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Inverse nodal problem for the quadratic pencil of the Sturm−Liouville equations with parameter-dependent nonlocal boundary condition

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Abstract: In this paper, we consider the inverse nodal problem for a quadratic pencil of the Sturm−Liouville equations with parameter-dependent Bitsadze−Samarskii type nonlocal boundary condition and we give an algorithm for the reconstruction of the potential functions by obtaining the asymptotics of the nodal points.

Key words: Quadratic pencil of the Sturm−Liouville equations, inverse nodal problem, nonlocal boundary condition

1. Introduction

We consider the boundary value problem generated by the differential equation

$$\ell y := -y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, \quad 0 < x < 1 \tag{1.1}$$

with the boundary conditions

$$U(y) := y(0) = 0, \quad V(y) := \lambda y(1) - y(\alpha) = 0, \tag{1.2}$$

where λ is the spectral parameter, α is rational number in $(0,1)$, the potential functions $q(x) \in W_1^1[0,1]$, $p(x) \in W_1^2[0,1]$ are real valued functions such that $p(x) \neq const.$ for $\beta = \alpha, 1$

$$\int_0^\beta p(x)dx = 0 \tag{1.3}$$

holds.

In what follows we denote the boundary problem $(1.1)-(1.2)$ by $L = L(p(x), q(x), \alpha)$.

In the present paper, we construct the functions $p(x)$ and $q(x)$ which are the potentials of operator L from nodal points of its eigenfunctions and give an algorithm for solving the inverse nodal problem.

Inverse nodal problem consists in reconstructing the operator from a given dense set of zeros of its eigenfunctions. McLaughlin gave firstly a solution for the inverse nodal problem for the specific Sturm−Liouville operator. She sought to recover the potential $q(x)$ by using the nodes (i.e. zeros) of the eigenfunctions (see [20]). Hald and McLaughlin showed that coefficients in a second-order differential equation can be uniquely determined from the positions of the nodes for the eigenfunctions. They proved unique results, derived approximate

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solutions, gave error bounds, and presented numerical experiments (see [12]). Yang gave an algorithm to reconstruct the potential and the boundary condition of the Sturm−Liouville problem from nodal points of its eigenfunctions (see [36]). Inverse nodal problems have been extensively studied by many researchers for various operators (see [3], [4], [5], [7], [8], [13], [15], [17], [18], [19], [28], [29], [30], [34], [35] and references therein).

There are two kinds of nonlocal boundary conditions such as integral type conditions and Bitsadze and Samarskii type conditions. These types of nonlocal boundary conditions appear in various fields such as mathematical physics, biology, biotechnology, and when data cannot be measured directly at the boundary (see [9], [11], [21], [27], [37] and references therein). Nonlocal boundary conditions were first applied to elliptic equations by Bitsadze and Samarskii (see [2]).

In recent years, some inverse problems for various types of operators with nonlocal boundary conditions have been investigated (see [1], [22], [23], [24], [33] and references therein). In addition, studies on inverse nodal problems with nonlocal boundary conditions can also be seen in [14], [16], [25], [26], [31], [32]. As far as we know, inverse nodal problem for quadratic pencil of the Sturm−Liouville equations with parameter-dependent Bitsadze−Samarskii-type nonlocal boundary condition has not been considered before.

2. Preliminaries

Let the functions $C = C(x,\lambda)$ and $S = S(x,\lambda)$ be the solutions of the equation (1.1) satisfying the initial conditions

$$C(0,\lambda) = 1,\ C'(0,\lambda) = 0 \text{ and } S(0,\lambda) = 0,\ S'(0,\lambda) = 1 \tag{2.1}$$

respectively.

