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Some fractional Dirac systems

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Abstract: In this work, including $\alpha\epsilon(0,1)$; we examined the Dirac system in the frame which includes α order right and left Reimann-Liouville fractional integrals and derivatives with exponential kernels, and the Dirac system which includes α order right and left Caputo fractional integrals and derivatives with exponential kernels. Furthermore, we have given some definitions and properties for discrete exponential kernels and their associated fractional sums and fractional differences, and we have studied discrete fractional Dirac systems.

Key words: Exponential kernels, fractional Dirac system, fractional derivatives, discrete fractional Dirac system

1. Introduction

Fractional derivative and fractional integral, which is a subbranch of mathematical analysis, is the extended form of derivative and integral to noninteger orders [10]. In the fractional derivative and integral fields, which are known to have emerged towards the end of the 17th century, many researchers such as Leibnitz, Riemann, Liouville, Weyl, Euler, Lagrange, Fourier, Greenwald, Letnikov, Laplace, Abel, Holmgren, Heaviside, Hadamard, Lacroix, and Caputo have done many studies. Mathematical models created with fractional differential equations have obtained more successful results than classical integer differential equations. Fractional calculus has frequently been used in modeling and applications of problems in the fields of engineering, mathematics, and science in recent years. With the spread of fractional calculations, many scientists have worked in this field [4, 10, 16, 21, 22, 24–26]. With the more widespread use of fractional calculations, continuous and discrete fractional differential equations have also started to be studied in many scientific fields. Many mathematicians have worked on discrete fractional calculation [2, 8, 9, 15]. In recent years, some researchers have defined nonsingular operators to have fractional operators with better-behaving kernels and have used these operators in modeling and solving some of their problems [1, 3, 6, 7, 11, 12]. New fractional derivatives with nonsingular kernels have been used to solve many problems [11, 14, 18, 23]. Nonlocal fractional operators enable the development of more efficient algorithms to solve fractional dynamical systems.

The Dirac equation was found in the first quarter of the 20th century while searching for a relative covariant wave equation of the Schrödinger form, and it has an important place today. When the literature is reviewed, it is seen that there is a need for new studies involving the fractional Dirac system. The authors in [5] examined regular q-fractional Dirac-type system in their studies and investigated the properties of the eigenvalues and eigenfunctions of the system. Using the fixed point theorem, they gave the eigenvalue condition

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for the existence and uniqueness of eigenfunctions. The authors in [20] defined fractional operators with Mittag-Leffler kernels to formulate and investigate fractional Sturm-Liouville problems and investigated the self-adjoint, eigenvalue, and eigenfunction properties of fractional Sturm-Liouville operators. In their work [1], Abdeljawad and Belanau defined integration with the part formula using the right fractional derivative and the right fractional integral corresponding to the exponential kernels and used the q-operator to validate the results. They also formulated the discrete correspondences of the results. The study in [13] dealt with the exponential Dirac system in the sense of Riemann-Liouville and Caputo and the fractional Dirac system with Mittag-Leffler core and obtained the representations of the solutions for Dirac systems by Laplace transforms. Mert et al. [19] worked on fractional derivatives with nonsingular nuclei in their articles. They formulated some fractional Sturm-Liouville problems with differential involving left- and right-sided derivatives by examining the Sturm-Liouville Equations in the framework of fractional operators with Mittag-Leffler kernels.

2. Fractional derivatives with exponential kernels for regular Dirac system

In this section, we have given the definitions and properties of fractional derivatives and fractional integrals with exponential kernels. Furthermore, including $\alpha \in (0,1)$, we examined the Dirac system which includes α order right and left Reimann-Liouville and Caputo fractional integrals and derivatives.

Definition 1 *([11]) Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$. Then the left Caputo fractional derivative with exponential kernel is defined by*

$$ {}_{a}^{CFC}D^{\alpha} f(x) = \frac{B(\alpha)}{1-\alpha} \int_{a}^{x} \exp\left(\frac{-\alpha}{1-\alpha}(x-\rho)\right) f'(\rho) d\rho, \tag{1} $$

where $B(\alpha) > 0$ is a normalization function with $B(0) = B(1) = 1$.

