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On the properties of solutions for nonautonomous third-order stochastic differential equation with a constant delay

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Abstract: In this work, complete Lyapunov functionals (LFs) are constructed and used for the established conditions on the nonlinear functions appearing in the main equation, to guarantee stochastically asymptotically stable (SAS), uniformly stochastically bounded (USB) and uniformly exponentially asymptotically stable (UEAS) in probability of solutions to the nonautonomous third-order stochastic differential equation (SDE) with a constant delay as

$$\dddot{x}(t) + a(t)f(x(t), \dot{x}(t))\ddot{x}(t) + b(t)\varphi(x(t))\dot{x}(t) + c(t)\psi(x(t-r))$$
$$+ g(t,x)\dot{\omega}(t) = p(t, x(t), \dot{x}(t), \ddot{x}(t)).$$

In Section 4, we give two numerical examples as an application to illustrate the results.

Key words: (DDE), (SDE), (SDDE), (SAS), (USB), (UEAS)

1. Introduction

Stochastic differential equation (SDE) is typically a dynamical system endowing random components that models the evolution over time of particular phenomena that is subject to uncertainty. For instance the evolution of a financial asset, risk assessment in insurance policy, etc.

To the best of our knowledge, SDE of the third-order with or without time-varying delays naturally appears in multiple applications, where deterministic models are perturbed by the white noise or its generalizations [13, 23, 27, 28]. In most cases, SDEs are understood as a continuous time limit of the corresponding SDEs. This understanding of SDEs is ambiguous and must be complemented by a proper mathematical definition of the corresponding integral. Such a mathematical definition was first proposed by Kiyosi It$\hat{o}$, leading to what is known today as the It$\hat{o}$ formula (IF). Mathematically, stochastic delay differential equations (SDDEs) were first introduced by It$\hat{o}$ and Nisio [15], in which the existence and uniqueness of the solutions have been investigated.

More than one hundred years ago, Lyapunov introduced the concept of stability of a dynamical system and created a very powerful tool known as the Lyapunov's second method (LSM) in the study of stability and boundedness. A manifest advantage of this method is that it does not require the knowledge of solutions for equations and thus has exhibited a great power in applications. In general, many results have been obtained on uniformly stochastically stable (USS) and USB of solutions for delay differential equations (DDEs) by using LSM. See, for example [1, 2, 9, 10, 14, 20, 29–32, 34–41], and the references cited in therein.

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In the last decades, SDDEs have attracted a great interest in the literature because of their applications in characterizing many problems in physics, mechanics, electrical engineering, biology, ecology and so on. See [17, 24–26]. Some related papers will be presented on the kind of SDDEs. See, for example [3–8, 11, 12, 16, 19, 21, 22, 42].

Recently, Mahmoud and Tunç [22] investigated new criteria for USB and UEAS for a certain third-order SDDE as

$$\dddot{x}(t) + \Phi(x,\dot{x})\ddot{x} + G(\dot{x}(t-r)) + F(x(t-r)) + \sigma x(t-h(t))\dot{\omega}(t) = P(t,x,\dot{x},\ddot{x}).$$

The purpose of this paper is to establish the sufficient conditions for SAS of the zero solution, USB and UESA of all solution to third-order SDE with a constant delay as

$$\begin{aligned} &\dddot{x}(t) + a(t)f(x(t),\dot{x}(t))\ddot{x}(t) + b(t)\varphi(x(t))\dot{x}(t) + c(t)\psi(x(t-r)) \\ &\quad + g(t,x)\dot{\omega}(t) = p(t,x(t),\dot{x}(t),\ddot{x}(t)), \end{aligned} \tag{1.1}$$

where $a(t)$, $b(t)$ and $c(t)$ are positive and continuously differentiable functions on $[0,\infty)$, also $f(x,y), \varphi(x), g(t,x)$, and $\psi(x)$ are continuous functions with $\varphi(0) = \psi(0) = 0$, $\omega(t) \in \mathbb{R}^n$ is standard Brownian motion, $\dot{\omega}(t) = \frac{d\omega}{dt}$. In this study we will consider the derivatives $a'(t) = \frac{da(t)}{dt}$, $b'(t) = \frac{db(t)}{dt}, c'(t) = \frac{dc(t)}{dt}$, $\psi' = \frac{d\psi}{dx}$ and $f_x(x,y) = \frac{\partial f(x,y)}{\partial x}$.

Remark 1.1 *We will give the following remarks:*

(*1*) *In [37], Sadek investigated the asymptotic stability of DDE. Comparing his equation to (1.1), we find $f(x(t),\dot{x}(t)) = 1$ and $\varphi(x(t)) = 1$ with $p = 0$, then (1.1) can be reduced to Sadek's equation without the stochastic term.*

(*2*) *The obtained results in [3, 6, 7, 12, 42] are on second-order SDDE.*

(*3*) *In (1.1), if we let $a(t) = a,\ b(t) = b,\ f(x(t),\dot{x}(t)) = 1,\ \varphi(x(t)) = 1,\ c(t) = 1$ and $g(t,x) = \sigma x(t)$, we note that the equation studied by Ademola [8] represents a special case from (1.1) in this study.*

(*4*) *In 2015, Abou El-Ela and his students have begun studding the behaviour of SDDE, see [3–6] and then Mahmoud et al. continued the studding of the SDDE, see [19, 21, 22]. Our paper represents a generalization of all the above studding. For example, in (1.1), let $a(t) = b(t) = c(t) = 1$, $\varphi(x(t))\dot{x}(t) = G(\dot{x}(t-r))$ and $g(t,x) = \sigma x(t-h(t))$, it tends to the SDDE studied in [22].*

2. Stability result

Let $B(t) = (B_1(t),\ldots,B_m(t))$ be an m-dimensional Brownian motion defined on the probability space. Consider a nonautonomous n-dimensional SDDE

$$dx(t) = M(t,x(t),x(t-r))dt + N(t,x(t),x(t-r))dB(t),\ \forall\, t \geq 0, \tag{2.1}$$

with initial data $\{x(\theta) : -r < \theta < 0\}, x_0 \in \mathcal{C}([-r,0];\mathbb{R}^n)$. Suppose that $M : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^n$ and $N : \mathbb{R}^+ \times \mathbb{R}^{2n} \to \mathbb{R}^{n\times m}$ satisfy the local Lipschitz condition and the linear growth condition. Hence, for any given initial value $x(0) = x_0 \in \mathbb{R}^n$, it is therefore known that Equation (2.1) has a unique continuous solution on

$t \geq 0$, which is known as $x(t;x_0)$ in this section. Suppose that $M(t,0,0)=0$ and $N(t,0,0)=0$, for all $t \geq 0$. Hence, the SDDE admits the zero solution $x(t;0) \equiv 0$ for any given initial value $x_0 \in \mathcal{C}([-r,0];\mathbb{R}^n)$.

Let $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$ denote the family of nonnegative functions $W(t,x_t)$ defined on $\mathbb{R}^+ \times \mathbb{R}^n$, which are once continuously differentiable in t and twice continuously differentiable in x.

By IF we have

$$dW(t,x_t) = \mathcal{L}W(t,x_t)dt + W_x(t,x_t)N(t,x_t)dB(t),$$

where

$$\begin{aligned}\mathcal{L}W(t,x_t) &= W_t(t,x_t) + W_x(t,x_t)M(t,x_t)\\ &\quad + \frac{1}{2}trace[N^T(t,x_t)W_{xx}(t,x_t)N(t,x_t)],\end{aligned} \tag{2.2}$$

where $W_x = (W_{x_1},\ldots,W_{x_n})$ and $W_{xx} = (W_{x_i x_j})_{n\times n},\ i,j=0,1,2,...,n$. Moreover, let $\mathcal{K}$ denote the family of all continuous nondecreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0)=0$ and $\rho(r)>0$, if $r>0$. Here, we will use the diffusion operator $\mathcal{L}W(t,x_t)$ defined in (2.2) to replace $W'(t,x_t) = \frac{d}{dt}W(t,x_t)$.

Theorem 2.1 [8, 13] *Assume that there exist $W \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$ and $\rho_1,\rho_2,\ \rho_3 \in \mathcal{K}$ such that*

$$\rho_1(\|x\|) \leq W(t,x_t) \leq \rho_2(\|x\|),$$

and

$$\mathcal{L}W(t,x_t) \leq -\rho_3(\|x\|),\quad \forall\,(t,x_t) \in \mathbb{R}^+ \times \mathbb{R}^n.$$

Then, the zero solution of the SDDE (2.1) is SAS.

Now, we can give the following main theorem.