From [4] and [6], the functions $C(x,\lambda)$ and $S(x,\lambda)$ satisfy the following asymptotic representations for $|\lambda| \to \infty$

$$C(x,\lambda) = \cos(\lambda x - Q(x)) + O\left(\frac{1}{\lambda}\exp|\mathrm{Im}\lambda|\,x\right), \tag{2.2}$$

$$\begin{aligned} S(x,\lambda) &= \frac{1}{\lambda}\sin(\lambda x - Q(x)) + \frac{1}{2\lambda^2}\left\{(p(x)+p(0))\sin(\lambda x - Q(x)) - c_1(x)\cos(\lambda x - Q(x))\right. \\ &+ \int_0^x \left(q(t)+p^2(t)\right)\cos\left[\lambda(x-2t) - Q(x) + 2Q(t)\right]dt \\ &\left.+ \int_0^x p'(t)\sin\left[\lambda(x-2t) - Q(x) + 2Q(t)\right]dt\right\} \\ &+ \frac{1}{4\lambda^3}\left\{c_3(x)\sin(\lambda x - Q(x)) - c_4(x)\cos(\lambda x - Q(x))\right\} + O\left(\frac{1}{\lambda^4}\exp|\mathrm{Im}\lambda|\,x\right) \end{aligned} \tag{2.3}$$

where $Q(x) = \int\limits_0^x p(t)dt,\ c_1(x) = \int\limits_0^x \left(q(t)+p^2(t)\right)dt,\ c_2(x) = \int\limits_0^x \left(q(t)+p^2(t)\right)p(t)\,dt,$

$$c_3(x) = p^2(x) + p^2(0) + \frac{(p(x)+p(0))^2}{2} - \frac{1}{2}\left(\int\limits_0^x \left(q(t)+p^2(t)\right)dt\right)^2,$$

$$c_4(x) = \int\limits_0^x \left(q(t)+p^2(t)\right)(p(x)+p(0)+2p(t))\,dt = (p(x)+p(0))\,c_1(x) + 2c_2(x).$$

The eigenvalues of the problem L coincide with the zeros of its characteristic function given by

$$\Delta(\lambda) = \begin{vmatrix} U(C) & U(S) \\ V(C) & V(S) \end{vmatrix} = \lambda S(1,\lambda) - S(\alpha,\lambda). \tag{2.4}$$

Thus, using the formulae (1.3), (2.2), (2.3), and (2.4), we obtain the following asymptotic formula for $\Delta(\lambda)$

$$\begin{aligned}
\Delta(\lambda) &= \sin\lambda + \frac{1}{2\lambda}\{(p(1)+p(0))\sin\lambda - c_1(1)\cos\lambda - 2\sin\lambda\alpha \\
&+ \int_0^1 \left(q(t)+p^2(t)\right)\cos\left[\lambda(1-2t)+2Q(t)\right]dt + \int_0^1 p'(t)\sin\left[\lambda(1-2t)+2Q(t)\right]dt\Big\} \\
&+ \frac{1}{4\lambda^2}\{c_3(1)\sin\lambda - c_4(1)\cos\lambda - 2(p(\alpha)+p(0))\sin\lambda\alpha + 2c_1(\alpha)\cos\lambda\alpha\} \\
&+ O\left(\frac{1}{\lambda^3}\exp|\mathrm{Im}\lambda|\right), \quad |\lambda| \to \infty.
\end{aligned} \tag{2.5}$$

By the method in [10], using (2.5) and Rouchè theorem and taking $\Delta(\lambda_n) = 0$ we can prove that the eigenvalues λ_n have the form

$$\begin{aligned}
\lambda_n = \quad & n\pi + \frac{c_1(1) - A_n^n + 2(-1)^n \sin n\alpha\pi}{2n\pi} \\
& + \frac{(p(1)+p(0))c_1(1) + 2c_2(1) + 2(-1)^n(p(\alpha)+p(0))\sin n\alpha\pi - 2(-1)^n c_1(\alpha)\cos n\alpha\pi}{4n^2\pi^2} \\
& + o\left(\frac{1}{n^2}\right), \quad |n| \to \infty,
\end{aligned} \tag{2.6}$$

where, for $n \in \mathbb{Z}\setminus\{0\}$, $x_n^0 = 0$, $x_n^n = 1$, $j \in \mathbb{Z}$,

$$A_n^j = \int_0^{x_n^j} \left(q(t)+p^2(t)\right)\cos(2n\pi t - 2Q(t))\,dt - \int_0^{x_n^j} p'(t)\sin(2n\pi t - 2Q(t))\,dt.$$

3. Main results

In this section, under condition (2.1) we obtain the asymptotics for the zeros of the function $\varphi(x,\lambda_n)$ called the nodal points of the problem L and develop a constructive procedure for solving the inverse nodal problem.

