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On the measure of noncompactness in $L_p(\mathbb{R}^+)$ and applications to a product of n-integral equations

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Abstract: In this article, we prove a new compactness criterion in the Lebesgue spaces $L_p(\mathbb{R}^+), 1 \leq p < \infty$ and use such criteria to construct a measure of noncompactness in the mentioned spaces. The conjunction of that measure with the Hausdroff measure of noncompactness is proved on sets that are compact in finite measure. We apply such measure with a modified version of Darbo fixed point theorem in proving the existence of monotonic integrable solutions for a product of n-Hammerstein integral equations $n \geq 2$.

Key words: Compactness criterion, measure of noncompactness, discontinuous solutions, Hammerstein integral equations, compact in finite measure

1. Introduction

There are two methodologies for tackling different types of integral, differential, partial differential, or functional equations and their applications, to be specific they are the traditional fixed point hypotheses including theories from topology and analysis and the fixed point hypotheses concerning the utilization of measures of noncompactness.

In [25], Kuratowski introduced the first definition of measures of noncompactness and Darbo utilized this idea to present his fixed point hypothesis in 1955 (cf. [17]). Such methodology was utilized to contemplate the presence of solutions of various types of problems in different function spaces, for example, in the space $C[a, b]$ (cf. [3, 16]), in the space $BC(\mathbb{R}^+)$ (cf. [5, 7]), in the space of Lebesgue integrable functions L_p on bounded or unbounded domains (cf. [15, 27, 30]), and in Orlicz spaces (cf. [28, 29]).

Recently, there have been many manuscripts that developed and constructed new measures of noncompactness in various function spaces and applying these results to study the existence theorems of numerous integral equations.

Let us mention that some new measures of noncompactness were defined and demonstrated in the Banach algebras $C(I)$, $BC(\mathbb{R}^+)$ satisfying condition (m) [6], in the space of regular functions on bounded and unbounded domains [13, 18, 26], in the space of Lebesgue integrable functions $L_1(\mathbb{R}^N)$ [10], in the space of all locally integrable functions $L^1_{loc}(\mathbb{R}^+)$ [32], in general Lebesgue spaces $L^p(\mathbb{R}^N)$ [1], and in the Sobolev space $W^{k,1}(I)$ [24]. Moreover, Erzakowa in [19, 20] uses the notion of compact in measure to define a measure of

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noncompactness in L_p-spaces, $1 \leq p < \infty$ on bounded intervals.

In this manuscript, we characterize new criteria and define a new measure of noncompactness on unbounded domain in the spaces $L_p(\mathbb{R}^+)$, $1 \leq p < \infty$ utilizing the concept of compactness in the finite measure. The conjunction of that measure with the Hausdorff measure of noncompactness is demonstrated, then an adjusted adaptation of the Darbo fixed point hypothesis identified with our outcomes is given. The technique presented in this paper is not difficult to apply for studying various problems in L_p-spaces under a general set of assumptions which permits us to skip some restrictions presented in the former literature.

The importance of studying L_p-spaces, $1 \leq p < \infty$ is that they are the most suitable spaces when we study the problems related to the integral equations where the studied operators need only to be integrable (not necessary to be continuous).

As an application, we apply our outcomes to examine the solutions of the equation

$$x(t) = h(t, x(t)) + \prod_{i=1}^{n} \left(g_i(t, x(t)) + f_i(t, x(t)) \cdot |x(t)|^{\frac{p}{q_i}} \cdot \int_0^\infty K_i(t, s) u_i(s, x(s))\, ds \right) \tag{1.1}$$

in the spaces $L_p(\mathbb{R}^+)$, $1 \leq p < q_i < \infty$, $n \geq 2$.

Recall that, in [11, 21], the authors discussed the presence and the uniqueness of a continuous solution to the equation

$$x(t) = k\left(p(t) + \int_0^t A(t-s)x(s)\, ds \right)\left(q(t) + \int_0^t B(t-s)x(s)\, ds \right),\ t > 0. \tag{1.2}$$

Model (1.2) emerges in the investigation of the spread of diseases that do not induce permanent immunity.

However, various models of infectious diseases contain data functions that are discontinuous, so it is better to inspect that model in L_p-spaces.

Additionally, the authors in [31] discussed the presence and uniqueness of a continuous solution to the equation

$$x(t) = \prod_{i=1}^{n} \left(g_i(t) + \int_a^t K_i(t, s, x(s))\, ds \right),\ t \in [a, b].$$

The authors in [23] discussed the presence of monotonic solutions in the space $L_1[0, \tau]$ for the equation

$$x(t) = \Big(h_1(t) + g(t) \cdot (Tx)(t) \Big) \cdot \left(h_2(t) + \frac{|x(t)|^{\frac{1}{p}}}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} \right),$$

where $T(x)$ is a general operator, by using a proper measure of noncompactness.

The existence of L_1-solution on unbounded interval was studied in [9] utilizing the measure of weak noncompactness for the equation

$$x(t) = f(t, x(t)) + \prod_{i=1}^{n} f_i\left(t,\ \int_a^t K_i(t, s, x(s))\, ds \right),\ t > 0.$$

This manuscript is motivated by extending and generalizing the outcomes introduced in the former literature to the case of σ-finite measures and applying these outcomes to examine the discontinuous monotonic solutions for a product of n-Hammerstein integral equations $n \geq 2$ in $L_p(\mathbb{R}^+)$. An example to show the applicability of our outcomes is included.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^+ = [0, \infty)$, and $J = [a, b]$.

We denote by $L_p = L_p(\mathbb{R}^+)$, $1 \leq p < \infty$ the Banach space of equivalence classes of measurable functions on $\mathbb{R}^+$ such that $\int_0^\infty |x(s)|^p \, ds < \infty$ with the norm

$$||x||_p = \left(\int_0^\infty |x(s)|^p \, ds \right)^{\frac{1}{p}}.$$

Definition 2.1 *[2] Assume that a function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in \mathbb{R}^+$. Then to every measurable function x, we may assign the function*

$$F_f(x)(t) = f(t, x(t)).$$

The operator F_f in such a way is called the superposition (Nemytskii) operator generated by the function f.

Theorem 2.2 *[2] Let f satisfy the Carathéodory conditions. The superposition operator F_f generated by the function f maps continuously the space L_p into L_q $(p, q \geq 1)$ if and only if*

$$|f(t, x)| \leq a(t) + b \cdot |x|^{\frac{p}{q}} \tag{2.1}$$

for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, where $a \in L_q$ and $b \geq 0$.

