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On solvability of homogeneous Riemann boundary value problems in Hardy-Orlicz classes

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Abstract: This work deals with the Orlicz space and the Hardy-Orlicz classes generated by this space, which consist of analytic functions inside and outside the unit disk. The homogeneous Riemann boundary value problems with piecewise continuous coefficients are considered in these classes. New characteristic of Orlicz space is defined which depends on whether the power function belongs to this space or not. Relationship between this characteristic and Boyd indices of Orlicz space is established. The concept of canonical solution of homogeneous problem is defined, which depends on the argument of the coefficient. In terms of the above characteristic, a condition on the jumps of the argument is found which is sufficient for solvability of these problems, and, in case of solvability, a general solution is constructed. It is established the basicity of the parts of exponential system in Hardy-Orlicz classes.

Key words: Orlicz space, Hardy-Orlicz classes, Riemann boundary value problems, basicity

1. Introduction

Theory of Riemann problems has a long history. These problems probably date back to the study of Riemann [43]. Later Hilbert [21, 22] also considered them and stated a problem which is now referred to as Riemann-Hilbert problem. In the context of applications to some problems of mechanics and mathematical physics, this field has been significantly developed over the years by well-known mathematicians and the theory of these problems has been well covered in the literature [18, 19, 35, 39]. The methods of this theory are also used in other fields of mathematics such as approximation theory, spectral theory of differential operators, etc. The method of boundary value problems is used in establishing basis properties of special function systems in Lebesgue spaces, is due to B.T.Bilalov [3–8, 13, 14]. This method allowed Bilalov to find Riesz basicity criterion for the well-known Kostyuchenko system (see [6–8]) in the space $L_2(0,\pi)$.

Note that the Riemann-Hilbert problems are still of great interest. As the harmonic analysis develops further and new function spaces arise, new statements of Riemann problem appear. For example, since recently there arose great interest in the nonstandard spaces of functions such as Lebesgue space with variable summability index, Morrey space, grand Lebesgue space, etc. (see, e.g., [1, 10, 17, 20, 29, 48]). Various issues of mathematical analysis (such as boundedness of singular integral operators, Riesz potentials, direct and inverse problems of approximation theory with respect to Faber polynomials [25], etc.) are being studied in such spaces. Riemann-Hilbert problems also began to be studied in these spaces in different statements (see, e.g.,

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[9, 12, 27, 36, 38, 40, 44–46]), though many issues in this field still remain unsolved.

In this work, we consider the Orlicz space and the Hardy-Orlicz classes generated by this space, which consist of analytic functions inside and outside the unit disk. The homogeneous Riemann boundary value problems with piecewise continuous coefficients are considered in these classes. We define new characteristic of Orlicz space which depends on whether the power function belongs to this space or not. Relationship between this characteristic and Boyd indices of Orlicz space is established. The concept of canonical solution of homogeneous problem is defined, which depends on the argument of the coefficient. In terms of the above characteristic, a condition on the jumps of the argument is found which is sufficient for solvability of these problems, and, in case of solvability, a general solution is constructed.

As far as the authors know, this is the first time the Riemann problem is considered in Hardy-Orlicz classes. See [26, 47] for more information about these problems.

2. Needful information

We will use the following notations. N will denote the set of positive integers, $Z_+ = \{0\} \bigcup N; Z = \{-N\} \bigcup Z_+$, $\chi_M(\cdot)$ will be the characteristic function of the set M; R will stand for the set of real numbers, by C we will denote the set of complex numbers, $\omega = \{z \in C : |z| < 1\}$ will denote a unit disk in C, $\partial\omega$ will be a unit circle, $\bar{M}$ will stand for the closure of the set M in the corresponding norm, and $(\bar{\cdot})$ will denote the complex conjugation. By $[X]$ we will denote the algebra of linear bounded operators acting in the Banach space X.

Definition 2.1 *Continuous convex function $M(u)$ in R is called an N-function if it is even and satisfies the conditions*

$$\lim_{u\to 0} \frac{M(u)}{u} = 0; \lim_{u\to\infty} \frac{M(u)}{u} = \infty.$$

Definition 2.2 *Let M be an N-function. The function*

$$M^*(v) = \max_{u\geq 0} [u|v| - M(u)],$$

is called an N-function complementary to $M(\cdot)$.

$M(\cdot)$ and $M^*(\cdot)$ can be represented as follows:

$$M(u) = \int_0^{|u|} p(t)\,dt; M^*(v) = \int_0^{|v|} q(s)\,ds,$$

where $p(\cdot)$ and $q(\cdot)$ are positive integrable functions on $(0, +\infty)$.

Definition 2.3 *N-function $M(\cdot)$ satisfies Δ_2-condition for large values of u, if $\exists k > 0 \wedge \exists u_0 \geq 0$:*

$$M(2u) \leq kM(u), \forall u \geq u_0.$$

Now let us define the Orlicz space. Let $M(\cdot)$ be some N-function, $G \subset R$ be a (Lebesgue) measurable set. Denote by $L_0(G)$ the set of all functions measurable in G. Let

$$\rho_M(u) = \int_G M[u(x)]\,dx,$$

and

$$L_M(G) = \{u \in L_0(M) : \rho_M(u) < +\infty\}.$$

$L_M(G)$ is called an Orlicz class.

More details about concerning facts can be found in [30, 42].