It is clear from (2.6) that for sufficiently large $|n|$, there is exactly one eigenvalue λ_n in the domain $\Gamma_n = \{\lambda \mid \ |\lambda - n\pi| \le 1\}$ and since the functions $p(x)$ and $q(x)$ are real-valued, λ_n are real. Thus, the functions $\varphi(x,\lambda_n)$ are real-valued and

$$\varphi(x,\lambda_n) = U(C(x,\lambda_n))\,S(x,\lambda_n) - U(S(x,\lambda_n))\,C(x,\lambda_n) = S(x,\lambda_n) \tag{3.1}$$

are the eigenfunctions corresponding to the eigenvalues λ_n for sufficiently large $|n|$.

From (3.1), (2.3), and (2.6), we get

$$\begin{aligned}
&\lambda_n \varphi(x,\lambda_n) = \sin(n\pi x - Q(x)) + \frac{1}{2n\pi}\{[(c_1(1) - A_n^n + 2(-1)^n \sin n\alpha\pi)x - c_1(x)]\cos(n\pi x - Q(x)) \\
&+ (p(x)+p(0))\sin(n\pi x - Q(x)) + \int_0^x \left(q(t)+p^2(t)\right)\cos[n\pi(x-2t) - Q(x) + 2Q(t)]\,dt \\
&+ \left.\int_0^x p'(t)\sin[n\pi(x-2t) - Q(x) + 2Q(t)]\,dt\right\} + \frac{1}{4n^2\pi^2}\{[((p(1)+p(0))c_1(1) + 2c_2(1) \\
&+2(-1)^n(p(\alpha)+p(0))\sin n\alpha\pi - 2(-1)^n c_1(\alpha)\cos n\alpha\pi)x \qquad (3.2)\\
&+ (p(x)+p(0))(c_1(1) + 2(-1)^n \sin n\alpha\pi)x - c_4(x)]\cos(n\pi x - Q(x)) \\
&+ \left.\left[c_1(x)(c_1(1)+2(-1)^n \sin n\alpha\pi)x - (c_1(1)+2(-1)^n\sin n\alpha\pi)^2 x^2 + c_3(x)\right]\sin(n\pi x - Q(x))\right\} \\
&+o\left(\frac{1}{n^2}\right),\ \ |n|\to\infty,
\end{aligned}$$

uniformly in $x\in[0,1]$.

We can see from (3.2) that for sufficiently large $|n|$ and $j\in\mathbb{Z}$, the eigenfunctions $\varphi(x,\lambda_n)$ have exactly $|n|-1$ nodal points x_n^j in $(0,1)$ as

$$0 < x_n^1 < x_n^2 < ... < x_n^{n-1} < 1 \quad \text{for } n>0$$

and

$$0 < x_n^{-1} < x_n^{-2} < ... < x_n^{n+1} < 1 \quad \text{for } n<0.$$

Lemma 3.1 *The numbers x_n^j satisfy the following asymptotic formula for sufficiently large $|n|$:*

$$\begin{aligned}
x_n^j &= \frac{j}{n} + \frac{Q(x_n^j)}{n\pi} + \frac{1}{2n^2\pi^2}\left[c_1\left(x_n^j\right) - c_1(1)x_n^j - \left(A_n^j - A_n^n x_n^j\right) - 2(-1)^n x_n^j \sin n\alpha\pi\right] \\
&+\frac{1}{2n^3\pi^3}\left[c_2\left(x_n^j\right) - \left(c_2(1) + \frac{(p(1)+p(0))c_1(1)}{2}\right)x_n^j + (-1)^n(p(\alpha)+p(0))x_n^j \sin n\alpha\pi \right. \qquad (3.3)\\
&\left. -(-1)^n c_1(\alpha)x_n^j\cos n\alpha\pi\right] + o\left(\frac{1}{n^3}\right),
\end{aligned}$$

uniformly with respect to j.