Definition 2 *([3]) Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$. Then, the left Reiman-Liouville fractional derivative with the exponential kernel is defined by*

$$ {}_{a}^{CFR}D^{\alpha} f(x) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dx} \int_{a}^{x} \exp\left(\frac{-\alpha}{1-\alpha}(x-\rho)\right) f(\rho) d\rho, \tag{2} $$

and the associated fractional integral by

$$ {}_{a}^{CF}I^{\alpha} f(x) = \frac{1-\alpha}{B(\alpha)} f(x) + \frac{\alpha}{B(\alpha)} \int_{a}^{x} f(\rho) d\rho. $$

Definition 3 *([3]) Let $f \in H^1(a,b)$, $a < b$, $\alpha \in [0,1]$. Then, the right Caputo fractional derivative with the exponential kernel is defined by*

$$ {}^{CFC}D_{b}^{\alpha} f(x) = -\frac{B(\alpha)}{1-\alpha} \int_{x}^{b} \exp\left(\frac{-\alpha}{1-\alpha}(\rho-x)\right) f'(\rho) d\rho, \tag{3} $$

and the right Reiman-Liouville one by

$$^{CFR}D_b^{\alpha}\ f\left(x\right)=-\frac{B(\alpha)}{1-\alpha}\frac{d}{dx}\int\limits_x^b \exp\left(\frac{-\alpha}{1-\alpha}(\rho-x)\right)f(\rho)d\rho. \tag{4}$$

In addition, the corresponding fractional integral is defined by

$$^{CF}I_b^{\alpha}\ f\left(x\right)=\frac{1-\alpha}{B(\alpha)}f(x)+\frac{\alpha}{B(\alpha)}\int_x^b f(\rho)d\rho.$$

Theorem 2.1 *([3]) (Integration by parts formula for CFR fractional derivatives) Let* $0<\alpha<1,\ p\geqslant 1,\ q\geqslant 1,$ *and* $\frac{1}{p}+\frac{1}{q}\leqslant 1+\alpha$ *(*$p\neq 1$ *and* $q\neq 1$ *in case* $\frac{1}{p}+\frac{1}{q}=1+\alpha$*).*

1. If $\varphi\in L_p(a.b)$ and $\psi\in L_q(a.b)$,

$$\int_a^b \varphi(x)_a^{CF}I^{\alpha}\psi(x)dx=\int_a^b \psi(x)_a^{CF}I^{\alpha}\varphi(x)dx$$

2. If $f\in^{CF} I_b^{\alpha}(L_p)$ and $g\in^{CF} I_b^{\alpha}(L_p)$,

$$\int_a^b f(x)_a^{CFR}D^{\alpha}g(x)dx=\int_a^b g(x)^{CFC}D_b^{\alpha}f(x)dx.$$

Theorem 2.2 *([3]) (İntegration by parts formula for CFC fractional derivatives) Let* $f,g\in H^1$ *and* $0<\alpha<1$*.*

1.

$$\int_a^b f(x)_a^{CFC}D^{\alpha}g(x)dx=\int_a^b g(x)^{CFR}D_b^{\alpha}f(x)dx+\frac{B(\alpha)}{1-\alpha}g(x)e_{\frac{-\alpha}{1-\alpha},b^-}f(x)|_a^b$$

2.

$$\int_a^b g(x)^{CFR}D_b^{\alpha}f(x)dx=\int_a^b f(x)_a^{CFC}D^{\alpha}g(x)dx-\frac{B(\alpha)}{1-\alpha}g(x)e_{\frac{-\alpha}{1-\alpha},a^+}f(x)|_a^b$$

Let $y:=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}$, and $p(x)$ and $r(x)$ are real-valued continuous functions defined on $[a,b]$

$$\begin{aligned}\digamma y&=\begin{pmatrix} 0 & {}_a^{CFC}D^{\alpha}\\ {}^{CFR}D_b^{\alpha} & 0\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}+\begin{pmatrix} p\left(x\right) & 0\\ 0 & r\left(x\right)\end{pmatrix}\begin{pmatrix} y_1\\ y_2\end{pmatrix}\\ &=\begin{pmatrix} {}_a^{CFC}D^{\alpha}y_2+p\left(x\right)y_1\\ {}_b^{CFR}D^{\alpha}y_1+r\left(x\right)y_2\end{pmatrix}.\end{aligned}$$

The fractional Dirac system is:

$$\digamma y(x)=\lambda\omega(x)y(x),\ a\leqslant x\leqslant b<\infty, \tag{5}$$

where λ is a complex spectral parameter, $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $\omega(x) = \begin{pmatrix} \omega_1(x) & 0 \\ 0 & \omega_2(x) \end{pmatrix}$ are real-valued continuous functions defined on $[a,b]$ and $w_i(x) > 0,\ \forall x \in [a,b],\ (i = 1,2)$. We consider the boundary conditions

$$c_{11} e_{\frac{-\alpha}{1-\alpha}b^-} y_1(a) + c_{12} y_2(a) = 0, \tag{6}$$

$$c_{21} e_{\frac{-\alpha}{1-\alpha}b^-} y_1(b) + c_{22} y_2(b) = 0, \tag{7}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$. Now, we introduce convenient Hilbert space $L_w^2((a,b);E)$ $\left(E := \mathbb{C}^2\right)$ of vector-valued functions using the inner product

$$\begin{aligned} \langle y, z \rangle = & \int_a^b y_1(x)\overline{z_1}(x)\omega_1(x)dx \\ & + \int_a^b y_2(x)\overline{z_2}(x)\omega_2(x)dx \end{aligned}$$

where

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}, \qquad z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix}$$

y_i, z_i, and ω_i are real-valued continuous functions defined on $[a,b]$ and $\omega_i(x) > 0,\ \forall x \in [a,b],\ (i = 1,2)$.

