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Geodesics and natural complex magnetic trajectories on tangent bundles

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Abstract: In this paper, we investigate geodesics of the tangent bundle TM of a Riemannian manifold (M,g) endowed with an arbitrary pseudo-Riemannian g-natural metric of Kaluza-Klein type. Then considering a class of naturally defined almost complex structures on TM, constructed by V. Oproiu, we construct a class of magnetic fields and we characterize the corresponding magnetic curves on TM, when (M,g) is a space form.

Key words: Tangent bundle, g-natural metric, metric of Kaluza-Klein type, geodesic, magnetic curve

1. Introduction

Magnetic curves represent, in physics, the trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field. A magnetic field F on a Riemannian manifold (M,g) is any closed $2-$form F and the Lorentz-force associated to F is an endomorphism field φ such that

$$F(X,Y) = g(\varphi(X),Y), \tag{1.1}$$

for all $X,Y \in \mathfrak{X}(M)$. The magnetic trajectories of F are curves γ in M that satisfy the Lorentz equation (called also the Newton equation)

$$\nabla_{\dot\gamma}\dot\gamma = \varphi(\dot\gamma), \tag{1.2}$$

which generalizes the equation of geodesics under arc length parametrization, namely $\nabla_{\dot\gamma}\dot\gamma = 0$. Here ∇ denotes the Levi-Civita connection associated to the metric g. Since the Lorentz equation implies that the speed (and hence the energy) of γ is constant, it is usual to focus on unit speed magnetic curves with a strength $q \in \mathbb{R}$, that is normal magnetic curves satisfying the Lorentz equation

$$\nabla_{\dot\gamma}\dot\gamma = q\varphi(\dot\gamma), \tag{1.3}$$

where by dot we denote the derivative with respect to the arc-length parameter s.

The topic of magnetic trajectories related to magnetic fields and their relationship with the geometry of the Riemannian manifold has been extensively studied in the two last decades in many geometric contexts (cf. [9] and the references therein). In the context of the geometry of tangent bundles, magnetic trajectories has been studied in the unit tangent bundle of a Riemannian manifold endowed with the Sasaki metric and the magnetic

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structure associated to the standard contact structure (cf. [7] and [8]). A generalization to the case when the unit tangent bundle is equipped with an arbitrary g-natural metric and the magnetic structure associated to a family of compatible contact (resp. paracontact) structures has been recently made by the authors and M.I. Munteanu [1] (resp. by the authors [2]).

Surprisingly, to the best of our knowledge, there is a lack in the literature of results concerning magnetic trajectories on the tangent bundle, even equipped by the Sasaki metric. In this paper, we will try to fill this gap by considering on the tangent bundle TM of a Riemannian metric (M,g) an arbitrary pseudo-Riemannian g-natural metric G of Kaluza-Klein type. At first, we will investigate geodesics, remarking that any curve in TM is, locally, either contained in a fiber or transverse to the fibers it meets. Any curve of TM transverse to fibers can be interpreted as vector field V along a curve x in M. It is said to be horizontal if V is parallel along x and oblique otherwise. According to this natural classification of curves, we will investigate vertical, horizontal, and oblique geodesics in TM. Using the fact that, for pseudo-Riemannian g-natural metrics of Kaluza-Klein type, the fibers of TM are totally geodesic submanifolds [3], we deduce that geodesics of vertical type on (TM,G) are exactly geodesics in the pseudo-Euclidean vector spaces $(T_xM, G_{\restriction T_xM})$, $x \in M$ (Proposition 3.1). We also prove that, under some conditions on the metric G, horizontal geodesics are parallel vector fields along geodesics on TM (Theorem 3.4) and that for some pseudo-Riemannian Kaluza-Klein metrics on TM, there is no horizontal geodesic on TM (Corollary 3.5). Concerning oblique geodesics, we restrict to the case when the base manifold is a space of constant sectional curvature and the g-natural metric G on TM is a Kaluza-Klein metric. We give necessary conditions for oblique curves to be geodesics (Proposition 3.9) and investigate the special cases of velocity vector fields (Proposition 3.11) and vector fields of constant norm along base curves (Proposition 3.13). We end our investigation on geodesics on TM by giving a classification of oblique geodesics on the tangent bundle TM^2 of a 2-dimensional manifold of constant Gaussian curvature, endowed with a pseudo-Riemannian Kaluza-Klein metric (Theorem 3.15).

In the second part of the paper, we construct a family of magnetic structures: considering the family of almost Kählerian structures on the tangent bundle of a Riemannian manifold of constant sectional curvature constructed by Oproiu [12], we define a family of magnetic structures that we call g-natural complex magnetic structures and we investigate the corresponding magnetic trajectories. We give a classification of nongeodesic horizontal g-natural complex magnetic trajectories (Theorem 4.2). As a particular case, we prove that, in the Sasaki metric case, horizontal geodesics are the only horizontal g-natural complex magnetic trajectories (Corollary 4.3). Finally, restricting to Kaluza-Klein metrics on the tangent bundle, we investigate two special kinds of nongeodesic oblique g-natural complex magnetic trajectories. On the one hand, we classify nongeodesic g-natural complex magnetic trajectories which are velocity vector fields (Proposition 4.4) and, on the other hand, we characterize, among vector fields of constant norm base curves making a constant angle with it, those which are g-natural complex magnetic trajectories (Theorem 4.5).

2. Preliminaries

2.1. Basic formulas on tangent bundles

Let (M,g) be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of g. Then the tangent space of TM at any point $(x,u) \in TM$ into the horizontal and vertical subspaces with respect to ∇:

$$(TM)_(x,u) = H(x,u) \oplus V(x,u). \tag{2.1}$$

If $(x,u) \in TM$ is given then, for any vector $X \in M_x$, there exists a unique vector $X^h \in H(x,u)$ such that $p^*X^h = X$, where $p : TM \longrightarrow M$ is the natural projection. We call X^h the horizontal lift of X to the point $(x,u) \in TM$. The vertical lift of a vector $X \in M_x$ to $(x,u) \in TM$ is a vector $X^v \in V(x,u)$ such that $X^v(df) = X(f)$, for all functions f on M. Here we consider 1-forms df on M as functions on TM (i.e. $(df)(x,u) = u(f)$). Note that the map $X \longrightarrow X^h$ is an isomorphism between the vector spaces M_x and $H(x,u)$. Similarly, the map $X \longrightarrow X^v$ is an isomorphism between the vector spaces M_x and $V(x,u)$. Obviously, each tangent vector $Z \in (TM)_{(x,u)}$ can be written in the form

$$Z = X^h + Y^v, \tag{2.2}$$

where $X, Y \in M_x$ are uniquely determined vectors. In an obvious way, we can define horizontal and vertical lifts of vector fields on M.