Theorem 2.3 *[22, Theorem 6.2] The operator $K_0x(t) = \int_0^\infty K(t, s)x(s) \, ds$ preserves the monotonicity of functions iff for any $l > 0$ the following condition holds true*

$$t_1 < t_2 \implies \int_0^l K(t_1, s) \, ds \geq \int_0^l K(t_2, s) \, ds, \; t_1, t_2 \in \mathbb{R}^+.$$

Lemma 2.4 [12] *Let $n \geq 2$. If $1 \leq p, p_i < \infty$ for $i = 1, \cdots, n$, then the following statements are equivalent:*

1. $\sum_{i=1}^n \frac{1}{p_i} = \frac{1}{p}$.
2. $\left\| \prod_{i=1}^n u_i \right\|_p \leq \prod_{i=1}^n \|u_i\|_{p_i}$ *for every* $u_i \in L_{p_i}$, $i = 1, \cdots, n$.
3. *For every* $u_i \in L_{p_i}$, *then* $\prod_{i=1}^n u_i \in L_p$.

Denote by B_r the closed ball centered at zero element θ and with radius r and let $S = S(J)$ be the set of measurable (in Lebesgue sense) functions on J. Identifying the functions equal almost everywhere the set S is furnished with the metric

$$d(x, y) = \inf_{a>0} [a + meas\{s \; : \; |x(s) - y(s)| \geq a\}]$$

becomes a complete metric space. Moreover, the convergence in measure on J is equivalent to the convergence with respect to the metric d (Proposition 2.14 in [34]). Concerning the case of $\mathbb{R}^+$, as the measure is σ-finite, a notion of *convergence in finite measure* is used and it means that (x_n) is convergent to x in finite measure if and only if it converges to x on each set $T \subset \mathbb{R}^+$ of finite measure. The compactness in such spaces is called a "compactness in measure" ("in finite measure").

Let X be a bounded subset of $L_p(J)$. Assume that there is a family of subsets $(\Omega_c)_{0\leq c\leq b-a}$ of the interval J such that $\operatorname{meas}\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \geq x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$. Such a family is equimeasurable (cf. [4]) and then the set X is compact in measure in $L_p(J)$. It is clear that by putting $\Omega_c = [0, c) \cup Z$ or $\Omega_c = [0, c) \setminus Z$, where Z is a set with measure zero, this family contains nonincreasing functions (possibly except for a set Z).

We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose an integrable and nonincreasing function y and all functions are equal a.e. to y satisfies the above condition. Thus, we can write that elements from $L_p(J)$ belong to this class of functions. Clearly, the same holds true for $\mathbb{R}^+$.

Theorem 2.5 *[15] Let X be a bounded subset of $L_p(J)$ consisting of functions that are a.e. nondecreasing (or a.e. nonincreasing) on the interval J. Then X is compact in measure in $L_p(J)$.*

Corollary 2.6 *Let X be a bounded subset of L_p consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on $\mathbb{R}^+$. Then X is compact in the finite measure in L_p.*

Proof Let us assume that $L_p(T)$ for σ-finite measure space T, then there is some equivalent finite measure ν $(\nu(\mathbb{R}^+) < \infty)$ (Proposition 2.1. in [34] or Corollary 2.20 in [34]). Then the convergence of sequences in S are the same for the metric d and

$$d_\nu(x, y) = \inf_{a>0} [a + \nu\{s : |x(s) - y(s)| \geq a\}]$$

(Proposition 2.2 in [33]). Let $(x_n) \subset X$ be an arbitrary bounded sequence. As a subset of a metric space X = $(L_p(\mathbb{R}^+), d_\nu)$ the sequence is compact in this metric space (Theorem 2.5). Then there exists a subsequence (x_{n_k}) of (x_n) which is convergent in the space X to some x, i.e.

$$d_\nu(x_{n_k}, x) \overset{k\to\infty}{\longrightarrow} 0.$$

Since the two metrics have the same convergent sequences, then

$$d(x_{n_k}, x) \overset{k\to\infty}{\longrightarrow} 0.$$

This means that X is compact in finite measure in L_p. □

Remark 2.7 *Let Q_r be the set of all functions $x \in L_p$ which are a.e. nonincreasing (or a.e. nondecreasing) on $\mathbb{R}^+$. Then Q_r is nonempty, bounded, closed and convex subset of L_p such that $\|x\|_p \leq r$, $r > 0$. Moreover, the set Q_r is compact in finite measure (cf. [8] and [14, Lemma 4.10]).*

3. Main results.

We need to present a new measure of noncompactness in L_p-spaces, so we shall prove the following criteria. We will start by expanding Erzakova results [19, 20] from the case of finite measure space to σ-finite measures.

Theorem 3.1 (cf. [20]) (Compactness criterion in L_p)
Fix any $1 \leq p < \infty$. A bounded subset X of L_p is relatively compact if and only if it is compact in finite measure and has uniformly equicontinuous norm in L_p, i.e. the following conditions hold true:

1. *for measurable subset $D \subset \mathbb{R}^+$*
$$\limsup_{\varepsilon \to 0} \sup_{mesD \leq \varepsilon} \sup_{x \in X} \|x \cdot \chi_D\|_p = 0,$$

2. $$\lim_{T \to \infty} \sup_{x \in X} \|x \cdot \chi_{[T,\infty)}\|_p = 0,$$

where χ_A denotes the characteristic function of a measurable subset A of $\mathbb{R}^+$.

Proof One implication is known: if a set X is relatively compact in L_p, then it is compact in finite measure. Indeed, if W is a measurable subset of $\mathbb{R}^+$ having finite measure and that convergence on W implies convergence in measure on W, so X is relatively compact in finite measure. Moreover, such a set X satisfies condition 1. (cf. [20, Theorem 1], [34]) and for condition 2. (cf. [8]).

