

## On conditions of regular solvability for two classes of third-order operator-differential equations in a fourth-order Sobolev-type space

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**Abstract:** In this paper, we study two classes of operator-differential equations of the third order with a multiple characteristic, considered on the whole axis. We introduce the concept of a smooth regular solution of order 1 and obtain sufficient conditions for the "smoothly" regular solvability of these equations.

**Key words:** Operator-differential equation, Sobolev-type space, smooth regular solution, operator pencils, eigenvalue

### 1. Introduction

When modeling some problems in mechanics and engineering, particularly, in filtration problems [6], in stability problems for plates made of a plastic material [16], in problems of the dynamics of arches and rings [15], etc., partial differential equations with real and real multiple characteristics are used. These equations can be reduced to operator-differential equations with a multiple characteristic.

In [1], [3], [5], [8], [10], the main attention is paid to various issues of well-posed and unique solvability of fourth-order operator-differential equations with a multiple characteristic in Sobolev-type spaces. Despite the considerable number of journal publications (see, for example, [4], [7], [11], [13], [14]) devoted to the study of various aspects of the theory of operator-differential equations of an odd order, there are comparatively few works in which operator-differential equations of the third order with a multiple characteristic are studied in a broad aspect (see, for example, [2]).

Consider a third-order operator-differential equation of the form

$$\left(-\frac{d}{dt} + A\right)^k \left(\frac{d}{dt} + A\right)^{3-k} u(t) + \sum_{j=1}^2 A_j u^{(3-j)}(t) = f(t), \quad t \in \mathbb{R} = (-\infty, +\infty), \quad (1.1)$$

where  $A$  is a self-adjoint positive-definite operator in a separable Hilbert space  $H$ ,  $A_j$ ,  $j = 1, 2$ , are linear unbounded operators in  $H$ ,  $f(t) \in W_2^1(\mathbb{R}; H)$ ,  $u(t) \in W_2^4(\mathbb{R}; H)$ ,  $k = 1$  or  $k = 2$ . Here by  $W_2^m(\mathbb{R}; H)$  for

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integers  $m \geq 1$  we understand a Hilbert space (see [12]):

$$W_2^m(\mathbb{R}; H) = \left\{ u(t) : \frac{d^m u(t)}{dt^m} \in L_2(\mathbb{R}; H), A^m u(t) \in L_2(\mathbb{R}; H) \right\}$$

with the norm

$$\|u\|_{W_2^m(\mathbb{R}; H)} = \left( \left\| \frac{d^m u}{dt^m} \right\|_{L_2(\mathbb{R}; H)}^2 + \|A^m u\|_{L_2(\mathbb{R}; H)}^2 \right)^{1/2},$$

where  $L_2(\mathbb{R}; H)$  denotes the Hilbert space of vector-functions  $f(t)$  defined in  $\mathbb{R}$  with the values in  $H$  (see [9]), and for which

$$\|f\|_{L_2(\mathbb{R}; H)} = \left( \int_{-\infty}^{+\infty} \|f(t)\|_H^2 dt \right)^{1/2} < +\infty.$$

Derivatives are understood in the sense of the theory of distributions (see [9]).

Writing an equation of the form (1.1) allows us to consider in one equation two classes of operator-differential equations of the third order with multiple characteristics. Note that equations of the form (1.1) cover equations with real and real multiple characteristics and can be applied, for example, in modeling of filtration problems [6] and problems of the dynamics of arches and rings [15].

**Definition 1.1** *If a vector-function  $u(t) \in W_2^4(\mathbb{R}; H)$  satisfies Equation (1.1) for all  $t \in \mathbb{R}$ , then we will call it a smooth regular solution of order 1 to Equation (1.1).*

**Definition 1.2** *If for any  $f(t) \in W_2^1(\mathbb{R}; H)$  there exists a smooth regular solution of order 1 to Equation (1.1) satisfying the inequality*

$$\|u\|_{W_2^4(\mathbb{R}; H)} \leq \text{const} \|f\|_{W_2^1(\mathbb{R}; H)},$$

*then Equation (1.1) will be called “smoothly” regularly solvable.*

In the present paper, coefficient conditions have been found that ensure the “smoothly” regular solvability of Equation (1.1). These conditions are sufficient. Such issues are studied in [5] for a class of fourth-order operator-differential equations with a multiple characteristic.

## 2. Boundedness of operators

Denote by  $H_\theta$  the scale of Hilbert spaces generated by the operator  $A$ , i.e.

$$H_\theta = \text{Dom}(A^\theta), \theta \geq 0, (x, y)_\theta = (A^\theta x, A^\theta y), x, y \in \text{Dom}(A^\theta).$$

Throughout the entire work,  $L(X, Y)$  is traditionally understood as the set of linear bounded operators acting from a Hilbert space  $X$  to another Hilbert space  $Y$ .

