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Generalization of statistical limit-cluster points and the concepts of statistical limit inferior-superior on time scales by using regular integral transformations

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Abstract: With the aid of regular integral operators, we will be able to generalize statistical limit-cluster points and statistical limit inferior-superior ideas on time scales in this work. These two topics, which have previously been researched separately from one another sometimes only in the discrete case and other times in the continuous case, will be studied at in a single study. We will investigate the relations of these concepts with each other and come to a number of new conclusions. On some well-known time scales, we shall analyze these ideas using examples.

Key words: Regular summability methods, statistical convergence, delta measure on time scales, time scales, statistical limit and cluster points, statistical limit superior and inferior

1. Introduction

In nature, researchers must deal with variables that contain both discrete and continuous situations. In some real-world problems, a variable may be continuous for some time before becoming discrete or vice versa. For instance, while modelling an insect population, we can need a continuous variable while in the season but this variable may become discrete in winter (see [9, 10]). As a result, the necessity to establish a new variable that encompasses both discrete and continuous cases as well as those in between them has arisen. In fact, the idea of combining discrete and continuous cases is quite old and may even be dated back to the origins of the Riemann-Stieltjes integral, which combines sums and integrals. Finally, Stefan Hilger filled this fundamental gap in the literature in 1988 in his PhD thesis [26] by introducing measure chains, which would later be called time scale calculus. Since any discipline that needs simultaneous modeling of discrete and continuous data can benefit from the use of time scale calculus, it has received intense attention from researchers from many different fields, from ecology to economics to control theory to population models (see, i.e.[1, 5–8, 27, 31, 37]). By 2012, the first use of time scales in summability theory was presented in [38]. Thus, the first convergence method, which is called statistical convergence on time scales was defined. Since then, many researchers have attempted to answer the question, "Is it possible to transfer the convergence methods studied in discrete analysis, i.e. in natural numbers, to any time scale?" Many studies have been inspired by this question (see [39–42]).

One of the important convergence methods in summability theory is the concept of statistical convergence which was introduced by Fast [19]. A nonnegative regular summability matrix A can be used to generalize the concept of statistical convergence for number sequences in the classical case. This idea was first mentioned by

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Buck [12] and further studied by different authors [14, 15, 17, 18, 33]. This generalized limit is then referred to as A-statistical convergence. In our most recent study, inspired by the previously mentioned generalization of natural numbers, we extended the concept of statistical convergence to functions defined on time scales with the help of some regular integral transformations instead of a matrix [42]. Our main purpose in [42] is to extend the concept of A-statistical convergence known for number sequences to functions defined on time scales. As a result, we are able to combine discrete and continuous cases that are known from the literature in [12, 14, 15, 20, 30, 33] as well as derive new convergence methods in our examples with the selection of suitable transformations and time scales.

The ideas of statistical limit points and statistical cluster points of a number sequence were first discussed by Fridy using the nonthin concept in [21]. If $\left(x_{n(j)}\right)$ is a subsequence of the number sequence $x = (x_n)$ and for the set $K = \{n(j) : j \in \mathbb{N}\}$ does not have density zero, then $\left(x_{n(j)}\right)$ is a nonthin subsequence of x. Using this nonthin concept, Fridy defined statistical limit points and statistical cluster points of the number sequence $x = (x_n)$ as follows:

A real number λ is called a statistical limit point of the number sequence x, if there exists a nonthin subsequence of x that converges to λ.

A real number γ is called a statistical cluster point of the number sequence x, if for every $\varepsilon > 0$ the set $\{n \in \mathbb{N} : |x_n - \gamma| < \varepsilon\}$ does not have natural density zero.

These two point types have been established and investigated in various ways by many researchers by modifying the density function (For more works one can examine [29, 32]). The concept of a statistical limit and cluster point of a number sequence was further expanded by Connor and Kline to A-statistical limit and $A-$statistical cluster points in [16]. In light of the traces in the classical case, we, therefore, came to the conclusion that we can derive a generalization for statistical limits and cluster points on time which are already defined in [34]. In the second part of this article, we will give general definitions for $k-$statistical limit and $k-$statistical cluster points on a time scale using kernel functions and regular integral operators and the results will be illustrated with examples. Furthermore, the connections between these points will be investigated, and some significant conclusions will be discovered for each point type.

After defining and analyzing the statistical limit and cluster points in natural numbers, researchers moved on to the ideas of statistical limit inferior and superior notions in several papers (for instance [23]). This topic is still of considerable interest to researchers [28]. Our motivation in the third section is to unify the statistical limit-cluster point and statistical limit inferior-superior studies, which were done independently for the classical case, in a single study by acquiring their more general form. In the third part of this paper, we focused on the concepts of statistical limit inferior and statistical limit superior which is introduced by Fridy and Orhan in [23]. For a real sequence x, they defined following sets $B_x = \{b \in \mathbb{R} : \delta\{n : x_n > b\}\}$ and $A_x = \{a \in \mathbb{R} : \delta\{n : x_n < a\}\}$. With the help of these sets, they construct the statistical limit inferior and the statistical limit superior concepts for a number sequence x as follows:

$$st - \limsup x = \begin{cases} \sup B_x, & \text{if } \; B_x \neq \varnothing \\ -\infty, \text{ if } & B_x = \varnothing \end{cases}$$

and

$$st - \liminf x = \begin{cases} \inf A_x, & \text{if } \; A_x \neq \varnothing \\ \infty, \text{ if } & A_x = \varnothing. \end{cases}$$

In [18], Demirci extended these concepts to A-statistical convergence using a nonnegative regular summability matrix A in place of C_1. This generalization for A-statistical convergence inspired us in the third part, as it did in the second. With the use of kernel function and regular integral operators, an analogous generalization for the statistical limit inferior and superior notions, which have already been developed on time scales in paper [35], will be established as $k-$statistical limit inferior and superior. The relationships of these concepts with each other and with $k-$statistical convergence will be investigated and the relationships of these concepts with cluster points will be demonstrated using a theorem.

To be able to discuss the main results in greater depth, we believe it is necessary to first introduce the following basic definitions and facilities for a reader unfamiliar with time scale calculus.