Theorem 2.2 *In addition to the basic fundamental assumptions imposed on the functions f, φ and ψ appearing in Equation (1.1), suppose that there exist the positive constants $a_1,\ a_2,\ b_1,\ b_2,\ f_1\ L_0,\ L,\ c_1,\varphi_2,\alpha,\sigma$ and ψ_0, and the negative constant a_0 such that the following conditions are satisfied:*

(i) $a_1 \leq a(t) \leq a_2,\ \ b_1 \leq b(t) \leq b_2 \leq 1,$ *and* $c_1 \leq c(t) \leq 1,$ *for all* $t \geq 0.$

(ii) $a'(t) \leq a_0,\ \ -\sigma \leq b'(t) \leq c'(t) \leq a'(t) \leq 0,\ t \geq 0.$

(iii) $1 \leq f(x,y) \leq f_1,$ *with* $f_x(x,y) < 0$ *for all* $x,y,\quad 1 \leq \varphi(x) \leq \varphi_2,$
$\frac{\psi(x)}{x} \geq L_0,\ \ x \neq 0,$ *and* $|\psi'(x)| \leq L,$ *such that* $\psi(x)signx > 0,\ \sup\{|\psi'(x)|\} = \frac{\psi_0}{2}$ *with* $a_1b_1 - \psi_0 > 0.$

(iv) $|b_2\varphi'(x)y| \leq \Delta$, *for all* $(x,y,z) \in D$ *where* D *is a domain in* xyz*-space containing the origin.*

(v) $\{\mu b_1 - \psi'(x)\} \geq \Delta = \mu b_1 - \frac{\psi_0}{2}$, *for some* $\Delta > 0,$ *with* $\mu = \frac{a_1b_1+\psi_0}{4b_1}.$

(vi) $g(t,x) \leq \alpha x,$ *such that* $\alpha^2 \leq 2c_1L_0 - a_1 - b_1 - 2.$

(vii) $\Delta \geq 3 + b_1 + \mu a_0 f_1$ *and* $2\Delta \geq b_1.$

Therefore, the zero solution of (1.1) *is SAS, provided that*

$$r < \min\left\{\frac{2c_1L_0 - a_1 - b_1 - 2 - \alpha^2}{2L}, \frac{\Delta - \mu a_0 f_1 - b_1 - 3}{4(\mu+1)L}, \frac{2\Delta - b_1}{2b_1}\right\}.$$

Proof of Theorem 2.2.

By considering $p = 0$, (1.1) is equivalent to the following

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -a(t)f(x,y)z - b(t)\varphi(x)y - c(t)\psi(x) + c(t)\int_{t-r}^{t}\psi'(x(s))y(s)ds \\ &- g(t,x)\dot{\omega}(t). \end{aligned} \tag{2.3}$$

Define LF $W_1 = W_1(x_t, y_t, z_t)$ of the system (2.3) as the following

$$\begin{aligned} W_1(x_t,y_t,z_t) =& \mu c(t)\int_0^x \psi(\xi)d\xi + c(t)\psi(x)y + \mu a(t)\int_0^y f(x,\eta)\eta d\eta + \mu yz \\ &+ \frac{z^2}{2} + xz + b(t)\varphi(x)\frac{y^2}{2} + x^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds, \end{aligned} \tag{2.4}$$

where λ is a positive constant to be determined later in the proof.

Differentiating (2.4) and applying IF, then the time derivative of the function $W_1(t, x_t, y_t, z_t)$, along the system (2.3), becomes

$$\begin{aligned} \mathcal{L}W_1(x_t,y_t,z_t) =& \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \mu a'(t)\int_0^y f(x,\eta)\eta d\eta \\ &+ \mu a(t)y\int_0^y f_x(x,\eta)\eta d\eta + c(t)\psi'(x)y^2 + \mu z^2 - \mu b(t)\varphi(x)y^2 \\ &- a(t)f(x,y)z^2 + yz - a(t)f(x,y)xz - b(t)\varphi(x)xy - c(t)\psi(x)x \\ &+ 2xy + c(t)\big(\mu y + z + x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds + b'(t)\varphi(x)\frac{y^2}{2} \\ &+ b(t)\varphi'(x)\frac{y^3}{2} + \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + \frac{1}{2}g^2(t,x). \end{aligned}$$

From the assumptions $(i) - (iii)$ and (vi), we have

$$\begin{aligned} \mathcal{L}W_1(x_t,y_t,z_t) \leq& \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + b'(t)\varphi(x)\frac{y^2}{2} + \mu z^2 + 2xy + yz \\ &- \big(\mu b_1 - \psi'(x)\big)y^2 + \frac{1}{2}b_2\varphi'(x)y^3 + \frac{1}{2}\mu a_0 f_1 y^2 - a_1 z^2 - a_1 xz \\ &- b_1 xy - c_1 L_0 x^2 + \big(\mu y + z + x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds \\ &+ \lambda r y^2 - \lambda\int_{t-r}^{t} y^2(s)ds + \frac{1}{2}\alpha^2 x^2. \end{aligned}$$

Using the assumptions (iv) and (v), we have $\{\mu b_1 - \psi'(x)\} \geq \Delta$ and $|b_2\varphi'(x)y| \leq \Delta$. Therefore, the above equation becomes

$$\begin{aligned}\mathcal{L}W_1(x_t, y_t, z_t) \leq & \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \frac{1}{2}b'(t)\varphi(x)y^2 + \mu z^2 + 2xy + yz \\ & - \frac{1}{2}\Delta y^2 + \frac{1}{2}\mu a_0 f_1 y^2 - a_1 z^2 - a_1 xz - b_1 xy - c_1 L_0 x^2 + \frac{1}{2}\alpha^2 x^2 \\ & + \left(\mu y + z + x\right)\int_{t-r}^t \psi'(x(s))y(s)ds + \lambda r y^2 - \lambda\int_{t-r}^t y^2(s)ds.\end{aligned}$$

Now, let the function $R(t, x, y)$ be known as

$$R = \mu c'(t)\int_0^x \psi(\xi)d\xi + c'(t)\psi(x)y + \frac{1}{2}b'(t)\varphi(x)y^2. \tag{2.5}$$

1. If $c'(t) = 0$, then Equation (2.5) becomes

$$R = \frac{1}{2}b'(t)\varphi(x)y^2 \leq 0, \text{by } (ii). \tag{2.6}$$

2. If $c'(t) < 0$, then $R(t, x, y)$ can be written as

$$R = \mu c'(t)R_1(t, x, y),$$

where

$$R_1 = \int_0^x \psi(\xi)d\xi + \frac{1}{\mu}\psi(x)y + \frac{b'(t)}{2\mu c'(t)}\varphi(x)y^2.$$

The function $R_1(t, x, y)$ can be represented as follows:

$$R_1 = \int_0^x \psi(\xi)d\xi + \frac{b'(t)\varphi(x)}{2\mu c'(t)}\left(y + \frac{c'(t)}{b'(t)\varphi(x)}\psi(x)\right)^2 - \frac{c'(t)}{2\mu b'(t)\varphi(x)}\psi^2(x).$$

By using assumption (ii), we have $0 \leq \frac{c'(t)}{b'(t)} \leq 1$, and also from (iii), we get

$$R_1 \geq \int_0^x (1 - \frac{\psi'(\xi)}{\mu})\psi(\xi)d\xi.$$

Since $\mu = \frac{a_1 b_1 + \psi_0}{4b_1}$, and by assumption (iii), we obtain

$$1 - \frac{\psi'(\xi)}{\mu} \geq \frac{1}{\mu}\left(\frac{a_1 b_1 + \psi_0}{4b_1} - \frac{\psi_0}{2}\right) = \left\{\frac{a_1 b_1 + \psi_0(1 - 2b_1)}{4\mu b_1}\right\}.$$

Since $b_1 \leq 1$, it follows that

$$R_1 \geq \frac{a_1 b_1 - \psi_0}{4b_1\mu}\int_0^x \psi(\xi)d\xi.$$

From the condition (iii), we have $a_1b_1-\psi_0>0$; therefore, we conclude that $R_1\geq 0$,

and since $c'(t)<0$, then the function $R(t,x,y)$ becomes

$$R=\mu c'(t)R_1<0.$$

Hence, from the two cases (1) and (2) it can be concluded that $R\leq 0$ for all x, y and $t\geq 0$. Thus, we have

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&\mu z^2+2xy+yz-\frac{1}{2}\Delta y^2+\frac{1}{2}\mu a_0f_1y^2-a_1z^2-a_1xz\\&-b_1xy-c_1L_0x^2+\big(\mu y+z+x\big)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\&+\lambda ry^2-\lambda\int_{t-r}^{t}y^2(s)ds+\frac{1}{2}\alpha^2x^2.\end{aligned}$$

By applying the inequality $2|pq|\leq p^2+q^2$, and using the assumption $|\psi'(x)|\leq L,$ we obtain