In the sequel, as G we will consider the interval $G \equiv [-\pi, \pi]$, and, for simplicity, we will always omit the letter G in the notations (for example, $L_M^*(G) = L_M^*$, etc). Later we will need some facts about Fourier analysis in Orlicz spaces. Let us first define the following characteristic of the space L_M. For the N-function $M(\cdot)$, we denote

$$\gamma_M = \inf\{\alpha : |t|^\alpha \in L_M\}. \tag{2.1}$$

Let us show that $\gamma(M) \geq -1$. In fact, let $\alpha < -1$ be an arbitrary number. Consider

$$\int_0^\pi M(t^\alpha)\,dt = \frac{1}{|\alpha|}\int_{\pi^\alpha}^\infty \frac{M(x)}{x} x^{\frac{1}{\alpha}}dx.$$

From $\lim\limits_{x\to\infty} \frac{M(x)}{x} = \infty$ it follows that $\exists x_0 > \pi^\alpha : \frac{M(x)}{x} \geq 1, \forall x \geq x_0$. Consequently

$$\int_0^\pi M(t^\alpha)\,dt = \frac{1}{|\alpha|}\int_{\pi^\alpha}^{x_0} M(x)\,x^{\frac{1}{\alpha}-1}dx + \frac{1}{|\alpha|}\int_{x_0}^\infty M(x)\,x^{\frac{1}{\alpha}-1}dx \geq$$

$$\geq \frac{1}{|\alpha|}\int_{\pi^\alpha}^{x_0} M(x)\,x^{\frac{1}{\alpha}-1}dx + \frac{1}{|\alpha|}\int_{x_0}^\infty x^{\frac{1}{\alpha}}dx = \infty.$$

It immediately follows $\gamma_M \geq -1$. Let us show that $\forall \alpha > \gamma_M$ the relation $|t|^\alpha \in L_M$ holds. Obviously, $\forall \alpha \geq 0 : |t|^\alpha \in L_M$. Therefore, it suffices to prove that if $|t|^{\alpha_1} \in L_M, \gamma_M \leq \alpha_1 < 0$, then $\forall \alpha_2 \in (\alpha_1, 0)$ the relation $|t|^{\alpha_2} \in L_M$ holds. We have

$$|\alpha_2|\int_0^\pi M(t^{\alpha_2})\,dt = \int_{\pi^{\alpha_2}}^\infty M(x)\,x^{\frac{1}{\alpha_2}-1}dx =$$

$$= \int_{\pi^{\alpha_2}}^1 M(x)\,x^{\frac{1}{\alpha_2}-1}dx + \int_1^\infty M(x)\,x^{\frac{1}{\alpha_1}-1}x^{-\frac{\alpha_2-\alpha_1}{\alpha_2\alpha_1}}dx \leq$$

$$\leq \int_{\pi^{\alpha_2}}^1 M(x)\,x^{\frac{1}{\alpha_2}-1}dx + \int_1^\infty M(x)\,x^{\frac{1}{\alpha_1}-1}dx < +\infty.$$

Thus, the following lemma is true.

Lemma 2.4 *Let $M(\cdot)$ be some N-function. Then for γ_M defined by (2.1) the relation $\gamma_M \in [-1, 0]$ holds. Moreover, $|t|^\alpha \in L_M, \forall \alpha > \gamma_M$.*

Using this lemma, it is easy to prove the following one.

Lemma 2.5 *Let $M(\cdot)$ be some N-function.Then the finite product*

$$f(t) = \prod_{k=0}^m |t - t_k|^{\alpha_k}$$

belongs to L_M if $\alpha_k > \gamma_M, \forall k = \overline{0, m}$, where $-\pi \leq t_0 < \ldots < t_m < \pi$ are arbitrary points.

The following corollary is also true.

Corollary 2.6 *For arbitrary points* $-\pi = s_0 < s_1 < \ldots < s_r < \pi$ *the finite product*

$$\mu(t) = \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{\alpha_k}, t \in (-\pi, \pi)$$

belongs to L_M *if* $\alpha_k > \gamma_M, \forall k = \overline{0, r}$.

Let $M(\cdot)$ be some N-function satisfying Δ_2-condition. Take $f \in L_M$ and consider

$$S_n[f](x) = \sum_{|k| \le n} c_k e^{ikx},$$

where

$$c_k = c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx, k \in Z ,$$

are Fourier coefficients of $f(\cdot)$. We have

$$S_n[f](x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t) dt,$$

where

$$D_n(x) = \frac{1}{2} \sum_{|k| \le n} e^{ikx} = \frac{\sin\left(n + \frac{1}{2}\right) x}{2 \sin \frac{x}{2}}, n \in Z_+,$$

is an n-th order Dirichlet kernel.

We will also need the following concept.

Definition 2.7 *We will say that the function* $M(\cdot)$ *satisfies* ∇_2*-condition if* $\lim_{u \to \infty} \inf \frac{M(2u)}{M(u)} > 2$, *i.e.* $\exists k > 2 \wedge \exists u_0 > 0$*:*

$$M(2u) \ge kM(u), \forall u \ge u_0.$$

The set of N-functions satisfying Δ_2-condition (∇_2-condition) will be denoted by $\Delta_2(\infty)$ ($\nabla_2(\infty)$).

Definition 2.8 *The operator* $T : L_0 \to L_0$ *is called quasilinear if* $|T(\lambda f)| = |\lambda| |T(f)|$, *and* $\exists c \ge 1$*:*

$$|T(f_1 + f_2)| \le c(|T(f_1)| + |T(f_2)|), \forall f; f_1; f_2 \in L_0, \forall \lambda \in C .$$

In case $c = 1$ it is called sublinear.

Ryan theorem below (see, e.g., [42, page 193]) plays a fundamental role in the theory of Fourier analysis in Orlicz spaces.

Theorem R1. *Let* $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$. *If the quasilinear operator* T *is bounded as an operator* $T : L_p(-\pi, \pi) \to L_p(-\pi, \pi)$, *for* $\forall p : 1 < p < +\infty$, *then it is bounded as an operator* $T : L_M \to L_M$.

The following Ryan theorem directly implies the basicity criterion for the exponential system $\left\{e^{int}\right\}_{n \in Z}$ in L_M.

Theorem R2. *Let* $M(\cdot)$ *be an* N*-function. Then the following properties are equivalent:*

i) L_M *is reflexive* $\Leftrightarrow M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$;

ii)$\exists C > 0: \left\|\tilde{f}\right\|_M \leq C \|f\|_M, \forall f \in L_M$,

where $\tilde{f}$ *is a conjugate function of* f*:*

$$\tilde{f}(x) = -\frac{1}{\pi}\int_0^\pi \frac{f(x+t) - f(x-t)}{2\tan\frac{t}{2}}dt;$$

iii)$\exists C > 0:$

$$\|S_n[f]\|_M \leq C\|f\|_M, \forall f \in L_M.$$

This theorem has the following direct corollary.