A time scale denoted by $\mathbb{T}$, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of real numbers (see [10] for details). Throughout the paper time scale is assumed to be unbounded above and bounded below, i.e.

$$\inf \mathbb{T} = t_0 \ \ (t_0 > 0) \quad \text{and} \quad \sup \mathbb{T} = \infty. \tag{1.1}$$

The definitions of two important operators that are frequently used when classifying time scales are given as follows: The forward jump operator σ on $\mathbb{T}$ is defined by

$$\sigma : \mathbb{T} \to \mathbb{T},\ \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

Similarly, the backward jump operator ρ on $\mathbb{T}$ is given by

$$\rho : \mathbb{T} \to \mathbb{T},\ \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

Another frequently used function on time scales is the graininess function μ given by

$$\mu : \mathbb{T} \to [0, \infty),\ \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called right-dense $\sigma(t) = t$ which also implies $\mu(t) = 0$; otherwise, it is called right-scattered. Similarly, the backward jump operator is used to define left-dense and left-scattered point.

With the help of forward jump and backward jump operators, we can summarize the type of points of time scales as follows:

- if $t < \sigma(t)$, then t is right-scattered
- if $t = \sigma(t)$, then t is right dense
- if $\rho(t) < t$, then t is t left-scattered
- if $\rho(t) = t$, then t is left dense
- if $\rho(t) < t < \sigma(t)$, then t is isolated
- if $\rho(t) = t = \sigma(t)$, then t is dense.

By the notation $[a, b]_{\mathbb{T}}$ we denote an interval entirely in $\mathbb{T}$ such that $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. Obviously, the other type intervals such as $[a, b)_{\mathbb{T}}$ and $(a, b]_{\mathbb{T}}$ can be defined similarly.

In this paper, like in our earlier ones (see [38–41]), we should use the Lebesgue Δ-measure μ_{Δ} that Guseinov presented in [24] (see also [4]). Additionally, Guseinov estimated the Lebesgue Δ-measure for all forms of the intervals on $\mathbb{T}$ in these articles, as shown below:

Let $a, b \in \mathbb{T}$ and $a \leq b$. Then

- $\mu_{\Delta}\left([a,b)_{\mathbb{T}}\right)=b-a,$
- $\mu_{\Delta}\left((a,b)_{\mathbb{T}}\right)=b-\sigma\left(a\right),$
- $\mu_{\Delta}\left((a,b]_{\mathbb{T}}\right)=\sigma\left(b\right)-\sigma\left(a\right),$
- $\mu_{\Delta}\left([a,b]_{\mathbb{T}}\right)=\sigma\left(b\right)-a.$

Now let us review the essential concepts of density and convergence methods on time scales.

Let Ω be a Δ-measurable subset of $\mathbb{T}$ and consider the following set:

$$\Omega\left(x\right)=\left\{y\in[t_0,x]_{\mathbb{T}}:y\in\Omega\right\}.$$

Then, the density of Ω is given by

$$\delta_{\mathbb{T}}\left(\Omega\right):=\lim_{x\to\infty}\frac{\mu_{\Delta}\left(\Omega\left(x\right)\right)}{\mu_{\Delta}\left([t_0,x]_{\mathbb{T}}\right)}$$

provided that the above limit exists. Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. We say that f is statistically convergent on $\mathbb{T}$ to a number L if, for every $\varepsilon>0$,

$$\delta_{\mathbb{T}}\left(\left\{x\in\mathbb{T}:\left|f\left(x\right)-L\right|\geqslant\varepsilon\right\}\right)=0 \tag{1.2}$$

holds (see [38] for details). We denote this statistical limit on a time scale $\mathbb{T}$ by

$$st_{\mathbb{T}}-\lim f=L.$$

Then, it is not hard to see that (1.2) is equivalent to the following limit:

$$\lim_{x\to\infty}\frac{\mu_{\Delta}\left(\left\{y\in[t_0,x]_{\mathbb{T}}:|f(y)-L|\geqslant\varepsilon\right\}\right)}{\mu_{\Delta}\left([t_0,x]_{\mathbb{T}}\right)}=0.$$

Let $k:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ be a $\Delta\times\Delta$-measurable function on the product time scale $\mathbb{T}\times\mathbb{T}$. Throughout this paper we also assume that, for every $y\in\mathbb{T}$, $k(x,y)$ is a Lebesgue Δ-integrable function on $\mathbb{T}$ (see [13, 24] for details about the Lebesgue Δ-integration on time scales). By Ψ we denote the family of all nonnegative kernel functions k satisfying the following three conditions:

- $\lim\limits_{x\to\infty}\int\limits_{[t_0,Y]_{\mathbb{T}}}|k(x,y)|\,\Delta y=0$ for every finite $Y\in\mathbb{T}$.
- $\lim\limits_{x\to\infty}\int\limits_{\mathbb{T}}k(x,y)\Delta y=1.$
- $\sup\limits_{x\in\mathbb{T}}\int\limits_{\mathbb{T}}|k(x,y)|\,\Delta y<\infty.$

Actually, these given conditions are Silverman-Toeplitz type conditions (see [11, 25, 36]) but give only sufficient conditions for the regularity of a kernel function on time scales. In this case, if $k\in\Psi$, then k is regular for bounded functions on $\mathbb{T}$. Then, the k-density of Δ-measurable subset Ω on $\mathbb{T}$, which is denoted by $\delta_{k-\mathbb{T}}\left(\Omega\right)$, is defined by

$$\delta_{k-\mathbb{T}}\left(\Omega\right):=\lim_{x\to\infty}\int\limits_{y\in\Omega}k\left(x,y\right)\Delta y$$

provided that the above limit exists. Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function on $\mathbb{T}$. We say that f is k-statistically convergent to a number L if, for every $\varepsilon>0$,

$$\delta_{k-\mathbb{T}}\left(\{y\in\mathbb{T}:|f(y)-L|\geq\varepsilon\}\right)=0$$

holds. Then, this limit is denoted by

$$st_{k-\mathbb{T}}-\lim f=L.$$

Then using $k-$density definition one can observe that $st_{k-\mathbb{T}}-\lim f=L$ if and only if

$$\lim_{x\to\infty}\int\limits_{y\in\mathbb{T}\,:\,|f(y)-L|\geq\varepsilon}k(x,y)\,\Delta y=0$$

which is time scale analogues of $A-$statistical convergence. More details see [42].

It is assumed that the function $k(x,y)$ will be one of the set Ψ's elements in all definitions that will be made throughout this article. Also, it has been verified that the $k(x,y)$ functions used in the examples satisfy the Ψ family's requirements.