Denote, respectively, by  $P_{0,k}$ ,  $P_{1,k}$ , and  $P^{(k)}$  the operators acting from the space  $W_2^4(\mathbb{R}; H)$  into the space  $W_2^1(\mathbb{R}; H)$  as follows:

$$P_{0,k} u(t) \equiv \left( -\frac{d}{dt} + A \right)^k \left( \frac{d}{dt} + A \right)^{3-k} u(t), u(t) \in W_2^4(\mathbb{R}; H),$$

$$P_{1,k}u(t) \equiv \sum_{j=1}^2 A_j u^{(3-j)}(t), \quad u(t) \in W_2^4(\mathbb{R}; H),$$

$$P^{(k)}u(t) \equiv P_{0,k}u(t) + P_{1,k}u(t), \quad u(t) \in W_2^4(\mathbb{R}; H).$$

The following two lemmas hold.

**Lemma 2.1** *Let  $A$  be a self-adjoint positive-definite operator in  $H$ . Then the operator  $P_{0,k}$  acts boundedly from the space  $W_2^4(\mathbb{R}; H)$  into the space  $W_2^1(\mathbb{R}; H)$ .*

**Proof** For any  $u(t) \in W_2^4(\mathbb{R}; H)$ , in case  $k = 1$ , we have:

$$\begin{aligned} \|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}^2 &= \left\| -\frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} + A^2\frac{du}{dt} + A^3u \right\|_{W_2^1(\mathbb{R}; H)}^2 = \\ &= \left\| -\frac{d^4u}{dt^4} - A\frac{d^3u}{dt^3} + A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)}^2 + \\ &+ \left\| -A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} + A^4u \right\|_{L_2(\mathbb{R}; H)}^2 \leq \\ &\left( \left\| \frac{d^4u}{dt^4} \right\|_{L_2(\mathbb{R}; H)} + \left\| A\frac{d^3u}{dt^3} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^2\frac{d^2u}{dt^2} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^3\frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)} \right)^2 + \\ &\left( \left\| A\frac{d^3u}{dt^3} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^2\frac{d^2u}{dt^2} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^3\frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^4u \right\|_{L_2(\mathbb{R}; H)} \right)^2. \end{aligned} \tag{2.1}$$

Using the theorem on intermediate derivatives [12]

$$\left\| A^j \frac{d^{4-j}u}{dt^{4-j}} \right\|_{L_2(\mathbb{R}; H)} \leq c_j \|u\|_{W_2^4(\mathbb{R}; H)}, \quad j = 0, 1, 2, 3, 4,$$

from inequality (2.1) we obtain

$$\|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)} \leq \text{const} \|u\|_{W_2^4(\mathbb{R}; H)}. \tag{2.2}$$

Carrying out the same reasoning in the case of  $k = 2$ , we have:

$$\begin{aligned} \|P_{0,2}u\|_{W_2^1(\mathbb{R}; H)}^2 &= \left\| \frac{d^3u}{dt^3} - A\frac{d^2u}{dt^2} - A^2\frac{du}{dt} + A^3u \right\|_{W_2^1(\mathbb{R}; H)}^2 = \\ &= \left\| \frac{d^4u}{dt^4} - A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} + A^3\frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)}^2 + \\ &+ \left\| A\frac{d^3u}{dt^3} - A^2\frac{d^2u}{dt^2} - A^3\frac{du}{dt} + A^4u \right\|_{L_2(\mathbb{R}; H)}^2 \leq \end{aligned}$$

$$\begin{aligned} & \left( \left\| \frac{d^4 u}{dt^4} \right\|_{L_2(\mathbb{R}; H)} + \left\| A \frac{d^3 u}{dt^3} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^2 \frac{d^2 u}{dt^2} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^3 \frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)} \right)^2 + \\ & \left( \left\| A \frac{d^3 u}{dt^3} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^2 \frac{d^2 u}{dt^2} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^3 \frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)} + \left\| A^4 u \right\|_{L_2(\mathbb{R}; H)} \right)^2 \leq \\ & \leq \text{const } \|u\|_{W_2^4(\mathbb{R}; H)}^2. \end{aligned} \tag{2.3}$$

□

**Lemma 2.2** *Let  $A$  be a self-adjoint positive-definite operator in  $H$ , and the operators  $A_j \in L(H_j, H) \cap L(H_{j+1}, H_1)$ ,  $j = 1, 2$ . Then the operator  $P_{1,k}$  acts boundedly from the space  $W_2^4(\mathbb{R}; H)$  into the space  $W_2^1(\mathbb{R}; H)$ .*