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&-\left\{\frac{2c_1L_0-a_1-b_1-2-\alpha^2}{2}-\frac{L}{2}r\right\}x^2\\&-\left\{\frac{\Delta-3-\mu a_0f_1-b_1}{2}-\frac{\mu L}{2}r-\lambda r\right\}y^2\\&-\left\{\frac{a_1}{2}-\mu-\frac{1}{2}-\frac{L}{2}r\right\}z^2\\&+\left\{\frac{1}{2}L(\mu+2)-\lambda\right\}\int_{t-r}^{t}y^2(s)ds.\end{aligned}\tag{2.7}$$

Take $\lambda=\frac{1}{2}L(\mu+2)$, and since $\mu=\frac{a_1b_1+\psi_0}{4b_1}$, so $\frac{a_1}{2}-\mu=\frac{a_1b_1-\psi_0}{4b_1}=\frac{\Delta}{b_1}>0$, by (i), then we get

$$\begin{aligned}\mathcal{L}W_1(x_t,y_t,z_t)\leq&-\left\{\frac{2c_1L_0-2-a_1-b_1-\alpha^2}{2}-\frac{c_2L}{2}r\right\}x^2\\&-\left\{\frac{\Delta-3-\mu a_0f_1-b_1}{2}-L(\mu+1)r\right\}y^2\\&-\left\{\frac{2\Delta-b_1}{2b_1}-\frac{L}{2}r\right\}z^2.\end{aligned}$$

Therefore, suppose that

$$r<\min\left\{\frac{2c_1L_0-a_1-b_1-2-\alpha^2}{2L},\frac{\Delta-\mu a_0f_1-b_1-3}{4(\mu+1)L},\frac{2\Delta-b_1}{2b_1}\right\}.$$

Then, for positive constant δ_1, we can write

$$\mathcal{L}W_1(t,x_t,y_t,z_t)\leq\delta_1(x^2+y^2+z^2).\tag{2.8}$$

Since $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is positive, we have

$$\begin{aligned} W_1(x_t,y_t,z_t) \geq &\mu c(t)\int_0^x \psi(\xi)d\xi + c(t)\psi(x)y + \mu a(t)\int_0^y f(x,\eta)\eta d\eta \\ &+ \mu yz + \frac{z^2}{2} + xz + b(t)\varphi(x)\frac{y^2}{2} + x^2. \end{aligned}$$

From the assumptions $(i)-(iii)$, we get

$$W_1(x_t,y_t,z_t) \geq \mu c_1 \int_0^x \psi(\xi)d\xi + c_1\psi(x)y + \mu a_1\frac{y^2}{2} + \mu yz + \frac{z^2}{2} + xz + b_1\frac{y^2}{2} + x^2.$$

Now, we can write the above equation as

$$\begin{aligned} W_1(x_t,y_t,z_t) \geq &\frac{1}{2b_1}\big\{b_1 y + c_1\psi(x)\big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}\big(a_1-2\mu\big)y^2 + \big(x+\frac{z}{2}\big)^2 \\ &+ \frac{c_1}{2b_1y^2}\left[4\int_0^x \psi(\xi)\left\{(\mu b_1 - c_1\psi'(\xi))\eta d\eta\right\}d\xi\right]. \end{aligned}$$

Since $\mu b_1 - c_1\psi'(\xi) = \frac{a_1b_1+(1-2c_1)\psi_0}{4} > \frac{a_1b_1-\psi_0}{4} > 0,$ by (i); therefore, we conclude

$$\begin{aligned} W_1(x_t,y_t,z_t) \geq &\frac{1}{2b_1}\big\{b_1 y + c_1\psi(x)\big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}\big(a_1-2\mu\big)y^2 \\ &+ \big(x+\frac{z}{2}\big)^2 + \frac{c_1(a_1b_1-\psi_0)}{4}\int_0^x \psi(\xi)d\xi. \end{aligned} \tag{2.9}$$

Since $a_1 - 2\mu = \frac{a_1b_1-\psi_0}{2b_1} > 0,$ by (iii), we get

$$W_1(x_t,y_t,z_t) \geq \delta_2(x^2+y^2+z^2) \ \ for\ some \ \ \delta_2 > 0. \tag{2.10}$$

By applying the assumptions $(i)-(iii)$, we have

$$\begin{aligned} W_1(x_t,y_t,z_t) \leq &\frac{1}{2}\mu L x^2 + Lxy + \frac{1}{2}\mu a_2 f_1 y^2 + \mu yz + \frac{z^2}{2} + xz + x^2 \\ &+ \frac{1}{2}b_2\varphi_2 y^2 + \lambda\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds. \end{aligned}$$

Since

$$\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \leq \|y\|^2\int_{t-r}^{t}(\theta - t + r)d\theta = \frac{r^2}{2}\|y\|^2. \tag{2.11}$$

Therefore, from the inequality $pq \leq \frac{1}{2}(p^2+q^2),$ we obtain

$$\begin{aligned} W_1(x_t,y_t,z_t) \leq &\frac{1}{2}\Big\{(\mu+1)L + 3\Big\}\|x\|^2 + \frac{1}{2}\Big\{L + \mu(a_2f_1+1) \\ &+ b_2\varphi_2 + \lambda r^2\Big\}\|y\|^2 + \frac{1}{2}(\mu+2)\|z\|^2. \end{aligned} \tag{2.12}$$

Then, we get

$$W_1(x_t, y_t, z_t) \le \delta_3(x^2 + y^2 + z^2) \ \mathit{for} \ \mathit{some}\ \delta_3 > 0. \tag{2.13}$$

Hence, from the results (2.8), (2.10) and (2.13), we find that all conditions of Theorem 2.1 are satisfied. The proof of Theorem 2.2 is now complete.

3. Boundedness result

Assuming Equation (2.1) and the function $W(t, x_t)$ chosen from $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfies the following.

Assumption 3.1 [18, 33] *We assume that for any solution $x(t)$ of (2.1) and for any fixed $0 \le t_0 \le T < \infty$, the following condition hold:*

$$E^{x_0}\left\{ \int_{t_0}^{T} W_{x_i}^2(t, x_t)) N_{ik}^2(t, x_t) dt \right\} < \infty, \ 1 \le i \le n, 1 \le k \le m. \tag{3.1}$$

Theorem 3.1 [18, 33] *Suppose that there exists a function $W(t, x_t)$ in $C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n; \mathbb{R}^+)$ satisfying Assumption 3.1, such that for all $(t, x_t) \in \mathbb{R}^+ \times \mathbb{R}^n$:*

$$(i)\|x\|^{q_1} \le W \le \|x\|^{q_2},$$

$$(ii)\mathcal{L}W \le -\nu(t)\|x\|^n + \beta(t),$$

$$(iii)W - W^{n/q_2} \le \gamma,$$

where $\nu, \beta \in C(\mathbb{R}^+; \mathbb{R}^+)$, q_1, q_2, n are positive constants, $q_1 \ge 1$ and γ is a nonnegative constant. Then all solutions of (2.1) satisfy

$$E^{x_0}\|x(t; t_0, x_0)\| \le \left\{ W(t_0, x_0) e^{-\int_{t_0}^{t} \nu(s) ds} + \int_{t_0}^{t} \big(\gamma \nu(u) + \beta(u)\big) e^{-\int_{u}^{t} \nu(s) ds} du \right\}^{1/q_1},$$

for all $t \ge t_0$.

Definition 3.1 [18, 33] *Let $M(t, 0, 0) = 0$ and $N(t, 0, 0) = 0$. We say that the zero solution of (2.1) is ν-UEAS in probability, if there exists a positive continuous function $\nu(t)$ such that $\int_{t_0}^{t} \nu(s) ds \to \infty$ as $t \to \infty$ and constants Γ, $C \in \mathbb{R}^+$ such that any solution $x(t; t_0, x_0)$ of (2.1) satisfies the following*

$$E^{x_0}\|x(t, t_0, x_0)\| \le C(\|x_0\|, t_0)\big(e^{-\int_{t_0}^{t} \nu(s) ds}\big)^{\Gamma}, \ \mathit{for} \ \ \mathit{all}\ t \ge t_0,$$

where the constant C may depend on t_0 and x_0. The zero solution of (2.1) is said to be ν-UEAS in probability, if C is independent of t_0.

Corollary 3.1 [18, 33] *Suppose that the hypotheses of Theorem 3.1 hold. In addition*

$$\int_{t_0}^{t} \{\gamma \nu(u) + \beta(u)\} e^{-\int_{u}^{t} \nu(s) ds} du \le \mathcal{M}, \ \mathit{for\ all}\ t \ge t_0 \ge 0, \tag{3.2}$$

for some positive constant $\mathcal{M}$, then all solutions of (2.1) are USB.