Note that each system of local coordinates $(U; x^i, i = 1, ..., n)$ in M induces on TM a system of local coordinates $(p^{-1}(U); x^i, u^i, i = 1, ..., n)$. If $X = \sum X^i (\frac{\partial}{\partial x^i})_x$ is the local expression in $(U; x^i, i = 1, ..., n)$ of a vector X in M_x, $x \in M$, then the horizontal lift X^h and the vertical lift X^v of X to $(x,u) \in TM$ are given, with respect to the induced coordinates, by:

$$X^h = \sum X^i (\frac{\partial}{\partial x^i})_{(x,u)} - \sum \Gamma^i_{jk} u^j X^k (\frac{\partial}{\partial u^i})_{(x,u)}, \tag{2.3}$$

$$X^v = \sum X^i (\frac{\partial}{\partial u^i})_{(x,u)}, \tag{2.4}$$

where (Γ^i_{jk}) denote the Christoffel's symbols of g. The canonical vertical vector field on TM is a vector field U defined, in terms of local coordinates, by
$U = \sum u^i \frac{\partial}{\partial u^i}$. Here U does not depend on the choice of local coordinates and it is defined globally on TM. For a vector $u = \sum u^i \frac{\partial}{\partial x^i}$, we see that $u^v = \sum u^i (\frac{\partial}{\partial x^i})^v = U$ and $u^h = \sum u^i (\frac{\partial}{\partial x^i})^h$ is the geodesic flow on TM. The Riemannian curvature R of g is defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{2.5}$$

2.2. g-natural metrics on tangent bundles

Let (M,g) be a Riemannian manifold. Using the concept of naturality and related notions, O. Kowalski and M. Sekizawa succeeded to construct a 6-parameter family of metrics whose coefficients are real functions defined on $\mathbb{R}^+$. We refer to [11] for the construction of such metrics called g-natural metrics. g-natural metrics on tangent bundles are characterized as follows:

Proposition 2.1 *[3] Let (M,g) be a Riemannian manifold and G be a g-natural metric on TM. Then there are functions $\alpha_i, \beta_i : [0,+\infty[\to \mathbb{R}, i = 1;2;3,$ such that for every $x \in M$ and $u, X, Y \in M_x$, we have*

$$\begin{cases} G_{(x,u)}(X^h, Y^h) &= (\alpha_1+\alpha_3)(r^2) g_x(X,Y) \\ & \quad +(\beta_1+\beta_3)(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^h) &= \alpha_2(r^2) g_x(X,Y) + \beta_2(r^2) g_x(X,u) g_x(Y,u), \\ G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2) g_x(X,Y) + \beta_1(r^2) g_x(X,u) g_x(Y,u), \end{cases} \tag{2.6}$$

where $r^2 = \|u\|^2 = g_x(u,u)$. For $dim(M) = 1$, the same holds with $\beta_j = 0,\ j = 1,2,3$.

We put:
- $\varphi_i(t) = \alpha_i(t) + t\beta_i(t)$,
- $\varphi(t) = \varphi_1(t)[\varphi_1 + \varphi_3] - \varphi_2^2$,
- $\alpha(t) = \alpha_1(t)[\alpha_1 + \alpha_3] - \alpha_2^2$,

for all $t \in [0,+\infty[$.

Lemma 2.2 *[3] The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold (M,g) to be nondegenerate are $\alpha(t) \neq 0$ and $\varphi(t) \neq 0$ for all $t \in [0,+\infty[$*
If $dim(M) = 1$ this is equivalent to $\alpha(t) \neq 0$ for all $t \in [0,+\infty[$.

The wide class of g-natural metrics includes several well known metrics (Riemannian and not) on TM. In particular:

- The *Sasaki metric* g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$.
- *Kaluza–Klein metrics*, as commonly defined on principal bundles [17] (see also [6]), are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$.
- *Metrics of Kaluza–Klein type* are defined by the geometric condition of orthogonality between horizontal and vertical distributions. Thus, a g-natural metric G is of Kaluza-Klein type if $\alpha_2 = \beta_2 = 0$.

Hereafter, we will consider pseudo-Riemannian g-natural metrics of Kaluza-Klein type on TM. In this case, the Levi-Civita connection is reduced to the following form (cf. [3]):

Proposition 2.3 *Let (M,g) be a Riemannian manifold, ∇ its Levi-Civita connection, and R its curvature tensor. If G is a pseudo-Riemannian g-natural metric of Kaluza-Klein type on TM, then the Levi-Civita connection $\bar{\nabla}$ of (TM,G) is given by*

$$
\begin{aligned}
(\bar{\nabla}_{X^h}Y^h)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)} + v\{-\tfrac{1}{2}R(X,Y)u \\
&\quad + B_1(r^2)[g(Y,u)X + g(X,u)Y] \\
&\quad + [B_2(r^2)g(X,Y) + B_3(r^2)g(X,u)g(Y,u)]u\},
\end{aligned}
$$

$$
\begin{aligned}
(\bar{\nabla}_{X^h}Y^v)_{(x,u)} &= (\nabla_X Y)^v_{(x,u)} + h\{C_1(r^2)R(Y,u)X + C_2(r^2)g(X,u)Y \\
&\quad + C_3(r^2)g(Y,u)X + [C_4(r^2)g(R(X,u)Y,u) \\
&\quad + C_5(r^2)g(X,Y) + C_6(r^2)g(X,u)g(Y,u)]u\},
\end{aligned}
$$

$$
\begin{aligned}
(\bar{\nabla}_{X^v}Y^h)_{(x,u)} &= h\{C_1(r^2)R(X,u)Y + C_2(r^2)g(Y,u)X \\
&\quad + C_3(r^2)g(Y,u)X + [C_4(r^2)g(R(X,u)Y,u) \\
&\quad + C_5(r^2)g(X,Y) + C_6(r^2)g(X,u)g(Y,u)]u\},
\end{aligned}
$$

$$
\begin{aligned}
(\bar{\nabla}_{X^v}Y^v)_{(x,u)} &= v\{F_1(r^2)[g(Y,u)X + g(X,u)Y] \\
&\quad + [F_2(r^2)g(X,Y) + F_3(r^2)g(X,u)g(Y,u)]u\},
\end{aligned}
$$

for all vector fields X, Y *and* $(x,u) \in TM$*, where* $r^2 = \|u\|^2$ *and* B_i*,* C_i*, and* F_i *are the functions defined from* $[0,+\infty[$ *to* $\mathbb{R}$ *by*

$$
B_1 = -\frac{\beta_1+\beta_3}{2\alpha_1}, \qquad B_2 = -\frac{(\alpha_1+\alpha_3)'}{\varphi_1}, \qquad B_3 = -\frac{(\beta_1+\beta_3)'}{\varphi_1} + \frac{\beta_1(\beta_1+\beta_3)}{\alpha_1\varphi_1},
$$

$$
C_1 = -\frac{\alpha_1}{2(\alpha_1+\alpha_3)}, \qquad C_2 = \frac{(\beta_1+\beta_3)}{2(\alpha_1+\alpha_3)}, \qquad C_3 = \frac{(\alpha_1+\alpha_3)'}{\alpha_1+\alpha_3},
$$

$$
C_4 = \frac{\alpha_1(\beta_1+\beta_3)}{2(\varphi_1+\varphi_3)(\alpha_1+\alpha_3)}, \qquad C_5 = \frac{\beta_1+\beta_3}{2(\varphi_1+\varphi_3)},
$$

$$
C_6 = \frac{1}{2(\varphi_1+\varphi_3)}\left[2(\beta_1+\beta_3)' - \frac{\beta_1+\beta_3}{\alpha_1+\alpha_3}((\beta_1+\beta_3)+2(\alpha_1+\alpha_3)')\right],
$$

$$
F_1 = \frac{\alpha_1'}{\alpha_1}, \qquad F_2 = \frac{\beta_1-\alpha_1'}{\varphi_1}, \qquad F_3 = \frac{1}{\varphi_1}(\beta_1' - 2\frac{\alpha_1'}{\alpha_1}\beta_1).
$$

2.3. Natural complex structures on tangent bundles

Following the same approach used by O. Kowalski and M. Sekizawa to define g-natural metrics on the tangent bundle of a Riemannian manifold as first order natural operators (cf. [11]), V. Oproiu defined the class of natural almost complex structures on the tangent bundle TM of a Riemannian manifold (M,g) as the family of complex structures parametrized by six functions (cf. [12]). In this paper, we shall use the terminology g-natural to refer to such almost complex structures and we will adopt the same spirit of notations used in the case of g-natural metrics.