2. Generalization of statistical limit and cluster points by using integral transformations on time scales

In classical summability theory, sets with zero density are crucial. The first thing that came to mind when considering how to generalize the ordinary limit concept was to use zero density sets to generalize it. As a result, a variety of convergence methods, such as statistical convergence in [19] and lacunary statistical convergence in [22], have been developed with the help of zero density sets. In this present chapter, firstly using a kernel function and integral transformation zero density sets will be defined. It should be noted here that if a suitable kernel function is selected, all the definitions in the literature on a fixed time scale will become examples for our definition.

Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function and Ω is a nonempty subset of $\mathbb{T}$. We say that Ω is k-nonthin if

$$\begin{aligned}\delta_{k-\mathbb{T}}(\Omega) &= \lim_{x\to\infty}\int\limits_{y\in\Omega}k(x,y)\,\Delta y \\ &\neq 0.\end{aligned} \tag{2.1}$$

It should be noted that Ω is k-nonthin subset of $\mathbb{T}$ either limit in (2.1) does not exist or the result of this limit is a positive number.

Using the concept of k-nonthin subsets we will define $k-$statistical limit points and $k-$statistical cluster points as follows:

Definition 2.1 *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. A real number L is called $k-$statistical limit point of the function f provided that there is a set $\Omega\subset\mathbb{T}$ which is $k-$ nonthin and*

$$\lim_{\substack{t\to\infty\\ t\in\Omega}}f(t)=L.$$

By Λ_{k-f} we denote the set of all $k-$statistical limit points of function f.

To clarify this definition let us examine the following example.

Example 2.2 *Let $\mathbb{T} = [a, \infty)$, $a > 0$ which is the continuous case. In this time scale, consider the following kernel function:*

$$k(x, y) = \begin{cases} \frac{1}{\mu_\Delta([a,x]_\mathbb{T})}, & \text{if } y \in [a, x]_\mathbb{T} \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Actually, this kernel function is familiar to us and it comes from the definition of statistical convergence on time scales in [42]. Using this kernel function, one can calculate the following function's $k-$statistical limit points:

$$f(t) = \begin{cases} 1, & \text{if } t \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Without loosing generality, we can assume $a < 1$,

$$\begin{aligned} \delta_{k-\mathbb{T}}(\mathbb{N}) &= \lim_{x\to\infty} \int\limits_{y\in\mathbb{N}} k(x, y)\,\Delta y \\ &= \lim_{x\to\infty} \frac{1}{\mu_\Delta([a, x])} \int\limits_{y\in[a,x]\cap\mathbb{N}} dy \\ &= \lim_{x\to\infty} \frac{1}{x-a} \left(\mu_\Delta(\{1, 2, .., \lfloor x \rfloor\})\right) \\ &= 0 \end{aligned}$$

where $\lfloor x \rfloor$ denotes the greatest integer function. From the property of k-density $\delta_{k-\mathbb{T}}(\mathbb{T} \setminus \mathbb{N}) = 1$ which implies $\mathbb{T} \setminus \mathbb{N}$ is $k-$nonthin and so 0 is only $k-$statistical limit point of function f which means $0 \in \Lambda_{k-f}$.

Let L_f denotes the limit points of function $f(t)$, then it is clear that $L_f = \{0, 1\}$.

It is clear that for any given Δ-measurable f function $\Lambda_{k-f} \subset L_f$.

As another example, let $\mathbb{T} = \mathbb{N}$ which is a discrete case. Using kernel function defined in (2.2) for natural numbers, one can find the $k-$statistical limit points of

$$f(t) = \begin{cases} 1, & \text{if } t = n^2 \text{ for } n = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases}$$

which is given in [21] as an Example 1. For this given f, it is obvious that $\Lambda_{\mathbb{N}-f} = \{0\}$ and $L_f = \{0, 1\}$ as stated in [21].

Definition 2.3 *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function. A real number L is called $k-$statistical cluster point of the function f provided that for every $\varepsilon > 0$, the set*

$$\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}$$

is a $k-$ nonthin set which means

$$\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}) \neq 0.$$

By Γ_{k-f} we denote the set of all $k-$statistical cluster points of function f.

Proposition 2.4 *If $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function, then $\Lambda_{k-f}\subset\,\Gamma_{k-f}$.*

Proof Suppose that $L\in\Lambda_{k-f}$, then there exists a k-nonthin subset of $\mathbb{T}$, say Ω, such that $\delta_{k-\mathbb{T}}(\Omega)\neq 0$ and

$$\lim_{\substack{t\to\infty\\ t\in\Omega}} f(t)=L. \tag{2.3}$$

From the definition of $k-$density

$$\limsup_{x\to\infty}\int\limits_{y\in\Omega} k(x,y)\,\Delta y=d>0.$$

For each $\varepsilon>0$, using the (2.3) we know that $A=\{t\in\nleq : |f(t)-L|\geq\varepsilon\}$ is a finite subset of Ω. So we have

$$\Omega\diagdown A\subset\{t\in\mathbb{T}:|f(t)-L|<\varepsilon\}.$$

Using monotonicity of $k-$density and knowledge of $k-$density of finite sets, we finally get

$$\limsup_{x\to\infty}\int\limits_{y\in\{t\in\mathbb{T}:|f(t)-L|<\varepsilon\}} k(x,y)\,\Delta y\geq d>0$$

which means $\delta_{k-\mathbb{T}}(\{t\in\mathbb{T}:|f(t)-L|<\varepsilon\})\neq 0.$ Hence, $L\in\,\Gamma_{k-f}$. □

Our experience with ordinary limit points in real numbers gives us the illusion that sets Λ_{k-f} and Γ_{k-f} are equal. However, this is not true. The inclusion given in Proposition 2.4 is proper. One can check the counter example in [21] which is obtained by choosing suitable time scales and kernel function on time scales.

Proposition 2.5 *Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. The set of all $k-$statistical cluster point of the function f, that is Γ_{k-f} is closed.*

Proof Let (L_n) be a real number sequence in Γ_{k-f} such that

$$\lim_{n\to\infty} L_n=L.$$

We need to show that $L\in\,\Gamma_{k-f}$. For a given $\varepsilon>0$, there exists a $\gamma\in(L_n)$ in the interval $(L-\varepsilon,L+\varepsilon)$. Choose $\varepsilon^{'}$ such that

$$\left(\gamma-\varepsilon^{'},\gamma+\varepsilon^{'}\right)\subset(L-\varepsilon,L+\varepsilon)$$

and

$$\left\{t\in\mathbb{T}:|f(t)-\gamma|<\varepsilon^{'}\right\}\subset\{t\in\mathbb{T}:|f(t)-L|<\varepsilon\}.$$

Since $\gamma\in(L_n)$ implies $\gamma\in\Gamma_{k-f}$,

$$\delta_{k-\mathbb{T}}\left(\left\{t\in\mathbb{T}:|f(t)-\gamma|<\varepsilon^{'}\right\}\right)\neq 0.$$

So we have

$$\delta_{k-\mathbb{T}}(\{t\in\mathbb{T}:|f(t)-L|<\varepsilon\})\neq 0$$

and $L\in\,\Gamma_{k-f}$. □

The next example will show us that unlike set Γ_{k-f}, set Λ_{k-f} is not closed.