Corollary 3.2 [18, 33] *Suppose* $M(t,0,0)=0$ *and* $N(t,0,0)=0$. *Assume*

$$\int_{t_0}^{t}\{\gamma\nu(u)+\beta(u)\}e^{\int_{t_0}^{u}\nu(s)ds}du \leq \mathcal{M}, \text{ for all } t \geq t_0 \geq 0, \tag{3.3}$$

for some positive constant $\mathcal{M}$ *and*

$$\int_{t_0}^{t}\nu(s)ds \to \infty, \text{ as } t \to \infty. \tag{3.4}$$

If the hypotheses of Theorem 3.1 hold, then the zero solution of (2.1) is ν*-UEAS in probability with* $\Gamma = 1/q_1$.

Theorem 3.2 *If the conditions* $(i)-(vii)$ *of Theorem 2.2 hold. In addition, we assume that the following conditions are satisfied:*

$(viii)$ $\alpha^2 \leq \frac{2c_1L_0(a_1b_1-\psi_0+1)-a_1-b_1-2}{a_1+1}$.

(ix) $|p(t,x(t),\dot{x}(t),\ddot{x}(t))| \leq m, \quad m>0.$

Then,

(1) *All solutions of Equation (1.1) are USB, provided that*

$$r < \min\Bigg\{\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2(a_1b_1-\psi_0+1)L},$$
$$\frac{\Delta+a_1\psi_0-\mu a_0 f_1-(b_1+3)}{2L(2a_1^2+a_1+\mu+2+a_1b_1-\psi_0)L}, \frac{2\Delta-b_1}{2Lb_1(a_1+1)}\Bigg\}.$$

(2) *The zero solution of (1.1) is* $\nu-$ *UEAS in probability.*

Proof of Theorem 3.2.
In this case, we have $p \neq 0$ and the equivalent system is

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= z,\\
\dot{z} &= -a(t)f(x,y)z - b(t)\varphi(x)y - c(t)\psi(x) + c(t)\int_{t-r}^{t}\psi'(x(s))y(s)ds\\
&\quad - g(t,x)\dot{\omega}(t) + p(t,x,y,z).
\end{aligned}$$

Consider the function

$$W(x_t,y_t,z_t) = W_1(x_t,y_t,z_t) + W_2(x_t,y_t,z_t),$$

where W_1 is defined as (2.4) and we can define Lyapunov functional W_2 as the following

$$\begin{aligned}
W_2(x_t,y_t,z_t) =& c(t)a_1^2\int_0^x \psi(\xi)d\xi + a(t)a_1^2\int_0^y f(x,\eta)\eta d\eta + a_1c(t)\psi(x)y\\
&+\frac{b_1}{2}(a_1b_1-\psi_0)x^2 \ + (a_1b_1-\psi_0)x(z+a_1y)\\
&+a_1^2yz + \frac{\psi_0}{2}c(t)y^2 + \frac{a_1}{2}z^2.
\end{aligned} \tag{3.5}$$

Differentiating (3.5) by applying IF (2.2) and using the assumptions of Theorem 2.2, we get

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \leq & c'(t)a_1^2\int_0^x \psi(\xi)d\xi + a_1c'(t)\psi(x)y + \frac{\psi_0}{2}c'(t)y^2 - \frac{a_1\psi_0}{2}y^2 \\ & + a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta - (a_1b_1-\psi_0)c_1L_0x^2 + \frac{a_1}{2}g^2(t,x) \\ & + \Big\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\Big\}\int_{t-r}^t \psi'(x(s))y(s)ds \\ & + \Big\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\Big\}p(t,x(t),\dot{x}(t),\ddot{x}(t)).\end{aligned}$$

Now, let the function $R_2(t,x,y)$ take the form

$$\begin{aligned}R_2 = & c'(t)a_1^2\int_0^x \psi(\xi)d\xi + a_1c'(t)\psi(x)y + \frac{\psi_0}{2}c'(t)y^2 \\ & + a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta.\end{aligned} \tag{3.6}$$

We have here two cases:

Case 1. If $c'(t) = 0$ and from (ii), then Equation (3.6) becomes

$$R_1 = a_1^2a'(t)\int_0^y f(x,\eta)\eta d\eta \leq a_1^2a_0\int_0^y f(x,\eta)\eta d\eta \ \leq 0.$$

Case 2. If $c'(t) < 0$, then $R_2(t,x,y)$ can be written as

$$R_2 = c'(t)R_3(t,x,y), \tag{3.7}$$

where

$$R_3 = a_1^2\int_0^x \psi(\xi)d\xi + a_1\psi(x)y + \frac{\psi_0}{2}y^2 + a_1^2\frac{a'(t)}{c'(t)}\int_0^y f(x,\eta)\eta d\eta.$$

It follows that

$$\begin{aligned}R_3 = & \frac{a_1^2a'(t)}{c'(t)}\int_0^y f(x,\eta)\eta d\eta + \frac{1}{2\psi_0}(\psi_0y + a_1\psi(x))^2 \\ & + \frac{a_1^2}{2\psi_0y^2}\Bigg[4\int_0^x \psi(\xi)d\xi\Big\{\int_0^y (\psi_0 - \psi'(\xi)\Big\}\eta d\eta\Bigg].\end{aligned}$$

By using the condition (ii), we have $0 \leq \frac{a'(t)}{c'(t)} \leq 1$, and since $\psi_0 - \psi'(\xi) \geq \frac{\psi_0}{2}$, by (iii), we find

$$R_3 \geq \frac{1}{2\psi_0}\Big(\psi_0y + a_1\psi(x)\Big)^2 + \frac{a_1^2}{2}\int_0^x \psi(\xi)d\xi \geq 0.$$

Then, we conclude

$$R_2 = c'(t)R_3(t,x,y) \leq 0.$$

Therefore, on combining the two cases for $c'(t)$, we obtain $R_2(t,x,y) \leq 0$, for all x, y, and $t \geq 0$. Thus, we get

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \leq &- \frac{a_1\psi_0}{2}y^2 - (a_1b_1-\psi_0)c_1L_0x^2 + \frac{a_1}{2}g^2(t,x)\\ &+\left\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\right\}\int_{t-r}^{t}\psi'(x(s))y(s)ds\\ &+\left\{(a_1b_1-\psi_0)x + a_1^2y + a_1z\right\}p(t,x,y,z).\end{aligned}$$

Now, from the assumptions of Theorem 3.2 and by applying the fact that $|pq| \leq \frac{1}{2}(p^2+q^2)$, we conclude

$$\begin{aligned}\mathcal{L}W_2(x_t,y_t,z_t) \leq &-\left\{\frac{2(a_1b_1-\psi_0)c_1L_0 - a_1\alpha^2}{2} - \frac{(a_1b_1-\psi_0)L}{2}\right\}x^2\\ &-\left\{\frac{a_1\psi_0}{2} - \frac{a_1^2L}{2}r\right\}y^2 + \frac{1}{2}a_1Lrz^2\\ &+\left\{\frac{(a_1b_1-\psi_0)L}{2} + \frac{(a_1^2+a_1)L}{2}\right\}\int_{t-r}^{t}y^2(s)ds\\ &+\left\{(a_1b_1-\psi_0)|x| + a_1^2|y| + a_1|z|\right\}m.\end{aligned} \tag{3.8}$$

Now, from Equation (1.1), Lyapunov functional W_1 in (2.4), and by using Equation (2.7), we conclude

$$\begin{aligned}\mathcal{L}W_1(t,x_t,y_t,z_t) \leq &-\left\{\frac{2c_1L_0 - a_1 - b_1 - 2 - \alpha^2}{2} - \frac{L}{2}r\right\}x^2\\ &-\left\{\frac{\Delta - 3 - \mu a_0f_1 - b_1}{2} - \frac{\mu L}{2}r - \lambda r\right\}y^2\\ &-\left\{\frac{a_1}{2} - \mu - \frac{1}{2} - \frac{L}{2}r\right\}z^2 + \left\{\frac{1}{2}L(\mu+2) - \lambda\right\}\int_{t-r}^{t}y^2(s)ds\\ &+\big(|x| + \mu|y| + |z|\big)m.\end{aligned} \tag{3.9}$$

By combining Equations (3.8) and (3.9), we obtain

$$\begin{aligned}\mathcal{L}W(t,x_t,y_t,z_t) \leq &-\left\{\frac{2c_1L_0(a_1b_1-\psi_0+1) - (a_1+b_1+2) - (a_1+1)\alpha^2}{2}\right.\\ &\left.-\frac{(a_1b_1-\psi_0+1)L}{2}r\right\}x^2 - \left\{\frac{a_1\psi_0 + \Delta - \mu a_0f_1 - (b_1+3)}{2}\right.\\ &\left.-\frac{a_1^2}{2}Lr - \lambda r\right\}y^2 - \left\{\frac{2\Delta - b_1}{2b_1} - \frac{(a_1+1)L}{2}r\right\}z^2\\ &+\left\{\frac{(a_1b_1-\psi_0)L + (a_1^2+a_1)L + (\mu+2)L}{2} - \lambda\right\}\int_{t-r}^{t}y^2(s)ds\\ &+\big\{(a_1b_1-\psi_0+1)|x| + (a_1^2+\mu)|y| + (a_1+1)|z|\big\}m.\end{aligned}$$