To prove the converse implication, it is sufficient to apply the Krasnoselskii theorem ([34, Theorem 3.19]. Indeed, as L_p is a regular ideal space and X has a uniformly equicontinuous norm, it is sufficient to prove that any sequence in X has a.e. convergent subsequence, but the last property is a consequence of the compactness in finite measure for σ-finte measure spaces ([34, Corollary 2.19]) and we are done. □

Due to observation made in the above proof, we get:

Corollary 3.2 *Theorem 3.1 can be extended to the case of regular ideal spaces, so it holds also for Lorentz spaces and Orlicz spaces whose generating function satisfies a Δ_2-condition.*

Corollary 3.3 *For a nonempty and bounded subset X of the space L_p, $\varepsilon > 0$ and let*

$$c(X) = \limsup_{\varepsilon \to 0} \sup_{mesD \leq \varepsilon} \sup_{x \in X} \|x \cdot \chi_D\|_p$$

and

$$d(X) = \lim_{T \to \infty} \sup_{x \in X} \|x \cdot \chi_{[T,\infty)}\|_p.$$

Then the measure

$$\mu(X) = c(X) + d(X) \tag{3.1}$$

has the following properties:

(a) $X \subset Y \implies \mu(X) \leq \mu(Y)$.

(b) $\mu(X \bigcup Y) = \max\{\mu(X), \mu(Y)\}$.

(c) $\mu(\bar{X}) = \mu(convX) = \mu(X)$, *where* $\bar{X}$ *and* $convX$ *refer to the closure and convex closure of* X, *respectively.*

(d) $\mu(\lambda X) = |\lambda|\ \mu(X)$, $for\ \lambda \in \mathbb{R}$.

(e) $\mu(X + Y) \leq \mu(X) + \mu(Y)$, *here* $X + Y = \{x + y : x \in X \text{ and } y \in Y\}$.

(f) $|\mu(X) - \mu(Y)| \leq k\ dist(X, Y)$, *where* $dist(X, Y)$ *denotes the Hausdorff distance, and the constant* k *does not depend on* X *and* Y.

(g) That X *is relatively compact implies that* $\mu(X) = 0$, *the converse is not true i.e.* $\mu(X) = 0$ *does not imply that* X *is relatively compact.*

Proof The proof can be directly done by applying equation (3.1) with the help of Theorem 3.1 (cf. [20]). □

Remark 3.4 *The measure* μ *satisfies* $\mu(B_r) = 2r$ *(cf. [8]).*

Definition 3.5 *[2] Let* $X \neq \emptyset$ *be a bounded subset of* L_p. *The Hausdroff measure of noncompactness* $\chi(X)$ *is defined as*

$$\chi(X) = \inf\{r > 0 : \textit{there exists a finite subset } Y \textit{ of } E \textit{ such that } X \subset Y + B_r\}.$$

Remark 3.6 *The measure* χ *has all properties of* μ *mentioned above; for measure-compact sets as distinct from the Hausdorff measure of noncompactness* $\chi(X) = 0 \Rightarrow X$ *is compact, while the equality* $\mu(X) = 0$ *is possible on noncompact sets.*

We are in a position to present the relation between the measures of noncompactness μ and χ.

Theorem 3.7 *Let* $X \neq \emptyset$ *be a bounded subset of* L_p, *which is also compact in finite measure, then*

$$\chi(X) \leq \mu(X) \leq 2\chi(X).$$

Proof Suppose $\chi(X) = r$ and $\varepsilon > 0$ is arbitrary. Then we can find a finite set $Y \subset L_p$ such that $X \subset Y + (r + \varepsilon)B_1$. From the properties of μ, we have

$$\mu(X) \leq \mu(Y) + (r + \varepsilon)\mu(B_1) = 2(r + \varepsilon)$$

and since ε is arbitrary, we get

$$\mu(X) \leq 2\chi(X). \tag{3.2}$$

Further, let X be a subset of L_p which is compact in finite measure and that $\chi(X) = r$ and $c(X) = r_1$, $d(X) = r_2$, where $r_1 + r_2 = r$.
Fix an arbitrary $\eta > 0$. Then there exist $T > 0$ and $\varepsilon > 0$ such that

$$\|x \cdot \chi_D\|_p \leq r_1 + \eta \tag{3.3}$$

and

$$\sup_{x \in X} \|x \cdot \chi_{[T,\infty)}\|_p \leq r_2 + \eta \tag{3.4}$$

for any $x \in X$ and for any measurable subset $D \subset [0,T]$ with $measD \leq \epsilon$.

Denote by

$$\Omega(x,h) = \{t \in [0,T] : |x(t)| \geq h\}$$

for an arbitrary $h \geq 0$ and $x \in X$. Since X is bounded, we have

$$\lim_{h\to\infty} \{\sup[meas\Omega(x,h) : x \in X]\} = 0.$$

Then, for any $x \in X$, we can choose $h_0 \geq 0$ such that

$$meas\Omega(x,h_0) \leq \varepsilon. \tag{3.5}$$

By applying (3.3) for an arbitrary $x \in X$, we have

$$\|x \cdot \chi_{\Omega(x,h_0)}\|_p \leq r_1 + \eta. \tag{3.6}$$

Next, for any $x \in X$ denote by x_{h_0}, the function

$$x_{h_0}(t) = \begin{cases} 0 & \text{for} \quad t \geq T \\ t & \text{for} \quad t \in [0,T] - \Omega(x,h_0) \\ h_0 \ sign \ x(t) & \text{for} \quad t \in \Omega(x,h_0). \end{cases}$$

Since X is compact in finite measure, we conclude that $X_{h_0} = \{x_{h_0} : x \in X\}$ is also compact in finite measure. Moreover, one can easily check that $c(X_{h_0}) = d(X_{h_0}) = 0$, which implies $\mu(X_{h_0}) = 0$. Thus, by Theorem 3.1, the set X_{h_0} is compact in L_p.

Consequently,

$$\chi(X_{h_0}) = 0.$$

Now, applying (3.4), we infer

$$\begin{aligned} \|x - x_{h_0}\|_p &= \|(x - x_{h_0}) \cdot \chi_{[0,T)}\|_p + \|x \cdot \chi_{[T,\infty)}\|_p \\ &\leq \|(x - x_{h_0}) \cdot \chi_{[0,T)}\|_p + r_2 + \eta. \end{aligned} \tag{3.7}$$

Moreover, by (3.6), we get

$$\|(x - x_{h_0}) \cdot \chi_{[0,T)}\|_p \leq \|x \cdot \chi_{\Omega(x,h_0)}\|_p \leq r_1 + \eta.$$

Hence, by (3.7), we have

$$\|x - x_{h_0}\|_p \leq r_1 + r_2 + 2\eta$$

and consequently

$$X \subset X_{h_0} + B_{r+2\eta}.$$

Thus,

$$\chi(X) \leq (r + 2\eta)\chi(B_1) = r + 2\eta$$

and since η is arbitrary, we have

$$\chi(X) \leq \mu(X).$$

Combining the above inequality with (3.2), we obtain the proof. □

Next, we present an adjusted adaptation of Darbo-type fixed point theorem.