Theorem 2.3 *The operator $\Pi = w^{-1}\digamma$ generated by fractional Dirac-type system (FD) defined by (5)-(7) is formally self-adjoint on $L_w^2((a,b);E)$.*

Proof Let $y(.),\ z(.)\, L_w^2((a,b);E)$. Then, we have

$$\begin{aligned} \langle \Pi y, z \rangle - \langle y, \Pi z \rangle = & \int_a^b \left({}_a^{CFC}D^\alpha y_2 + p(x)\, y_1 \right) \overline{z_1} dx \\ & + \int_a^b \left({}^{CFR}D_b^\alpha y_1 + r(x)\, y_2 \right) \overline{z_2} dx \\ & - \int_a^b y_1 \left(\overline{{}_a^{CFC}D^\alpha z_2 + p(x)\, z_1} \right) dx \\ & - \int_a^b y_2 \left(\overline{{}^{CFR}D_b^\alpha z_1 + r(x)\, z_2} \right) dx \\ = & \int_a^b {}_a^{CFC}D^\alpha y_2 \overline{z_1} dx + \int_a^b {}^{CFR}D_b^\alpha y_1 \overline{z_2} dx \\ & - \int_a^b y_1 \overline{({}_a^{CFC}D^\alpha z_2)} dx - \int_a^b y_2 \left(\overline{{}^{CFR}D_b^\alpha z_1} \right) dx. \end{aligned}$$

Since

$$\int_a^b {}_a^{CFC}D^\alpha y_2 \overline{z_1} dx = \int_a^b y_2 \left(\overline{{}^{CFR}D_b^\alpha z_1} \right) dx + \frac{B(\alpha)}{1-\alpha} y_2(x) e_{\frac{-\alpha}{1-\alpha}, b^-} \overline{z_1}|_a^b$$

and

$$\int_a^b y_1 \overline{({}_a^{CFC}D^{\alpha} z_2)}dx = \int_a^b \overline{z_2}{}^{CFR}D_b^{\alpha} y_1 dx + \frac{B(\alpha)}{1-\alpha} \overline{z_2} e_{\frac{-\alpha}{1-\alpha}, b^-} y_1(x)|_a^b$$

We get

$$\langle \Pi y, z \rangle - \langle y, \Pi z \rangle = [y, z]_b - [y, z]_a \tag{8}$$

where

$$[y, z]_x = \frac{B(\alpha)}{1-\alpha} \overline{z_2(x)} e_{\frac{-\alpha}{1-\alpha}, b^-} y_1(x) - \frac{B(\alpha)}{1-\alpha} y_2(x) e_{\frac{-\alpha}{1-\alpha} b^-} \overline{z_1(x)}.$$

$\langle \Pi y, z \rangle = \langle y, \Pi z \rangle$ for any $y(.)$, $z(.)\, L_w^2((a, b); E)$. We have $[y, z]_b = 0$ and $[y, z]_a = 0$ from the boundary (6)-(7). Consequently, we get

$$\langle \Pi y, z \rangle = \langle y, \Pi z \rangle.$$

□

Theorem 2.4 *All eigenvalues of the problem (5)-(7) are real.*

Proof By Theorem 2.6, we have

$$\begin{aligned}
\langle f, \digamma y \rangle &= \int_a^b f_1(x) \left({}_a^{CFC}D^{\alpha} y_2 + p(x) y_1 \right) dx \\
&+ \int_a^b f_2(x) \left({}^{CFR}D_b^{\alpha} y_1 + r(x) y_2 \right) dx \\
&= \int_a^b f_1(x) p(x) y_1 dx + \int_a^b f_2(x) {}_a^{CFC}D^{\alpha} y_2 dx \\
&+ \int_a^b f_1(x) r(x) y_2 dx + \int_a^b f_2(x) {}^{CFR}D_b^{\alpha} y_1 dx \\
&= \int_a^b f_1(x) p(x) y_1 dx + \int_a^b y_2 \; {}^{CFR}D_b^{\alpha} f_2(x) dx \\
&+ \frac{B(\alpha)}{1-\alpha} y_2(x) e_{\frac{-\alpha}{1-\alpha} b^-} f_1(x)|_a^b \\
&+ \int_a^b f_1(x) r(x) y_2 dx + \int_a^b y_1 \; {}_a^{CFC}D^{\alpha} f_2(x) dx \\
&- \frac{B(\alpha)}{1-\alpha} y_1(x) e_{\frac{-\alpha}{1-\alpha} b^-} f_2(x)|_a^b
\end{aligned}$$