Choose $\lambda = \frac{\{(a_1b_1-\psi_0)+a_1^2+a_1+\mu+2\}L}{2}$; therefore, the above equation becomes

$$\begin{aligned}\mathcal{L}W(t,x_t,y_t,z_t) \le -\bigg\{&\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2}\\&-\frac{(a_1b_1-\psi_0+1)L}{2}r\bigg\}x^2-\bigg\{\frac{a_1\psi_0+\Delta-\mu a_0f_1-(b_1+3)}{2}\\&-\frac{(2a_1^2+a_1b_1-\psi_0+a_1+\mu+2)L}{2}r\bigg\}y^2\\&-\bigg\{\frac{2\Delta-b_1}{2b_1}-\frac{(a_1+1)L}{2}r\bigg\}z^2\\&+\big\{(a_1b_1-\psi_0+1)|x|+(a_1^2+\mu)|y|+(a_1+1)|z|\big\}m.\end{aligned}$$

If

$$\begin{aligned}r<\min\bigg\{&\frac{2c_1L_0(a_1b_1-\psi_0+1)-(a_1+b_1+2)-(a_1+1)\alpha^2}{2(a_1b_1-\psi_0+1)L},\\&\frac{\Delta+a_1\psi_0-\mu a_0f_1-(b_1+3)}{2L(2a_1^2+a_1+\mu+2+a_1b_1-\psi_0)L},\frac{2\Delta-b_1}{2Lb_1(a_1+1)}\bigg\}.\end{aligned}$$

Thus, we have the following

$$\begin{aligned}\mathcal{L}W &\le -H(x^2+y^2+z^2)+kH(|x|+|y|+|z|)\\&=-\frac{H}{2}(x^2+y^2+z^2)-\frac{H}{2}\{(|x|-k)^2+(|y|-k)^2+(|z|-k)^2\}+\frac{3H}{2}k^2\\&\le -\frac{H}{2}(x^2+y^2+z^2)+\frac{3H}{2}k^2,\ \text{ for some } k,H>0,\end{aligned}$$

where

$$k=m\max\{a_1b_1-\psi_0+1,a_1^2+\mu,a_1+1\}.$$

By the conditions $(i)-(iii)$ of Theorem 2.2, we have

$$\begin{aligned}W_2(x_t,y_t,z_t)\ge &a_1^2c_1\int_0^x\psi(\xi)d\xi+\frac{c_1}{2\psi_0}\big(\psi_0y+a_1\psi(x)\big)^2-\frac{a_1^2c_1}{2\psi_0}\psi^2(x)+\frac{a_1}{2}(z+a_1y)^2\\&+\frac{(a_1b_1-\psi_0)}{2b_1}\bigg\{b_1x+(z+a_1y)\bigg\}^2-\frac{(a_1b_1-\psi_0)}{2b_1}(z+a_1y)^2.\end{aligned}$$

Therefore, we get

$$\begin{aligned}W_2(x_t,y_t,z_t)\ge &\frac{c_1a_1^2}{2\psi_0y^2}\bigg[4\int_0^x\psi(\xi)d\xi\bigg\{\int_0^y\big(\psi_0-\psi'(\xi)\big)\bigg\}\eta d\eta\bigg]+\frac{c_1}{2\psi_0}(\psi_0y+a_1\psi(x))^2\\&+\frac{(a_1b_1-\psi_0)}{2b_1}\bigg\{b_1x+(z+a_1y)\bigg\}^2+\frac{\psi_0}{2b_1}(z+a_1y)^2.\end{aligned}$$

From the condition (iii), we obtain

$$\begin{aligned} W_2(x_t, y_t, z_t) \geq & \frac{c_1 a_1^2}{2} \int_0^x \psi(\xi) d\xi + \frac{c_1}{2\psi_0} \big(\psi_0 y + a_1 \psi(x)\big)^2 \\ & + \frac{(a_1 b_1 - \psi_0)}{2b_1} \big\{b_1 x + (z + a_1 y)\big\}^2 + \frac{\psi_0}{2b_1}(z + a_1 y)^2. \end{aligned} \tag{3.10}$$

Therefore, by both Equations (2.9) and (3.10), we conclude

$$\begin{aligned} W(x_t, y_t, z_t) \geq & \frac{1}{2b_1} \Big\{ b_1 y + c_1 \psi(x) \Big\}^2 + \big(\mu y + \frac{z}{2}\big)^2 + \frac{\mu}{2}(a_1 - 2\mu) y^2 + (x + \frac{z}{2})^2 \\ & + \frac{c_1(a_1 b_1 - \psi_0 + 2a_1^2)}{4} \int_0^x \psi(\xi) d\xi + \frac{c_1}{2\psi_0} \Big(\psi_0 y + a_1 \psi(x) \Big)^2 \\ & + \frac{(a_1 b_1 - \psi_0)}{2b_1} \Big\{ b_1 x + (z + a_1 y) \Big\}^2 + \frac{\psi_0}{2b_1}(z + a_1 y)^2. \end{aligned}$$

Therefore, for positive constant δ_4, we have

$$W(x_t, y_t, z_t) \geq \delta_4 (x^2 + y^2 + z^2). \tag{3.11}$$

From (3.5) and using the conditions $(i) - (iii)$ of Theorem 2.2, we find

$$\begin{aligned} W_2(x_t, y_t, z_t) \leq & a_1^2 L x^2 + a_2 a_1^2 f_1 y^2 + a_1 L xy + \frac{b_1}{2}(a_1 b_1 - \psi_0) x^2 \\ & + (a_1 b_1 - \psi_0) x (z + a_1 y) + a_1^2 yz + \frac{\psi_0}{2} y^2 + \frac{a_1}{2} z^2. \end{aligned}$$

By using the fact that $2pq \leq (p^2 + q^2)$, then the last inequality becomes

$$\begin{aligned} W_2(x_t, y_t, z_t) \leq & \Bigg\{ \frac{(a_1^2 + a_1)L + (a_1 b_1 - \psi_0)(a_1 + b_1 + 1)}{2} \Bigg\} \|x\|^2 \\ & + \Bigg\{ \frac{a_1^2 a_2 f_1 + a_1 L + \psi_0 + a_1^2 + a_1(a_1 b_1 - \psi_0)}{2} \Bigg\} \|y\|^2 \\ & + \Bigg\{ \frac{(a_1 b_1 - \psi_0) + a_1 + a_1^2}{2} \Bigg\} \|z\|^2. \end{aligned} \tag{3.12}$$

Hence, by combining the two inequalities (2.12) and (3.12), we obtain

$$\begin{aligned} W(x_t, y_t, z_t) \leq & \Bigg\{ \frac{(\mu + 1 + a_1^2 + a_1)L + (a_1 b_1 - \psi_0)(a_1 + b_1 + 1) + 3}{2} \Bigg\} \|x\|^2 \\ & + \Bigg\{ \frac{a_1^2 a_2 f_1 + (a_1 + 1)L + \psi_0 + \mu(a_2 f_1 + 1)}{2} \\ & + \frac{a_1^2 + b_2 \varphi_2 + a_1(a_1 b_1 - \psi_0) + \lambda r^2}{2} \Bigg\} \|y\|^2 \\ & + \Bigg\{ \frac{(a_1 b_1 - \psi_0) + (a_1 + a_1^2) + (\mu + 2)}{2} \Bigg\} \|z\|^2. \end{aligned}$$

Consequently, for positive constant δ_5, we conclude

$$W \leq \delta_5(\|x\|^2 + \|y\|^2 + \|z\|^2). \tag{3.13}$$

Thus, the assumptions (ii) of Theorem 3.1 is satisfied by taking $\nu(t) = H/2$, $\beta(t) = (3H/2)k^2$ and $n = 2$. From inequalities (3.11) and (3.13), we see that the LF $W(x_t, y_t, z_t)$ also satisfies the condition (i) of Theorem 3.1. As well as we can test that the condition (iii) of Theorem 3.1 is satisfied with $q_1 = q_2 = n = 2$ with $\gamma = 0$. Then, all conditions of Theorem 3.1 hold.