Example 2.6 *Let $q > 1$ and consider $\mathbb{T} = q^{\mathbb{N}}$. We should note that this time scale known as the quantum scale takes great attention from different fields such as fluid mechanics, combinatorics (for instance see [3] and references therein). In this time scale, we use the following kernel function:*

$$k(q^n, q^m) = \begin{cases} \frac{1}{\mu_\Delta\left([1,q^n]_{q^{\mathbb{N}}}\right)}, & \text{if } q^m \in [1, q^n]_{q^{\mathbb{N}}} \\ 0, & \text{otherwise} \end{cases}$$

and choose the following function:

$$f(q^n) = \begin{cases} 1, \text{ if } n = 2s+1 \\ \frac{1}{p}, \text{ if } n = 2s \end{cases}$$

where $p-1$ is the number of factors of 2 in prime factorization of n. Now let us show that 1 is a $k-$statistical limit point of f:

$$\begin{aligned} \delta_{k-q^{\mathbb{N}}}\left(\left\{q^n \in q^{\mathbb{N}} : f(q^n) = 1\right\}\right) &= \delta_{k-q^{\mathbb{N}}}\left(\left\{q, q^3, q^5, ...\right\}\right) \\ &= \lim_{n\to\infty} \int\limits_{q^m \in \{q,q^3,q^5,...\}} \frac{1}{\mu_\Delta\left([1,q^n]_{q^{\mathbb{N}}}\right)} \Delta q^m. \end{aligned}$$

Without loosing generality we can assume that n is odd and continue to calculate as follows:

$$\begin{aligned} \delta_{k-q^{\mathbb{N}}}\left(\left\{q, q^3, q^5, ...\right\}\right) &= \lim_{n\to\infty} \frac{1}{q^{n+1}-1} \mu_\Delta\left(\left\{q, q^3, q^5, ..., q^n\right\}\right) \\ &= \lim_{n\to\infty} \frac{(q-1)\left(q + q^3 + ... + q^n\right)}{q^{n+1}-1} \\ &= \frac{1}{1+q} = \frac{1}{[2]} \end{aligned}$$

where $[2]$ shows the $q-$integer. This result, obtained here with a specially chosen kernel function, is proven as Lemma 13 in [2]. Now let us show that for each p number, $\frac{1}{p}$ is an element of set Λ_{k-f}. Firstly we assume that

$$n = q^{2^{(p-1)}m}$$

where m is an odd number, then

$$\begin{aligned}
& \delta_{k-q^{\mathbb{N}}}\left(\left\{q^n \in q^{\mathbb{N}} : f\left(q^n\right) = \frac{1}{p}\right\}\right) \\
= \; & \delta_{k-q^{\mathbb{N}}}\left(\left\{q^{2^{(p-1)}(2s+1)} : s = 0,1,2,...\right\}\right) \qquad (2.4)\\
= \; & \lim_{n\to\infty} \int\limits_{q^m \in \left\{q^{2^{(p-1)}}, q^{3.2^{(p-1)}}, q^{5.2^{(p-1)}},...\right\}} \frac{1}{\mu_\Delta\left(\left[1,q^n\right]_{q^{\mathbb{N}}}\right)} \Delta q^m \\
= \; & \lim_{n\to\infty} \frac{1}{q^{n+1}-1} \mu_\Delta\left(\left\{q^{2^{(p-1)}(2s+1)} : s = 0,1,..., \frac{n-2^{(p-1)}}{2^p}\right\}\right) \\
= \; & \lim_{n\to\infty} \frac{(q-1)\left(q^{2^{(p-1)}} + q^{3.2^{(p-1)}} + ... + q^n\right)}{q^{n+1}-1} \\
= \; & \lim_{n\to\infty} \frac{(q-1)\sum\limits_{k=0}^{\frac{n-2^{(p-1)}}{2^p}} q^{2^{(p-1)}(2s+1)}}{q^{n+1}-1} \\
= \; & (q-1) \lim_{n\to\infty} \frac{\frac{q^{2^{(p-1)}}}{q^{2^p}-1}\left(q^{n+2^{(p-1)}}-1\right)}{q^{n+1}-1} \\
= \; & \frac{(q-1)\, q^{2^{(p-1)}}}{q^{2^p}-1} q^{\left(2^{(p-1)}-1\right)} = \frac{q^{2^p-1}}{\left[2^p\right]}.
\end{aligned}$$

It is obvious that for each $p \geq 2$ natural number, $k-$density defined in (2.4) is not zero. At this point, it is worth emphasizing that the results obtained in this part are $q-$like of Example 3 in [21]. Now we assert see $0 \in \Gamma_{k-f}$. To see this, we need to show

$$\delta_{k-q^{\mathbb{N}}}\left(\left\{q^n \in q^{\mathbb{N}} : 0 < f\left(q^n\right) < \frac{1}{p}\right\}\right) \neq 0.$$