For any six real-valued smooth functions $a_i, b_i, i = 1,2,3$ defined on $[0,+\infty[$, we can define a $(1,1)$-tensor field on TM by

$$
\begin{cases}
JX^h = (a_1+a_3)(r^2)X^v + (b_1+b_3)(r^2)g(X,u)u^v \\
\qquad\quad -a_2(r^2)X^h - b_2(r^2)g(X,u)u^h, \\
\\
JX^v = a_2(r^2)X^v + b_2(r^2)g(X,u)u^v \\
\qquad\quad -a_1(r^2)X^h - b_1(r^2)g(X,u)u^h,
\end{cases}
\tag{2.7}
$$

for every $x \in M$ and $u, X \in M_x$, where vertical and horizontal lifts are taken at u and $r^2 = g_x(u,u)$. It is easy to check that J is an almost complex structure on TM if and only if

$$a_1(a_1+a_3)-a_2^2=1, \qquad A_1(A_+A_3)-A_2^2=1, \tag{2.8}$$

where A_i is the real function defined on $[0,+\infty[$ by $A_i(t)=a_i(t)+tb_i(t)$, $i=1,2,3$.

Definition 2.4 *An almost complex structure J on TM defined by (2.7), with the conditions (2.8), is called g-natural.*

When $a_2=b_2=0$, then we obtain the class of "natural almost complex structures" defined in [16] and if, furthermore, $a_3=b_1=b_3=0$ and $a_1=1$, we get the canonical complex structure on TM.

The integrability of g-natural almost complex structures is characterized by the following:

Proposition 2.5 *A g-natural almost complex structure J is integrable if and only if the following conditions hold*

i) (M,g) *has constant sectional curvature* k;

ii) $(a_1+a_3)'=\frac{1}{a_1+tb_1}[(a_1+a_3)(b_1+b_3)+k(1-3a_2^2-4ta_2b_2)]$;

iii) $a_1^{'}=\frac{1}{a_1+tb_1}[2a_2b_2-a_1(b_1+b_3)-ka_1^2]$;

iv) $a_2^{'}=\frac{1}{a_1+tb_1}[(a_1+a_3)b_2-2ka_1(a_2+tb_2)]$.

Now, we shall characterize almost Hermitian (resp. almost Kählerian) g-natural metrics with respect to an arbitrary g-natural almost complex structure. Let G be a pseudo-Riemannian g-natural metric on TM and J be a g-natural almost complex structure on TM. In [12], its shown that if G is Riemannian, then (TM,G,J) is almost Hermitian if and only if there are C^∞-functions λ and μ on $\mathbb{R}^+$ such that $a_i=\lambda\alpha_i$ and $A_i=\mu\varphi_i$, $i=1,2,3$. The result can be obviously generalized to pseudo-Riemannian metrics. However, by virtue of (2.8), we easily deduce that $\lambda^2=\alpha$ and $\mu^2=\varphi$. It follows that $\alpha>0$, $\varphi>0$, $\lambda=\pm\sqrt{\alpha}$ and $\mu=\pm\sqrt{\varphi}$. Hence, we have:

Proposition 2.6 *Let G be a pseudo-Riemannian g-natural metric on TM and J be a g-natural almost complex structure on TM. Then*

1. (TM,G,J) *is almost Hermitian if and only if*
   - *i)* $a_i=\lambda\alpha_i$, $i=1,2,3$;
   - *ii)* $A_i=\mu\varphi_i$, $i=1,2,3$.
2. (TM,G,J) *is almost Kählerian if and only if there are C^∞-functions λ and μ on $\mathbb{R}^+$ such that*
   - *i)* $a_i=\lambda\alpha_i$, $i=1,2,3$;
   - *ii)* $A_i=\mu\varphi_i$, $i=1,2,3$;
   - *iii)* $\mu=(t\lambda)'$, *i.e.* $\sqrt{\alpha\varphi}=\pm(\alpha+t\alpha')$.

Remarks 2.7 *1. By (2.8), a_1 and $a_1 + a_3$ (resp. A_1 and $A_1 + A_3$) do not vanish and have the same sign. Then, for almost Hermitian g-natural structures, λ and μ do not vanish and α_1 and $\alpha_1 + \alpha_3$ (resp. φ_1 and $\varphi_1 + \varphi_3$) do not vanish and have the same sign.*

2. Examples of pseudo-Riemannian metrics on TM which satisfy the condition iii) of the previous proposition are the Sasaki metric and more generally, the Oproiu metrics (cf. [13]). Indeed, Oproiu metrics are the g-natural metrics of Kaluza-Klein type characterized by the condition $\alpha = \varphi = 1$ (cf. [4]).

3. By virtue of (2.8), the class of g-natural almost complex structures is a 4-parameter family with coefficients real functions on $\mathbb{R}^+$. Since λ and μ can be expressed by means of the functions α_i and φ_i, then almost Hermitian (resp. almost Kählerian) structures constitute a 4-parameter (resp. 3-parameter) family with coefficients real functions on $\mathbb{R}^+$.

3. Geodesics on the tangent bundle endowed with a pseudo-Riemannian Kaluza-Klein type metric

Let (M, g) be a Riemannian manifold. In Theorem 4.3 in [3], the first author and M. Sarih gave the classification of Riemannian g-natural metrics on TM for which the fibers are totally geodesic. A particular subclass of such metrics is that of Kaluza-Klein type metrics. This result can be straightforwardly generalized to pseudo-Riemannian g-natural metrics. In particular, if G is a pseudo-Riemannian metric of Kaluza-Klein type on TM, then any tangent space T_xM, $x \in M$, is totally geodesic in (TM, G). We deduce then that if a geodesic γ of TM has a vertical velocity at a point, then it is has vertical velocity everywhere, i.e. γ is a vertical curve in the sense that it belongs completely to a fiber of the bundle TM. Hence, any geodesic curve in (TM, G) is either vertical or everywhere transverse to the fibers of TM.

Generally speaking, let $\gamma = (x(t), V(t))$ be a curve in TM. It is locally expressed as $\gamma(t) = \sum_{i=1}^{n} V^i(t) \left(\frac{\partial}{\partial x^i}\right)_{x(t)}$.

- γ is a vertical curve if $x(t) = x_0$ is a constant and, in this case, γ is a curve in the pseudo-Euclidean space $(T_{x_0}M, G_{\restriction T_{x_0}M})$.
- γ is transverse if $\dot{x} \neq 0$ everywhere. In this case, γ can be regarded as a vector field V along the curve x and its velocity vector field $\dot{\gamma}$ is given by

$$\dot{\gamma}(s) = \dot{x}(s)^h_{\gamma(s)} + (\nabla_{\dot{x}} V)^v_{\gamma(s)}. \tag{3.1}$$

When V is parallel along x, then the curve is called horizontal and it is called oblique otherwise.