Let λ be an eigenvalues of (5)-(7) and $y(x) = (y_1(x), y_2(x))^T$ be the corresponding eigenfunction. Then y and its complex conjugate $\overline{y(x)} = \left(\overline{y_1(x)}, \overline{y_2(x)} \right)^T$ satisfy

$$\digamma y(x) = \lambda w(x) y(x) \tag{9}$$

$$c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_1(a) + c_{12} y_2(a) = 0 \tag{10}$$

$$c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_1(b) + c_{22} y_2(b) = 0 \tag{11}$$

and

$$\digamma \overline{y(x)} = \overline{\lambda} w(x) \overline{y(x)} \tag{12}$$

$$c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} \overline{y_1(a)} + c_{12} \overline{y_2(a)} = 0 \tag{13}$$

$$c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} \overline{y_1(b)} + c_{22} \overline{y_1(b)} = 0 \tag{14}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$.
Then, we obtain

$$\begin{aligned}(\lambda - \overline{\lambda})\langle y, y\rangle &= \langle \lambda y, y\rangle - \langle \overline{\lambda} y, y\rangle \\ &= \langle \digamma y, y\rangle - \langle \digamma \overline{y}, y\rangle \\ &= \frac{B(\alpha)}{1-\alpha} y_1(b) \left[\begin{array}{c} c_{22} y_2\left(b\right) c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} \overline{y_1\left(b\right)} \\ -c_{22} \overline{y_2\left(b\right)} c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_1\left(b\right) \end{array} \right] \\ &\quad - \frac{B(\alpha)}{1-\alpha} y_2(a) \left[\begin{array}{c} c_{12} \overline{y_2\left(a\right)} c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_1\left(a\right) \\ -c_{12} y_2\left(a\right) c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} \overline{y_1\left(a\right)} \end{array} \right].\end{aligned}$$

From the equality (10),(11),(13), and (14), we have

$$(\lambda - \overline{\lambda})\langle y, y\rangle = 0$$

since $y(x)$ is nontrivial and $w > 0$, we have $\lambda = \overline{\lambda}$. □

Lemma 2.5 *If λ_1 and λ_2 are two different eigenvalues of the fractional Dirac system defined by (5)-(7), then the corresponding eigenfunctions y_{λ_1}, y_{λ_2} are orthogonal in the space $L_w^2\left((a,b);E\right)$.*

Proof Assume λ_1, λ_2 are distinct eigenvalues of (5)-(7) and y_{λ_1} and y_{λ_2} are the eigenfunctions. Then, we have

$$\digamma y_{\lambda_1}(x) = \lambda_1 w(x) y_{\lambda_1}(x) \tag{15}$$

$$c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{11}}\left(a\right) + c_{12} y_{\lambda_{12}}\left(a\right) = 0 \tag{16}$$

$$c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{11}}\left(b\right) + c_{22} y_{\lambda_{12}}\left(b\right) = 0 \tag{17}$$

and

$$\digamma y_{\lambda_2}(x) = \lambda_2 w(x) y_{\lambda_2}(x) \tag{18}$$

$$c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{21}}\left(a\right) + c_{12} y_{\lambda_{22}}\left(a\right) = 0 \tag{19}$$

$$c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{21}}\left(b\right) + c_{22} y_{\lambda_{22}}\left(b\right) = 0 \tag{20}$$

Therefore, we obtain,

$$\begin{aligned}(\lambda_1 - \lambda_2)\langle y_{\lambda_1}, y_{\lambda_2}\rangle &= \frac{B(\alpha)}{1-\alpha} y_2(b) \left[\begin{array}{c} c_{22} y_{\lambda_{22}}\left(b\right) c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{11}}(b) \\ -c_{22} y_{\lambda_{12}}(b) c_{21} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{21}}\left(b\right) \end{array} \right] \\ &\quad + \frac{B(\alpha)}{1-\alpha} y_1(a) \left[\begin{array}{c} c_{12} y_{\lambda_{22}}\left(a\right) c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{11}}\left(a\right) \\ -c_{12} y_{\lambda_{12}}\left(a\right) c_{11} e_{\frac{-\alpha}{1-\alpha} b^-} y_{\lambda_{21}}\left(a\right) \end{array} \right].\end{aligned}$$

From the boundary conditions (16),(17),(19), and (20), we have that

$$(\lambda_1 - \lambda_2)\langle y_{\lambda_1}, y_{\lambda_2}\rangle = 0$$

; hence, as $\lambda_1 \neq \lambda_2$, $\langle y_{\lambda_1}, y_{\lambda_2}\rangle = 0$. □

