \Delta_4\mathbf{j} + \Delta_1\mathbf{k} \\ \mathbf{ji} &= -\omega_{21} - \Delta_5\mathbf{i} - \Delta_4\mathbf{j} - \Delta_1\mathbf{k} \\ \mathbf{jk} &= -\omega_{32} + \Delta_3\mathbf{i} + \Delta_6\mathbf{j} + \Delta_5\mathbf{k} \\ \mathbf{kj} &= -\omega_{32} - \Delta_3\mathbf{i} - \Delta_6\mathbf{j} - \Delta_5\mathbf{k} \\ \mathbf{ki} &= -\omega_{31} + \Delta_6\mathbf{i} + \Delta_2\mathbf{j} + \Delta_4\mathbf{k} \\ \mathbf{ik} &= -\omega_{31} - \Delta_6\mathbf{i} - \Delta_2\mathbf{j} - \Delta_4\mathbf{k}. \end{aligned}$$

Step 9. Find $c = \cos(\theta/2) = \sqrt{\dfrac{\cos\theta + 1}{2}}$ and $s = \sqrt{1 - c^2}$. Define the quaternion

$$q = \cos(\theta/2) + \mathbf{u}\sin(\theta/2) = c + su_1\mathbf{i} + su_2\mathbf{j} + su_3\mathbf{k}$$

where θ is the elliptical rotation angle and $\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is the rotation axis, obtained in Step 6 and Step 7.

Step 10. Find the elliptical rotation matrix corresponding to $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ that rotates $\mathbf{x}$ to $\mathbf{y}$

elliptically on the ellipsoid using the matrix $R(q,\Omega,\Delta)=[q_{ij}]_{3\times 3}$ where

$$\begin{aligned}
q_{11} &= 2q_0^2-1+2(q_1(Aq_1+Dq_2+Eq_3)+q_0(\Delta_6 q_3-\Delta_5 q_2))\\
q_{21} &= 2(q_2(Aq_1+Dq_2+Eq_3)+q_0(\Delta_2 q_3-\Delta_4 q_2))\\
q_{31} &= 2(q_3(Aq_1+Dq_2+Eq_3)+q_0(\Delta_4 q_3-\Delta_1 q_2))\\
q_{12} &= 2(q_1(Dq_1+Bq_2+Fq_3)+q_0(\Delta_5 q_1-\Delta_3 q_3))\\
q_{22} &= 2q_0^2-1+2(q_2(Dq_1+Bq_2+Fq_3)+q_0\Delta(\Delta_4 q_1-\Delta_6 q_3))\\
q_{32} &= 2(q_3(Dq_1+Bq_2+Fq_3)+q_0(\Delta_1 q_1-\Delta_5 q_3))\\
q_{13} &= 2(q_1(Eq_1+Fq_2+Cq_3)+q_0(\Delta_3 q_2-\Delta_6 q_1))\\
q_{23} &= 2(q_2(Eq_1+Fq_2+Cq_3)+q_0(\Delta_6 q_2-\Delta_2 q_1))\\
q_{33} &= 2q_0^2-1+2(q_3(Eq_1+Fq_2+Cq_3)+q_0(\Delta_5 q_2-\Delta_4 q_1)).
\end{aligned}$$

References

[1] Abrate M. Quadratic formulas for generalized quaternions. Journal of Algebra and its Applications 2009; 8 (3): 289-306. https://doi.org/10.1142/S0219498809003308

[2] Agrawal OP. Hamilton operators and dual-number-quaternions in spatial kinematics. Mechanism and Machine theory 1987; 22 (6): 569-575. https://doi.org/10.1016/0094-114X(87)90052-8

[3] Akyar B. Dual quaternions in spatial kinematics in an algebraic sense. Turkish Journal of Mathematics 2008; 32 (4): 373-391.

[4] Arribas M, Elipe A, Palacios M. Quaternions and the rotation of a rigid body. Celestial Mechanics and Dynamical Astronomy 2006; 96 (3): 239-251. https://doi.org/10.1007/s10569-006-9037-6

[5] Bottema O, Roth B. Theoretical Kinematics. New York, USA: Dover Publications, 2012.

[6] Buchheim A. A memoir on biquaternions. American Journal of Mathematics 1885; 7 (4): 293-326. https://doi.org/10.2307/2369176

[7] Chelnokov IN. On integration of kinematic equations of a rigid body's screw-motion. Journal of Applied Mathematics and Mechanics 1980; 44 (1): 19-23. https://doi.org/10.1016/0021-8928(80)90168-9

[8] Cockle J. III. On a new imaginary in algebra. The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science 1849; 34 (226): 37-47. https://doi.org/10.1080/14786444908646169

[9] Cohen A, Shoham M. Application of hyper-dual numbers to multibody kinematics. Journal of Mechanisms and Robotics 2016; 8 (1): 1-4. https://doi.org/10.1115/1.4030588