In this section, we will study vertical, horizontal and oblique geodesics on (TM, G).

3.1. Vertical geodesics

As said before, vertical curves of TM are curves belonging to a fiber of TM. Locally, a vertical curve $\gamma = (x, V(s))$ of TM is expressed as $\gamma(t) = \sum_{i=1}^{n} V^i(t) \left(\frac{\partial}{\partial x^i}\right)_x$. Vertical geodesics on (TM, G) are vertical curves of TM, which are geodesics with respect to G. They are characterized as follows:

Proposition 3.1 *Let (M, g) be a Riemannian manifold and G be a pseudo-Riemannian metric of Kaluza-Klein type on TM. Geodesics of vertical type on (TM, G) are exactly geodesics in the pseudo-Euclidean vector spaces*

$(T_xM, G_{\restriction T_xM})$, $x \in M$. More precisely, a vertical curve $\gamma = (x, V(s))$ is a geodesic if and only if the following equation holds

$$\ddot{V} + 2F_1(\rho)g(V,\dot{V})\dot{V} + \{F_2(\rho)\|\dot{V}\|^2 + F_3(\rho)g(V,\dot{V})^2\}V = 0. \tag{3.2}$$

Proof Let γ be a curve on $T_{x_0}M$. In this case, we have

$$\dot{\gamma}(t) = \sum_{i=1}^{n} \dot{V}^i(t)\left(\frac{\partial}{\partial u^i}\right)_{\gamma(t)} = \sum_{i=1}^{n} \dot{V}^i(t)\left(\frac{\partial}{\partial x^i}\right)^v_{(x_0,V(t))}.$$

γ is a geodesic if and only if $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, which is equivalent, by Proposition 2.3, to equation (3.2). □

Now, we shall characterize geodesics of vertical type with nonzero constant length.

Proposition 3.2 *Let (M, g) be a Riemannian manifold and G be a pseudo-Riemannian metric of Kaluza-Klein type on TM. Let $x_0 \in M$ and $\gamma(s) = (x_0; V(s))$ be a nonconstant curve in $T_{x_0}M$ such that $\rho := \|V\|^2$ is constant. Then γ is a geodesic in (TM, G) if and only if the two following conditions hold:*

1. *$\alpha_1(\rho) + \rho\alpha_1^{'}(\rho) = 0$,*
2. *γ is a great circle of radius $\sqrt{\rho}$ in $T_{x_0}M$.*

Furthermore, γ is timelike (resp. spacelike) if $\alpha_1(\rho) < 0$ (resp. $\alpha_1(\rho) > 0$).

Proof Suppose then that $\rho = \|V\|^2$ is a nonzero constant so that $g(V, \dot{V}) = 0$. In particular, we have

$$G(\dot{\gamma},\dot{\gamma}) = \alpha_1(\rho)\|\dot{V}\|^2 + \beta_1(\rho)(g(\dot{V},V))^2 = \alpha_1(\rho)\|\dot{V}\|^2. \tag{3.3}$$

From (3.2), γ is a geodesic if and only if

$$\ddot{V} + F_2(\rho)\|\dot{V}\|^2 V = 0. \tag{3.4}$$

The scalar product of the last differential equation by $\dot{V}$ yields $g(\ddot{V},\dot{V}) = 0$. However, $g(\ddot{V},\dot{V}) = \frac{1}{2}(\frac{d}{dt}\|\dot{V}\|^2)$. Hence, $\|\dot{V}\| = cte$.

On the other hand, the scalar product of equation (3.4) by V gives $g(\ddot{V}, V) + \rho F_2(\rho)\|\dot{V}\|^2 = 0$, but $g(\ddot{V}, V) = \frac{d}{dt}(g(\dot{V}, V)) - \|\dot{V}\|^2 = -\|\dot{V}\|^2$. Then we obtain

$$(1 - \rho F_2(\rho))\|\dot{V}\|^2 = 0. \tag{3.5}$$

We claim that $\|\dot{V}\| \neq 0$. Indeed, if not, we have $\dot{V} = 0$ and then V is constant, which contradicts the fact that γ is a nonconstant curve. Thus, by virtue of (3.3), γ is either timelike (if $\alpha_1(\rho) < 0$) or spacelike (if $\alpha_1(\rho) > 0$). It follows also from (3.5) that $1 - \rho F_2(\rho) = 0$, i.e. $\alpha_1(\rho) + \rho\alpha_1^{'}(\rho) = 0$. Putting $G(\dot{\gamma},\dot{\gamma}) = c$, where c is a constant of the same sign as $\alpha_1(\rho)$, we have $\|\dot{V}\|^2 = \frac{c}{\alpha_1(\rho)}$. Hence, by (3.4), γ is a geodesic curve if and only if

$$\ddot{V} + \frac{c}{\rho\alpha_1(\rho)}V = 0, \tag{3.6}$$

if and only if

$$V(t) = \cos(\sqrt{\frac{c}{\rho\alpha_1(\rho)}}t).A + \sin(\sqrt{\frac{c}{\rho\alpha_1(\rho)}}t).B, \tag{3.7}$$

where $A,\ B \in T_{x_0}M$ such that $\|A\|^2 = \|B\|^2 = \rho$ and $g(A,B) = 0$. □

It is known that, for the Sasaki metrics, all straight lines in fibers are geodesics. The following Proposition gives the classification of all pseudo-Riemannian g-natural metrics of Kaluza-Klein type on TM which possess this property:

Proposition 3.3 *Let (M,g) be a Riemannian manifold and G be a pseudo-Riemannian metric of Kaluza-Klein type on TM. Then all the straight lines in the fibers of TM are geodesics if and only if the function φ_1 is constant.*

Proof Let γ be given by $\gamma(t) = tV_0$, where $V_0 \in T_{x_0}M$, $\|V_0\| = 1$. In this case, $\rho = t^2$ and $g(V,\dot{V}) = t$. Then γ is a geodesic if and only if

$$2F_1(t^2) + F_2(t^2) + t^2F_3(t^2) = 0, \tag{3.8}$$

i.e. $\varphi_1^{'}(t^2) = 0$, for all t. Hence, φ_1 is constant. □

3.2. Horizontal geodesics

Theorem 3.4 *Let (M,g) be a Riemannian manifold and G be a pseudo-Riemannian metric of Kaluza-Klein type on TM. A curve $\gamma(s) = (x(s);V(s))$ in TM is a horizontal geodesic curve on (TM,G) if and only if x is a geodesic on (M,g) of constant speed $\sqrt{\sigma}$ and V is a parallel vector field of (constant) squared norm ρ along x and one of the following cases occurs:*