Proof From (3.2), taking $\varphi\left(x_n^j,\lambda_n\right)=0$, we get

$$\begin{aligned}
&\sin\left(n\pi x_n^j - Q\left(x_n^j\right)\right) + \frac{1}{2n\pi}\left\{\left[(c_1(1) - A_n^n + 2(-1)^n\sin n\alpha\pi)x_n^j - c_1\left(x_n^j\right)\right]\cos\left(n\pi x_n^j - Q\left(x_n^j\right)\right)\right. \\
&+ \left(p\left(x_n^j\right)+p(0)\right)\sin\left(n\pi x_n^j - Q\left(x_n^j\right)\right) + \textstyle\int_0^{x_n^j}\left(q(t)+p^2(t)\right)\cos\left[n\pi\left(x_n^j - 2t\right) - Q\left(x_n^j\right) + 2Q(t)\right]dt \\
&+ \left.\textstyle\int_0^{x_n^j} p'(t)\sin\left[n\pi\left(x_n^j-2t\right) - Q\left(x_n^j\right) + 2Q(t)\right]dt\right\} + \frac{1}{4n^2\pi^2}\{[((p(1)+p(0))c_1(1) + 2c_2(1) \\
&+2(-1)^n(p(\alpha)+p(0))\sin n\alpha\pi - 2(-1)^n c_1(\alpha)\cos n\alpha\pi)x_n^j
\end{aligned}$$

$$+\left(p\left(x_n^j\right)+p(0)\right)\left(c_1\left(1\right)+2\left(-1\right)^n\sin n\alpha\pi\right)x_n^j-c_4\left(x_n^j\right)\right]\cos\left(n\pi x_n^j-Q\left(x_n^j\right)\right)$$

$$+\left[c_1\left(x_n^j\right)\left(c_1\left(1\right)+2\left(-1\right)^n\sin n\alpha\pi\right)x_n^j-\left(c_1\left(1\right)+2\left(-1\right)^n\sin n\alpha\pi\right)^2\left(x_n^j\right)^2+c_3\left(x_n^j\right)\right]\sin\left(n\pi x_n^j-Q\left(x_n^j\right)\right)\Big\}$$

$$+o\left(\frac{1}{n^2}\right)=0,\ |n|\to\infty.$$

This implies

$$\tan\left(n\pi x_n^j-Q\left(x_n^j\right)\right)=\frac{1}{2n\pi}\left[c_1\left(x_n^j\right)-c_1\left(1\right)x_n^j-\left(A_n^j-A_n^n x_n^j\right)-2\left(-1\right)^n x_n^j\sin n\alpha\pi\right]$$

$$+\frac{1}{2n^2\pi^2}\left[c_2\left(x_n^j\right)-\left(c_2\left(1\right)+\frac{\left(p(1)+p(0)\right)c_1\left(1\right)}{2}\right)x_n^j+\left(-1\right)^n\left(p(\alpha)+p(0)\right)x_n^j\sin n\alpha\pi\right.$$

$$\left.-\left(-1\right)^n c_1\left(\alpha\right)x_n^j\cos n\alpha\pi\right]+o\left(\frac{1}{n^2}\right),\ |n|\to\infty.$$

Using Taylor's expansion formula for the arctangent, we get

$$n\pi x_n^j-Q\left(x_n^j\right)=j\pi+\frac{1}{2n\pi}\left[c_1\left(x_n^j\right)-c_1\left(1\right)x_n^j-\left(A_n^j-A_n^n x_n^j\right)-2\left(-1\right)^n x_n^j\sin n\alpha\pi\right]$$

$$+\frac{1}{2n^2\pi^2}\left[c_2\left(x_n^j\right)-\left(c_2\left(1\right)+\frac{\left(p(1)+p(0)\right)c_1\left(1\right)}{2}\right)x_n^j+\left(-1\right)^n\left(p(\alpha)+p(0)\right)x_n^j\sin n\alpha\pi\right.$$

$$\left.-\left(-1\right)^n c_1\left(\alpha\right)x_n^j\cos n\alpha\pi\right]+o\left(\frac{1}{n^2}\right),\ |n|\to\infty.$$