**Proof** For any  $u(t) \in W_2^4(\mathbb{R}; H)$ , we have:

$$\begin{aligned} \|P_{1,k} u\|_{W_2^1(\mathbb{R}; H)}^2 &= \left\| \sum_{j=1}^2 A_j u^{(4-j)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| \sum_{j=1}^2 AA_j u^{(3-j)} \right\|_{L_2(\mathbb{R}; H)}^2 \leq \\ & \left( \sum_{j=1}^2 \left\| A_j A^{-j} A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}; H)} \right)^2 + \left( \sum_{j=1}^2 \left\| AA_j A^{-(j+1)} A^{j+1} u^{(3-j)} \right\|_{L_2(\mathbb{R}; H)} \right)^2 \leq \\ & \left( \sum_{j=1}^2 \left\| A_j A^{-j} \right\| \left\| A^j u^{(4-j)} \right\|_{L_2(\mathbb{R}; H)} \right)^2 + \left( \sum_{j=1}^2 \left\| AA_j A^{-(j+1)} \right\| \left\| A^{j+1} u^{(3-j)} \right\|_{L_2(\mathbb{R}; H)} \right)^2. \end{aligned}$$

Since  $A_j \in L(H_j, H) \cap L(H_{j+1}, H_1)$ ,  $j = 1, 2$ , then the operators  $A_j A^{-j}$  and  $AA_j A^{-(j+1)}$ ,  $j = 1, 2$ , are bounded in  $H$ . Taking into account again the theorem on intermediate derivatives [12], we obtain

$$\|P_{1,k} u\|_{W_2^1(\mathbb{R}; H)} \leq \text{const } \|u\|_{W_2^4(\mathbb{R}; H)}. \tag{2.4}$$

□

Lemmas 2.1 and 2.2 imply the validity of the following theorem.

**Theorem 2.3** *Let the conditions of Lemma 2.2 be satisfied. Then the operator  $P^{(k)}$  acts boundedly from the space  $W_2^4(\mathbb{R}; H)$  into the space  $W_2^1(\mathbb{R}; H)$ .*

**Proof** For any  $u(t) \in W_2^4(\mathbb{R}; H)$ , from the inequalities (2.2), (2.3), and (2.4), it follows that

$$\left\| P^{(k)} u \right\|_{W_2^1(\mathbb{R}; H)} \leq \|P_{0,k} u\|_{W_2^1(\mathbb{R}; H)} + \|P_{1,k} u\|_{W_2^1(\mathbb{R}; H)} \leq \text{const } \|u\|_{W_2^4(\mathbb{R}; H)}.$$

□

**3. Solvability of the equation**  $P_{0,k}u(t) = f(t)$

Now we study the solvability of the main part of Equation (1.1).

The following theorem holds.

**Theorem 3.1** *The equation  $P_{0,k}u(t) = f(t)$  has a unique smooth solution regular of order 1,  $u(t)$ , for any  $f(t) \in W_2^1(\mathbb{R}; H)$ , and the following inequality holds*

$$\|u\|_{W_2^4(\mathbb{R}; H)} \leq \text{const} \|f\|_{W_2^1(\mathbb{R}; H)}.$$

**Proof** Let  $f(t) \in W_2^1(\mathbb{R}; H)$ , i.e. the following norm is finite

$$\left\| \frac{df}{dt} \right\|_{L_2(\mathbb{R}; H)}^2 + \|Af\|_{L_2(\mathbb{R}; H)}^2 = \|f\|_{W_2^1(\mathbb{R}; H)}^2.$$

Then it follows from Parseval's equality that

$$\|i\lambda \widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 + \|A\widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 < +\infty,$$

where  $\widehat{f}(\lambda)$  is the Fourier transform of the function  $f(t)$ . Denoting by  $\widehat{u}(\lambda)$  the Fourier transform of the function  $u(t)$ , from the equation  $P_{0,k}u(t) = f(t)$  we have

$$P_{0,k}(i\lambda) \widehat{u}(\lambda) = \widehat{f}(\lambda)$$

or

$$\widehat{u}(\lambda) = P_{0,k}^{-1}(i\lambda) \widehat{f}(\lambda), \lambda \in \mathbb{R}. \tag{3.1}$$

From here, we determine

$$u(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P_{0,k}^{-1}(i\lambda) \widehat{f}(\lambda) e^{i\lambda t} d\lambda.$$

Let us show that  $u(t)$  is a smooth regular solution of order 1 of the equation  $P_{0,k}u(t) = f(t)$ .