Then by induction for each $p \geq 2$, we have

$$\delta_{k-q^{\mathbb{N}}}\left(\left\{q^n \in q^{\mathbb{N}} : 0 < f\left(q^n\right) < \frac{1}{p}\right\}\right) \leq \frac{q}{\left[2^p\right]}$$

which implies $0 \in \Gamma_{k-f}$. We have that

$$\Gamma_{k-f} = \{0\} \cup \left\{\frac{1}{p} : p = 1,2,...\right\}.$$

In our last step, we will see $0 \notin \Lambda_{k-f}$. To see this, we can choose a Δ-measurable subset $\Omega \subset q^{\mathbb{N}}$ such that

$$\lim_{\substack{m\to\infty \\ q^m \in \Omega}} f\left(q^m\right) = 0$$

and we need to show that $\delta_{k-q^{\mathbb{N}}}(\Omega)=0$. *For each* p,

$$\begin{aligned}\delta_{k-q^{\mathbb{N}}}(\Omega) &= \delta_{k-q^{\mathbb{N}}}\left(\left\{q^m\in[1,q^n]_{q^{\mathbb{N}}}:q^m\in\Omega\right\}\right)\\ &= \delta_{k-q^{\mathbb{N}}}\left(\left\{q^m\in[1,q^n]_{q^{\mathbb{N}}}:f(q^m)\geq\frac{1}{p}\right\}\right)\\ &\quad+\delta_{k-q^{\mathbb{N}}}\left(\left\{q^m\in[1,q^n]_{q^{\mathbb{N}}}:0<f(q^m)<\frac{1}{p}\right\}\right)\\ &\leq O(1)+\delta_{k-q^{\mathbb{N}}}\left(\left\{q^m\in q^{\mathbb{N}}:0<f(q^m)<\frac{1}{p}\right\}\right)\\ &\leq \frac{q}{[2^p]}.\end{aligned}$$

Since p *is arbitrary this implies* $\delta_{k-q^{\mathbb{N}}}(\Omega)=0$. *Then we can conclude* Λ_{k-f} *is not a closed set.*

In the next theorem, we will show that changing the nonthin subset does not affect the $k-$statistical limit points and $k-$statistical cluster points as stated before in [21] by choosing a special kernel function for discrete case.

Theorem 2.7 *If* $f:\mathbb{T}\to\mathbb{R}$ *and* $g:\mathbb{T}\to\mathbb{R}$ *are two* Δ*-measurable functions such that* $f(t)=g(t)$ *for almost all* $t\in\mathbb{T}$, *then*

$$\Lambda_{k-f}=\Lambda_{k-g} \tag{2.5}$$

and

$$\Gamma_{k-f}=\Gamma_{k-g}. \tag{2.6}$$

Proof Firstly, using the hypothesis, we immediately get

$$A=\{t\in\mathbb{T}:f(t)\neq g(t)\} \tag{2.7}$$

and

$$\delta_{k-\mathbb{T}}(A)=0. \tag{2.8}$$

To obtain (2.5), let $L\in\Lambda_{k-f}$. In this case, we have a nonthin $\Omega\subset\mathbb{T}$ such that

$$\delta_{k-\mathbb{T}}(\Omega)\neq 0 \text{ and } \lim_{\substack{t\to\infty\\ t\in\Omega}} f(t)=L.$$

Since we have (2.8), it follows that

$$\delta_{k-\mathbb{T}}(\{t\in\Omega:f(t)=g(t)\})\neq 0.$$

This means the set $\Omega'=\{t\in\Omega:f(t)=g(t)\}$ is a nonthin set we try to obtain for $k-$statistical limit point of g. So,

$$\lim_{\substack{t\to\infty\\ t\in\Omega'}} f(t)=\lim_{\substack{t\to\infty\\ t\in\Omega'}} g(t)=L$$

which implies $L\in\Lambda_{k-g}$. Then we have $\Lambda_{k-f}\subset\Lambda_{k-g}$. Using the same technique, the opposite side of inclusion can be simply proved from symmetry and it completes the proof of (2.5).

To obtain (2.6), let $\gamma \in \Gamma_{k-f}$ and we have

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : |f(t) - \gamma| < \varepsilon\}\right) \neq 0.$$

Using the set A defined in (2.7),

$$\begin{aligned}
\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : |g(t) - \gamma| < \varepsilon\}\right) &= \delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} \setminus \mathbb{A} : |g(t) - \gamma| < \varepsilon\}\right) \\
&\quad + \delta_{k-\mathbb{T}}\left(\{t \in \mathbb{A} : |g(t) - \gamma| < \varepsilon\}\right) \\
&= \delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} \setminus \mathbb{A} : |f(t) - \gamma| < \varepsilon\}\right) \\
&\neq 0.
\end{aligned}$$

We finally get $\gamma \in \Gamma_{k-f}$ which means $\Gamma_{k-f} \subset \Gamma_{k-g}$. Using the same technique, the opposite side of inclusion can be simply proved from symmetry and it completes the proof of (2.6). □

In order to prove the following theorem, we need to extend the additive property for sets of zero natural density (APO) that was already introduced by Freedman and Sember [20] for natural numbers. Further, they extended it for the sets of zero A-density. From this point of view, we believe that by altering the definition of the density function, this property may be used on various sets. Here we will introduce additive property for sets of zero k-density.

Definition 2.8 *The $k-$ density is said to satisfy condition additive property for sets of zero k-density (APkO) if any given countable collection of mutually disjoint sets $\{\Omega_i\}_{i\in\mathbb{N}}$ in $\mathbb{T}$ with $\delta_{k-\mathbb{T}}(\Omega_i) = 0$ for all i, then there exists a collection of sets $\{\Phi_i\}_{i\in\mathbb{N}}$ in $\mathbb{T}$ such that $\Omega_i \Delta \Phi_i$, where Δ is a symmetric difference of two sets is finite for each i and*

$$\delta_{k-\mathbb{T}}\left(\bigcup_{i\in\mathbb{N}} \Phi_i = \Phi\right) = 0.$$

The following theorem indicates a meaningful connection between limit points and $k-$statistical cluster points.

Theorem 2.9 *If $f : \mathbb{T} \to \mathbb{R}$ is a Δ-measurable function, then there exists a $g : \mathbb{T} \to \mathbb{R}$ Δ-measurable function such that*

$$L_g = \Gamma_{k-f}$$

and $f(t) = g(t)$ for almost all $t \in \mathbb{T}$.