Theorem 2.6 *The Wronskian of any solutions of Eq. (5) is independent of x.*

Proof Let $v(x)$ and $\omega(x)$ be two solutions of Eq. (5). By Green's formula (8),

$$\langle \Pi v, \omega\rangle - \langle v, \Pi\omega\rangle = [v,\omega](b) - [v,\omega](a).$$

Since $\Pi v = \lambda v$ and $\Pi\omega = \lambda\omega$, we get

$$\langle \lambda v, \omega\rangle - \langle v, \lambda\omega\rangle = [v,\omega](b) - [v,\omega](a),$$

$$(\lambda - \overline{\lambda})(v,\omega) = [v,\omega](b) - [v,\omega](a).$$

Since $\lambda \in \mathbb{R}$, we have $[v,\omega](b) = [v,\omega](a) = W(v,\overline{\omega})(a)$, i.e. the Wronskian is independent of x. □

Theorem 2.7 *Any two solutions of the Eq. (5) are linearly dependent if and only if their Wronskian is zero.*

Proof Assume $y(x)$ and $\omega(x)$ are two linearly dependent solutions of equation (5). Then, there exists a constant $\eta > 0$ such that $y(x) = \eta.\omega(x)$. Hence,

$$W(y,\omega) = \begin{vmatrix} e_{\frac{-\alpha}{1-\alpha},b^-} y_1(x) & y_2(x) \\ e_{\frac{-\alpha}{1-\alpha},b^-} \omega_1(x) & \omega_2(x) \end{vmatrix} = \begin{vmatrix} \eta e_{\frac{-\alpha}{1-\alpha},b^-} \omega_1(x) & \eta\omega_2(x) \\ e_{\frac{-\alpha}{1-\alpha},b^-} \omega_1(x) & \omega_2(x) \end{vmatrix} = 0$$

Moreover, if the Wronskian $W(y,\omega)(x)$ is zero for some x in $[a,b]$ and $y(x) = \eta.\omega(x)$ are found. From this, it can be seen that $y(x)$ and $\omega(x)$ are linearly dependent on $[a,b]$. □

3. Dirac systems in the frame of CFR fractional derivatives and their discrete counterparts

In this section, new integration for discrete fractional Dirac systems will be presented, giving some definitions and properties for discrete exponential kernels and their associated fractional sums and fractional differences. The functions we consider will be defined on sets of the form

$$\mathbb{N}_a = \{a, a+1, a+2, ...\}, \qquad {}_b\mathbb{N} = \{..., b-2, b-1, b\},$$

where the form $a, b \in \mathbb{R}$ or set of the form

$$\mathbb{N}_{a,b} = \{a, a+1, a+2, ..., b\},$$

where the form $a, b \in \mathbb{R}$ and $b - a$ is a positive integer. From ([15]), the nabla discrete exponential function is defined by

$$\widehat{\exp}_\lambda(x,\rho) = \left(\frac{1}{1-\lambda}\right)^{x-\rho}, \qquad \lambda \neq 1,$$

In particular, when $\lambda = \frac{-\alpha}{1-\alpha}$, $\alpha \in (0,1)$,

$$\widehat{\exp}_\lambda(x,\rho) = (1-\alpha)^{x-\rho}. \tag{21}$$

Definition 4 *([3]) Assume* $f:\mathbb{N}_a\to\mathbb{R}$ *and* $0<\alpha<1$. *Then, the nabla discrete left Caputo fractional difference with the discrete exponential kernel is defined by*

$$\begin{aligned} {}_{a}^{CFC}\nabla^{\alpha}f(x) &= \frac{B(\alpha)}{1-\alpha}\sum_{\rho=a+1}^{x}(1-\alpha)^{x-\tau(\rho)}\nabla f(\rho) \\ &= B(\alpha)\sum_{\rho=a+1}^{x}(1-\alpha)^{x-\rho}\nabla f(\rho), \qquad x\in\mathbb{N}_{a+1}, \end{aligned}$$

and the left Reimannn-Liouville one by

$$\begin{aligned} {}_{a}^{CFR}\nabla^{\alpha}f(x) &= \frac{B(\alpha)}{1-\alpha}\nabla_x\sum_{\rho=a+1}^{x}(1-\alpha)^{x-\tau(\rho)}f(\rho) \\ &= B(\alpha)\nabla_x\sum_{\rho=a+1}^{x}(1-\alpha)^{x-\rho}f(\rho), \qquad x\in\mathbb{N}_{a+1}, \end{aligned}$$

where $B(\alpha)>0$ is a normalization function with $B(0)=B(1)=1$ as the associated fractional sum function

$${}_{a}^{CF}\nabla^{-\alpha}f(x)=\frac{1-\alpha}{B(\alpha)}f(x)+\frac{\alpha}{B(\alpha)}\sum_{\rho=a+1}^{x}f(\rho), \qquad x\in\mathbb{N}_{a+1}.$$