1. *$(\alpha_1+\alpha_3)'(\rho) = 0$ and V is orthogonal to x;*
2. *$(\alpha_1+\alpha_3)'(\rho) = (\beta_1+\beta_3)'(\rho) = (\beta_1+\beta_3)(\rho) = 0$;*
3. *$(\alpha_1+\alpha_3)'(\rho) = (\varphi_1+\varphi_3)'(\rho) = 0$, $(\beta_1+\beta_3)(\rho) \neq 0$, $\beta_1(\rho) \neq \frac{\alpha_1(\rho)(\beta_1+\beta_3)'(\rho)}{(\beta_1+\beta_3)(\rho)}$ and $V = \pm\sqrt{\frac{\rho}{\sigma}}\dot{x}$;*
4. *$(\varphi_1+\varphi_3)'(\rho) = 0$, $(\alpha_1+\alpha_3)'(\rho) \neq 0$ and $V = \sqrt{\frac{\rho}{\sigma}}\dot{x}$.*

Proof Recall that a horizontal curve in TM is a curve $\gamma(t) = (x(t),V(t))$ such that $\dot{x} \neq 0$ and $\nabla_{\dot{x}}V = 0$ everywhere. In particular, V is of constant norm along x. Denote $\rho := \|V\|^2$. From Proposition 2.3, γ is a horizontal geodesic, i.e. $\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = 0$, if and only if the following system holds:

$$\begin{cases} 0 = \nabla_{\dot{x}}\dot{x}; \\ 0 = 2B_1(\rho)g(\dot{x},V)\dot{x} + \left[B_2(\rho)\|\dot{x}\|^2 + B_3(\rho)g(\dot{x},V)^2\right]V. \end{cases} \tag{3.9}$$

From the first equation of (3.9), x is a geodesic of (M,g) and in particular has a constant speed. We denote $\sigma := \|\dot{x}\|^2$. Making the scalar product of the second equation of (3.9) by V, we obtain:

$$[2B_1(\rho) + \rho.B_3(\rho)]g(\dot{x},V)^2 + \sigma\rho.B_2(\rho) = 0. \tag{3.10}$$

- If $2B_1(\rho) + \rho.B_3(\rho) = 0$, then $B_2(\rho) = 0$, i.e. $(\alpha_1 + \alpha_3)'(\rho) = 0$. However, $2B_1(\rho) + \rho.B_3(\rho) = \frac{(\alpha_1+\alpha_3)'(\rho)-(\varphi_1+\varphi_3)'(\rho)}{\varphi_1(\rho)} = 0$. Then $(\varphi_1 + \varphi_3)'(\rho) = 0$. In this case, the second equation of (3.9) is equivalent to

  either $g(\dot{x}, V) = 0$ in an open interval, i.e. $\dot{x}$ and V are orthogonal;

  or $g(\dot{x}, V) \neq 0$ except may be at a finite number of isolated points and $B_1(\rho) = B_2(\rho) = 0$, i.e. $(\beta_1 + \beta_3)(\rho) = (\beta_1 + \beta_3)'(\rho) = 0$;

  or $g(\dot{x}, V) \neq 0$ except may be at a finite number of isolated points and $B_3(\rho) \neq 0$. It follows from the second equation of (3.9) that $B_1(\rho) \neq 0$, i.e. $(\beta_1 + \beta_3)(\rho) \neq 0$. On the other hand, making the scalar product of the second equation of (3.9) by $\dot{x}$, we obtain $2B_1(\rho)\sigma + B_3(\rho)g(\dot{x}, V)^2 = 0$, i.e. $g(\dot{x}, V)^2 = \rho\sigma$, since $2B_1(\rho) + \rho.B_3(\rho) = 0$. Hence, the second equation of (3.9) is equivalent to $V = \pm\sqrt{\frac{\rho}{\sigma}}\dot{x}$.

- If $2B_1(\rho) + \rho.B_3(\rho) \neq 0$, i.e.

$$(\varphi_1 + \varphi_3)'(\rho) - (\alpha_1 + \alpha_3)'(\rho) \neq 0. \tag{3.11}$$

  Then, by virtue of (3.10), $g(\dot{x}, V)^2$ is constant equal to

$$g(\dot{x}, V)^2 = \frac{-\rho.B_2(\rho)\sigma}{2B_1(\rho) + \rho.B_3(\rho)}. \tag{3.12}$$

  – If $B_2(\rho) = 0$, i.e. $(\alpha_1 + \alpha_3)'(\rho) = 0$, then $g(\dot{x}, V) = 0$ and the second equation of (3.9) is satisfied;

  – If $B_2(\rho) \neq 0$, i.e. $(\alpha_1 + \alpha_3)'(\rho) \neq 0$, then (3.12) implies that

$$B_2(\rho)[2B_1(\rho) + \rho.B_3(\rho)] < 0. \tag{3.13}$$

  Substituting from (3.12) into the second equation of (3.9), we obtain

$$V = \sqrt{\frac{-\rho[2B_1(\rho) + \rho.B_3(\rho)]}{\sigma B_2(\rho)}}\dot{x}.$$

  Taking the equalities of the norms of the two sides of the last identity, we obtain $B_2(\rho) + 2B_1(\rho) + \rho.B_3(\rho) = 0$, i.e. $(\varphi_1 + \varphi_3)'(\rho) = 0$. By (3.11), $(\alpha_1 + \alpha_3)'(\rho) \neq 0$. In particular, equation (3.13) is satisfied since $B_2(\rho)[2B_1(\rho) + \rho.B_3(\rho)] = -\left[\frac{(\alpha_1+\alpha)'(\rho)}{\varphi(\rho)}\right]^2 < 0$.

□

As corollary, we have the following result which characterizes the horizontal geodesics with respect to pseudo-Riemannian Kaluza-Klein metric on TM.

Corollary 3.5 *Let (M, g) be a Riemannian manifold and G be a pseudo-Riemannian Kaluza-Klein metric on TM. A curve $\gamma(s) = (x(s); V(s))$ in TM is a horizontal geodesic curve on (TM, G) if and only if x is a geodesic on (M, g) of constant speed $\sqrt{\sigma}$ and V is a parallel vector field of (constant) squared norm ρ along x, where ρ is a critical point of $\alpha_1 + \alpha_3$.*

Consequently, we have

- *If G is a pseudo-Riemannian Kaluza-Klein metric on TM such that $\alpha_1+\alpha_3$ is a constant function (in particular if G is the Sasaki metric or the Cheeger-Gromoll metric), then $\gamma(s)=(x(s);V(s))$ is a horizontal geodesic curve on (TM,G) if and only if x is a geodesic on (M,g) and V is a parallel vector field along x.*

- *If G is a pseudo-Riemannian Kaluza-Klein metric on TM such that $(\alpha_1+\alpha_3)'$ does not vanish, then there is no horizontal geodesic on (TM,G).*

We have also the following characterization of horizontal geodesic velocity curves which is an immediate corollary of Theorem 3.4:

Corollary 3.6 *Let (M,g) be a Riemannian manifold and G be a pseudo-Riemannian metric of Kaluza-Klein type on TM. A velocity curve $\gamma(s)=(x(s);\dot{x}(s))$ in TM is a horizontal geodesic curve on (TM,G) if and only if x is a geodesic on (M,g) and $\|\dot{x}\|^2$ is a critical point of $\varphi_1+\varphi_3$.*

Remark 3.7 *If $\alpha_1+\alpha_3$ is a positive (resp. negative) function, then any horizontal geodesic is spacelike (resp. timelike). Indeed, $G(\dot{\gamma},\dot{\gamma})=(\alpha_1+\alpha_3)(\rho)\sigma$.*