Corollary 3.8 *Let Q be a nonempty, bounded, closed, and convex subset of L_p. Also, assume Q consists of functions which are a.e. nondecreasing (or a.e. nonincreasing) on $\mathbb{R}^+$. Suppose $H: Q \to Q$ is a continuous operator and takes a.e. nondecreasing (or a.e. nonincreasing) functions on $\mathbb{R}^+$ into functions of the same type. Finally suppose there exists a constant k, $0 \le k < \frac{1}{2}$ with*

$$\mu(H(X)) \le k\mu(X)$$

for any nonempty subset X of Q. Then H has at least one fixed point in Q.

Proof Let X be a subset of Q. Note from Remark 2.7 that X and $H(X)$ are compact in finite measure in L_p. Then from Theorem 3.7, we have

$$\mu(HX) \le 2\chi(HX) \le 2k \cdot \chi(X) \le 2k \cdot \mu(X).$$

Now, by applying the classical Darbo fixed point theorem with $0 \le k < \frac{1}{2}$, we get the thesis. □

4. Applications

We will apply the results presented in Section 3 to prove the existence of monotonic integrable solutions for equation (1.1).

Rewrite equation (1.1) as follows

$$x(t) = Hx(t) = F_h x(t) + \prod_{i=1}^{n} H_i x(t), \quad \text{where}$$

$$H_i(x) = F_{g_i}(x) + A_i(x), \ A_i(x) = F_{f_i}(x) \cdot U_i(x),$$

$$U_i(x) = |x|^{\frac{p}{q_i}} \cdot K_{0_i} F_{u_i}(x), \quad \text{and} \quad K_{0_i} = \int_0^\infty K_i(t,s)x(s)\, ds,$$

such that F_h, F_{g_i}, F_{f_i}, and F_{u_i} are the superposition operators as in Definition 2.1.

Assume that $\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}$, where $\frac{1}{p_i} = \frac{1}{q_i} + \frac{1}{q_i'}$ associated with the next assumptions:

(i) The functions $h, g_i, f_i, u_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ satisfy Carathéodory conditions. Moreover, assume that $h(t,x) \ge 0$, $g_i(t,x) \ge 0$, $f_i(t,x) \ge 0$, $u_i(t,x) \ge 0$, for a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and h, g_i, f_i, u_i, $i = 1, \cdots, n$ are nonincreasing with respect to both variables t and x separately.

(ii) There exist positive constants d, d_i, c_i, l_i and functions $b \in L_p$, $b_i \in L_{q_i'}, a_i \in L_{p_i}$ and $e_i \in L_{q_i}$ such that

$$|h(t,x)| \le b(t) + d|x|, \quad |f_i(t,x)| \le b_i(t) + d_i|x|^{\frac{p}{q_i'}}$$

and

$$|g_i(t,x)| \le a_i(t) + c_i|x|^{\frac{p}{p_i}}, \quad |u_i(t,x)| \le e_i(s) + l_i|x|^{\frac{p}{q_i}}, \ i = 1, \cdots, n.$$

(iii) Assume that the functions K_i are measurable in (t,s) and assume that the linear integral operators K_{0_i} with kernels $K_i(t,s)$ map $L_{q_i} \to L_\infty$ and K_{0_i} are continuous with

$$\|K_{0_i}\|_\infty = ess \sup_{0\le t<\infty} \left(\int_0^\infty |K_i(t,s)|^{q_i''}\, ds \right)^{\frac{1}{q_i''}},$$

where $\frac{1}{q_i} + \frac{1}{q_i''} = 1,\ i = 1, \cdots, n$.

(iv) For any $l > 0$ and $i = 1, \cdots, n$, we have

$$t_1 < t_2 \Longrightarrow \int_0^l K_i(t_1,s)\, ds \ge \int_0^l K_i(t_2,s)\, ds, \quad t_1, t_2 \in \mathbb{R}^+.$$

(v) Assume there exists a number $r > 0$ fulfills

$$d + \prod_{i=1}^n \left(c_i + d_i \cdot \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot r^{\frac{p}{q_i}} \right) \right) < \frac{1}{2}$$

and

$$\|b\|_p + d \cdot r + \prod_{i=1}^n \Bigg[\|a_i\|_{p_i} + \big(c_i + d_i \|K_{0_i}\|_\infty \|e_i\|_{q_i}\big) r^{\frac{p}{p_i}} \tag{4.1}$$

$$+ \|K_{0_i}\|_\infty \|b_i\|_{q_i'} \|e_i\|_{q_i} r^{\frac{p}{q_i}} + l_i \|K_{0_i}\|_\infty \|b_i\|_{q_i'} r^{\frac{2p}{q_i}} + d_i l_i \|K_{0_i}\|_\infty r^{\frac{p}{p_i} + \frac{p}{q_i}} \Bigg] \le r.$$

Remark 4.1 *Equation (4.1) takes the form* $a + d \cdot r + \prod_{i=1}^n \left(B_i + C_i r^{\frac{p}{p_i}} + D_i r^{\frac{p}{q_i}} + E_i r^{\frac{2p}{q_i}} + G_i r^{\frac{p}{p_i} + \frac{p}{q_i}} \right) \le r$. *For example, for* $r = 1$, *we would need* $a + d + \prod_{i=1}^n \left(B_i + C_i + D_i + E_i + G_i \right) \le 1$.