Corollary 2.9 *Let* $M(\cdot)$ *be some* N*-function. Exponential system* $\left\{e^{int}\right\}_{n\in Z}$ *forms a basis for* L_M *if and only if* $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$.

In fact, if the system $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for L_M, then the basicity criterion implies the satisfaction of condition (iii) in Theorem R2, and Theorem R2 implies the satisfaction of condition (i). The converse is also true by Corollary 9 of [42, page 107]).

Therefore, let $M(\cdot)$ be some N-function and $M^{-1}(\cdot)$ be its inverse on $[0, +\infty)$. Let

$$h(t) = \lim_{x\to\infty}\sup\frac{M^{-1}(x)}{M^{-1}(tx)},\ t > 0,$$

and define the numbers

$$\alpha_M = -\lim_{t\to\infty}\frac{\log h(t)}{\log t};\ \ \beta_M = -\lim_{x\to 0+}\frac{\log h(t)}{\log t}.$$

The numbers α_M and β_M are called upper and lower Boyd indices for the Orlicz space L_M. The following relations hold

$$0 \leq \alpha_M \leq \beta_M \leq 1;$$

$$\alpha_M + \beta_{M^*} \equiv 1;\ \alpha_{M^*} + \beta_M = 1.$$

The space L_M is reflexive if and only if $0 < \alpha_M \leq \beta_M < 1$. If $1 \leq q < \frac{1}{\beta_M} \leq \frac{1}{\alpha_M} < p \leq \infty$, then the continuous embeddings $L_p(-\pi, \pi) \subset L_M \subset L_q(-\pi, \pi)$ hold. More details regarding these concepts can be found in [15, 37]. We will also need the class of Muckenhoupt weights A_p, so let us define it. Let $p \in (1, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The following interesting fact is true (see, e.g., [2, 16]).

Theorem 2.10 *[16] Let* $1 < q < p < +\infty$. *If the linear operator* $T \in [L_p] \wedge T \in [L_q]$, *then* $T \in [L_M]$ *for arbitrary Orlicz space* L_M *with Boyd indices* $\alpha_M, \beta_M : \frac{1}{p} < \alpha_M \leq \beta_M < \frac{1}{q}$.

Let us state some facts about γ_M. It is not difficult to show that if $M \in \Delta_2(\infty)$, then $\gamma_M < 0$, i.e. the following lemma is true.

Lemma 2.11 *Let the* N*-function* $M(\cdot)$ *belong to the class* $\Delta_2(\infty)$*. Then* $\gamma_M < 0$*.*

Proof In fact, if $M \in \Delta_2(\infty)$, then, by the results of [42, Theorem 4.1, page 37], $\exists \beta;\ t_0 > 0:$

$$\frac{tp(t)}{M(t)} < \beta,\ \forall t \geq t_0, \tag{2.2}$$

where $p(\cdot)$ is a right-hand derivative of the function $M(\cdot)$. It is clear that $\beta > 1$. Let $\gamma < 0$ be some number. We have

$$\int_0^\pi M(t^\gamma)\,dt = \left| t^\gamma = x;\ dt = \frac{1}{\gamma} x^{\frac{1}{\gamma}-1} \right| = \frac{1}{|\gamma|} \int_{\pi^{\frac{1}{\gamma}}}^\infty x^{\frac{1}{\gamma}-1} M(x)\,dx.$$

From (2.2) it directly follows that $\exists c > 0:$

$$M(t) \leq ct^\beta,\ \forall t \geq t_0. \tag{2.3}$$

Consequently (assuming $t_0 > \pi^{\frac{1}{\gamma}}$), the integral $\int_0^\pi M(t^\gamma)\,dt$ exists if and only if the integral $\int_{t_0}^\infty x^{\frac{1}{\gamma}-1} M(x)\,dx$ exists. Considering (2.3), we obtain

$$\int_{t_0}^\infty x^{\frac{1}{\gamma}-1} M(x)\,dx \leq c \int_{t_0}^\infty x^{\frac{1}{\gamma}-1} x^\beta dx.$$

It follows that $\forall \gamma > -\frac{1}{\beta}$ the integral $\int_0^\pi M(t^\gamma)\,dt$ exists, and, therefore, $\gamma_M \leq -\frac{1}{\beta}$. The lemma is proved. □

Let $M(\cdot) \in \Delta_2(\infty)$, $M^*(\cdot)$ be a complementary function to M, $p(\cdot)$ and $q(\cdot)$ be the corresponding right-hand derivatives. Let $p(\cdot)$ and $q(\cdot)$ be continuous. By Lemma 4.1 of [30, page 39], the inequality (2.2) holds if and only if for large values of t, i.e. $\exists t_0 > 0$ the relation

$$\frac{tq(t)}{M^*(t)} > \frac{\beta}{\beta - 1},\ \forall t \geq t_0,$$

holds. On integrating this relation, we have

$$\int_{t_0}^x \frac{q(t)\,dt}{M^*(t)} \geq \frac{\beta}{\beta - 1} \int_{t_0}^x \frac{dt}{t} \Rightarrow M^*(t) \geq ct^{\frac{\beta}{\beta-1}},\ \forall t \geq t_0, \tag{2.4}$$

where $c > 0$ is some constant. Using (2.4), we show that for $\forall \gamma \leq -1 + \frac{1}{\beta}$ the integral $\int_0^\pi M^*(t^\gamma)\,dt$ is divergent, and, consequently, $\gamma_{M^*} \geq -1 + \frac{1}{\beta}$.

The following lemma is also true.