Definition 5 *([3]) Assume* $f:\mathbb{N}_a\to\mathbb{R}$ *and* $0<\alpha<1$. *Then, the nabla discrete right Caputo fractional difference with discrete exponential kernel is defined by*

$$\begin{aligned} {}^{CFC}\nabla_b^{\alpha}f(x) &= \frac{-B(\alpha)}{1-\alpha}\sum_{\rho=x}^{b-1}(1-\alpha)^{\rho-\tau(x)}\triangle f(\rho) \\ &= -B(\alpha)\sum_{\rho=x}^{b-1}(1-\alpha)^{\rho-x}\triangle f(\rho), \qquad x\in{}_{b-1}\mathbb{N}, \end{aligned}$$

and the right Reimann-Liouville one by

$$\begin{aligned} {}^{CFR}\nabla_b^{\alpha}f(x) &= \frac{-B(\alpha)}{1-\alpha}\triangle_x\sum_{\rho=x}^{b-1}(1-\alpha)^{\rho-\tau(x)}f(\rho) \\ &= -B(\alpha)\triangle_x\sum_{\rho=x}^{b-1}(1-\alpha)^{\rho-x}f(\rho), \qquad x\in{}_{b-1}\mathbb{N}. \end{aligned}$$

In addition, the associated fractional sum is defined by

$${}^{CF}\nabla_b^{-\alpha}f(x)=\frac{1-\alpha}{B(\alpha)}f(x)+\frac{\alpha}{B(\alpha)}\sum_{\rho=x}^{b-1}f(\rho), \qquad x\in{}_{b-1}\mathbb{N}.$$

Theorem 3.1 *([3]) (Integration by parts formula for CFR fractional differences)*
Assume $f,g:\mathbb{N}_{a,b}\to\mathbb{R}$ *and* $0<\alpha<1$.

$$\sum_{x=a+1}^{b-1}g(x)\,{}_{a}^{CF}\nabla^{-\alpha}f(x)=\sum_{x=a+1}^{b-1}f(x)\,{}^{CF}\nabla_b^{-\alpha}g(x)$$

and

$$\sum_{x=a+1}^{b-1} g(x)_a^{CFR}\nabla^\alpha f(x) = \sum_{x=a+1}^{b-1} f(x)^{CFR}\nabla_b^\alpha g(x).$$

Theorem 3.2 *(Integration by parts formula for CFR fractional differences)*
Assume $f, g : \mathbb{N}_{a,b} \to \mathbb{R}$ *and* $0 < \alpha < 1$.

$$\sum_{x=a+1}^{b-1} f(x)_a^{CFR}\nabla^\alpha g(x) = \sum_{x=a+1}^{b-1} g(x-1)^{CFR}\nabla_b^\alpha f(x-1) + \frac{B(\alpha)}{1-\alpha} g(x)\hat{e}_{\frac{-\alpha}{1-\alpha},b^-} f(x)|_a^{b-1}.$$

In the above, it is easy to see that $\hat{e}_{\frac{-\alpha}{1-\alpha},b^-} f(b-1) = (1-a)f(b-1)$.

Now, consider the nabla discrete fractinal Dirac systems

$$\begin{aligned} \digamma y &= \begin{pmatrix} 0 & {}_a^{CFC}\nabla^\alpha \\ {}^{CFR}\nabla_b^\alpha & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} {}_a^{CFC}\nabla^\alpha y_2 + p(x) y_1 \\ {}^{CFR}\nabla_b^\alpha y_1 + r(x) y_2 \end{pmatrix}, \end{aligned}$$

where λ is a complex spectral parameter the fractional Dirac type system is:

$$\digamma y(x) = \lambda \omega(x) y(x), \; x \in N_{a,b-1}, \tag{22}$$

The functions $y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, $p(x)$, $r(x)$, $w(x) = \begin{pmatrix} w_1(x) & 0 \\ 0 & w_2(x) \end{pmatrix}$ are real-valued functions defined on $N_{a,b-1}$ and $w_i(x) > 0, \; \forall x \in N_{a,b-1}, \; (i = 1, 2)$. We consider the boundary conditions

$$c_{11} \left[{}^{CFR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} \hat{e}_{\frac{-\alpha}{1-\alpha},b^-} \right] y_1(a) + c_{12} y_2(a) = 0, \tag{23}$$

$$c_{21} \left[{}^{CFR}\nabla_b^\alpha - \frac{B(\alpha)}{1-\alpha} \hat{e}_{\frac{-\alpha}{1-\alpha},b^-} \right] y_1(b-1) + c_{22} y_2(b-1) = 0, \tag{24}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$.