Therefore, the proof is concluded by the last equality. □

Let X be the set of nodal points and $\alpha=\frac{k}{\ell},\ k,\ell\in\mathbb{Z}$. It is obvious from (3.3) that the set X of all nodal points is dense in the interval $[0,1]$. We can choose a sequence $\{j_n\}\subset X$ so that $\lim\limits_{|n|\to\infty}x_n^{j_n}=x$. Clearly the subsequence $\left\{x_m^{j_m}\right\}$ converges also to x for $m=2n\ell$. Then, there exist finite limits and corresponding equalities hold:

$$\pi\lim_{|m|\to\infty}\left(mx_m^{j_m}-j_m\right):=Q(x), \tag{3.4}$$

$$2\pi\lim_{|m|\to\infty}m\left[\pi\left(mx_m^{j_m}-j_m\right)-Q\left(x_m^{j_m}\right)\right]:=f(x), \tag{3.5}$$

$$\pi\lim_{|m|\to\infty}m\left\{2m\pi\left[\pi\left(mx_m^{j_m}-j_m\right)-Q\left(x_m^{j_m}\right)\right]-f\left(x_m^{j_m}\right)\right. \tag{3.6}$$

$$\left.+A_m^{j_m}-A_m^m x_m^{j_m}+2\left(-1\right)^m x_m^{j_m}\sin m\alpha\pi\right]:=g(x)$$

and

$$f(x)=c_1\left(x\right)-c_1\left(1\right)x, \tag{3.7}$$

$$g(x)=c_2\left(x\right)-\left(c_2\left(1\right)-\frac{\left(p(1)+p(0)\right)c_1\left(1\right)}{2}\right)x-c_1\left(\alpha\right)x. \tag{3.8}$$

Thus, the following theorem for the solution of the inverse nodal problem can be proved.

Theorem 3.2 *Given the specification of any dense subset of nodal points* $X_0 \subset X$ *uniquely determines the functions* $p(x)$ *and* $q(x)$ *which can be found by the following algorithm.*

Step 1. *Denote* $m = 2n\ell$ *and for each fixed* $x \in [0,1]$, *choose a sequence* $\left(x_m^{j_m}\right) \subset X_0$ *such that* $\lim\limits_{|m|\to\infty} x_m^{j_m} = x$,

Step 2. *Find the function* $Q(x)$ *via (3.4) and calculate*

$$p(x) = Q'(x), \tag{3.9}$$

Step 3. *Find the function* $f(x)$ *via (3.5) and determine*

$$r(x) := q(x) - \int_0^1 q(t)dt = f'(x) - p^2(x) + \int_0^1 p^2(t)dt, \tag{3.10}$$

Step 4. *For each fixed* $x \in [0,1]$ *and* $2Q(x) - (p(1)+p(0))\,x - 2\alpha x \neq 0$, *find* $g(x)$ *via (3.6) and calculate*

$$\int_0^1 q(t)dt = \frac{2}{2Q(x) - (p(1)+p(0))\,x - 2\alpha x}\left[g(x) - \int_0^x \left(r(t) + p^2(t)\right)p(t)dt \right. \tag{3.11}$$

$$\left. + x\int_0^1 \left(r(t)+p^2(t)\right)p(t)dt + \frac{(p(1)+p(0))\,x}{2}\int_0^1 \left(r(t)+p^2(t)\right)dt\right],$$

Step 5. *Calculate the function* $q(x)$ *from the formula*

$$q(x) = r(x) + \int_0^1 q(t)dt. \tag{3.12}$$

Proof It is clear from the formula $Q(x) = \int\limits_0^x p(t)dt$ that the formula (3.9) is provided. If we differentiate (3.7), we get $f'(x) = q(x) + p^2(x) - \int_0^1 \left(q(t) + p^2(t)\right)dt$. If we denote $r(x) := q(x) - \int_0^1 q(t)dt$, we obtain immediately the formula (3.10). Substituting the function $q(x) = r(x) - \frac{1}{\pi}\int_0^\pi q(t)dt$ in (3.8) and taking (1.3) into account, we get the formula (3.11). Finally, from (3.10) and (3.11), we arrive at the formula (3.12). ◻