Lemma 2.12 *Let* $M(\cdot) \in \Delta_2(\infty)$*,* $M^*(\cdot)$ *be a complementary function to M,* $p(\cdot)$ *and* $q(\cdot)$ *be the corresponding right-hand derivatives which are continuous. Then the following relations are true*

$$-1 \leq \gamma_M \leq -\frac{1}{B_M},\ \ 0 \geq \gamma_{M^*} \geq -1 + \frac{1}{B_M},$$

where

$$B_M = \limsup_{t \to \infty} \frac{tp(t)}{M(t)}.$$

Proof It is clear that $\forall\varepsilon > 0$, $\exists t_0 > 0$:

$$\frac{tp(t)}{M(t)} < B_M + \varepsilon\,,\ \forall t \geq t_0.$$

Then from Lemma 2.11 we obtain $\gamma_M \leq -\frac{1}{B_M+\varepsilon}$. By the arbitrariness of ε, it follows $\gamma_M \leq -\frac{1}{B_M}$. Similarly we have $\gamma_{M^*} \geq -1 + \frac{1}{B_M+\varepsilon}$, and, as a result, $\gamma_{M^*} \geq -1 + \frac{1}{B_M}$. The lemma is proved. □

In the sequel, we will need the following result of [34] (see also [26]). Let α_M and β_M be upper and lower Boyd indices of Orlicz space.

Theorem 2.13 *[34] For every p and q such that*

$$1 \leq q < \frac{1}{\beta_M} \leq \frac{1}{\alpha_M} < p \leq \infty,$$

we have

$$L_p \subset L_M \subset L_q,$$

with the inclusion maps being continuous.

Let $p > \frac{1}{\alpha_M}$ be an arbitrary number. Then it is clear that $\forall\alpha > -\frac{1}{p}$: $t^\alpha \in L_p(0,\pi)$, and it follows from Theorem 2.13 that $t^\alpha \in L_M(0,\pi)$. Then the relation $-\alpha_M < -\frac{1}{p} < \alpha$ and the arbitrariness of p and α imply $\gamma_M \leq -\alpha_M$. If $1 \leq q < \frac{1}{\beta_M}$ is an arbitrary number, then the function t^α does not belong to the space $L_q(0,\pi)$ for $\alpha = -\frac{1}{q}$, and, by Theorem 2.13, $t^\alpha \notin L_M$. It immediately follows $\gamma_M \geq -\frac{1}{q}$. Then the relation $-\frac{1}{q} < -\beta_M$ and the arbitrariness of q imply $\gamma_M \geq -\beta_M$. Thus, the following statement is true.

Proposition 2.14 *Let $M \in \Delta_2(\infty)$ be some N-function with Boyd indices α_M and β_M. Then $\gamma_M \in [-\beta_M, -\alpha_M]$.*

Corollary 2.15 *Let $M \in \Delta_2(\infty)$ be some N- function for which the Boyd indices coincide, i.e. $\alpha_M = \beta_M$. Then $\gamma_M = -\alpha_M$ and $\gamma_{M^*} = -\beta_{M^*}$, and it is clear that $\gamma_M + \gamma_{M^*} = -1$.*

In fact, it is known that the following relations hold

$$\alpha_M + \beta_{M^*} = 1, \beta_M + \alpha_{M^*} = 1.$$

It follows that $\alpha_{M_*} = \beta_{M^*}$, and therefore $\gamma_{M^*} = -\beta_{M^*}$. Moreover, $\gamma_M + \gamma_{M^*} = -\alpha_M - \beta_{M^*} = -1$.

3. Hardy-Orlicz classes and bases for them

Let $M(\cdot)$ be some $N-$function. In the sequel, by $M(f)$ we will mean $M(|f|)$, i.e. $M(f) =: M(|f|)$ (for complex-valued function $f(\cdot)$, too). As usual, by H_M^+ we denote the Hardy-Orlicz class of analytic functions $F(\cdot)$ inside ω equipped with the norm

$$\|F\|_{H_M^+} = \sup_{0<r<1}\ \sup_{\rho_{M^*}(\nu)\leq 1} |(F_r(\cdot)\,;\nu(\cdot))| = \sup_{0<r<1} \|F_r(\cdot)\|_M\,,$$

where $F_r(t) = F\left(re^{it}\right)$ and $(f;g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$.

These classes were first studied in [28, 31, 32]. Some problems of approximation in these classes have been considered in [23, 24, 28]. We will deal with the basicity problems of the parts of exponential system in these classes. In what follows, we will need some facts and concepts related to this area.

By $\mathcal{A}$ we will denote the set of analytic functions $F(\cdot)$ in ω which satisfy

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ \left|F\left(re^{it}\right) dt\right| < +\infty,$$

where $\log^+ u = \log \max\{1; u\}$, $u \geq 0$. About these classes one can see, e.g., [32, 41].

As is known, the nonzero function $F(\cdot)$ belongs to the class $\mathcal{A}$ if and only if it is representable in the form

$$F(z) = B(z) \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} dh(t)\right), \tag{3.1}$$

where $B(\cdot)$ is a Blaschke function, and $h(\cdot)$ is a function of bounded variation on $[0, 2\pi]$.

By $\mathcal{A}'$ we will denote the class of functions $F \in \mathcal{A}$ such that the function $h(\cdot)$ in (3.1) is absolutely continuous on $[0, 2\pi]$.

For $F \in \mathcal{A}$ we denote

$$\rho_M(\mathcal{A}; F) = \sup_{0<r<1} \rho_M(F_r) = \sup_{0<r<1} \int_0^{2\pi} M\left(F\left(re^{it}\right)\right) dt.$$

The following theorem is true.

Theorem 3.1 *[32] If the function F is analytic in ω and $F : \rho_M(\mathcal{A}; F) < +\infty$, then $F \in \mathcal{A}'$, and conversely, if $F \in \mathcal{A}' \wedge F^+ \in L_M$, then $\rho_M(\mathcal{A}; F) < +\infty$, where $F^+(\cdot)$ are the nontangential boundary values of $F(\cdot)$ on $\partial\omega$.*

Consider the singular operator

$$S(f)(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - \tau} d\xi, \ \tau \in \partial\omega,$$

where $f(\cdot) \in L_1(\partial\omega)$ is some function. The following theorem is true.