Proof We already have

$$\Gamma_{k-f} \subset L_f.$$

So, for each $L \in L_f \diagdown \Gamma_{k-f}$ choose $\varepsilon > 0$ and define an open interval in $\mathbb{R}$

$$I_L = (L - \varepsilon, L + \varepsilon)$$

such that it is obvious

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : f(t) \in I_L\}\right) = 0.$$

The collection of all such I_L's for each L is a open cover for the set $L_f \diagdown \Gamma_{k-f}$ which means

$$L_f \diagdown \Gamma_{k-f} \subset \bigcup_{L \in L_f \diagdown \Gamma_{k-f}} I_L$$

by the Lindelöf property there exists a countable set such that

$$L_f \diagdown \Gamma_{k-f} = \underset{i\in\mathbb{N}}{\cup} I_{L_i}.$$

For every $i \in \mathbb{N}$, let

$$A_i = \{t \in \mathbb{T} : f(t) \in I_{L_i}\}$$

with $\delta_{k-\mathbb{T}}(A_i) = 0$. Since k-density has the property APkO, there exists a collection of sets $\{\Phi_i\}_{i\in\mathbb{N}}$ in $\mathbb{T}$ such that

$$\delta_{k-\mathbb{T}}\left(\underset{i\in\mathbb{N}}{\cup}\Phi_i = \Phi\right) = 0$$

and $A_i \setminus \Phi$ is a finite set for each i. Let $\mathbb{T} \setminus \Phi := \{s : s \in \mathbb{T}\}$ and define the following function:

$$g(t) := \begin{cases} f_{|\mathbb{T}\setminus\Phi}(t), \text{ if } t \in \Phi \\ f(t), \text{ if } t \in \mathbb{T} \setminus \Phi. \end{cases}$$

Obviously $\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) \neq g(t)\}) = 0$ and from Theorem 2.7 we have

$$\Gamma_{k-f} = \Gamma_{k-g}.$$

Now we need to see that $\Gamma_{k-g} = L_g$. We already have $\Gamma_{k-g} \subset L_g$. To see the other side inclusion, suppose that there exists a l number such that $l \in L_g \setminus \Gamma_{k-g}$. Then there exists a subset $B \subset \mathbb{T}$ such that

$$\lim_{\substack{t\to\infty \\ t\in B}} g(t) = l$$

and for the set B, we have $\delta_{k-\mathbb{T}}(B) = 0$. However, $g_{|\Phi}$ has no limit point. Therefore, no such l can exist which completes the proof. □

3. Generalization of limit inferior and limit superior concepts by using integral transformations on time scales

In this section, we define the concept of $k-$statistical limit superior and $k-$statistical limit inferior of Δ-measurable functions defined on time scales and demonstrate through an example how to compute these points.

Definition 3.1 *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and define the following sets*

$$B_f = \{b \in \mathbb{R} : \delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) > b\}) \neq 0\} \tag{3.1}$$

and likewise

$$A_f = \{a \in \mathbb{R} : \delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) < a\}) \neq 0\} \tag{3.2}$$

then $k-$statistical limit superior of function f is given by

$$st_{k-\mathbb{T}} - \limsup_{t\to\infty} f(t) = \begin{cases} \sup B_f, & \text{if } B_f \neq \varnothing \\ -\infty, & \text{if } B_f = \varnothing \end{cases} \tag{3.3}$$

and also $k-$statistical limit inferior of function f is given by

$$st_{k-\mathbb{T}} - \liminf_{t\to\infty} f(t) = \begin{cases} \inf A_f, & \text{if } A_f \neq \varnothing \\ \infty, & \text{if } A_f = \varnothing. \end{cases} \tag{3.4}$$

We should note the special case of Definition 3.1 was examined in [35] when the kernel function has been chosen especially as indicated below:

$$k(x,y)=\begin{cases}\frac{1}{\mu_\Delta\left([t_0,x]_{\mathbb{T}}\right)}, & \text{if } y\in[t_0,x]_{\mathbb{T}}\\ 0, & \text{otherwise.}\end{cases} \tag{3.5}$$

Definition 3.2 *The $f:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function it is said to be $k-$statistically bounded if there exists a M real number such that*

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>M\}\right)=0.$$

The next example can help the reader to clarify the ideas that were just defined.

Example 3.3 *Let us take $\mathbb{T}=P_{1,1}=\cup_{n=0}^{\infty}[2n,2n+1]$. This is an interesting example of time scales since each interval's right endpoint scatters to the left endpoint of the following interval. In fact, the reason we pick this time scale, in particular, is that it includes both dense and scattered points. For this time scale, let us use the kernel function defined in (3.5) and the function as follows:*

$$f(t)=\begin{cases}2n, & \text{if } t=2n \text{ for } n=1,2,3,\ldots \text{ (for left-scattered points)}\\ 1, & \text{if } t=2n+1 \text{ for } n=1,2,3,\ldots \text{ (for right-scattered points)}\\ \frac{1}{2}, & \text{if } t\in\left(2n,2n+\frac{1}{2}\right] \text{ for } n=1,2,3,\ldots\text{(for some of dense points)}\\ 0, & \text{if } t\in\left(2n+\frac{1}{2},2n+1\right) \text{ for } n=1,2,3,\ldots\text{(for some of dense points).}\end{cases}$$

In this case $f(t)$ is unbounded above, but it is $k-$statistically bounded since the set of $\{2n:n=0,1,\ldots\}$ has zero denstiy on $P_{1,1}$. Let us show it:

$$\begin{aligned}\delta_{k-P_{1,1}}\left(\{t\in P_{1,1}:f(t)>1\}\right) &= \delta_{k-P_{1,1}}\left(\{2,4,6,\ldots\}\right)\\ &= \lim_{x\to\infty}\int\limits_{y\in\{2,4,6,\ldots\}} k(x,y)\,\Delta y\\ &= \lim_{x\to\infty}\int\limits_{y\in[0,x]_{P_{1,1}}\cap\{2,4,6,\ldots\}} \frac{1}{\mu_\Delta\left([0,x]_{P_{1,1}}\right)}\Delta y\\ &= \lim_{x\to\infty}\frac{1}{\sigma(x)}\mu_\Delta\left([0,x]_{P_{1,1}}\cap\{2,4,6,\ldots\}\right).\end{aligned} \tag{3.6}$$

For sufficiently large x, without losing generality we can assume that $\lfloor x\rfloor$ is an even number where $\lfloor x\rfloor$ denotes the greatest integer function. Even if $\lfloor x\rfloor$ is not an even number, $\lfloor x\rfloor-1$ will be an even number and it does not affect limit calculation in (3.6). Then, we have

$$\begin{aligned}\delta_{k-P_{1,1}}\left(\{t\in P_{1,1}:f(t)>1\}\right) &= \lim_{x\to\infty}\frac{1}{\sigma(x)}\mu_\Delta\left(\{2,4,\ldots,\lfloor x\rfloor\}\right)\\ &= 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\delta_{k-P_{1,1}}\left(\{t\in\mathbb{T}:f(t)>0\}\right) &= \delta_{k-P_{1,1}}\left(P_{1,1}\diagdown\cup_{n=0}^{\infty}\left(2n+\frac{1}{2},2n+1\right)\right)\\ &= 1-\delta_{k-P_{1,1}}\left(\cup_{n=0}^{\infty}\left(2n+\frac{1}{2},2n+1\right)\right)\\ &\neq 0\end{aligned}$$