Remark 4.2 *Assumptions (ii) and (iii) imply that the operators* $U_i : L_p \to L_{q_i}$ *are continuous and satisfying*

$$\|U_i(x)\|_{q_i} \le \|K_{0_i}\|_\infty \big(\|e_i\|_{q_i} + l_i \|x\|_p^{\frac{p}{q_i}} \big) \|x\|_p^{\frac{p}{q_i}},\ i = 1, \cdots, n.$$

Indeed, for $x \in L_p,\ i = 1, \cdots, n$, *we have*

$$\begin{aligned}
\|U_i(x)\|_{q_i} &= \left\| |x(t)|^{\frac{p}{q_i}} \int_0^\infty K_i(t,s) u(s,x(s))\, ds \right\|_{q_i} \\
&\le \left\| |x(t)|^{\frac{p}{q_i}} \int_0^\infty K_i(t,s) \big(e_i(s) + l_i \cdot |x(s)|^{\frac{p}{q_i}} \big)\, ds \right\|_{q_i} \\
&\le \left\| |x(t)|^{\frac{p}{q_i}} \|K_i(t,\cdot)\|_{q''} \|e_i + l_i \cdot |x|^{\frac{p}{q_i}} \|_{q_i} \right\|_{q_i} \\
&\le \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot \|x^{\frac{p}{q_i}}\|_{q_i} \right) \left\| |x|^{\frac{p}{q_i}} \right\|_{q_i} \\
&= \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot \|x\|_p^{\frac{p}{q_i}} \right) \|x\|_p^{\frac{p}{q_i}},
\end{aligned}$$

where $\|x^{\frac{p}{q_i}}\|_{q_i} = \|x\|_p^{\frac{p}{q_i}}$.

Similarly, if $D \subset \mathbb{R}^+$ *, we have*

$$\|U_i(x)\cdot\chi_D\|_{q_i} \le \|K_{0_i}\|_\infty\left(\|e_i\|_{q_i} + l_i\cdot\|x\|_p^{\frac{p}{q_i}}\right)\|x\cdot\chi_D\|_p^{\frac{p}{q_i}}.$$

Theorem 4.3 *Let assumptions (i) - (v) be satisfied, then there exists a solution* $x \in L_p$ *of (1.1) which is a.e. nonincreasing on* $\mathbb{R}^+$ *.*

Proof Step I. In what follows, let $i = 1, \cdots, n$. Assumption (i), (ii) and Theorem 2.2 imply that the operators $F_h : L_p \to L_p$, $F_{g_i} : L_p \to L_{p_i}$, $F_{f_i} : L_p \to L_{q_i'}$ and $F_{u_i} : L_p \to L_{q_i}$ are continuous. The operators U_i map L_p into L_{q_i} continuously (thanks to Remark 4.2). From the Hölder inequality, the operators $A_i : L_p \to L_{p_i}$ are continuous, which implies that $H_i : L_p \to L_{p_i}$ are continuous. By using Corollary 2.4, we can deduce that the operator $H = F_h + \prod_{i=1}^n H_i$ maps continuously L_p into itself.

Step II. For $x \in L_p$ and by using assumptions (i) - (iii) and Remark 4.2, we have

$$\begin{aligned}
\|H_i(x)\|_{p_i} &\le \|F_{g_i}\|_{p_i} + \|A_i x\|_{p_i}\\
&\le \| a_i + c_i\cdot|x|^{\frac{p}{p_i}}\|_{p_i} + \|F_{f_i}(x)U_i(x)\|_{p_i}\\
&\le \|a_i\|_{p_i} + c_i\cdot\|x^{\frac{p}{p_i}}\|_{p_i} + \|F_{f_i}(x)\|_{q_i'}\|U_i(x)\|_{q_i}\\
&\le \|a_i\|_{p_i} + c_i\cdot\|x\|_p^{\frac{p}{p_i}} + \left\|b_i + d_i\cdot|x|^{\frac{p}{q_i'}}\right\|_{q_i'}\|K_{0_i}\|_\infty\left(\|e_i\|_{q_i} + l_i\cdot\|x\|_p^{\frac{p}{q_i}}\right)\|x\|_p^{\frac{p}{q_i}}\\
&\le \|a_i\|_{p_i} + c_i\cdot\|x\|_p^{\frac{p}{p_i}} + \|K_{0_i}\|_\infty\left(\|b_i\|_{q_i'} + d_i\cdot\|x\|_p^{\frac{p}{q_i'}}\right)\left(\|e_i\|_{q_i} + l_i\cdot\|x\|_p^{\frac{p}{q_i}}\right)\|x\|_p^{\frac{p}{q_i}}.
\end{aligned}$$

By using Corollary 2.4, we have

$$\begin{aligned}
&\|H(x)\|_p \le \left\|F_h(x)\right\|_p + \left\|\prod_{i=1}^n H_i(x)\right\|_p\\
\le\ & \left\|b + d\cdot|x|\right\|_p + \prod_{i=1}^n\left\|H_i(x)\right\|_{p_i}\\
\le\ & \|b\|_p + d\cdot\|x\|_p + \prod_{i=1}^n\left[\|a_i\|_{p_i} + c_i\cdot\|x\|_p^{\frac{p}{p_i}} + \|K_{0_i}\|_\infty\left(\|b_i\|_{q_i'} + d_i\cdot\|x\|_p^{\frac{p}{q_i'}}\right)\left(\|e_i\|_{q_i} + l_i\cdot\|x\|_p^{\frac{p}{q_i}}\right)\|x\|_p^{\frac{p}{q_i}}\right].
\end{aligned}$$

Thus, $H : L_p \to L_p$. Let r be as in equation (4.1) and let $x \in B_r$, where $B_r = \{m \in L_p : \ \|m\|_p \le r\}$, then

$$\|b\|_p + d\cdot r + \prod_{i=1}^n\left[\|a_i\|_{p_i} + c_i\cdot r^{\frac{p}{p_i}} + \|K_{0_i}\|_\infty\left(\|b_i\|_{q_i'} + d_i\cdot r^{\frac{p}{q_i'}}\right)\left(\|e_i\|_{q_i} + l_i\cdot r^{\frac{p}{q_i}}\right)r^{\frac{p}{q_i}}\right] \le r,$$

which indicate that $H : B_r \to B_r$ is continuous.

Step III. Let Q_r be a subset of B_r containing all functions which are a.e. nonincreasing on $\mathbb{R}^+$. This set is a nonempty, bounded, convex, and closed set in L_p. Moreover, by Remark 2.7, the set Q_r is compact in finite measure.

Step IV. Now, we show that H preserves the monotonicity of functions. Take $x \in Q_r$, then x is a.e. nonincreasing on $\mathbb{R}^+$ and by assumption (ii) the functions F_h, F_{g_i}, F_{f_i}, and F_{u_i} are also a.e. nonincreasing. Further, assumption (iv) implies that $U_i(x)$ are a.e. nonincreasing on $\mathbb{R}^+$, then we have A_i, H_i are also a.e. nonincreasing on $\mathbb{R}^+$. This implies that $H : Q_r \to Q_r$ is continuous.