Indeed, from Parseval's equality, taking into account equality (3.1), we obtain:

$$\begin{aligned} \|u\|_{W_2^4(\mathbb{R}; H)}^2 &= \left\| \frac{d^4u}{dt^4} \right\|_{L_2(\mathbb{R}; H)}^2 + \|A^4u\|_{L_2(\mathbb{R}; H)}^2 = \\ &= \|\lambda^4 \widehat{u}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 + \|A^4 \widehat{u}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 = \\ &= \|\lambda^4 P_{0,k}^{-1}(i\lambda) \widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 + \|A^4 P_{0,k}^{-1}(i\lambda) \widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 \leq \\ &= \sup_{\lambda \in \mathbb{R}} \left\| -i\lambda^3 P_{0,k}^{-1}(i\lambda) \right\|_{H \rightarrow H}^2 \|i\lambda \widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2 + \\ &= \sup_{\lambda \in \mathbb{R}} \left\| A^3 P_{0,k}^{-1}(i\lambda) \right\|_{H \rightarrow H}^2 \cdot \|A\widehat{f}(\lambda)\|_{L_2(\mathbb{R}; H)}^2. \end{aligned} \tag{3.2}$$

On the other hand, from the spectral expansion of the operator  $A$ , we have:

$$\sup_{\lambda \in \mathbb{R}} \left\| -i\lambda^3 P_{0,k}^{-1}(i\lambda) \right\|_{H \rightarrow H} = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| -i\lambda^3 (i\lambda + \sigma)^{-3+k} (-i\lambda + \sigma)^{-k} \right| =$$

$$\sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| -i\lambda^3 (i\lambda + \sigma)^{-3+2k} (\lambda^2 + \sigma^2)^{-k} \right| = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \frac{\lambda^3}{(\lambda^2 + \sigma^2)^{3/2}} \leq 1, \tag{3.3}$$

$$\sup_{\lambda \in \mathbb{R}} \left\| A^3 P_{0,k}^{-1}(i\lambda) \right\|_{H \rightarrow H} = \sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \left| \sigma^3 (i\lambda + \sigma)^{-3+k} (-i\lambda + \sigma)^{-k} \right| =$$

$$\sup_{\lambda \in \mathbb{R}} \sup_{\sigma \in \sigma(A)} \frac{\sigma^3}{(\lambda^2 + \sigma^2)^{3/2}} \leq 1, \tag{3.4}$$

where  $\sigma(A)$  denotes the spectrum of the operator  $A$ . Taking (3.3) and (3.4) into account in the inequality (3.2), we obtain:

$$\|u\|_{W_2^4(\mathbb{R};H)}^2 \leq \left\| i\lambda \widehat{f}(\lambda) \right\|_{L_2(\mathbb{R};H)}^2 + \left\| A \widehat{f}(\lambda) \right\|_{L_2(\mathbb{R};H)}^2 = \|f\|_{W_2^1(\mathbb{R};H)}^2.$$

Obviously,  $u(t)$  satisfies the equation  $P_{0,k}u(t) = f(t)$ . Therefore,  $u(t)$  is a smooth regular solution of order 1 of the equation  $P_{0,k}u(t) = f(t)$ . □

**Corollary 3.2** *The norms  $\|P_{0,k}u\|_{W_2^1(\mathbb{R};H)}$  and  $\|u\|_{W_2^4(\mathbb{R};H)}$  are equivalent in the space  $W_2^4(\mathbb{R};H)$ .*

#### 4. Estimation of the norms of intermediate derivatives operators

By the theorem on intermediate derivatives [12], according to Corollary 3.2, the following numbers are finite:

$$n_{j,k} = \sup_{0 \neq u \in W_2^4(\mathbb{R};H)} \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R};H)} \cdot \|P_{0,k}u\|_{W_2^1(\mathbb{R};H)}^{-1}, \quad j = 1, 2.$$

There arises a problem of calculating  $n_{j,k}$ ,  $j = 1, 2$ . Before turning to this problem, we prove the following lemma.