which implies

$$B_f=\left(-\infty,\frac{1}{2}\right)$$

and

$$st_{k-P_{1,1}}-\limsup_{t\to\infty}f(t)=\sup B_f=\frac{1}{2}.$$

Similarly one can see

$$A_f=(0,\infty)$$

and

$$st_{k-P_{1,1}}-\liminf_{t\to\infty}f(t)=\inf A_f=0.$$

Also we can even find $k-$ *statistical cluster points of* f. *For every* $\varepsilon>0$,

$$\begin{aligned}\delta_{k-P_{1,1}}\left(\{t\in\mathbb{T}:|f(t)|<\varepsilon\}\right) &= \lim_{x\to\infty}\frac{1}{\mu_\Delta\left([0,x]_{P_{1,1}}\right)}\int_{y\in\{t\in\mathbb{T}:|f(t)|<\varepsilon\}\cap[0,x]_{P_{1,1}}}\Delta y\\ &= \lim_{x\to\infty}\frac{\mu_\Delta\left(\{0\leq t\leq x:f(t)=0\}\right)}{\mu_\Delta\left([0,x]_{P_{1,1}}\right)}\\ &\neq 0\end{aligned}$$

and

$$\begin{aligned}\delta_{k-P_{1,1}}\left(\left\{t\in\mathbb{T}:\left|f(t)-\frac{1}{2}\right|<\varepsilon\right\}\right) &= \lim_{x\to\infty}\frac{1}{\mu_\Delta\left([0,x]_{P_{1,1}}\right)}\int_{y\in\left\{t\in\mathbb{T}:\left|f(t)-\frac{1}{2}\right|<\varepsilon\right\}\cap[0,x]_{P_{1,1}}}\Delta y\\ &= \lim_{x\to\infty}\frac{\mu_\Delta\left(\left\{0\leq t\leq x:f(t)=\frac{1}{2}\right\}\right)}{\mu_\Delta\left([0,x]_{P_{1,1}}\right)}\\ &\neq 0\end{aligned}$$

which means

$$\Gamma_{k-f}=\left\{0,\frac{1}{2}\right\}.$$

At this time, we would like to call your attention to the fact that this set also includes $k-$ *statistical limit inferior and* $k-$ *statistical limit superior points. This fact gives us the main idea of the following theorem. However, let us first establish the two lemmas that will be needed in the theorem that results from this idea.*

Lemma 3.4 *Let* $f:\mathbb{T}\rightarrow\mathbb{R}$ *be a* Δ*-measurable function.* $st_{k-\mathbb{T}}-\limsup_{t\rightarrow\infty} f(t)=\beta$ *is a finite real number if and only if for every* $\varepsilon>0$

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>\beta-\varepsilon\}\right)\neq 0 \tag{3.7}$$

and

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>\beta+\varepsilon\}\right)=0. \tag{3.8}$$

Proof Suppose that $st_{k-\mathbb{T}}-\limsup_{t\rightarrow\infty} f(t)=\beta$ is a finite real number. By Definition 3.1, $B_f\neq\varnothing$ and $\sup B_f=\beta$. Using properties of supremum in real numbers, for every $\varepsilon>0$ there exists a $N\in B_f$ such that $N>\beta-\varepsilon$. With the help of this last inequality obtained, we have

$$\{t\in\mathbb{T}:f(t)>N\}\subset\{t\in\mathbb{T}:f(t)>\beta-\varepsilon\}$$

and we immediately get (3.7). Now assume that (3.8) does not hold which means

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>\beta+\varepsilon\}\right)\neq 0.$$

Then $\beta+\varepsilon\in B_f$ and this contradicts with $\sup B_f=\beta$. We also have (3.8).

Conversely suppose that (3.7) and (3.8) hold. From (3.7), β is an upper bound for the set B_f. If there is another upper bound for B_f, it must be greater than or equal to β. However using (3.8) we have $\sup B_f=\beta$ and

$$st_{k-\mathbb{T}}-\limsup_{t\rightarrow\infty} f(t)=\beta.$$

□

We may formulate a parallel lemma for $k-$statistical limit inferior using the same reasoning. Since the proof for the upcoming lemma may be produced just as simple as for the prior lemma, it will be given without proof.

Lemma 3.5 *Let* $f:\mathbb{T}\rightarrow\mathbb{R}$ *be a* Δ*-measurable function.* $st_{k-\mathbb{T}}-\liminf_{t\rightarrow\infty} f(t)=\alpha$ *is a finite real number if and only if for every* $\varepsilon>0$

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)<\alpha+\varepsilon\}\right)\neq 0 \tag{3.9}$$

and

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)<\alpha-\varepsilon\}\right)=0. \tag{3.10}$$

The theorem we just mentioned is ready to be presented now. In the following theorem, we will give the relationship observed in Example 3.3 between set of $k-$statistical cluster points and the concepts of $k-$statistical limit superior and $k-$statistical limit inferior.

Theorem 3.6 *Let* $f:\mathbb{T}\rightarrow\mathbb{R}$ *be a* Δ*-measurable function, then*

(i) $st_{k-\mathbb{T}}-\limsup_{t\rightarrow\infty} f(t)=\sup\Gamma_{k-f}$

(ii) $st_{k-\mathbb{T}}-\liminf_{t\rightarrow\infty} f(t)=\inf\Gamma_{k-f}$

where Γ_{k-f} *denotes* $k-$*statistical cluster points of function* f.