Step V. Assume that $\emptyset \neq X \subset Q_r$ is nonempty set and let $\varepsilon > 0$ be arbitrary fixed constant. Then for an arbitrary $x \in X$ and for a set $D \subset \mathbb{R}^+$, meas $D \leq \varepsilon$, we obtain

$$\begin{aligned}
&\|H_i(x)\cdot\chi_D\|_{p_i} \leq \|F_{g_i}\cdot\chi_D\|_{p_i} \ + \ \|A_i\cdot\chi_D\|_{p_i}\\
\leq\ & \|F_{g_i}\cdot\chi_D\|_{p_i} \ + \ \|F_{f_i}\cdot\chi_D\|_{q_i'}\cdot\|U_i(x)\cdot\chi_D\|_{q_i}\\
\leq\ & \|(a_i + c_i\cdot|x|^{\frac{p}{p_i}})\cdot\chi_D\|_{p_i} + \|\big(b_i + d_i\cdot|x|^{\frac{p}{q_i'}}\big)\cdot\chi_D\|_{q_i'}\|K_{0_i}\|_\infty\left(\|e_i\|_{q_i} + l_i\cdot\|x\|_p^{\frac{p}{q_i}}\right)\|x\cdot\chi_D\|_p^{\frac{p}{q_i}}\\
\leq\ & \|a_i\cdot\chi_D\|_{p_i} + c_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{p_i}} + \|K_{0_i}\|_\infty\left(\|b_i\cdot\chi_D\|_{q_i'} + d_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{q_i'}}\right)\left(\|e_i\|_{q_i} + l_i\cdot r^{\frac{p}{q_i}}\right)\|x\cdot\chi_D\|_p^{\frac{p}{q_i}}.
\end{aligned}$$

By using Corollary 2.4, we have

$$\begin{aligned}
\|H(x)\cdot\chi_D\|_p \ &\leq \ \left\|F_h(x)\cdot\chi_D\right\|_p + \prod_{i=1}^n \left\|H_i(x)\cdot\chi_D\right\|_{p_i}\\
&\leq \ \|b\cdot\chi_D\|_p + d\cdot\|x\cdot\chi_D\|_p + \prod_{i=1}^n\Bigg[\|a_i\cdot\chi_D\|_{p_i} + c_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{p_i}}\\
&\quad + \|K_{0_i}\|_\infty\left(\|b_i\cdot\chi_D\|_{q_i'} + d_i\cdot\|x\cdot\chi_D\|_p^{\frac{p}{q_i'}}\right)\left(\|e_i\|_{q_i} + l_i\cdot r^{\frac{p}{q_i}}\right)\|x\cdot\chi_D\|_p^{\frac{p}{q_i}}\Bigg].
\end{aligned}$$

Hence, taking into account that $b \in L_p$, $a_i \in L_{p_i}$, and $b_i \in L_{q_i'}$, then

$$\lim_{\varepsilon\to 0}\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|b\cdot\chi_D\|_p\}]\} = 0, \quad \lim_{\varepsilon\to 0}\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|a_i\cdot\chi_D\|_{p_i}\}]\} = 0,$$

$$\text{and} \quad \lim_{\varepsilon\to 0}\{\sup_{mes\ D\leq\varepsilon}[\sup_{x\in X}\{\|b_i\cdot\chi_D\|_{q_i'}\}]\} = 0, \ i = 1,\cdots,n.$$

By using the definition of $c(x)$, we get

$$c(H(X)) \leq \left[d + \prod_{i=1}^n\left(c_i + d_i\cdot\|K_{0_i}\|_\infty\left(\|e_i\|_{q_i} + l_i\cdot r^{\frac{p}{q_i}}\right)\right)\right]c(X). \tag{4.2}$$

For $T > 0$ and $x \in X$, we have the following estimate

$$\begin{aligned} \|H(x).\chi_{[T,\infty)}\|_p \leq & \|b.\chi_{[T,\infty)}\|_p + d\|x.\chi_{[T,\infty)}\|_p \\ & + \prod_{i=1}^{n} \Bigg[\|a_i.\chi_{[T,\infty)}\|_{p_i} + c_i \|x.\chi_{[T,\infty)}\|_p^{\frac{p}{p_i}} + \|K_{0_i}\|_\infty \Bigg(\|b_i \cdot \chi_{[T,\infty)}\|_{q_i'} \\ & + d_i \cdot \|x \cdot \chi_{[T,\infty)}\|_p^{\frac{p}{q_i'}} \Bigg) \Bigg(\|e_i\|_{q_i} + l_i \cdot r^{\frac{p}{q_i}} \Bigg) \|x \cdot \chi_{[T,\infty)}\|_p^{\frac{p}{q_i}} \Bigg]. \end{aligned}$$

Now as $T \to \infty$, we get

$$d(H(X)) \leq \left[d + \prod_{i=1}^{n} \left(c_i + d_i \cdot \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot r^{\frac{p}{q_i}} \right) \right) \right] d(X). \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\mu(H(X)) \leq \left[d + \prod_{i=1}^{n} \left(c_i + d_i \cdot \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot r^{\frac{p}{q_i}} \right) \right) \right] \mu(X).$$

The above inequality with

$$d + \prod_{i=1}^{n} \left(c_i + d_i \cdot \|K_{0_i}\|_\infty \left(\|e_i\|_{q_i} + l_i \cdot r^{\frac{p}{q_i}} \right) \right) < \frac{1}{2}$$

and the properties of H on Q_r allow us to apply Corollary 3.8. This fulfills the proof. □

5. Example

Finally, we illustrate an example to show the applicability of our results.