**Lemma 4.1** *Let  $\beta \in [0, \frac{27}{4})$ . Then the operator pencils*

$$\widetilde{P}_j(\lambda; \beta; A) = (-\lambda^2 E + A^2) P_j(\lambda; \beta; A), \quad j = 1, 2, \tag{4.1}$$

where

$$P_j(\lambda; \beta; A) = (-\lambda^2 E + A^2)^3 - \beta (i\lambda)^{2j} A^{6-2j}, \quad j = 1, 2,$$

( $E$  is a unit operator) that depends on the parameter  $\beta$ , are invertible on the imaginary axis and there exist points  $\xi_{0,j} \in \mathbb{R}$ ,  $j = 1, 2$ , such that the characteristic polynomials

$$\widetilde{P}_j(i\xi; \beta; \sigma) = (\xi^2 + \sigma^2) ((\xi^2 + \sigma^2)^3 - \beta \xi^{2j} \sigma^{6-2j}), \quad j = 1, 2, \quad \sigma \in \sigma(A),$$

satisfy the following properties:

$$\tilde{P}_j(i\xi_{0,j}; \beta; \sigma) > 0 \text{ at } \beta \in [0, \frac{27}{4}), j = 1, 2, \sigma \in \sigma(A);$$

$$\tilde{P}_j(i\xi_{0,j}; \beta; \sigma) = 0 \text{ at } \beta = \frac{27}{4}, j = 1, 2, \sigma \in \sigma(A);$$

$$\tilde{P}_j(i\xi_{0,j}; \beta; \sigma) < 0 \text{ at } \beta > \frac{27}{4}, j = 1, 2, \sigma \in \sigma(A).$$

**Proof** Since  $A$  is a self-adjoint positive-definite operator, i.e.  $A = A^* \geq cE$ ,  $c > 0$ , then for  $\sigma \in \sigma(A)$  ( $\sigma \geq \sigma_0 \geq c > 0$ ) the characteristic polynomials of the operator pencils  $\tilde{P}_j(\lambda; \beta; A)$ ,  $j = 1, 2$ , have the form

$$\tilde{P}_j(\lambda; \beta; \sigma) = (-\lambda^2 + \sigma^2) \left( (-\lambda^2 + \sigma^2)^3 - \beta(i\lambda)^{2j} \sigma^{6-2j} \right), j = 1, 2.$$

For  $\lambda = i\xi$ ,  $\xi \in \mathbb{R}$ , we have

$$\begin{aligned} \tilde{P}_j(i\xi; \beta; \sigma) &= (\xi^2 + \sigma^2) \left( (\xi^2 + \sigma^2)^3 - \beta \xi^{2j} \sigma^{6-2j} \right) = \\ &(\xi^2 + \sigma^2)^4 \left( 1 - \beta \frac{\xi^{2j} \sigma^{6-2j}}{(\xi^2 + \sigma^2)^3} \right) \geq (\xi^2 + \sigma^2) \left[ 1 - \beta \frac{\xi^{2j}}{\left(1 + \frac{\xi^2}{\sigma^2}\right)^3} \right] \geq \\ &(\xi^2 + \sigma^2) \left[ 1 - \beta \sup_{r \geq 0} \frac{r^j}{(r+1)^3} \right] \geq \sigma_0^2 \left( 1 - \beta \frac{4}{27} \right), j = 1, 2. \end{aligned}$$

Therefore, for  $\beta \in [0, \frac{27}{4})$ , we have  $\tilde{P}_j(i\xi; \beta; \sigma) > 0$ ,  $j = 1, 2$ . Then it follows from the spectral expansion of the operator  $A$  that  $\tilde{P}_j(i\xi; \beta; A) > 0$ ,  $j = 1, 2$ , for any  $\xi \in \mathbb{R}$ , i.e. the operator pencils  $\tilde{P}_j(\lambda; \beta; A)$ ,  $j = 1, 2$ , are invertible on the imaginary axis. Obviously, the minima of the characteristic functions  $\tilde{P}_j(i\xi; \beta; \sigma)$ ,  $j = 1, 2$ , are positive for all  $\beta \in [0, \frac{27}{4})$ . Indeed, as shown above,

$$\tilde{P}_j(i\xi; \beta; \sigma) \geq (\xi^2 + \sigma^2) \left( 1 - \beta \sup_{r \geq 0} \frac{r^j}{(r+1)^3} \right), j = 1, 2.$$

But the functions  $\varphi_j(r) = \frac{r^j}{(r+1)^3}$ ,  $j = 1, 2$ , attain their maximum values for some  $r_{0,j} = \frac{\xi_{0,j}^2}{\sigma^2}$ ,  $j = 1, 2$ . Therefore, for these values  $\xi_{0,j}$ ,  $j = 1, 2$ ,

$$\tilde{P}_j(i\xi_{0,j}; \beta; \sigma) \geq (\xi_{0,j}^2 + \sigma^2) \left( 1 - \beta \frac{4}{27} \right), j = 1, 2.$$