Theorem 3.2 *[26] Let L_M be a reflexive Orlicz space. Then the singular operator S is bounded in L_M, i.e. $\exists M > 0$:*

$$\|Sf\|_M \leq M \|f\|_M, \ \forall f \in L_M.$$

Reflexivity of L_M is equivalent to the condition $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$. It is absolutely clear that the inclusion $L_M \subset L_1$ holds and the relation

$$\|f\|_{L_1} \leq C \|f\|_M, \forall f \in L_M, \tag{3.2}$$

is true, where $C > 0$ is an absolute constant.

Throughout this work we will assume that $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$. Let $f \in H_M^+$. From (3.2) it follows that $H_M^+ \subset H_1^+$. Denote by $f^+(\cdot)$ the nontangential boundary values of f on $\partial\omega : f^+ = f/_{\partial\omega}$. Let us expand the function $f(\cdot)$ in a Taylor series in the neighborhood of the point $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} f_n^+ z^n, \quad |z| < 1.$$

It is known that $f^+(\cdot) \in L_M$. By Riesz theorem we have

$$\int_{-\pi}^{\pi} \left| f\left(re^{it}\right) - f^+\left(e^{it}\right) \right| dt \to 0, \; r \to 1 - 0.$$

It directly follows

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f^+\left(e^{it}\right) e^{-int} dt = \begin{cases} f_n^+, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Therefore, from the basicity of the system $\left\{e^{int}\right\}_{n \in Z}$ we have the expansion in L_M :

$$f^+\left(e^{it}\right) = \sum_{n=0}^{\infty} f_n^+ e^{int}. \tag{3.3}$$

Denote the restriction of the class H_M^+ to $\partial\omega$ by $L_M^+ : L_M^+ = H_M^+/_{\partial\omega}$. L_M^+ is a subspace of L_M. The minimality of $\left\{e^{int}\right\}_{n \in Z}$ in L_M implies the minimality of the system $\left\{e^{int}\right\}_{n \in Z_+}$ in L_M^+, and, consequently, the expansion (3.3) is unique.

Therefore, the following statement is true.

Proposition 3.3 *Let* $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$. *Then the system* $\{z^n\}_{n \in Z_+}$ $\left(\left\{e^{int}\right\}_{n \in z_+}\right)$ *forms a basis for* H_M^+ *(for* L_M^+*).*

By the uniqueness theorem for analytic functions from the Hardy classes H_δ^+, $\delta > 0$, we can equate the spaces H_M^+ and L_M^+ to each other.

Similar to classical case, we define the Hardy-Orlicz class ${}_mH_M^-$ of analytic functions outside the unit disk which have a finite order at infinity. Let the function $f(\cdot)$, analytic outside ω, have a Laurent decomposition of the form

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n, z \to \infty,$$

in the vicinity of the infinitely remote point. Therefore, for $m > 0$ the point $z = \infty$ is a pole of order m, and for $m \leq 0$ the point $z = \infty$ is a zero of order $(-m)$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(\cdot)$ is the principal part, and $f_1(\cdot)$ is the regular part of Laurent decomposition in the vicinity of $z = \infty$. If the function $g(z) = \overline{f_0\left(\frac{1}{\bar{z}}\right)}$, $|z| < 1$ belongs to the class H_M^+, then we will say that the function $f(\cdot)$ belongs to the class ${}_mH_M^-$. Absolutely similar to the case H_M^+, we can prove the following statement.

Proposition 3.4 *Let* $M \in \Delta_2(\infty) \bigcap \nabla_2(\infty)$. *Then the system* $\{z^n\}_{-\infty}^{m}$ $\left(\left\{e^{int}\right\}_{-\infty}^{m}\right)$ *forms a basis for* ${}_mH_M^-$ *(for* ${}_mL_M^-$ *), where* ${}_mL_M^- = {}_m H_M^-/_{\partial\omega}$.

More detailed about these and other facts one can see the work [11].

When considering Riemann baoundary value problems, we will significantly use the following statement by Zygmund (see, e.g., [18, 41]).

Proposition 3.5 *If* $f(\cdot)$ *is a real function with* $\|f\|_\infty < +\infty$, *then the analytic function in* ω

$$\Phi(z) = \exp\left(\pm\frac{i}{2\pi}\int_{-\pi}^{\pi} f(s)\frac{e^{is}+z}{e^{is}-z}ds\right),$$

belongs to the class Hardy H_δ^+ *for* $\delta > 0$ *sufficiently small.*

4. Homogeneous Riemann problem in Hardy-Orlicz classes

Consider the homogeneous Riemann problem

$$\begin{array}{l} F^+(\tau) - G(\tau)F^-(\tau) = 0, \tau \in \partial\omega, \\ F^+(\cdot) \in H_M^+; F^-(\cdot) \in {}_mH_M^-, \end{array} \tag{4.1}$$

with a complex-valued coefficient $G\left(e^{it}\right) \equiv \left|G\left(e^{it}\right)\right| e^{i\theta(t)}, t \in [-\pi,\pi]$. By the solution of the problem (4.1) we mean a pair of analytic functions $(F^+;F^-) \in H_M^+ \times {}_mH_M^-$, whose nontangential boundary values satisfy Equation (4.1) a.e. on $\partial\omega$. We will solve this problem following the method of [9]. Consider the following piecewise continuous functions on the complex plane with a cut $\partial\omega$:

$$\begin{array}{l} Z_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e^{it}-z}dt\right\}, \\ Z_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}, z \notin \partial\omega. \end{array}$$

Let

$$Z_\theta(z) = Z_1(z)Z_2(z), z \notin \partial\omega.$$

Integral of the form

$$\Phi(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)H(s;z)ds, \tag{4.2}$$

is called Schwarz integral, where $f(\cdot) \in L_1(-\pi,\pi)$ is a density, and $H(s;z) = \frac{e^{is}+z}{e^{is}-z}-$ is a Schwarz kernel. The following Sokhotski-Plemelj formulae are true for Schwarz integral (4.2):

$$\Phi^\pm\left(e^{i\sigma}\right) = \pm f(\sigma) + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)H\left(s;e^{i\sigma}\right)ds.$$