Example 5.1 *Consider the following product of integral equations in* $L_2(\mathbb{R}^+)$

$$\begin{aligned} x(t) = & \frac{1}{50} e^{\frac{-t}{2}} + \frac{1}{50} \cdot \frac{|x(t)|}{1 + x^2(t)} \qquad (5.1) \\ + & \left[\left(\frac{1}{50} e^{\frac{-t}{4}} + \frac{1}{50} \frac{x^{\frac{2}{4}}(t)}{1 + x^2(t)} \right) + \left(\frac{e^{\frac{-t}{8}}}{50} + \frac{1}{50} \frac{x^{\frac{2}{8}}(t)}{1 + x^2(t)} \right) \left(|x(t)|^{\frac{2}{8}} \int_0^\infty e^{\frac{-7(s+t)}{8}} \left(\frac{1}{50(1+t^2)^{\frac{1}{8}}} + \frac{1}{50} \frac{x^{\frac{2}{8}}(t)}{1 + x^2(t)} \right) \right) \right] \\ \times & \left[\left(\frac{1}{50} e^{\frac{-t}{8}} + \frac{1}{50} \frac{x^{\frac{2}{8}}(t)}{1 + x^2(t)} \right) + \left(\frac{1}{50} e^{\frac{-t}{16}} + \frac{1}{50} \frac{x^{\frac{2}{16}}(t)}{1 + x^2(t)} \right) \left(|x(t)|^{\frac{2}{16}} \int_0^\infty e^{\frac{-15(s+t)}{16}} \left(\frac{1}{50(1+t^2)^{\frac{1}{16}}} + \frac{1}{50} \frac{x^{\frac{2}{16}}(t)}{1 + x^2(t)} \right) \right) \right]^2. \end{aligned}$$

Let $p_1 = 4,\ q_1 = q_1' = 8,\ q_1'' = \frac{8}{7}$ *and* $p_2 = p_3 = 8$ *with* $q_2 = q_3 = q_2' = q_3' = 16,\ q_2'' = q_3'' = \frac{16}{15}$*, then we have*

1. $|h(t,x)| \leq \frac{1}{50} e^{\frac{-t}{2}} + \frac{1}{50} \cdot |x|$ *with* $d = \frac{1}{50}$ *and* $\|b\|_2 = \|\frac{1}{50} e^{\frac{-t}{2}}\|_2 = \frac{1}{50}$.
2. $|g_1(t,x)| \leq \frac{1}{50} e^{\frac{-t}{4}} + \frac{1}{50} |x|^{\frac{2}{4}}$, $|g_2(t,x)| = |g_3(t,x)| \leq \frac{1}{50} e^{\frac{-t}{8}} + \frac{1}{50} |x|^{\frac{2}{8}}$ *with* $c_1 = c_2 = c_3 = \frac{1}{50}$ *and* $\|a_1\|_4 = \|\frac{1}{50} e^{\frac{-t}{4}}\|_4 = \frac{1}{50}$, $\|a_2\|_8 = \|a_3\|_8 = \|\frac{1}{50} e^{\frac{-t}{8}}\|_8 = \frac{1}{50}$.

3. $|f_1(t,x)| \leq \frac{1}{50}e^{\frac{-t}{8}} + \frac{1}{50}|x|^{\frac{2}{8}}$, $|f_2(t,x)| = |f_3(t,x)| \leq \frac{1}{50}e^{\frac{-t}{16}} + \frac{1}{50}|x|^{\frac{2}{16}}$ *with* $d_1 = d_2 = d_3 = \frac{1}{50}$ *and* $\|b_1\|_8 = \|\frac{1}{50}e^{\frac{-t}{8}}\|_8 = \frac{1}{50}$, $\|b_2\|_{16} = \|b_3\|_{16} = \|\frac{1}{50}e^{\frac{-t}{16}}\|_{16} = \frac{1}{50}$.

4. $|u_1(t,x)| \leq \frac{1}{50(1+t^2)^{\frac{1}{8}}} + \frac{1}{50}|x|^{\frac{2}{8}}$, $|u_2(t,x)| = |u_3(t,x)| \leq \frac{1}{50(1+t^2)^{\frac{1}{16}}} + \frac{1}{50}|x|^{\frac{2}{16}}$ *with* $l_1 = l_2 = l_3 = \frac{1}{50}$ *and* $\|e_1\|_8 = \left\|\frac{1}{50(1+t^2)^{\frac{1}{8}}}\right\|_8 = \frac{1}{50}\sqrt[8]{\frac{\pi}{2}}$, $\|e_2\|_{16} = \|e_3\|_{16} = \left\|\frac{1}{50(1+t^2)^{\frac{1}{16}}}\right\|_{16} = \frac{1}{50}\sqrt[16]{\frac{\pi}{2}}$.

5. $K_1 = e^{\frac{-7(s+t)}{8}}$, $K_2 = K_3 = e^{\frac{-15(s+t)}{16}}$ *with* $\left\|K_{0_1}\right\|_\infty \leq 1$, $\left\|K_{0_2}\right\|_\infty \leq 1$, $\left\|K_{0_3}\right\|_\infty \leq 1$.

6. Let $r = 1$ *and note*

$$d + \prod_{i=1}^{3}\left(c_i + d_i\|K_{0_i}\|_\infty\left(\|e_i\|_{q_i} + l_i\right)\right)$$
$$\leq \frac{1}{50}\left[1 + \frac{1}{2500}\left(1 + \frac{1}{50}(\sqrt[8]{\frac{\pi}{2}} + 1)\right)\left(1 + \frac{1}{50}(\sqrt[16]{\frac{\pi}{2}} + 1)\right)^2\right] < \frac{1}{2}.$$

Also note

$$\|b\|_2 + d + \prod_{i=1}^{3}\left(\|a_i\|_{p_i} + c_i + \|K_{0_i}\|_\infty\left(\|b_i\|_{q_i'} + d_i\right)\left(\|e_i\|_{q_i} + l_i\right)\right)$$
$$\leq \frac{2}{50}\left[1 + \frac{1}{625}\left(1 + \frac{1}{50}(\sqrt[8]{\frac{\pi}{2}} + 1)\right)\left(1 + \frac{1}{50}(\sqrt[16]{\frac{\pi}{2}} + 1)\right)^2\right] \leq 1,$$

so assumption (v) holds.

Hence, Theorem 4.3 implies that (5.1) has a solution $x \in L_2$ *which is a.e. nonincreasing on* $\mathbb{R}^+$.

References

[1] Aghajani A, O'Regan D, Shole Haghighi A. Measure of noncompactness on $L^p(\mathbb{R}^N)$ and applications. CUBO A Mathematical Journal 2015; 17 (1): 85-97. https://doi.org/10.4067/S0719-06462015000100007

[2] Appell J, Zabreiko PP. Nonlinear Superposition Operators. Cambridge University Press, Cambridge, 1990.