□

**Theorem 4.2** *The following equalities hold  $n_{j,k} = \frac{2}{3^{3/2}}$ ,  $j = 1, 2$ .*

**Proof** Let the functions  $u(t) \in W_2^4(\mathbb{R}; H)$  have compact supports and be infinitely differentiable. Then, in case  $k = 1$ , by Parseval's equality, we have:

$$\|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}^2 - \beta \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)}^2 =$$

$$\begin{aligned}
 & \left\| -\frac{d^3 u}{dt^3} - A \frac{d^2 u}{dt^2} + A^2 \frac{du}{dt} + A^3 u \right\|_{W_2^1(\mathbb{R}; H)}^2 - \\
 & \beta \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)}^2 = \left\| -\frac{d^4 u}{dt^4} - A \frac{d^3 u}{dt^3} + A^2 \frac{d^2 u}{dt^2} + A^3 \frac{du}{dt} \right\|_{L_2(\mathbb{R}; H)}^2 + \\
 & \left\| -A \frac{d^3 u}{dt^3} - A^2 \frac{d^2 u}{dt^2} + A^3 \frac{du}{dt} + A^4 u \right\|_{L_2(\mathbb{R}; H)}^2 - \beta \left( \left\| A^{3-j} u^{(j+1)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| A^{4-j} u^{(j)} \right\|_{L_2(\mathbb{R}; H)}^2 \right) = \\
 & \left\| \left( -(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi) \right\|_{L_2(\mathbb{R}; H)}^2 + \\
 & \left\| \left( -(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi) \right\|_{L_2(\mathbb{R}; H)}^2 - \\
 & \beta \left( \left\| (i\xi)^{j+1} A^{3-j} \widehat{u}(\xi) \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| (i\xi)^j A^{4-j} \widehat{u}(\xi) \right\|_{L_2(\mathbb{R}; H)}^2 \right) = \\
 & \int_{-\infty}^{+\infty} \left( \left( -(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi), \left( -(i\xi)^4 E - (i\xi)^3 A + (i\xi)^2 A^2 + (i\xi) A^3 \right) \widehat{u}(\xi) \right)_H d\xi + \\
 & \int_{-\infty}^{+\infty} \left( \left( -(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi), \left( -(i\xi)^3 A - (i\xi)^2 A^2 + (i\xi) A^3 + A^4 \right) \widehat{u}(\xi) \right)_H d\xi - \\
 & \beta \int_{-\infty}^{+\infty} \left( (i\xi)^{j+1} A^{3-j} \widehat{u}(\xi), (i\xi)^{j+1} A^{3-j} \widehat{u}(\xi) \right)_H d\xi - \beta \int_{-\infty}^{+\infty} \left( (i\xi)^j A^{4-j} \widehat{u}(\xi), (i\xi)^j A^{4-j} \widehat{u}(\xi) \right)_H d\xi = \\
 & \int_{-\infty}^{+\infty} \left( (\xi^2 E + A^2) ((\xi^2 E + A^2)^3 - \beta \xi^{2j} A^{6-2j}) \widehat{u}(\xi), \widehat{u}(\xi) \right)_H d\xi, \quad j = 1, 2.
 \end{aligned}$$

Thus, for any  $\beta \geq 0$  and sufficiently smooth  $u(t)$ , we have the equalities

$$\|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}^2 - \beta \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)}^2 = \int_{-\infty}^{+\infty} \left( \widetilde{P}_j(i\xi; \beta; A) \widehat{u}(\xi), \widehat{u}(\xi) \right)_H d\xi, \quad j = 1, 2, \tag{4.2}$$

where  $\widetilde{P}_j(i\xi; \beta; A)$  are defined in (4.1). By Lemma 4.1, for  $\beta \in [0, \frac{27}{4})$ , the operator pencils  $\widetilde{P}_j(i\xi; \beta; A) > 0$ ,  $j = 1, 2$ . Therefore, for  $\beta \in [0, \frac{27}{4})$ , it follows from (4.2) that

$$\|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}^2 - \beta \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)}^2 > 0, \quad j = 1, 2.$$