[10] Cohen A, Shoham M. Application of hyper-dual numbers to rigid bodies equations of motion. Mechanism and Machine Theory 2017; 111: 76-84. https://doi.org/10.1016/j.mechmachtheory.2017.01.013

[11] Cohen A, Shoham M. Hyper dual quaternions representation of rigid bodies kinematics. Mechanism and Machine Theory 2020; 150: 1-9. https://doi.org/10.1016/j.mechmachtheory.2020.103861

[12] Çolakoğlu HB. A generalization of the Minkowski distance and new definitions of the central conics. Turkish Journal of Mathematics 2020; 44: 319-333. https://doi.org/10.3906/mat-1904-56

[13] Çolakoğlu HB, Öztürk İ, Özdemir M. Non-parabolic conical rotations. Journal of Computational and Applied Mathematics 2023; 420: 114766. https://doi.org/10.1016/j.cam.2022.114766

[14] Dam EB, Koch M, Lillholm M. Quaternions, Interpolation and Animation. CiteSeer, 1998.

[15] Figueredo LFC, Adorno BV, Ishihara JY, Borges GA. Robust kinematic control of manipulator robots using dual quaternion representation. Institute of Electrical and Electronics Engineers International Conference on Robotics and Automation 2013; 1949-1955. https://ieeexplore.ieee.org/document/6630836

[16] Gonzalez GA, Aragon JL, Rodríguez-Andrade MA. The decomposition of an orthogonal transformation as a product of reflections. American Institute of Physics Journal of Mathematical Physics 2006; 47 (1): 1-10. https://doi.org/10.1063/1.2161072

[17] Gouasmi M, Ouali M, Brahim F. Robot kinematics using dual quaternions. International Journal of Robotics and Automation 2012; 1 (1): 13-30. https://doi.org/10.11591/ijra.v1i1.275

[18] Gu YL, Luh J. Dual-number transformation and its applications to robotics. Institute of Electrical and Electronics Engineers Journal on Robotics and Automation 1987; 3 (6): 615-623. https://doi.org/10.1109/JRA.1987.1087138

[19] Hamilton WR. On quaternions, or on a new system of imaginaries in algebra. Philosophical Magazine 1844; 25: 489-495. https://doi.org/10.1080/14786444408645047

[20] Hart JC, George K, Francis GK, Kauffman LH. Visualizing quaternion rotation. Association for Computing Machinery Transactions on Graphics 1994; 13 (3): 256-276. https://doi.org/10.1145/195784.197480

[21] Jafari M, Habib M. Spherical linear interpolation and bezier curves. General Scientific Researches 2014; 2 (1): 13-17. https://doi.org/10.12737/7052

[22] Jafari M, Yaylı Y. Generalized quaternions and their algebraic properties. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2015; 64 (1): 15-27. https://doi.org/10.1501/Commua1_0000000724

[23] Jantzen RT. Generalized quaternions and spacetime symmetries. Journal of Mathematical Physics 1982; 23 (10): 1741-1746. https://doi.org/10.1063/1.525223

[24] Kotelnikov AP. Screw calculus and some applications to geometry and mechanics. Annals of the Imperial University of Kazan, 1895.

[25] Kösal IA. A note on hyperbolic quaternions. Universal Journal of Mathematics and Applications 2018; 1 (3): 155-159. https://doi.org/10.32323/ujma.380645

[26] Kula L, Yaylı Y. Split quaternions and rotations in semi Euclidean space E_2^4. Journal of the Korean Mathematical Society 2007; 44 (6): 1313-1327. https://doi.org/10.4134/JKMS.2007.44.6.1313

[27] Leeney M. Fast quaternion slerp. International Journal of Computer Mathematics 2009; 86 (1): 79-84. https://doi.org/10.1080/00207160801923064

[28] Lin HY, Cahay M, Vellambi BN, Morris D. A generalization of quaternions and their applications. Symmetry 2022; 14 (3): 599. https://doi.org/10.3390/sym14030599

[29] Macfarlane A. Hyperbolic quaternions. Proceedings of the Royal Society of Edinburgh 1902; 23: 169-180. https://doi.org/10.1017/S0370164600010385

[30] Mackey DS, Mackey N, Mehl C, Mehrmann V. Möbius transformations of matrix polynomials. Linear Algebra and its Applications 2015; 470: 120-184. https://doi.org/10.1016/j.laa.2014.05.013

[31] Mackey DS, Mackey N, Tisseur F. G-reflectors: Analogues of householder transformations in scalar product spaces. Linear algebra and its Applications 2004; 385: 187-213. https://doi.org/10.1016/j.laa.2003.07.009

[32] Mamagani AB, Jafari M. On properties of generalized quaternion algebra. Journal of Novel Applied Science 2013; 2 (12): 683-689.

[33] McAulay A. Octonions: A Development of Clifford's Bi-quaterions. UK: University Press, 1898.