Example 3.8 *The base manifold is considered $M=\mathbb{S}^2(R)$, $R>0$, equipped with the metric induced by the standard scalar product on $\mathbb{R}^3$. We consider two examples of horizontal geodesics each representing a situation in Theorem 3.4:*

1. ***Situation 3- in Theorem 3.4:*** *Let $T\mathbb{S}^2(R)$ be endowed with a metric G such that $\varphi_1+\varphi_3$ is constant and $\beta_1+\beta_3$ does not vanish. If we consider the geodesic curve $x(s)=(R\cos(s),R\sin(s),0)$ in $\mathbb{S}^2(R)$ and $V=\frac{\rho}{R^2}\dot{x}$, where $\rho\in\mathbb{R}$, then $\gamma(s)=(x;V)$ is a horizontal geodesic curve in $(T\mathbb{S}^2(R),G)$. Figure 1 below illustrates the case when $R=\rho=2$.*

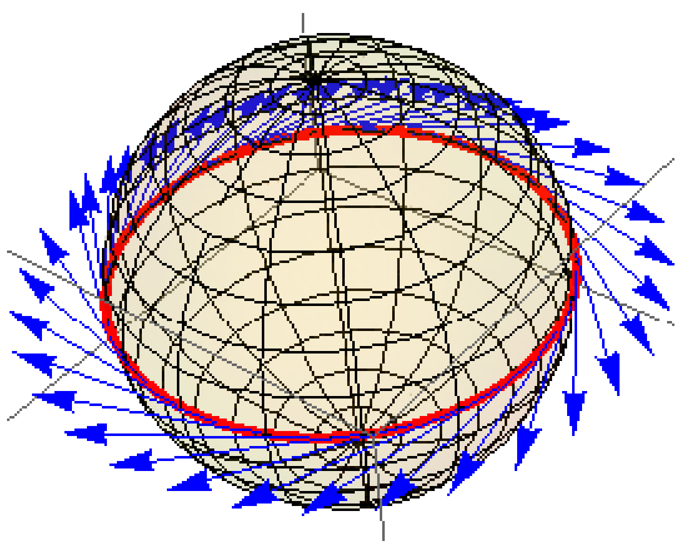

Figure 1. Horizontal geodesic curve in $(T\mathbb{S}^2(R),G)$ with $R=\rho=2$.

2. ***Situation 2- in Theorem 3.4:*** *Let $T\mathbb{S}^2(R)$ be endowed with a metric G such that $\alpha_1 + \alpha_3$ is constant and $\beta_1 + \beta_3$ vanishes (the Sasaki metric and the Cheeger-Gromoll metric are examples of such metrics). With the same expression of x as in the first case, if we put $V(s) = (V_1(s), V_2(s), V_3(s))$, then V is parallel vector field if and only if*

$$\begin{cases} 0 = \dot{V}_1 + (\cos(s)V_2 - \sin(s))\cos(s) \\ 0 = \dot{V}_2 + (\cos(s)V_2 - \sin(s))\sin(s) \\ 0 = \dot{V}_3 \end{cases} \tag{3.14}$$

In particular $V(s) = (-\sqrt{2}\sin(s), \sqrt{2}\cos(s), 2)$ is a solution of (3.14). In this case, $\gamma(s) = (x; V)$ is a horizontal geodesic curve in $(T\mathbb{S}^2(R), G)$. Figure 2 below illustrates the case when $R = 2$ and $\rho = 6$.

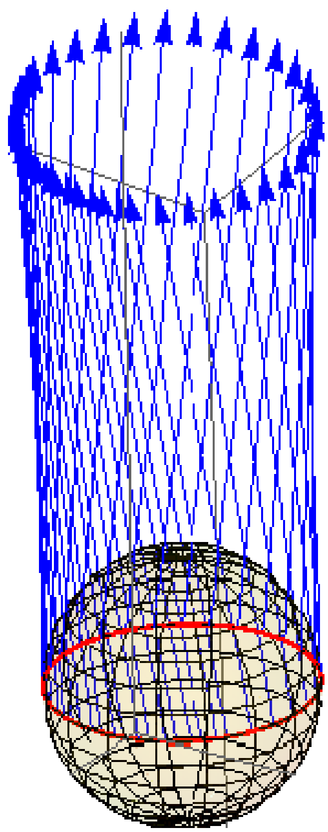

Figure 2. Horizontal geodesic curve in (TM, G) with $R = 2$ and $\rho = 6$.

3.3. Oblique geodesics

Recall that oblique geodesics correspond to the case $\|\dot{V}\| \neq 0$ everywhere. A close analysis of the differential equation system giving oblique geodesics shows that it is hard to solve it in the full generality of arbitrary Riemannian manifold and arbitrary pseudo-Riemannian Kaluza-Klein type metrics on its tangent bundle. Thus, we shall restrict ourselves to the case where (M, g) is a space of constant sectional curvature c and G is a pseudo-Riemannian Kaluza-Klein-metric, i,e. α_2, β_2 and $\beta_1 + \beta_3$ are identically zero.

Let $\gamma = (x(s), V(s))$ be an oblique curve of TM, i.e. $\rho := \|\dot{V}\|^2 \neq 0$ everywhere. In this case, we have $\dot{\gamma}(s) = \dot{x}(s)^h_{\gamma(s)} + (\nabla_{\dot{x}} V)^v_{\gamma(s)}$. It follows from Proposition 2.3 that the γ is a geodesic in (TM, G) if and only if

$$\begin{cases} 0 = \ddot{x} + 2c.C_1(\rho)[g(\dot{x}, V)\dot{V} - g(\dot{x}, \dot{V})V] + 2C_3(\rho)g(V, \dot{V})\dot{x}; \\ 0 = \ddot{V} + 2F_1(\rho)g(\dot{V}, V)\dot{V} + [B_2(\rho)\|\dot{x}\|^2 + F_2(\rho)\|\dot{V}\|^2 \\ \quad +F_3(\rho)g^2(\dot{V}, V)]V. \end{cases} \tag{3.15}$$

The following proposition gives necessary conditions for oblique curves to be geodesics:

Proposition 3.9 *Let (M, g) be a space of constant sectional curvature c and G be a pseudo-Riemannian g-natural Kaluza-Klein metric on TM. Let $\gamma(t) = (x(t), V(t))$ be an oblique geodesic curve on (TM, G) (i.e. $\dot{x} \neq 0$ everywhere) and put $\rho = \|V\|^2$. Then*

1. *There is a real constant $K_1 \neq 0$ of the same sign as $(\alpha_1 + \alpha_3) \circ \rho$ such that*

$$\|\dot{x}\| = \frac{K_1}{(\alpha_1 + \alpha_3) \circ \rho}. \tag{3.16}$$

In particular $\|\dot{x}\| = cte$ if and only if either $\|V\|$ is a constant or $\alpha_1 + \alpha_3$ is a constant on the set $\sum_\rho = \{\rho = \|V(t)\|^2, t \in I\}$.

2. *If V vanishes nowhere and $W := \frac{1}{\sqrt{\rho}}V$ then*

$$\|\dot{W}\| = \frac{K_2}{\rho.\alpha_1 \circ \rho}, \tag{3.17}$$

where K_2 is a real constant of the same sign as the function $\alpha_1 \circ \rho$.