Proceeding here to the limit as  $\beta \rightarrow \frac{27}{4}$ ,  $j = 1, 2$ , we have

$$\left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)} \leq \frac{2}{3^{3/2}} \|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}, \quad j = 1, 2,$$

i.e.  $n_{j,1} \leq \frac{2}{3^{3/2}}$ ,  $j = 1, 2$ . To prove the equalities  $n_{j,1} = \frac{2}{3^{3/2}}$ ,  $j = 1, 2$ , we define the functional

$$E(u) = \|P_{0,1}u\|_{W_2^1(\mathbb{R}; H)}^2 - \beta \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)}^2$$



in the space  $W_2^4(\mathbb{R}; H)$  and  $\forall \varepsilon > 0$ , we look for the vector-function  $u_\varepsilon(t) = g_\varepsilon(t)\psi_\varepsilon$ , for which  $E(u_\varepsilon) < 0$ , where  $\psi_\varepsilon \in \text{Dom}(A^8)$ ,  $g_\varepsilon(t)$  is a scalar function. For this purpose, we write the inequality  $E(u_\varepsilon) < 0$  as

$$E(u_\varepsilon) = \int_{-\infty}^{+\infty} \left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) |\widehat{g}_\varepsilon(\xi)|^2 d\xi < 0.$$

If  $A$  has at least one eigenvalue  $\sigma$ , then we choose the corresponding eigenvector  $\psi_\varepsilon$ :  $A\psi_\varepsilon = \sigma\psi_\varepsilon$ ,  $\|\psi_\varepsilon\| = 1$ . Then it is obvious that

$$\left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right)_H = \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; \sigma \right),$$

but by the lemma  $\tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; \sigma \right) < 0$ ,  $\forall \varepsilon > 0$  at  $\xi = \xi_{0,j}$ . If the operator  $A$  does not have an eigenvalue, then for any  $\sigma \in \sigma(A)$  and for any  $\delta > 0$ , we can find a vector  $\psi_\delta$  ( $\|\psi_\delta\| = 1$ ) such that for any  $s > 0$

$$A^s \psi_\delta = \sigma^s \psi_\delta + o(1, \delta) \text{ as } \delta \rightarrow 0, s > 0.$$

Then

$$\left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\delta, \psi_\delta \right)_H = \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; \sigma \right) + o(1, \delta) \text{ as } \delta \rightarrow 0.$$

For sufficiently small  $\delta > 0$

$$\tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; \sigma \right) + o(1, \delta) < 0.$$

Thus, for some  $\xi = \xi_{0,j}$  and  $\psi_\varepsilon \in \text{Dom}(A^8)$ ,  $\varepsilon > 0$

$$\left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) < 0. \tag{4.3}$$

Since  $\left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right)$  is a continuous function of the argument  $\xi$ , the inequality (4.3) holds for some  $\xi \in (\eta_1, \eta_2)$ . Now we construct the function  $g_\varepsilon(t)$  as follows. Let  $\widehat{g}_\varepsilon(t)$  be an infinitely differentiable function with the support in the interval  $(\eta_1, \eta_2)$ . Denote by

$$g_\varepsilon(t) = \frac{1}{\sqrt{2\pi}} \int_{\eta_1}^{\eta_2} \widehat{g}_\varepsilon(\xi) e^{i\xi t} d\xi.$$

Obviously,  $g_\varepsilon(t) \in W_2^4(\mathbb{R})$ . Then

$$E(u_\varepsilon) = E(g_\varepsilon(t)\psi_\varepsilon) = \int_{\eta_1}^{\eta_2} \left( \tilde{P}_j \left( i\xi; \frac{27}{4} + \varepsilon; A \right) \psi_\varepsilon, \psi_\varepsilon \right) |\widehat{g}_\varepsilon(\xi)|^2 d\xi < 0,$$

it is hereby proved that  $n_{j,1} = \frac{2}{3^{3/2}}$ ,  $j = 1, 2$ .

The proof in the case of  $k = 2$  is carried out in a similar way. □

**5. Solvability conditions for Equation (1.1)**

We now formulate conditions on the “smoothly” regular solvability of Equation (1.1).

**Theorem 5.1** *Let  $A$  be a self-adjoint positive-definite operator in  $H$  and the operators*

$$A_j \in L(H_j, H) \cap L(H_{j+1}, H_1), \quad j = 1, 2,$$

moreover, the following inequality holds

$$\sum_{j=1}^2 \max \left\{ \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}, \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H} \right\} < \frac{3^{3/2}}{2}.$$

Then Equation (1.1) is “smoothly” regularly solvable.