Therefore, with $\nu(t) = H/2, \beta(t) = (3H/2)k^2$ and $n = 2$, with $\gamma = 0$, we find that

$$\int_{t_0}^{t} \{\gamma\nu(u) + \beta(u)\}e^{-\int_u^t \nu(s)ds}du = (3H/2)k^2 \int_{t_0}^{t} e^{-\frac{H}{2}\int_u^t ds}du \leq 3k^2,$$

for all $t \geq t_0 \geq 0$. Thus, condition (3.2) holds. Now, since

$$\begin{aligned}
&g^T = (00\ \ -g(t,x)),\\
&W_x = (W_1)_x + (W_2)_x\\
&\ \ = \mu c(t)\psi(x) + 2x + z + c(t)\psi'(x)y + b(t)\varphi'(x)\frac{y^2}{2} + c(t)a_1^2\psi(x)\\
&\ \ + a_1c(t)\psi'(x)y + b_1(a_1b_1 - \psi_0)x + (a_1b_1 - \psi_0)(z + a_1y),\\
&W_y = (W_1)_y + (W_2)_y\\
&\ \ = c(t)\psi(x) + \mu a(t)f(x,y)y + \mu z + b(t)\varphi(x)y + a_1^2a(t)f(x,y)y\\
&\ \ + a_1c(t)\psi(x) + (a_1b_1 - \psi_0)a_1x + a_1^2z + \psi_0c(t)y,\\
&W_z = (W_1)_z + (W_2)_z = \mu y + z + x + (a_1b_1 - \psi_0)x + a_1^2y + a_1z.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|W_{x_i}(t,x)N_{ik}(t,x)| \leq &\alpha\Bigg[\left\{\frac{\mu + 3 + 2(a_1b_1 - \psi_0) + a_1^2 + a_1}{2}\right\}x^2 + \left(\frac{\mu + a_1^2}{2}\right)y^2\\
&+ \left(\frac{a_1 + 1}{2}\right)z^2\Bigg] := \vartheta(t).
\end{aligned}$$

Therefore, condition (3.1) is satisfied. Hence, by Corollary 3.1 all solutions of (1.1) are USB and satisfy

$$E^{x_0}\|x(t,t_0,x_0)\| \leq \{Cx_0^2 + 3k^2\}^{1/2}, \text{ for all } t \geq t_0 \geq 0,$$

where C is a constant. Next,

$$\begin{aligned}
\int_{t_0}^{t}\{\gamma\nu(u) + \beta(u)\}e^{\int_{t_0}^u \nu(s)ds}du &= (3H/2)k^2 \int_{t_0}^{t} e^{\frac{H}{2}\int_{t_0}^u ds}du\\
&= 3k^2(e^{t-t_0} - 1) \leq \mathcal{M},
\end{aligned}$$

for all $t \geq t_0 \geq 0$, where $\mathcal{M}$ is a positive constant. Hence condition (3.3) is satisfied. We can see that condition (3.4) is also satisfied. By Corollary 3.2, we find that the zero solution of (1.1) is ν-UEAS in probability with $\Gamma = 1/2$.

4. Examples

Example 4.1 *With $p = 0$, consider the following SDDE of third-order*

$$\begin{aligned}&\dddot{x}(t) + (-2\sin t + 13.5)\Big(1 + \frac{3}{5 + 2x^5 + \dot{x}^4}\Big)\ddot{x}(t) + \big(1 - \frac{1}{10 - t^4}\big)\big(1 + \frac{1}{8}e^{-x}\big)\dot{x}(t)\\ &+ \big(1 - \frac{1}{8 - t^2}\big)\Big\{20x^{\frac{1}{5}}(t - r) + \frac{1}{4}\sin x(t - r)\Big\} + \frac{1}{4}xe^{-t}\dot{w}(t) = 0.\end{aligned} \tag{4.1}$$

The equivalent system of (4.1) is

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -(-2\sin t + 13.5)\big(1 + \frac{3}{5 + 2x^5 + y^4}\big)z\\ &\quad - \big(1 - \frac{1}{10 - t^4}\big)\big(1 + \frac{1}{8}e^{-x}\big)y - \big(1 - \frac{1}{8 - t^2}\big)\big(20x^{\frac{1}{5}} + \frac{1}{4}\sin x\big)\\ &\quad + \big(1 - \frac{1}{8 - t^2}\big)\int_{t-r}^{t}\Big(4x^{\frac{-4}{5}}(s) + \frac{1}{4}\cos x(s)\Big)y(s)ds\\ &\quad - \frac{1}{4}xe^{-t}\dot{w}(t).\end{aligned} \tag{4.2}$$

If we compare system 2.3 with system 4.2, we get the following

$$a(t) = -2\sin t + 13.5,$$

we notice that

$$11.5 \leq -2\sin t + 13.5 \leq 15.5, \quad \textit{then } a_1 = 11.5, \ a_2 = 15.5.$$

And also

$$a'(t) = -2\cos t, \ \textit{it follows that } -2 \leq a'(t) \leq -1, \textit{so } \ a_0 = -1.$$

Figures 1 and 2 show the bounds of $a(t)$ with $t \in [-8\pi, 8\pi]$ and $a'(t)$ with $t \in [0, \frac{\pi}{3}]$.

The function

$$f(x, y) = 1 + \frac{3}{5 + 2x^5 + y^4}, \ \textit{since } 0 \leq \frac{3}{5 + 2x^5 + y^4} \leq \frac{3}{5},$$

therefore, we find

$$1 \leq f(x, y) \leq \frac{8}{5}, \ \textit{then } f_1 = \frac{8}{5},$$

and

$$f_x(x, y) = \frac{-30x^4}{(5 + 2x^5 + y^4)^2} < 0.$$

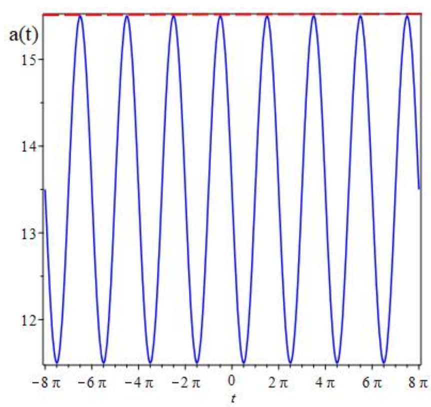

Figure 1. Trajectory of $a(t)$.

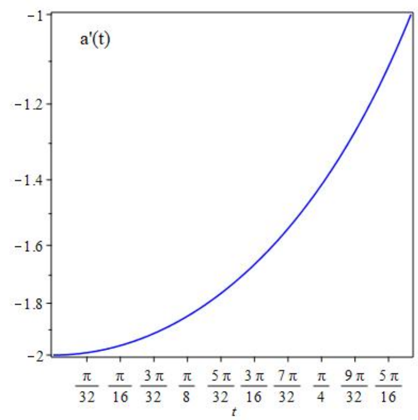

Figure 2. Trajectory of $a'(t)$.

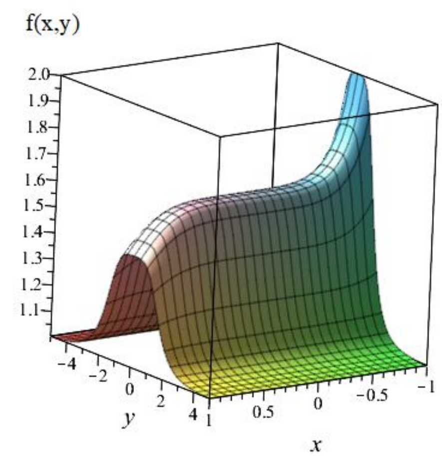

Figure 3. Trajectory of $f(x, y)$.

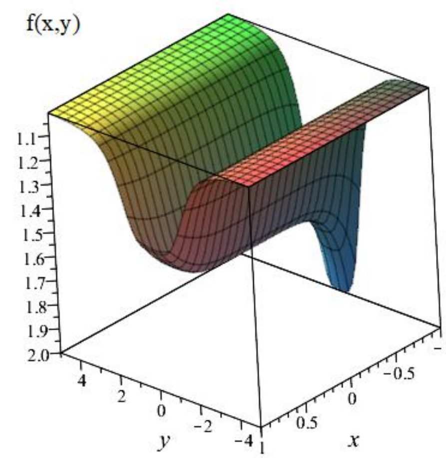

Figure 4. Trajectory of $f(x, y)$.

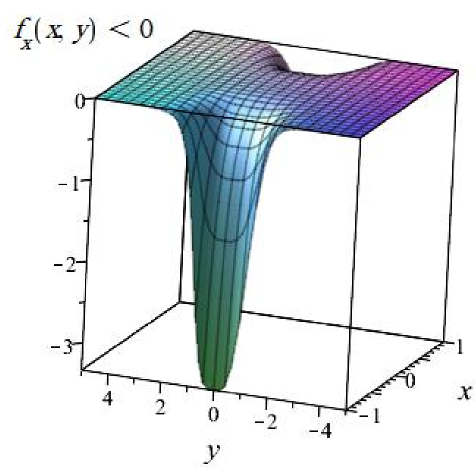

Figure 5. Trajectory of $f_x(x, y)$.