Proof

1. Making the scalar product of the first equation of (3.15) by $\dot{x}$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\dot{x}\|^2) + 2C_3(\rho)\|\dot{x}\|^2 g(V, \dot{V}) = 0 \tag{3.18}$$

i.e. $\frac{\frac{d}{dt}(\|\dot{x}\|^2)}{\|\dot{x}\|^2} = -2C_3(\rho)\frac{d}{dt}(\|V\|^2) = -2\rho' C_3(\rho) = -2\rho' \frac{(\alpha_1+\alpha_3)'}{\alpha_1+\alpha_3}(\rho)$.
Integrating, we obtain

$$\|\dot{x}\| = \frac{K_1}{(\alpha_1 + \alpha_3) \circ \rho}, \tag{3.19}$$

where K_1 is a real constant of the same sign as $(\alpha_1 + \alpha_3)(\rho)$.

2. Putting $r := \sqrt{\rho}$, we have from $W = \frac{1}{\sqrt{\rho}}V = \frac{1}{r}V$

$$\dot{V} = r'W + r\dot{W} \quad \text{and} \quad \ddot{V} = r''W + 2r'\dot{W} + r\ddot{W}. \tag{3.20}$$

Substituting into the second equation of (3.15), we obtain

$$r\ddot{W} + 2r'[1 + r^2 F_1(r^2)]\dot{W} + \{r'' + r(r')^2[2F_1(r^2) + F_2(r^2) + r^2 F_3(r^2)] + r^3 F_2(r^2)\|\dot{W}\|^2 + rB_2(r^2)\|\dot{x}\|^2\}W = 0. \tag{3.21}$$

Applying the scalar product of (3.21) with $\dot{W}$ and using $\frac{d}{dt}(g(\dot{W},\dot{W})) = 2g(\ddot{W},\dot{W})$, we obtain

$$r\frac{d}{dt}(g(\dot{W},\dot{W})) + 4r'[1 + r^2\frac{\alpha_1'(r^2)}{\alpha_1(r^2)}]g(\dot{W},\dot{W}) = 0,$$

which is equivalent to the equation

$$\frac{d}{dt}(\|\dot{W}\|) + 4rr'\frac{(t\alpha_1)'(r^2)}{r^2\alpha_1(r^2)}\|\dot{W}\| = 0.$$

Integrating the last equation, we obtain:

$$\|\dot{W}\| = \frac{K_2}{r^2\alpha_1(r^2)} = \frac{K_2}{\rho\alpha_1(\rho)},$$

where K_2 is a real constant of the same sign of $\alpha_1(\rho)$

□

Remark 3.10 *Suppose that* $K_2 = 0$, *i.e.* W *is a parallel vector field. Then, substituting from* (3.20) *into* (3.15) *and using* (3.16), *it is easy to see that* γ *is a geodesic curve on* (TM, G) *if and only if*

$$\begin{cases} \ddot{x} + [ln(|(\alpha_1 + \alpha_3)| \circ r^2)]'\dot{x} = 0; \\ r'' + r(r')^2\frac{\varphi_1'(r^2)}{\varphi_1(r^2)} - \frac{rK_1^2(\alpha_1+\alpha_3)'(r^2)}{\varphi_1(r^2)(\alpha_1+\alpha_3)^2} = 0. \end{cases} \tag{3.22}$$

Denoting the unit tangent vector and the principal normal vector of x *by* ξ_1 *and* ξ_2, *respectively, and the first curvature of* x *by* κ_1, *we have*

$$\dot{x} = \sqrt{\sigma}\xi_1 \quad \text{and} \quad \ddot{x} = (\sqrt{\sigma})'\xi_1 + \sqrt{\sigma}\kappa_1\xi_2. \tag{3.23}$$

Substituting from (3.23) *into the first equation of* (3.22), *we find that* $\kappa_1 = 0$, *i.e.* x *is a geodesic. In particular,* σ *is constant and, by* (3.16), $\alpha_1 + \alpha_3$ *is constant on* $\sum_\rho = \{\rho = \|V(t)\|^2, t \in I\}$.

On the other hand we have, using the fact that $\|\dot{\gamma}\|^2 =: l$ *is constant* (γ *is a geodesic) and equations* (3.20) *and* (3.16), *we have:*

$$l = (\alpha_1 + \alpha_3)(r^2)\|\dot{x}\|^2 + (r')^2\varphi_1(r^2) = \frac{K^2}{(\alpha_1 + \alpha_3)(r^2)} + (r')^2\varphi_1(r^2).$$

We deduce that

$$(r')^2 = \frac{l}{\varphi_1(r^2)} - \frac{K_1^2}{\varphi_1(r^2)(\alpha_1 + \alpha_3)(r^2)}. \tag{3.24}$$

Deriving the last equation we find that

either $r' = 0$ *on some interval* $J \subset I$, *i.e.* ρ *is constant on* J.

or $r' \neq 0$ *almost everywhere and*

$$r'' = r\frac{\varphi_1'(r^2)}{\varphi_1^2(r^2)}\Big[-l + \frac{K_1^2}{(\alpha_1+\alpha_3)(r^2)}\Big]. \tag{3.25}$$

Using (3.24) *and* (3.25), *we find that the second equation of* (3.22) *is satisfied.*

3.3.1. Oblique velocity geodesic curves

From Corollary 3.6, a velocity of a curve $x(t)$ of M is a horizontal geodesic only if $x(t)$ is a geodesic. More generally, a velocity of a curve $x(t)$ of M is an oblique curve of TM if and only if x is not a geodesic on (M,g). We shall give a characterization of oblique geodesic velocity curves on (TM,G).

Proposition 3.11 *Let* (M,g) *be a space of constant sectional curvature* c *and* G *be a pseudo-Riemannian Kaluza-Klein metric on* TM, *such that the function* $t \mapsto t(\alpha_1+\alpha_3)^2(t)$ *has isolated critical points. The velocity of a nongeodesic curve* $x(s)$ *of* (M,g) *is a geodesic curve in* (TM,G) *if and only if*

(i) x *is a Riemannian circle of constant speed* $\sqrt{\rho}$;

(ii) $c = \frac{(\alpha_1+\alpha_3)(\rho)}{\rho\alpha_1(\rho)}$;

(iii) one of the following assertions holds

- $(\alpha_1+\alpha_3)'(\rho) = \alpha_1(\rho) + \rho\alpha_1'(\rho) = 0$;
- $(\alpha_1+\alpha_3)'(\rho)[\alpha_1(\rho)+\rho\alpha_1'(\rho)] < 0$ *and the first curvature* κ_1 *of* x *is equal to* $\rho\sqrt{\frac{-(\alpha_1+\alpha_3)'(\rho)}{\alpha_1(\rho)+\rho\alpha_1'(\rho)}}$.