Theorem 3.3 *The operator* $\Pi = w^{-1}\digamma$ *generated by fractional Dirac type system (FD) defined by (22)-(24) is formally self-adjoint on* $L^2_{w\nabla}((N_{a,b-1}); E)$.

Proof Let $y(.), z(.)L^2_{w\nabla}((N_{a,b-1});E)$.

$$\begin{aligned}
\langle \Pi y, z\rangle - \langle y, \Pi z\rangle &= \sum_{x=a+1}^{b-1}\left({}_{a}^{CFC}\nabla^{\alpha}\ y_2 + p(x)\,y_1\right)\overline{z_1} + \sum_{x=a+1}^{b-1}\left({}^{CFR}\nabla_b^{\alpha}\ y_1 + r(x)\,y_2\right)\overline{z_2} \\
&- \sum_{x=a+1}^{b-1} y_1\left(\overline{{}_{a}^{CFC}\nabla^{\alpha} z_2 + p(x)\,z_1}\right) - \sum_{x=a+1}^{b-1} y_2\left(\overline{{}^{CFR}\nabla_b^{\alpha}\ z_1 + r(x)\,z_2}\right) \\
&= \sum_{x=a+1}^{b-1} {}_{a}^{CFC}\nabla^{\alpha} y_2\ \overline{z_1} + \sum_{x=a+1}^{b-1} p(x)y_1(x)\overline{z_1(x)} \\
&+ \sum_{x=a+1}^{b-1} {}^{CFR}\nabla_b^{\alpha}\ y_1\overline{z_2} + \sum_{x=a+1}^{b-1} r(x)y_2(x)\overline{z_2(x)} \\
&- \sum_{x=a+1}^{b-1} y_1\overline{({}_{a}^{CFC}\nabla^{\alpha} z_2)} - \sum_{x=a+1}^{b-1} p(x)y_1(x)\overline{z_1(x)} \\
&- \sum_{x=a+1}^{b-1} y_2\left(\overline{{}^{CFR}\nabla_b^{\alpha}\ z_1}\right) - \sum_{x=a+1}^{b-1} r(x)y_2(x)\overline{z_2(x)} \\
&= \sum_{x=a+1}^{b-1} {}_{a}^{CFC}\nabla^{\alpha} y_2\ \overline{z_1} + \sum_{x=a+1}^{b-1} {}^{CFR}\nabla_b^{\alpha}\ y_1\overline{z_2} \\
&- \sum_{x=a+1}^{b-1} y_1\overline{({}_{a}^{CFC}\nabla^{\alpha} z_2)} - \sum_{x=a+1}^{b-1} y_2\left(\overline{{}^{CFR}\nabla_b^{\alpha}\ z_1}\right) \\
&= \sum_{x=a+1}^{b-1} y_2\,(x-1)^{CFR}\,\nabla_b^{\alpha}\ \overline{z_1}\,(x-1)\ + \frac{B(\alpha)}{1-\alpha} y_2(x)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\overline{z_1}(x)\Big|_a^{b-1} \\
&+ \sum_{x=a+1}^{b-1}\left[{}^{CFR}\nabla_b^{\alpha} y_1\,(x)\right]\overline{z_2}\,(x) - \sum_{x=a+1}^{b-1}\overline{z_2}\,(x-1)^{CFR}\,\nabla_b^{\alpha} y_1\,(x-1) \\
&- \frac{B(\alpha)}{1-\alpha}\overline{z_2}(x)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-} y_1(x)\Big|_a^{b-1} - \sum_{x=a+1}^{b-1} y_2\,(x)\left(\overline{{}^{CFR}\nabla_b^{\alpha}\ z_1}\,(x)\right) \\
&= y_2\,(a)^{CFR}\,\nabla_b^{\alpha}\ \overline{z_1}\,(a) - y_2\,(b-1)^{CFR}\,\nabla_b^{\alpha}\ \overline{z_1}\,(b-1) \\
&+ \frac{B(\alpha)}{1-\alpha} y_2(b-1)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\overline{z_1}(b-1) - \frac{B(\alpha)}{1-\alpha} y_2(a)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\overline{z_1}(a) \\
&+ \overline{z_2}\,(b-1)^{CFR}\,\nabla_b^{\alpha}\ y_1\,(b-1) - \overline{z_2}\,(a)^{CFR}\,\nabla_b^{\alpha}\ y_1\,(a) \\
&+ \frac{B(\alpha)}{1-\alpha}\overline{z_2}(a)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-} y_1(a) - \frac{B(\alpha)}{1-\alpha}\overline{z_2}(b-1)\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-} y_1(b-1) \\
&= -\left[\overline{z_2}\,(a)\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_1(a) - y_2\,(a)\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right]\ \overline{z_1}\,(a)\right] \\
&+ \left[\overline{z_2}\,(b-1)\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_1\,(b-1) - y_2\,(b-1)\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right]\overline{z_1}(b-1)\right]
\end{aligned}$$