Figures 3 and 4 illustrate the path of the function $f(x,y)$*, with* $x \in [-1,1]$ *and* $y \in [-5.5]$ *and also Figure 5 shows the behaviour of the function* $f_x(x,y)$ *with* $x \in [-1,1]$ *and* $y \in [-5.5]$*.*

The function

$$b(t) = 1 - \frac{1}{10 - t^4}, \quad \textit{then } \frac{9}{10} \le b(t) \le 1 \textit{ and } b_1 = 0.9.$$

The derivative of $b(t)$ *in terms to* t *is*

$$b'(t) = \frac{-4t^3}{(10 - t^4)^2} \le 0.$$

Now, the function

$$\psi(x) = 20x^{\frac{1}{5}} + \frac{1}{4}\sin x, \quad \textit{so we get } \frac{\psi(x)}{x} = 20x^{\frac{-4}{5}} + \frac{1}{4x}\sin x.$$

We know that

$$-\frac{1}{4} \le \frac{1}{4x}\sin x \le \frac{1}{4},$$

thus, we get

$$\frac{\psi(x)}{x} \ge 8.31 = L_0.$$

And also

$$|\psi'(x)| = |4x^{-\frac{4}{5}} + \frac{1}{4}\cos x| \le 4.25 = L; \textit{ therefore, we find}$$

$$\sup\{|\psi'(x)|\} = 4.25 = \frac{\psi_0}{2}, \textit{ then } \psi_0 = 8.5.$$

Now, Figures 6 and 7 show the path of the functions $\psi'(x)$ *and* $\frac{\psi(x)}{x}$ *with* $x \in [1,3]$*.*

We have also the function

$$c(t) = 1 - \frac{1}{8 - t^2} \textit{ so, } \frac{7}{8} \le c(t) \le 1, \quad c_1 = \frac{7}{8}, \quad \textit{with } c'(t) = \frac{-2t}{(8 - t^2)^2} \le 0.$$

The paths of $b(t)$ *and* $c(t)$ *for* $t \in [-10,10]$ *are depicted in Figure 8.*

The function

$$g(t,x) = \frac{1}{4}xe^{-t}, \textit{ then } g^2(t,x) = \frac{1}{16}x^2e^{-2t} \le \frac{1}{16}x^2.$$

Figures 9 and 10 show the trajectory of $g^2(t,x)$ *for all* t *and* $x,t \in [-4.4]$*.*

First, we get

$$2c_1L_0 - a_1 - b_1 - 2 = 0.14 > \frac{1}{16} = \alpha^2.$$

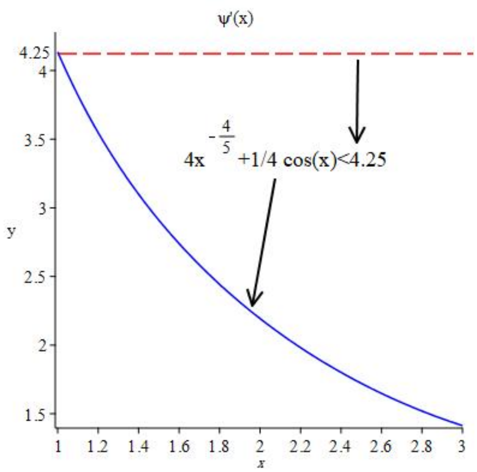

Figure 6. The path of $\psi'(x)$.

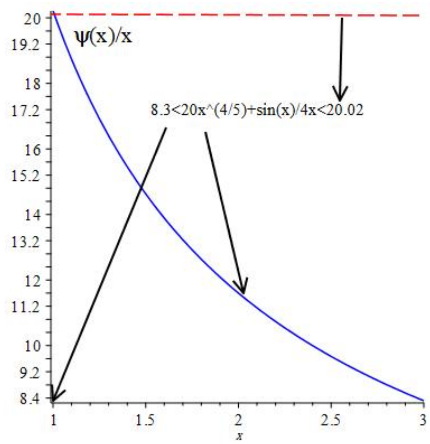

Figure 7. The path of $\frac{\psi(x)}{x}$.

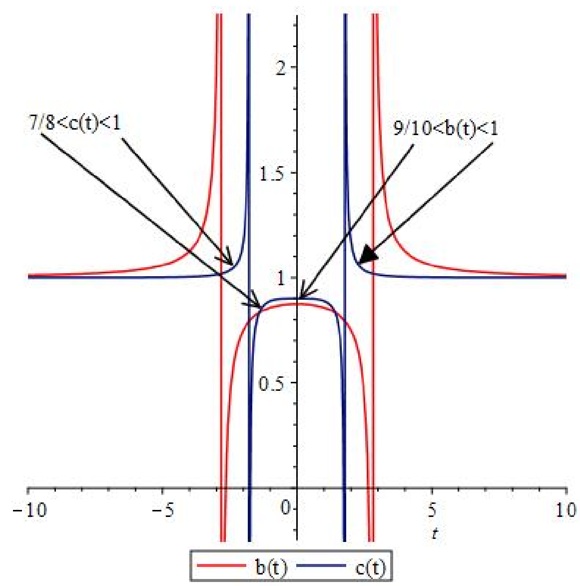

Figure 8. Trajectory of $b(t)$ and $c(t)$.

Second, with $\mu = \frac{a_1 b_1 + \psi_0}{4 b_1} = 5.24$, *then we obtain*

$$3 + b_1 + \mu a_0 f_1 = -4.48, \;\; \textit{and} \;\; \Delta = \frac{a_1 b_1 - \psi_0}{4} = 0.46 > 0,$$

it follows that

$$\Delta > 3 + b_1 + \mu a_0 f_1.$$

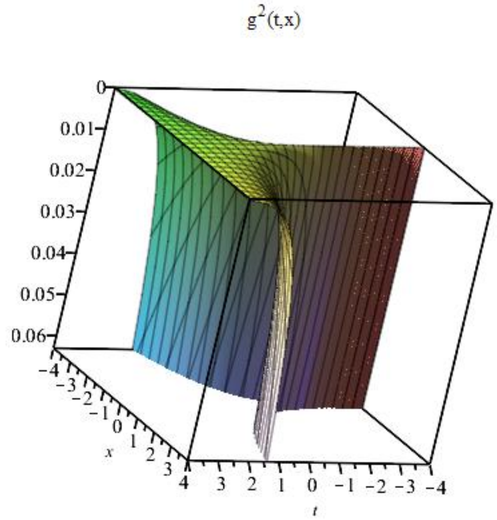

Figure 9. Trajectory of the function $g^2(t,x)$.

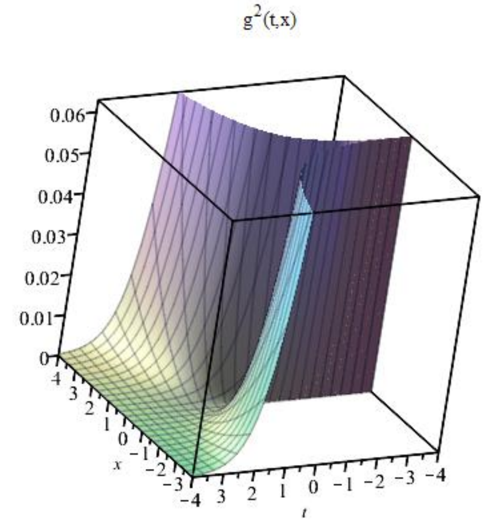

Figure 10. Trajectory of the function $g^2(t,x)$.

Finally, we conclude

$$\begin{aligned}\mathcal{L}W_1(t,x_t,y_t,z_t) \leq &-(0.04-2.13r)x^2 - \{2.47-(\frac{\mu L}{2}+\lambda)r\}y^2 \\ &-\{0.01-2.63r\}z^2 + \left\{\frac{1}{2}L(\mu+2)-\lambda\right\}\int_{t-r}^{t} y^2(s)ds.\end{aligned}$$

Take $\lambda = \frac{L}{2}(\mu+2) = 15.39 > 0$; *therefore, we get*

$$\mathcal{L}W_1(t,x_t,y_t,z_t) \leq -\big(0.04-2.13r\big)x^2 - (2.47-29.15r)y^2 + \{0.01-2.63r\}z^2.$$

Provided that

$$r < \min\{0.01, 0.04, 0.002\} \cong 0.002.$$

Hence, all conditions of Theorem 2.2 are satisfied, then the zero solution of (4.1) is SAS.