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References

[1] Albeverio S, Hryniv RO, Nizhnik LP. Inverse spectral problems for non-local Sturm−Liouville operators. Inverse problems 2007; 23 (2): 523-535. https://doi.org/10.1088/0266-5611/23/2/005

[2] Bitsadze AV, Samarskii AA. Some elementary generalizations of linear elliptic boundary value problems. Doklady Akademii Nauk SSSR 1969; 185 (4): 739-740.

[3] Browne PJ, Sleeman BD. Inverse nodal problems for Sturm−Liouville equations with eigenparameter dependent boundary conditions. Inverse Problems 1996; 12 (4): 377-381. https://doi.org/10.1088/0266-5611/12/4/002

[4] Buterin SA, Shieh CT. Inverse nodal problem for differential pencils. Applied Mathematics Letters 2009; 22 (8): 1240-1247. https://doi.org/10.1016/j.aml.2009.01.037

[5] Buterin SA, Shieh CT. Incomplete inverse spectral and nodal problems for differential pencils. Results in Mathematics 2012; 62: 167-179. https://doi.org/10.1007/s00025-011-0137-6

[6] Buterin SA. On half inverse problem for differential pencils with the spectral parameter in the boundary conditions. Tamkang Journal of Mathematics 2011; 42 (3): 355-364. https://doi.org/10.5556/j.tkjm.42.2011.355-364

[7] Chen X, Cheng YH, Law CK. Reconstructing potentials from zeros of one eigenfunction. Transactions of the American Mathematical Society 2011; 363 (9): 4831-4851.

[8] Çakmak Y. Inverse nodal problem for a conformable fractional diffusion operator. Inverse Problems in Science and Engineering 2021; 29 (9): 1308-1322. https://doi.org/10.1080/17415977.2020.1847103

[9] Day WA. Extensions of a property of the heat equation to linear thermoelasticity and order theories. Quarterly of Applied Mathematics 1982; 40 (3): 319-330.

[10] Freiling G, Yurko VA. Inverse Sturm – Liouville problems and their applications. New York: Nova Science Pub Inc, 2001.

[11] Gordeziani N. On some non-local problems of the theory of elasticity. Bulletin of TICMI 2000; 4: 43-46.

[12] Hald OH, McLaughlin JR. Solutions of inverse nodal problems. Inverse Problems 1989; 5 (3): 307-347. https://doi.org/10.1088/0266-5611/5/3/008

[13] Hald OH, McLaughlin JR. Inverse problems: recovery of BV coefficients from nodes. Inverse Problems 1998; 14 (2): 245-273. https://doi.org/10.1088/0266-5611/14/2/003

[14] Hu YT, Yang CF, Xu XC. Inverse nodal problems for the Sturm – Liouville operator with nonlocal integral conditions. Journal of Inverse and Ill-Posed Problems 2017; 25 (6): 799-806. https://doi.org/10.1515/jiip-2017-0017

[15] Keskin B, Ozkan AS. Inverse nodal problems for impulsive Sturm – Liouville equation with boundary conditions depending on the parameter. Advances in Analysis 2017; 2 (3): 151-156. https://doi.org/10.22606/aan.2017.23002

[16] Keskin B. Inverse nodal problems for Dirac type integro differential system with a nonlocal boundary condition. Turkish Journal of Mathematics 2022; 46 (6): 2430-2439. https://doi.org/10.55730/1300-0098.3278

[17] Koyunbakan H. A new inverse problem for the diffusion operator. Applied Mathematics Letters 2006; 19 (10): 995-999. https://doi.org/10.1016/j.aml.2005.09.014

[18] Law CK, Yang CF. Reconstructing the potential function and its derivatives using nodal data. Inverse Problems 1998; 14 (2): 299-312. https://doi.org/10.1088/0266-5611/14/2/006