From the boundary condition (23) and (24), we get the desired result. □

Theorem 3.4 *All eigenvalues of the problem (22)-(24) are real.*

Proof By Theorem 3.5,

$$\begin{aligned}\langle f, \digamma y\rangle &= \sum_{x=a+1}^{b-1} f_1(x)\left({}_{a}^{CFC}\nabla^{\alpha} y_2 + p(x)\, y_1\right)\\ &+ \sum_{x=a+1}^{b-1} f_2(x)\left({}^{CFR}\nabla_b^{\alpha}\, y_1 + r(x)\, y_2\right)\\ &= \sum_{x=a+1}^{b-1} f_1(x)p(x)\, y_1 + \sum_{x=a+1}^{b-1} f_1(x){}_{a}^{CFR}\nabla^{\alpha} y_2\\ &+ \sum_{x=a+1}^{b-1} f_2(x)r(x)\, y_2 + \sum_{x=a+1}^{b-1} f_2(x){}^{CFC}\nabla_b^{\alpha}\, y_1\end{aligned}$$

Let λ and $\overline{\lambda}$ be eigenvalues of (22)-(24) and $y(x) = (y_1(x), y_2(x))^T$ be the corresponding eigenfunction. Then, y and its complex conjugate $\overline{y(x)} = \left(\overline{y_1(x)}, \overline{y_2(x)}\right)^T$ satisfy.

$$\digamma y(x) = \lambda w(x) y(x) \tag{25}$$

$$c_{11}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha}, b^-}\right] y_1(a) + c_{12} y_2(a) = 0, \tag{26}$$

$$c_{21}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha}, b^-}\right] y_1(b-1) + c_{22} y_2(b-1) = 0, \tag{27}$$

and

$$\digamma \overline{y(x)} = \overline{\lambda} w(x) y\overline{(x)} \tag{28}$$

$$c_{11}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha}, b^-}\right]\overline{y_1(a)} + c_{12}\overline{y_2(a)} = 0, \tag{29}$$

$$c_{21}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha}, b^-}\right]\overline{y_1(b-1)} + c_{22}\overline{y_2(b-1)} = 0 \tag{30}$$

with $c_{11}^2 + c_{12}^2 \neq 0$ and $c_{21}^2 + c_{22}^2 \neq 0$.
Therefore, we obtain

$$\begin{aligned}(\lambda - \overline{\lambda})\langle y, y\rangle &= \langle \lambda y, y\rangle - \langle \overline{\lambda} y, y\rangle\\ &= \langle \digamma y, y\rangle - \langle \digamma \overline{y}, y\rangle\end{aligned}$$

From the equality (26),(27),(29), and (30), we have

$$(\lambda - \overline{\lambda})\langle y, y\rangle = 0$$

since $y(x)$ is nontrivial and $w > 0$, we have $\lambda = \overline{\lambda}$. □

Lemma 3.5 *If λ_1 and λ_2 are two different eigenvalues of the Fractional Dirac system defined by (22)-(24), then the corresponding eigenfunctions y_{λ_1} and y_{λ_2} are orthogonal in the space $L^2_{w,\nabla}((a,b);\mathbb{C}^2)$.*

Proof Assume λ_1 and λ_2 are distinct eigenvalues of (22)-(24) and y_{λ_1} and y_{λ_2} are the eigenfunctions.

$$\digamma y_{\lambda_1}(x) = \lambda_1 w(x) y_{\lambda_1}(x) \tag{31}$$

$$c_{11}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_1(a) + c_{12} y_1(a) = 0, \tag{32}$$

$$c_{21}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_1(b-1) + c_{22} y_1(b-1) = 0 \tag{33}$$

and

$$\digamma y_{\lambda_2}(x) = \lambda_2 w(x) y_{\lambda_2}(x) \tag{34}$$

$$c_{11}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_2(a) + c_{12} y_2(a) = 0, \tag{35}$$

$$c_{21}\left[{}^{CFR}\nabla_b^{\alpha} - \frac{B(\alpha)}{1-\alpha}\widehat{e}_{\frac{-\alpha}{1-\alpha},b^-}\right] y_2(b-1) + c_{22} y_2(b-1) = 0 \tag{36}$$

From the boundary conditions (32),(33),(35), and (36), we have that

$$(\lambda_1 - \lambda_2)\langle y_{\lambda_1}, y_{\lambda_2}\rangle = 0$$

; hence, as $\lambda_1 \neq \lambda_2$, $\langle y_{\lambda_1}, y_{\lambda_2}\rangle = 0$. □

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