From these formulae it immediately follows that

$$\left|G\left(e^{it}\right)\right| = \frac{Z_1^+\left(e^{it}\right)}{Z_1^-\left(e^{it}\right)},\ e^{i\theta(t)} = \frac{Z_2^+\left(e^{it}\right)}{Z_2^-\left(e^{it}\right)},\ \text{a.e. } t \in [-\pi,\pi].$$

Consequently

$$Z^{+}(\tau)-G(\tau)Z^{-}(\tau)=0,\ \text{a.e. } \tau\in\partial\omega. \tag{4.3}$$

As $\arg G(\cdot)$ is a multivalued function, it is clear that the integral $Z_2(\cdot)$ depends on the chosen branch, and, therefore, the piecewise analytic function $Z(\cdot)$ depends on the chosen branch, i.e. on $\theta(\cdot)$, and we denote it by $Z_\theta(\cdot): Z_\theta(z)=Z_1(z)Z_2(z), z\notin\partial\omega$.

We will call $Z_\theta(\cdot)$ *a canonical solution of homogeneous problem* (4.1)*, corresponding to the argument* $\theta(\cdot)$.

Considering (4.3) in (4.1), we obtain $\frac{F^{+}(\tau)}{Z_\theta^{+}(\tau)}=\frac{F^{-}(\tau)}{Z_\theta^{-}(\tau)}$, a.e. $\tau\in\partial\omega$. Introduce the piecewise analytic function

$$\Phi(z)=\frac{F(z)}{Z_\theta(z)}, z\notin\partial\omega.$$

We have

$$\Phi^{+}(\tau)=\Phi^{-}(\tau),\quad \text{a.e. } \tau\in\partial\omega.$$

Let us show that the function $\Phi(\cdot)$ satisfies all conditions of the uniqueness theorem. It is absolutely clear that the function $Z_\theta(\cdot)$ has no zeros or poles when $z\notin\partial\omega$. Therefore, the functions $\Phi(\cdot)$ and $F(\cdot)$ have the same order at infinity. Let us find the conditions which guarantee that the piecewise analytic function $\Phi(\cdot)=(\Phi^{+}(\cdot);\Phi^{-}(\cdot))$ belongs to the class $H_1^{+}\times{}_mH_1^{-}$. We will assume that the coefficient $G(\cdot)$ satisfies the following conditions:

i) $G^{\pm1}(\cdot)\in L_\infty(-\pi,\pi)$;

ii) $\theta(t)=\arg G\left(e^{it}\right)$ is a piecewise Hölder function on $[-\pi,\pi]$ with the jumps $h_k=\theta(s_k+0)-\theta(s_k-0), k=\overline{1,r}$, at the points of discontinuity $\{s_k\}_1^r: -\pi<s_1<\ldots<s_r<\pi$.

Let us represent the function $\theta(\cdot)$ as

$$\theta(t)=\theta_0(t)+\theta_1(t),$$

where $\theta_0(\cdot)$ is its continuous (Hölder) part, and $\theta_1(\cdot)$ is a jump function defined by

$$\theta_1(-\pi)=0,\ \theta_1(s)=\sum_{k:-\pi<s_k<s}h_k\ ,\ \forall s\in(-\pi,\pi].$$

If the conditions (i,ii) hold, then, by the results of [18], there exists a sufficiently small number $\delta>0$ such that the function $\Phi^{+}(z)$ belongs to the space H_δ^{+} (also, $\Phi^{-}(z)$ belongs to ${}_{\tilde m}H_\delta^{-}$ for some $\tilde m\in Z_+$). Indeed, from the classical facts it follows that the function $\mathrm{X}_2^{\pm1}(z)$ belongs to H_σ^{+} for sufficiently small $\delta>0$. As for the function $\mathrm{X}_1^{\pm1}(z)$, using Jensen's integral inequality of the form

$$\exp\left\{\frac{1}{\int_a^b|p(s)|\,ds}\int_a^b|p(s)f(s)|\,ds\right\}\le\frac{1}{\int_a^b|p(s)|\,ds}\int_a^b|p(s)|\exp|f(s)|\,ds,$$

we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\mathrm{X}_1^{\pm1}\left(\rho e^{i\sigma}\right)\right|^p d\sigma=$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \exp \frac{\pm p}{4\pi} \int_{-\pi}^{\pi} \ln \left| G\left(e^{is}\right) \right| P_{\rho}\left(\sigma - s\right) ds \right\} d\sigma \le$$

$$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G\left(e^{is}\right) \right|^{\pm \frac{p}{2}} P_{\rho}\left(\sigma - s\right) ds \right\} d\sigma \le \|G\|_{\infty}^{\pm \frac{p}{2}},$$

where $P_r(\cdot)$ is a Poisson kernel. It follows that if the condition (i) holds, then the function $\mathrm{X}_1^{\pm}(z)$ belongs to all classes H_p^+, $\forall p > 0$. Applying Hölder's inequality, we see that the function $X^{\pm 1}(z)$ belongs to the Hardy class H_δ^+ for sufficiently small $\delta > 0$. From the representation $\Phi(z) = F(z)\left[X(z)\right]^{-1}$ it follows that the same conclusion is true about the function $\Phi(z)$.