Example 4.2 *Consider here* $p \neq 0$, *then the SDDE (4.1) becomes*

$$\begin{aligned}&\dddot{x}(t) + (-2\sin t + 13.5)\left(1+\frac{3}{5+2x^5+\dot{x}^4}\right)\ddot{x}(t) + \big(1-\frac{1}{10-t^4}\big)\big(1+\frac{1}{8}e^{-x}\big)\dot{x}(t) \\ &\quad + \big(1-\frac{1}{8-t^2}\big)\left\{20x^{\frac{1}{5}}(t-r)+\frac{1}{4}\sin x(t-r)\right\} + \frac{1}{4}xe^{-t}\dot{w}(t) = p(t,x,\dot{x}(t),\ddot{x}(t)).\end{aligned} \tag{4.3}$$

We have

$$\frac{2c_1L_0(a_1b_1-\psi_0+1)-a_1-2}{a_1+1} = 2.43 > \frac{1}{16} = \alpha^2.$$

Thus, we obtain

$$\begin{aligned}\mathcal{L}W \leq &-(13.58-6.06r)x^2\\ &-\{51.35-(24.4+\lambda)r\}y^2-\{0.01-26.56r\}z^2\\ &+\left\{\frac{(a_1b_1-\psi_0)L+(a_1^2+a_1)L+(\mu+2)L}{2}-\lambda\right\}\int_{t-r}^{t}y^2(s)ds\\ &+\{(a_1b_1-\psi_0+1)|x|+(a_1^2+\mu)|y|+(a_1+1)|z|\}m.\end{aligned}$$

Take $\lambda=\frac{\{(a_1b_1-\psi_0)+a_1^2+a_1+\mu+2\}L}{2}=324.79>0$ *and let* $m=0.01$, *then we find*

$$\begin{aligned}\mathcal{L}W \leq &-\big(13.58-6.06r\big)x^2\\ &-(51.35-349.23r)y^2-\big(0.01-26.56r\big)z^2\\ &+0.03|x|+1.37|y|+0.13|z|,\end{aligned}$$

with

$$r<\min\{1.12,0.07,0.0002\}.$$

If we take $H=0.3$ *and* $m=0.01$, *then we obtain*

$$k=0.01\max\{2.58,137.49,12.5\}\cong 1.37.$$

Now, we can satisfy the condition (ii) of Theorem 3.1 by taking

$$\nu=0.15 \ \textit{and}\ \beta(t)=0.84, \ \ \textit{with}\ n=2.$$

Then, since $q_1=q_2=n=2$, *we get all assumptions of Theorem 3.1 are satisfied. It follows from the above estimates, the following inequality holds*

$$\int_{t_0}^{t}\{\gamma\nu(u)+\beta(u)\}e^{\int_{t_0}^{u}\nu(s)ds}du\leq 5.63, \quad \textit{for all } t\geq t_0\geq 0.$$

Furthermore,

$$|W_{x_i}(t,x)N_{ik}(t,x)|\leq\frac{1}{4}\bigg(77.85x^2+68.75y^2+6.25z^2\bigg):=\vartheta(t).$$

Next,

$$E^{x_0}\|x(t,t_0,x_0)\|\leq\{x_0^2+5.63\}^{1/2}, \ \textit{for all } t\geq t_0\geq 0. \tag{4.4}$$

Hence condition (3.3) is satisfied. By Corollary 3.2, we conclude that the zero solution of (4.3) is ν*-UEAS in probability with* $\Gamma=1/2$.

Simulation of the solutions: Here, by the numerical methods we will simulate the solutions of (4.1) and (4.3). Consequently, Figures 11 and 12 show the behaviour of the stochastic stability of the solution for (4.1) with the noise $\alpha = 0.25$ and $\alpha = 10$, respectively. Furthermore, Figures 13–15 illustrate the behaviour of the boundedness of the solutions for (4.3), with the noise $\alpha = 0.25, \alpha = 10$, and $\alpha = 100$, respectively. Hence, we get from our figures, the simulated solutions are SAS and USB which justifies our given results.

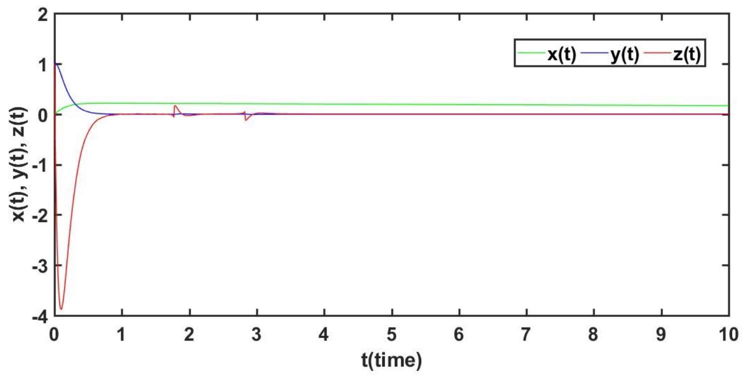

Figure 11. The behaviour for the stability of the solutions for (4.1), with $\alpha = 0.25$..

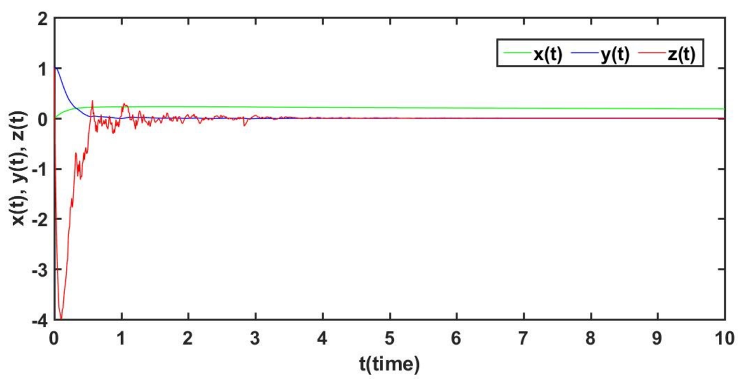

Figure 12. The behaviour for the stability of the solutions for (4.1), with $\alpha = 10$..

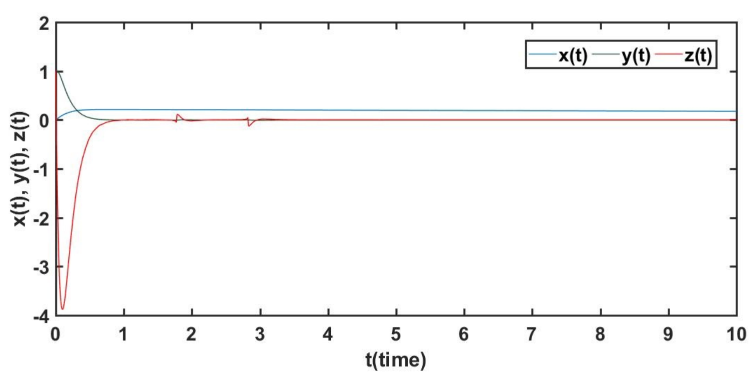

Figure 13. The path of the bonundedness for the solutions for (4.3), with $\alpha = 0.25$..

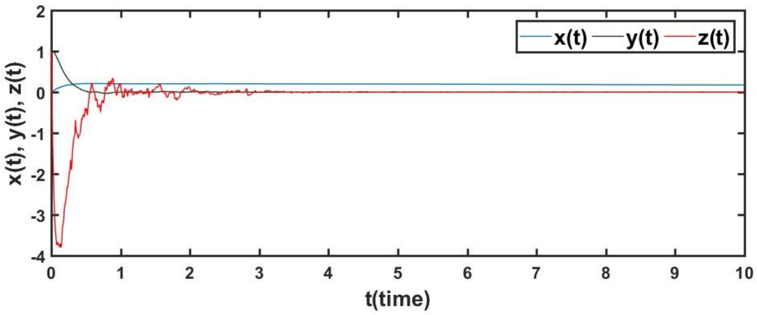

Figure 14. The path for the bonundedness of the solutions for (4.3), with $\alpha = 10.$.

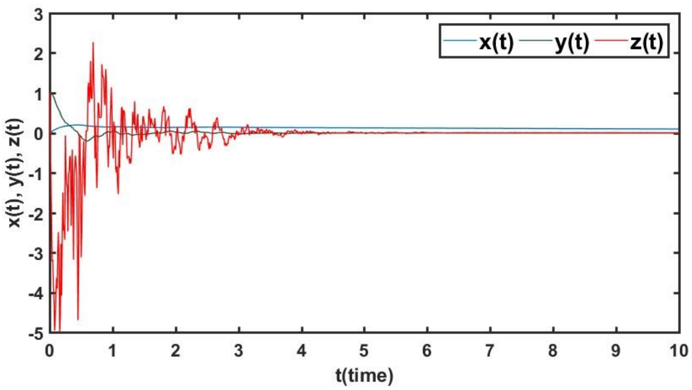

Figure 15. The path of the bonundedness of the solutions for (4.3), with $\alpha = 100.$.

5. Conclusion

The main results of the paper have discussed the following objects:
First: with $p = 0$, sufficiency criteria were established to study the SAS of the zero solution for (1.1).
Next: with $p \neq 0$, we established the sufficient conditions of the USB and UEAS in probability of solutions for (1.1).
Finally: two examples were given to illustrate feasibility of the established results and correctness of the main results. Lyapunove direct method was employed to set up the results.

The results obtained in this investigation extend many existing and exciting results on nonlinear nonautonomous third-order SDDE.

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