**Proof** Using the notation introduced earlier, we represent Equation (1.1) as an operator equation

$$P_{0,k} u(t) + P_{1,k} u(t) = f(t), \tag{5.1}$$

where  $f(t) \in W_2^1(\mathbb{R}; H)$ ,  $u(t) \in W_2^4(\mathbb{R}; H)$ . By Theorem 2.3, the operator  $P^{(k)} = P_{0,k} + P_{1,k}$  is a bounded operator from the space  $W_2^4(\mathbb{R}; H)$  to the space  $W_2^1(\mathbb{R}; H)$ , and, by Theorem 3.1, the operator  $P_{0,k}$  maps  $W_2^4(\mathbb{R}; H)$  isomorphically onto  $W_2^1(\mathbb{R}; H)$ . Then there is a bounded inverse  $P_{0,k}^{-1}$  acting from  $W_2^1(\mathbb{R}; H)$  to  $W_2^4(\mathbb{R}; H)$ . If in Equation (5.1) we make the substitution  $u(t) = P_{0,k}^{-1} v(t)$ , where  $v(t) \in W_2^1(\mathbb{R}; H)$ , then we obtain

$$(E + P_{1,k} P_{0,k}^{-1}) v(t) = f(t).$$

Let us show that, under the conditions of the theorem, the norm of the operator  $P_{1,k} P_{0,k}^{-1}$  is less than one. We have

$$\begin{aligned} \left\| P_{1,k} P_{0,k}^{-1} v \right\|_{W_2^1(\mathbb{R}; H)} &= \left\| P_{1,k} u \right\|_{W_2^1(\mathbb{R}; H)} \leq \sum_{j=1}^2 \left\| A_j u^{(3-j)} \right\|_{W_2^1(\mathbb{R}; H)} = \\ &= \sum_{j=1}^2 \left( \left\| A_{3-j} u^{(j+1)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| A A_{3-j} u^{(j)} \right\|_{L_2(\mathbb{R}; H)}^2 \right)^{\frac{1}{2}} \leq \\ &= \sum_{j=1}^2 \left( \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}^2 \left\| A^{3-j} u^{(j+1)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H}^2 \left\| A^{4-j} u^{(j)} \right\|_{L_2(\mathbb{R}; H)}^2 \right)^{\frac{1}{2}} \leq \\ &= \sum_{j=1}^2 \max \left\{ \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}, \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H} \right\} \times \\ &= \left( \left\| A^{3-j} u^{(j+1)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| A^{4-j} u^{(j)} \right\|_{L_2(\mathbb{R}; H)}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)} = \left( \left\| A^{3-j} u^{(j+1)} \right\|_{L_2(\mathbb{R}; H)}^2 + \left\| A^{4-j} u^{(j)} \right\|_{L_2(\mathbb{R}; H)}^2 \right)^{\frac{1}{2}},$$

then

$$\begin{aligned} & \left\| P_{1,k} P_{0,k}^{-1} v \right\|_{W_2^1(\mathbb{R}; H)} \leq \\ & \sum_{j=1}^2 \max \left\{ \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}, \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H} \right\} \times \left\| A^{3-j} u^{(j)} \right\|_{W_2^1(\mathbb{R}; H)} \leq \\ & \frac{2}{3^{3/2}} \sum_{j=1}^3 \max \left\{ \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}, \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H} \right\} \times \|P_{0,k} u\|_{W_2^1(\mathbb{R}; H)} = \\ & \frac{2}{3^{3/2}} \sum_{j=1}^2 \max \left\{ \left\| A_{3-j} A^{-(3-j)} \right\|_{H \rightarrow H}, \left\| A A_{3-j} A^{-(4-j)} \right\|_{H \rightarrow H} \right\} \times \|v\|_{W_2^1(\mathbb{R}; H)}. \end{aligned}$$

Thus,  $\left\| P_{1,k} P_{0,k}^{-1} \right\|_{W_2^1(\mathbb{R}; H) \rightarrow W_2^1(\mathbb{R}; H)} < 1$ . Therefore, the operator  $E + P_{1,k} P_{0,k}^{-1}$  has an inverse in the space  $W_2^1(\mathbb{R}; H)$  and, hence,  $u(t)$  can be defined by the formula

$$u(t) = P_{0,k}^{-1} \left( E + P_{1,k} P_{0,k}^{-1} \right)^{-1} f(t),$$

and

$$\begin{aligned} \|u\|_{W_2^4(\mathbb{R}; H)} & \leq \left\| P_{0,k}^{-1} \right\|_{W_2^1(\mathbb{R}; H) \rightarrow W_2^4(\mathbb{R}; H)} \left\| \left( E + P_{1,k} P_{0,k}^{-1} \right)^{-1} \right\|_{W_2^1(\mathbb{R}; H) \rightarrow W_2^1(\mathbb{R}; H)} \times \\ & \|f\|_{W_2^1(\mathbb{R}; H)} \leq \text{const} \|f\|_{W_2^1(\mathbb{R}; H)}. \end{aligned}$$

□

**Corollary 5.2.** *The operator  $P^{(k)}$  is an isomorphism between the spaces  $W_2^4(\mathbb{R}; H)$  and  $W_2^1(\mathbb{R}; H)$ .*

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