Let us find out under which conditions the function $\Phi(\cdot)$ belongs to the class H_1^+. To do so, it suffices to find out under which conditions the boundary values $\Phi^+(\tau)$ belong to $L_1(-\pi, \pi)$ (the rest will follow from the Smirnov theorem). Let

$$h_0 = \theta(-\pi) - \theta(\pi), \quad h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi),$$

and

$$u_0(t) = \left| \sin \frac{t + \pi}{2} \right|^{-\frac{h_0^{(0)}}{2\pi}} \exp\left(-\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0(\tau) \cot \frac{t - \tau}{2} dt \right).$$

Denote

$$u(t) = \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{\frac{h_k}{2\pi}},$$

where $s_0 = -\pi$. Applying Sokhotski-Plemelj formulae to $Z_1(z)$, we have

$$Z_1^{\pm}\left(e^{i\sigma}\right) = \exp\left\{ \pm \frac{1}{2} \ln \left| G\left(e^{i\sigma}\right) \right| + \frac{1}{4\pi} \int_{-\pi}^{\pi} \ln \left| G\left(e^{is}\right) \right| \frac{e^{is} + e^{i\sigma}}{e^{is} - e^{i\sigma}} ds \right\}.$$

It directly follows that

$$\sup_{(-\pi,\pi)} vrai \left\{ \left| Z_1^{-}\left(e^{it}\right) \right|^{\pm 1} \right\} < +\infty.$$

By the results of [18], the boundary values $\left| Z_2^{-}(\tau) \right|$ are expressed by the formula

$$\left| Z_2^{-}\left(e^{it}\right) \right| = u_0(t)\, u^{-1}(t) = u_0(t) \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{-\frac{h_k}{2\pi}}.$$

Consequently, for the boundary values of the canonical solution $Z_\theta(\cdot)$ we obtain

$$Z_\theta{}^{-}\left(e^{it}\right) = \left| Z_1^{-}\left(e^{it}\right) \right| \left| u_0(t) \right| \prod_{k=0}^{r} \left| \sin \frac{t - s_k}{2} \right|^{-\frac{h_k}{2\pi}}.$$

Taking into account the expression

$$\frac{e^{is} + e^{i\sigma}}{e^{is} - e^{i\sigma}} = \frac{e^{i\frac{s-\sigma}{2}} + e^{i\frac{\sigma-s}{2}}}{e^{i\frac{s-\sigma}{2}} - e^{i\frac{\sigma-s}{2}}} = \frac{\cos \frac{s-\sigma}{2}}{i \sin \frac{s-\sigma}{2}} = i \cot \frac{\sigma - s}{2},$$

for $Z_1^{\pm}(\cdot)$ we have

$$Z_1^{\pm}\left(e^{i\sigma}\right)=\exp\left\{\pm\frac{1}{2}\ln\left|G\left(e^{i\sigma}\right)\right|+\frac{i}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{is}\right)\right|\cot\frac{\sigma-s}{2}ds\right\}.$$

Consequently

$$\left|Z_1^{\pm}\left(e^{i\sigma}\right)\right|=\left|G\left(e^{i\sigma}\right)\right|^{\pm\frac{1}{2}},$$

and hence

$$\left|Z_{\theta}{}^{-}\left(e^{it}\right)\right|=\left|G\left(e^{it}\right)\right|^{-\frac{1}{2}}\left|u_0\left(t\right)\right|\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}. \tag{4.4}$$

It is absolutely clear that $\theta_0(\cdot)$ is a Hölder function on $[-\pi,\pi]$. Then, again by the results of [18], we obtain

$$\sup_{[-\pi,\pi]} vrai\ \left|u_0\left(t\right)\right|^{\pm 1}<+\infty.$$

We have

$$\Phi^{-}\left(e^{it}\right)=F^{-}\left(e^{it}\right)\left[Z_{\theta}^{-}\left(e^{it}\right)\right]^{-1}. \tag{4.5}$$

Thus, by the definition of solution, we have the inclusion $F^{-}(\cdot)\in L_M$. Therefore, by (4.5), to show the function $\Phi^{-}(\cdot)$ belongs to $L_1(-\pi,\pi)$, it suffices to show $\left|Z_{\theta}{}^{-}(\cdot)\right|^{-1}\in L_{M^*}$, where $M(\cdot)$ and $M^*(\cdot)$ are N-functions complementary to each other. And for this, in turn, it suffices to show the validity of inclusion $u(\cdot)\in L_{M^*}$.

Considering Corollary 2.6, we see that if the inequalities

$$\frac{h_k}{2\pi}>\gamma_{M^*}, k=\overline{0,r},$$

hold, then $\left|Z^{-}(\cdot)\right|^{-1}\in L_{M^*}$, and therefore $\Phi^{-}(\cdot)\in L_1(-\pi,\pi)$. Then from Smirnov theorem it follows that $\Phi\in H_1^{+}$. Similarly we obtain $\Phi\in {}_mH_1^{-}$. As $\Phi^{+}(\tau)=\Phi^{-}(\tau)$ a.e. $\tau\in\partial\omega$, from the uniqueness theorem it follows that $\Phi(\cdot)$ is a polynomial $P_k(\cdot)$ of degree $k\leq m$ (for $m<0$ we assume $P_k(z)\equiv 0$). So we get the following representation for the function $F(\cdot)$:

$$F\left(z\right)\equiv Z_{\theta}\left(z\right)P_k\left(z\right), k\leq m, \tag{4.6}$$

where $Z_{\theta}(\cdot)$ is a canonical solution of homogeneous problem corresponding to the argument $\theta(\cdot)$. Let us find the conditions under which the function (4.6) belongs to the Hardy-Orlicz classes $H_M^{+}\times_m H_M^{-}$. It is absolutely clear that $F(\cdot)\in H_{\delta}^{+}$ for sufficiently small $\delta>0$ (similar assertions hold true for the exterior of ω), and hence $F(\cdot)\in\mathcal{A}'$. Therefore it suffices to prove that $F^{+}(\cdot)\in L_M$, and for this, in turn, it suffices to show that $F^{-}(\cdot)\in L_M$. It is absolutely clear that if $Z_{\theta}^{-}(\cdot)\in L_M$, then $F^{-}(\cdot)\in L_M$. Corollary 2.6 implies that if the inequalities

$$-\frac{h_k}{2\pi}>\gamma_M,\ k=\overline{0,r},$$

hold, then $Z_{\theta}^{-}\in L_M\Rightarrow F^{-}\in L_M$, and hence it is clear that $(F^{+};F^{-})\in H_M^{+}\times_m H_M^{-}$. Therefore, we have proved the following theorem.