[3] Awad H, Darwish M, Metwali M. On a cubic integral equation of Urysohn type with linear perturbation of second kind. Journal of Mathematics and Applications 2018; 41: 29-38.

[4] Banaś J. On the superposition operator and integrable solutions of some functional equations. Nonlinear Analysis. Theory, Methods & Applications 1988; 12 (8): 777–784. https://doi.org/10.1016/0362-546X(88)90038-7

[5] Banaś J, Dhage BC. Global asymptotic stability of solutions of a functional integral equation. Nonlinear Analysis 2008; 69 (7): 1945-1952. https://doi.org/10.1016/j.na.2007.07.038

[6] Banaś J, Olszowy L. On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations. Zeitschrift für Analysis und ihre Anwendungen 2009; 28 (4): 475-498. https://doi.org/10.4171/ZAA/1394

[7] Banaś J, Rzepka B. An application of a measure of noncompactness in the study of asymptotic stability. Applied Mathematics Letters 2003; 16 (1): 1-6. https://doi.org/10.1016/S0893-9659(02)00136-2

[8] Banaś J, Knap Z. Measures of weak noncompactness and nonlinear integral equations of convolution type. Journal of Mathematical Analysis and Applications 1990; 146: 353-362. https://doi.org/10.1016/0022-247X(90)90307-2

[9] Boulfoul B, Bellour A, Djebali S. Solvability of nonlinear integral equations of product type. Electronic Journal of Differential Equations 2018; 19: 1-20.

[10] Boulfoul B, Djebali S. A measure of weak noncompactness in $L^1(\mathbb{R}^N)$ and applications. Mediterranean Journal of Mathematics 2022; 19 (64). https://doi.org/10.1007/s00009-022-01991-x

[11] Brestovanska E. Qualitative behavior of an integral equation related to some epidemic model. Demonstratio Mathematica 2003; 36 (3): 603-609. https://doi.org/10.1515/dema-2003-0312

[12] Brezis H. Analyse Fonctionnelle Théorie et Application. Masson, Paris, 1983 (in French).

[13] Cichoń K, Cichoń M, Metwali M. On some parameters in the space of regulated functions and their applications. Carpathian Journal of Mathematics 2018; 34 (1): 17-30.

[14] Cichoń M, Metwali M. On a fixed point theorem for the product of operators. Journal Fixed Point Theory Applications 2016; 18: 753-770. https://doi.org/10.1007/s11784016-0319-7.

[15] Cichoń M, Metwali M. On monotonic integrable solutions for quadratic functional integral equations. Mediterranean Journal of Mathematics 2013; 10: 909-926. https://doi.org/10.1007/s00009-012-0218-0.

[16] Das A, Hazarika B, Parvaneh V, Mursaleen M. Solvability of generalized fractional order integral equations via measures of noncompactness. Mathematical Sciences 2021; 15: 241-251. https://doi.org/10.1007/s40096-020-00359-0

[17] Darbo G. Punti uniti in trasformazioni a codominio non compatto. Rendiconti del Seminario Matematico della Universitá di Padova 1955; 24: 84-92.

[18] Dudek S, Olszowy L. Measures of noncompactness in the space of regulated functions on an unbounded interval. Annals of Functional Analysis 2022; 13 (63): 1-13. https://doi.org/10.1007/s43034-022-00206-4

[19] Erzakova N. Compactness in measure and measure of noncompactness. Siberian Mathematical Journal 1997; 38 (5): 926-928. https://doi.org/10.1007/BF02673034

[20] Erzakova N. On measures of non-compactness in regular spaces. Zeitschrift für Analysis und ihre Anwendungen 1996; 15: 299-307.

[21] Gripenberg G. On some epidemic models. Quarterly of Applied Mathematics 1981; 39 (3): 317-327.

[22] Krzyż J. On monotonicity-preserving transformations. Annales Universitatis Mariae Curie-Skłodowska 1952; 6: 91-111.

[23] Darwish M, Metwali M, O'Regan D. Unique solvability of fractional quadratic nonlinear integral equations. Differential Equations and Applications 2021; 13: 1-13. https://doi.org/10.7153/dea-2021-13-01

[24] Khanehgir M, Allahyari R, Gholamian N. Construction of a measure of noncompactness in Sobolev spaces with an application to functional integral-differential equations. Mathematical Sciences 2018; 12: 17-24. https://doi.org/10.1007/s40096-017-0240-2

[25] Kuratowski K. Sur les espaces complets. Fundamenta Mathematicae 1930; 15: 301-309 (in French).

[26] Mehravaran H, Amiri Kayvanloo H, Allahyari R. Measures of noncompactness in the space of regulated functions $R(J, R^\infty)$ and its application to some nonlinear infinite systems of fractional differential equations. Mathematical Sciences (2022). https://doi.org/10.1007/s40096-022-00464-2

[27] Metwali M. On a class of quadratic Urysohn-Hammerstein integral equations of mixed type and initial value problem of fractional order. Mediterranean Journal of Mathematics 2016; 13: 2691-2707. https://doi.org/10.1007/s00009-015-0647-7

[28] Metwali M. Nonlinear quadratic Volterra-Urysohn functional-integral equations in Orlicz spaces. Filomat 2021; 35 (9): 2963-2972.

[29] Metwali M. On perturbed quadratic integral equations and initial value problem with nonlocal conditions in Orlicz spaces. Demonstratio Mathematica 2020; 53: 86-94. https://doi.org/10.1515/DEMA-2020-0052

[30] Metwali M. Solvability of Gripenberg's equations of fractional order with perturbation term in weighted L_p-spaces on $\mathbb{R}^+$. Turkish Journal of Mathematics 2022; 46: 481-498. https://doi.org/10.3906/mat-2106-84

[31] Olaru IM. Generalization of an integral equation related to some epidemic models. Carpathian Journal of Mathematics 2010; 26 (1): 92-96.

[32] Olszowy L. A family of measures of noncompactness in the space $L^1_{loc}(\mathbb{R}^+)$ and its application to some nonlinear Volterra integral equation. Mediterranean Journal of Mathematics 2014; 11: 687-701. https://doi.org/10.1007/S00009-013-0375-9

[33] Väth M. Continuity of single and multivalued superposition operators in generalized ideal spaces of measurable functions. Nonlinear functional analysis and applications 2006; 11: 607-646.

[34] Väth M. Volterra and Integral Equations of Vector Functions. Marcel Dekker, New York-Basel, 2000.