[34] Mehl C, Rodman L. Symmetric matrices with respect to sesquilinear forms. Linear Algebra and its Applications 2002; 349: 55-75. https://doi.org/10.1016/S0024-3795(01)00598-5

[35] Nešović E. On rotation about lightlike axis in three-dimensional Minkowski space. Advances in Applied Clifford Algebras 2016; 26 (1): 237-251. https://doi.org/10.1007/s00006-015-0601-6

[36] Özdemir M. An alternative approach to elliptical motion. Advances in Applied Clifford Algebras 2016; 26 (1): 279-304. https://doi.org/10.1007/s00006-015-0592-3

[37] Özdemir M, Ergin AA. Rotations with unit timelike quaternions in Minkowski 3-space. Journal of Geometry and Physics 2006; 56 (2): 322-336. https://doi.org/10.1016/j.geomphys.2005.02.004

[38] Özdemir M. Introduction to hybrid numbers. Advances in Applied Clifford Algebras 2018; 28 (1): 1-32. https://doi.org/10.1007/s00006-018-0833-3

[39] Özdemir M. Kuaterniyonlar ve Geometri. İzmir, TR: Altın Nokta, 2020.

[40] Özdemir Z, Güner G. Elliptical kinematics of the accretive surface growth. Turkish Journal of Mathematics 2022; 46 (3): 728-745. https://doi.org/10.55730/1300-0098.3119

[41] Özen KE, Tosun M. Elliptic biquaternion algebra. American Institute of Physics Conference Proceedings 2018; 1926: 1-6. https://doi.org/10.1063/1.5020481

[42] Pall G. On generalized quaternions. Transactions of the American Mathematical Society 1946; 59 (2): 280-332. https://doi.org/10.1090/S0002-9947-1946-0018702-X

[43] Pennestri E, Valentini EP. Dual quaternions as a tool for rigid body motion analysis: A tutorial with an application to biomechanics. Archive of Mechanical Engineering 2010; 57 (2): 187-205. https://doi.org/10.2478/v10180-010-0010-2

[44] Plebański JF, Przanowski M. Generalizations of the quaternion algebra and lie algebras. Journal of Mathematical Physics 1988; 29 (3): 529-535. https://doi.org/10.1063/1.528045

[45] Rätz J. On isometries of generalized inner product spaces. Society for Industrial and Applied Mathematics Journal on Applied Mathematics 1970; 18 (1): 6-9. https://doi.org/10.1137/0118002

[46] Rodríguez-Andrade MA, Aragón-González A, Aragón JL, Verde-Star L. An algorithm for the Cartan-Dieudonne theorem on generalized scalar product spaces. Linear Algebra and its Applications 2011; 434 (5): 1238-1254. https://doi.org/10.1016/j.laa.2010.11.005

[47] Sangwine SJ, Ell TA, Bihan NL. Fundamental representations and algebraic properties of biquaternions or complexified quaternions. Advances in Applied Clifford Algebras 2011; 21 (3): 607-636. https://doi.org/10.1007/s00006-010-0263-3

[48] Schmidt J, Niemann H. Using quaternions for parametrizing 3-d rotations in unconstrained nonlinear optimization. In: Vision, Modeling and Visualization 2001, Proceedings; Stuttgart, DE; 2001. pp. 399-406.

[49] Segre C. Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. Mathematische Annalen 1892; 40 (3): 413-467. https://doi.org/10.1007/BF01443559

[50] Selig JM. Rational interpolation of rigid-body motions. In: Levine J, Müllhaupt P. (editors) Advances in the theory of control, signals and systems with physical modeling. DE: Springer, Berlin, Heidelberg, 2010, pp. 213-224. https://doi.org/10.1007/978-3-642-16135-3_18

[51] Selig JM. Geometric Fundamentals of Robotics. New York, USA: Springer, 2005.

[52] Shoemake K. Animating rotation with quaternion curves. In: Proceedings of the 12th annual conference on Computer graphics and interactive techniques; California, USA; 1985. 245-254. https://doi.org/10.1145/325165.325242

[53] Simsek H, Özdemir M. Generating hyperbolical rotation matrix for a given hyperboloid. Linear Algebra and Its Applications 2016; 496: 221-245. https://doi.org/10.1016/j.laa.2016.01.038

[54] Thiruvengadam S, Tan JS, Miller K. A generalised quaternion and Clifford algebra based mathematical methodology to effect multi-stage reassembling transformations in parallel robots. Advances in Applied Clifford Algebras 2021; 31 (3): 1-69. https://doi.org/10.1007/s00006-020-01114-3

[55] Valverde A, Tsiotras P. Spacecraft robot kinematics using dual quaternions. Robotics 2018; 7 (4): 1-24. https://doi.org/10.3390/robotics7040064

[56] Vicci L. Quaternions and rotations in 3-space: The algebra and its geometric interpretation. North Carolina, USA: University of North Carolina, 2001.

[57] Wang X, Han D, Yu C, Zheng Z. The geometric structure of unit dual quaternion with application in kinematic control. Journal of Mathematical Analysis and Applications 2012; 389 (2): 1352-1364. https://doi.org/10.1016/j.jmaa.2012.01.016