Theorem 4.1 *Let $M \in \Delta_2(\infty)$ be some N-function, and $M^*(\cdot)$ be an N-function complementary to M. Suppose the coefficient $G(\cdot)$ of the problem (4.1) satisfies the conditions $i)$, $ii)$ and $Z_\theta(\cdot)$ is a canonical solution corresponding to the argument $\theta(\cdot)$. Let the jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$, where $h_0 = \theta(-\pi) - \theta(\pi)$, satisfy the inequalities*

$$\gamma_{M^*} < \frac{h_k}{2\pi} < -\gamma_M, \; k = \overline{0,r}. \tag{4.7}$$

Then,

$\alpha)$ *for $m \geq 0$ the homogeneous Riemann problem (4.1) has a general solution of the form*

$$F(z) = Z_\theta(z) P_k(z), \tag{4.8}$$

in the Hardy-Orlicz classes $H_M^+ \times_m H_M^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$; $\beta)$ for $m < 0$ this problem has only a trivial, i.e. zero solution.

This theorem has the following direct corollary.

Corollary 4.2 *Let all the conditions of Theorem 4.1 hold. Then the homogeneous problem (4.1) under condition $F(\infty) = 0$ has only a trivial solution in the Hardy-Orlicz classes $H_M^+ \times_m H_M^-$.*

To further generalize this result, let us show that the relation $-1 \leq \gamma_M + \gamma_{M^*} \leq 0$ holds for the N-functions $M(\cdot)$ and $M^*(\cdot)$ complementary to each other. Right-hand side of this inequality is obvious. Let us prove the left-hand side. Let $M \in \Delta_2(\infty)$ and $t^\alpha \in L_M$ for some $\alpha \geq \gamma_M$. By the definition, the space L_M consists of functions $f(\cdot)$ such that

$$\left| \int_{-\pi}^{\pi} f(x) v(x) dx \right| < +\infty, \; \forall v : \rho_{M^*}(v) < +\infty.$$

In particular, if we assume $f(x) = t^\alpha$ and $v(x) = t^\beta$, then we obtain

$$I_{\alpha;\beta} = \int_0^\pi t^{\alpha+\beta} dt < +\infty, \; \forall \beta : \rho_{M^*}\left(t^\beta\right) < +\infty.$$

Consequently, this inequality holds $\forall \beta > \gamma_{M^*}$. On the other hand, the integral $I_{\alpha;\beta}$ is finite if and only if $\alpha + \beta > -1$. It follows $\alpha + \inf\limits_{\beta > \gamma_{M^*}} \beta = \alpha + \gamma_{M^*} \geq -1$, $\forall \alpha : \rho_M(t^\alpha) < +\infty$. Then we obtain $\alpha + \gamma_{M^*} \geq -1$, $\forall \alpha > \gamma_M$, and hence $\gamma_M + \gamma_{M^*} \geq -1$. Therefore, the following lemma is true.

Lemma 4.3 *Let $M \in \Delta_2(\infty)$ and $M^*(\cdot)$ be an N-function complementary to M. Then $-1 \leq \gamma_M + \gamma_{M^*} \leq 0$.*

Back to the homogeneous problem. Define the argument $\theta(\cdot)$ of the coefficient $G(\cdot)$ as follows:

$$\tilde{\theta}(t) = \begin{cases} \theta(t), \; -\pi < t < s_1, \\ \theta(t) + 2\pi n, \; s_1 < t < s_2, \\ \vdots \\ \theta(t) + 2\pi n_r, \; s_r < t < \pi, \end{cases}$$

where $\{n_k\}_1^r \subset Z$ are some integers. Assume

$$\tilde{G}(t) = |G(t)|\, e^{i\tilde{\theta}(t)}\ ,\ \ t \in (-\pi, \pi)\,.$$

It is absolutely clear that $G(t) \equiv \tilde{G}(t)$. Therefore, in (4.1) we can consider $\tilde{G}(\cdot)$ instead of $G(\cdot)$. Denote the jumps of the function $\tilde{\theta}(\cdot)$ at the points s_k, $k=\overline{1,r}$ by $\tilde{h}_k$, $k=\overline{1,r}$. We have

$$\tilde{h}_1 = h_1 + 2\pi n;\ \tilde{h}_k = h_k + 2\pi\left(n_k - n_{k-1}\right)\ ,\ \ k=\overline{2,r}\,;\tilde{h}_0 = h_0 - 2\pi n_r\,.$$

Applying Theorem 4.1 to the problem (4.1) with the coefficient $\tilde{G}(\cdot)$, we obtain the following result.

Theorem 4.4 *Let $M \in \Delta_2(\infty)$ be some N-function, and $M^*(\cdot)$ be an N-function complementary to M. Let the coefficient $G(\cdot)$ of the problem (4.1) satisfy the conditions (i, ii) and these exist the integers $\{n_k\}_1^r \subset Z$ such that the jumps $\{h_k\}_1^r$ of the argument $\theta(\cdot)$ satisfy the inequalities*

$$\left.\begin{array}{l} \gamma_{M^*} < \frac{h_1}{2\pi} + n_1 < -\gamma_M\,; \\ \gamma_{M^*} < \frac{h_k}{2\pi} + n_k - n_{k-1} < -\gamma_M\ ,\ \ k=\overline{2,r}\,; \\ \gamma_{M^*} < \frac{h_0}{2\pi} - 2\pi n_r < -\gamma_M\,. \end{array}\right\} \tag{4.9}$$

Then,

$\alpha)$ for $m \geq 0$ the problem (4.1) has a general solution of the form

$$F(Z) = Z_{\tilde{\theta}}(z)\, P_k(z)$$

in the Hardy-Orlicz classes $H_M^+ \times_m H_M^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$, and $Z_{\tilde{\theta}}(\cdot)$ is a canonical solution of the homogeneous problem corresponding to the argument $\tilde{\theta}(\cdot)$;

$\beta)$ for $m < 0$ this problem has only a trivial solution.

Remark 4.5 *It follows from Lemma 4.3 that the integers $\{n_k\}_1^r$ in (4.9) are defined uniquely.*

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Conflict of interest

The authors declare that they have no conflict of interest.

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