[19] Law CK, Shen CL, Yang CF. The inverse nodal problem on the smoothness of the potential function. Inverse Problems 1999; 15 (1): 253-263. Errata: Inverse Problems. 2001; 17 (2): 361-363. https://doi.org/10.1088/0266-5611/15/1/024

[20] McLaughlin JR. Inverse spectral theory using nodal points as data-A uniqueness result. Journal of Differential Equations 1988; 73 (2): 354-362. https://doi.org/10.1016/0022-0396(88)90111-8

[21] Nakhushev AM. Equations of Mathematical Biology. Moscow: Vysshaya Shkola, 1995 (in Russian)

[22] Nizhnik LP. Inverse eigenvalue problems for nonlocal Sturm – Liouville operators. Methods of Functional Analysis and Topology 2009; 15 (1): 41-47.

[23] Nizhnik LP. Inverse nonlocal Sturm – Liouville problem. Inverse Problems 2010; 26 (12): 125006. https://doi.org/10.1088/0266-5611/26/12/125006

[24] Nizhnik LP. Inverse eigenvalue problems for nonlocal Sturm – Liouville operators on a star graph. Methods of Functional Analysis and Topology 2012; 18 (1): 68-78.

[25] Ozkan AS, Adalar İ. Inverse nodal problems for Sturm − Liouville equation with nonlocal boundary conditions. Journal of Mathematical Analysis and Applications 2023; 520 (1): 126907.

[26] Qin X, Gao Y, Yang C. Inverse nodal problems for the Sturm − Liouville operator with some nonlocal integral conditions. Journal of Applied Mathematics and Physics 2019; 7 (1): 111-122. https://doi.org/10.4236/jamp.2019.71010

[27] Schügerl K. Bioreaction engineering: Reactions involving microorganisms and cells. Chichester: John Wiley and Sons, 1987. https://doi.org/10.1002/jctb.280420209

[28] Wang YP, Yurko VA. On the inverse nodal problems for discontinuous Sturm − Liouville operators. Journal of Differential Equations 2016; 260 (5): 4086-4109. https://doi.org/10.1016/j.jde.2015.11.004

[29] Wang YP, Shieh CT, Wei X. Partial inverse nodal problems for differential pencils on a star-shaped graph. Mathematical Methods in the Applied Sciences 2020; 43 (15): 8841–8855. https://doi.org/10.1002/mma.6574

[30] Wei X, Miao H, Ge C, Zhao C. An inverse problem for Sturm − Liouville operators with nodal data on arbitrarily-half intervals. Inverse Problems in Science and Engineering 2020; 29 (3): 305–317. https://doi.org/10.1080/17415977.2020.1779711

[31] Xu XJ, Yang CF. Inverse nodal problem for nonlocal differential operators. Tamkang Journal of Mathematics 2019; 50 (3): 337-347. https://doi.org/10.5556/j.tkjm.50.2019.3361

[32] Yang CF. Inverse nodal problem for a class of nonlocal Sturm − Liouville operator. Mathematical Modelling and Analysis 2010; 15 (3): 383-392. https://doi.org/10.3846/1392-6292.2010.15.383-392

[33] Yang CF, Yurko V. Recovering Dirac operator with nonlocal boundary conditions. Journal of Mathematical Analysis and Applications 2016; 440 (1): 155-166. https://doi.org/10.1016/j.jmaa.2016.03.021

[34] Yang CF. Reconstruction of the diffusion operator with nodal data. Zeitschrift für Naturforschung A 2010; 65: 100-106. https://doi.org/10.1515/zna-2010-1-211

[35] Yang CF. An inverse problem for a differential pencil using nodal points as data. Israel Journal of Mathematics 2014; 204: 431-446. https://doi.org/10.1007/s11856-014-1097-9

[36] Yang XF. A solution of the nodal problem. Inverse Problems 1997; 13 (1): 203-213. https://doi.org/10.1088/0266-5611/13/1/016

[37] Yin YF. On nonlinear parabolic equations with nonlocal boundary conditions. Journal of Mathematical Analysis and Applications 1994; 185 (1): 161-174. https://doi.org/10.1006/jmaa.1994.1239