Proof (i) In the first part of the proof, we will indicate that

$$\sup B_f = \sup \Gamma_{k-f}$$

where B_f defined in (3.1). Let take $\sup \Gamma_{k-f} = L_1$ and $st_{k-\mathbb{T}} - \limsup_{t\to\infty} f(t) = \sup B_f = L_2$. By Lemma 3.4, we have

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : f(t) > L_2 - \varepsilon\}\right) \neq 0 \tag{3.11}$$

and

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : f(t) > L_2 + \varepsilon\}\right) = 0. \tag{3.12}$$

We firstly see that $L_2 \in \Gamma_{k-f}$. To see this, we will use following idea:

$$\begin{aligned} & \{t \in \mathbb{T} : |f(t) - L_2| < \varepsilon\} \\ = \; & \{t \in \mathbb{T} : f(t) > L_2 - \varepsilon\} \diagdown \{t \in \mathbb{T} : f(t) > L_2 + \varepsilon\} \end{aligned}$$

and using (3.12)

$$\begin{aligned} \delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : |f(t) - L_2| < \varepsilon\}\right) &= \delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : f(t) > L_2 - \varepsilon\}\right) \\ &\neq 0 \end{aligned}$$

which implies $L_2 \in \Gamma_{k-f}$. Since $\sup \Gamma_{k-f} = L_1$, we have

$$L_2 \leq L_1. \tag{3.13}$$

Now assume that $L_1 > L_2$. Since Γ_{k-f} is closed and $\sup \Gamma_{k-f} = L_1$, we know that $L_1 \in \Gamma_{k-f}$ and this implies for every $\varepsilon > 0$,

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : |f(t) - L_1| < \varepsilon\}\right) \neq 0. \tag{3.14}$$

Let us choose ε_1 as

$$\varepsilon_1 = \frac{L_1 - L_2}{2}.$$

For this choosen ε_1, we know that

$$\{t \in \mathbb{T} : |f(t) - L_1| < \varepsilon_1\} \subset \{t \in \mathbb{T} : f(t) > L_2 + \varepsilon_1\}.$$

Using (3.12), we get

$$\delta_{k-\mathbb{T}}\left(\{t \in \mathbb{T} : |f(t) - L_1| < \varepsilon_1\}\right) = 0$$

which contradicts with (3.14). This means $L_2 \leq L_1$ and finally we obtain our desired result

$$L_1 = L_2.$$

(ii) This part of the proof can be demonstrated in the same way as the above. □

Theorem 3.7 *Let $f : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function, then*

$$st_{k-\mathbb{T}} - \liminf_{t\to\infty} f(t) \leq st_{k-\mathbb{T}} - \limsup_{t\to\infty} f(t).$$

Proof

- The case of $st_{k-\mathbb{T}} - \limsup\limits_{t\to\infty} f(t) = \infty$ is obvious.

- In the case of $st_{k-\mathbb{T}} - \limsup\limits_{t\to\infty} f(t) = -\infty$, we have $B_f = \varnothing$ and it means for each $b \in \mathbb{R}$,

$$\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) > b\}) = 0.$$

In other words,

$$\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) \leq b\}) = 1.$$

So for every $a \in \mathbb{R}$

$$\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) < a\}) \neq 0.$$

From last part, we get

$$st_{k-\mathbb{T}} - \liminf_{t\to\infty} f(t) = -\infty.$$

- We can assume that $\beta := st_{k-\mathbb{T}} - \limsup\limits_{t\to\infty} f(t)$ is finite. Let $\alpha := st_{k-\mathbb{T}} - \liminf\limits_{t\to\infty} f(t)$. For every $\varepsilon > 0$, from Lemma 3.4

$$\delta_{k-\mathbb{T}}\left(\left\{t \in \mathbb{T} : f(t) > \beta + \frac{\varepsilon}{2}\right\}\right) = 0$$

and we have

$$\delta_{k-\mathbb{T}}\left(\left\{t \in \mathbb{T} : f(t) \leq \beta + \frac{\varepsilon}{2}\right\}\right) = 1$$

which implies

$$\delta_{k-\mathbb{T}}(\{t \in \mathbb{T} : f(t) < \beta + \varepsilon\}) = 1.$$

So we get $\beta + \varepsilon \in A_f$. Since $st_{k-\mathbb{T}} - \liminf\limits_{t\to\infty} f(t) = \inf A_f = \alpha$, we know for every $\varepsilon > 0$

$$\alpha \leq \beta + \varepsilon$$

which means

$$\alpha \leq \beta.$$

□

As in [18, 23], combining the notion of k-statistical limit inferior-superior on time scales with the previous theorem yields the following conclusion:

$$\liminf_{t\to\infty} f(t) \leq st_{k-\mathbb{T}} - \liminf_{t\to\infty} f(t) \leq st_{k-\mathbb{T}} - \limsup_{t\to\infty} f(t) \leq \limsup_{t\to\infty} f(t).$$

The following theorem demonstrates that a fundamental property known for convergent sequences in classical analysis is also applicable to k-statistical convergence on any time scale.

Theorem 3.8 *Let $f : \mathbb{T} \to \mathbb{R}$ be a $k-$statistically bounded function. Then*

$$st_{k-\mathbb{T}} - \lim_{t\to\infty} f(t) = L \tag{3.15}$$

if and only if

$$st_{k-\mathbb{T}} - \liminf_{t\to\infty} f(t) = st_{k-\mathbb{T}} - \limsup_{t\to\infty} f(t) = L. \tag{3.16}$$

Proof Firstly assume that $st_{k-\mathbb{T}}-\lim\limits_{t\to\infty} f(t)=L$. Then for every $\varepsilon>0$,

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:|f(t)-L|\geq\varepsilon\}\right)=0. \tag{3.17}$$

Using last equality, one can observe

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>L+\varepsilon\}\right)=0 \tag{3.18}$$

and

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)<L-\varepsilon\}\right)=0. \tag{3.19}$$

Let $\beta=st_{k-\mathbb{T}}-\limsup\limits_{t\to\infty} f(t)$ and $\alpha=st_{k-\mathbb{T}}-\liminf\limits_{t\to\infty} f(t)$. From (3.18), we have

$$\beta\leq L. \tag{3.20}$$

Also from (3.19), we have

$$L\leq\alpha. \tag{3.21}$$

Now we combine (3.20) and (3.21) with Theorem 3.7, we can obtain (3.16).

Next assume that (3.16) holds. If we choose $\varepsilon>0$ and using (3.8) and (3.10), we have

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)>L+\varepsilon\}\right)=0$$

and

$$\delta_{k-\mathbb{T}}\left(\{t\in\mathbb{T}:f(t)<L-\varepsilon\}\right)=0$$

which implies

$$st_{k-\mathbb{T}}-\lim_{t\to\infty} f(t)=L.$$

□

4. Concluding remarks

The study of summability theory concepts on time scales is a recent development. Some of the concepts in summability theory have already been studied. However, using regular transformations on time scales instead of matrices in classical case is a new approach. This idea can be generalized for other convergence methods. We will have the opportunity to receive the most general version of the results that have previously been presented in other independent research, similar to the results given in this study. Consequently, depending on the time scale and kernel function used, all prior studies will come to be seen as examples of our study. If possible, we believe it would be fascinating to transfer other A-statistical convergence on natural number notions to time scales in upcoming work.

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