Proof Let $x(t)$ be a nongeodesic curve in M and $\gamma(t) = (x(t), \dot{x}(t))$ its velocity vector field. Put $\rho = \|\dot{x}\|^2$. Then, from (3.15), γ is a geodesic curve in (TM,G) if and only if

$$\begin{cases} 0 = [1+2c\rho C_1(\rho)]\ddot{x} + 2g(\dot{x},\ddot{x})[C_3(\rho) - cC_1(\rho)]\dot{x}; \\ 0 = \dddot{x} + 2F_1(\rho)g(\dot{x},\ddot{x})\ddot{x} + [\rho B_2(\rho) + F_2(\rho)\|\ddot{x}\|^2 + F_3(\rho)g(\dot{x},\ddot{x})^2]\dot{x}. \end{cases} \tag{3.26}$$

Making the scalar product of the first equation of (3.26) by $\dot{x}$, we obtain $\rho'[1+2\rho C_3(\rho)] = 0$, i.e.

$$\rho'[2\rho(\alpha_1+\alpha_3)'(\rho) + (\alpha_1+\alpha_3)(\rho)] = 0,$$

which gives, by the fact that the function $t \mapsto t(\alpha_1+\alpha_3)^2(t)$ has isolated critical points and by continuity of ρ', $\rho' = 0$ identically, i.e. ρ is constant. We deduce that $g(\dot{x},\ddot{x}) = 0$ and (3.26) becomes

$$\begin{cases} 0 = [1+2c\rho C_1(\rho)]\ddot{x}; \\ 0 = \dddot{x} + [\rho B_2(\rho) + F_2(\rho)\|\ddot{x}\|^2]\dot{x}. \end{cases} \tag{3.27}$$

Since x is a nongeodesic curve, then $\ddot{x} \neq 0$ everywhere. Hence, $1 + 2c\rho C_1(\rho) = 0$, i.e. $c = \frac{(\alpha_1+\alpha_3)(\rho)}{\rho\alpha_1(\rho)}$. On the other hand, since x is not a geodesic, then there exist a unit vector field ν_1 along x such that $\ddot{x} = \kappa_1\nu_1$, where the first curvature κ_1 of x is nonzero everywhere. We deduce that $\dddot{x} = \kappa_1 \nabla_{\dot{x}} \nu_1 + \kappa_1^{'} \nu_1$. Making the scalar product of the second equation of (3.27) with $\dot{x}$, we obtain:

$$\kappa_1^2 = \|\ddot{x}\|^2 = \frac{d}{dt}[g(\ddot{x}, \dot{x})] - g(\dddot{x}, \dot{x}) = \rho[\kappa_1^2 F_2(\rho) + \rho B_2(\rho)],$$

i.e. $(1 - \rho F_2(\rho))\kappa_1^2 = \rho^2 B_2(\rho)$. We have then two cases

Case 1: $1 - \rho F_2(\rho) = B_2(\rho) = 0$, i.e. $(\alpha_1 + \alpha_3)'(\rho) = \alpha_1(\rho) + \rho \alpha_1^{'}(\rho) = 0$. In this case, the second equation of (3.27) becomes $\dddot{x} + \frac{1}{\rho}\dot{x} = 0$, i.e. $\kappa_1 \nabla_{\dot{x}} \nu_1 + \kappa_1^{'} \nu_1 + \frac{1}{\rho}\dot{x} = 0$. Making the scalar product of the last equation by ν_1 and taking into account that $g(\nabla_{\dot{x}} \nu_1, \nu_1) = g(\dot{x}, \nu_1) = 0$, we obtain $\kappa' = 0$, i.e. κ is constant.

Case 2: $1 - \rho F_2(\rho) \neq 0$. Then $B_2(\rho) \neq 0$ and $B_2(\rho)[1 - \rho F_2(\rho)] > 0$, i.e. $(\alpha_1 + \alpha_3)'(\rho)[\alpha_1(\rho) + \rho \alpha_1^{'}(\rho)] < 0$. In this case, κ is also constant equal to $\rho \sqrt{\frac{-(\alpha_1+\alpha_3)'(\rho)}{\alpha_1(\rho)+\rho\alpha_\rho^{'}}}$.

The converse part of the Proposition is trivial. □

Remark 3.12 *Contrary to horizontal geodesics which could not be lightlike, there are some situations where oblique geodesics are lightlike. Indeed, for oblique geodesic velocity curves $(x(s), \dot{x}(s))$ on (TM, G), we have:*

$$G(\dot{\gamma}, \dot{\gamma}) = \rho(\alpha_1 + \alpha_3)(\rho) + \kappa^2 \alpha_1(\rho).$$

Thus, γ is lightlike if and only if $\kappa = \sqrt{-\frac{\rho(\alpha_1+\alpha_3)(\rho)}{\alpha_1(\rho)}}$. Notice that lightlike geodesics do exist only if the metric G in not Riemannian, i.e. if $\alpha_1(\alpha_1 + \alpha_3) < 0$. As an example, we consider the family of Kaluza-Klein metrics on TM given by the generating functions which are C^∞ on $\mathbb{R}^+$ and defined on an interval I of $\mathbb{R}_*^+$ by

$$(\alpha_1 + \alpha_3)(t) = ae^{\lambda t}, \qquad \alpha_1(t) = \frac{b}{t} e^{\lambda t}, \tag{3.28}$$

where a, b, and λ are real constants such that $ab < 0$. Then the velocity of any Riemannian circle x on M of speed $\sqrt{\rho}$ and first curvature $\rho\sqrt{\frac{-b}{a}}$ is a lightlike oblique geodesic on (TM, G). Note that the case $\lambda = 0$ (resp. $\lambda \neq 0$) corresponds to the first (resp. second) assertion in (iii) of Proposition 3.11.

3.3.2. Oblique geodesics of constant norm

We end this section by giving a characterization of oblique geodesics (x, V) on (TM, G) of constant norm, i.e. with $\rho := \|V\|^2$ constant along x.

Proposition 3.13 *Let (M, g) be a space of constant sectional curvature c and G be a pseudo-Riemannian g-natural Kaluza-Klein metric on TM. Let $\gamma(s) = (x(s); V(s))$ be an oblique curve in TM such that $\rho := \|V\|^2$ is a constant. If γ is a geodesic in (TM, G), then it possesses the following properties:*

1. *$\|\dot{x}\|$ and $\|\dot{V}\|$ are constant and are related by the identity*

$$[\alpha_1(\rho) + \rho\alpha_1'(\rho)]\|\dot{V}\|^2 + \rho(\alpha_1 + \alpha_3)'(\rho)\|\dot{x}\|^2 = 0. \tag{3.29}$$

2. *Let* $\lambda := 1 - \frac{c\rho\alpha_1(\rho)}{(\alpha_1+\alpha_3)(\rho)}$ *and let* $\theta \in [0,\pi]$ *be the angle between* $\dot{x}$ *and* V, *i.e.* $g(\dot{x},V) = \|\dot{x}\|\|V\|\cos\theta$. *Then*

$$\cos\theta = A\cos\left(\frac{\lambda\|\dot{V}\|}{\sqrt{\rho}}t+\sigma\right) + B\sin\left(\frac{\lambda\|\dot{V}\|}{\sqrt{\rho}}t+\sigma\right), \tag{3.30}$$

where A, B, σ *are real constants.*

3. *If* $\lambda = 0$, *i.e.* $c = \frac{(\alpha_1+\alpha_3)(\rho)}{\rho\alpha_1(\rho)}$, *then* x *is a Riemannian circle and* θ *is constant.*

Proof Let $\gamma(s) = (x(s); V(s))$ be an oblique geodesic in TM such that $\rho := \|V\|^2$ is a constant. Then $g(\dot{V},V) = 0$ and (3.15) becomes:

$$\begin{cases} \ddot{x} + 2c.C_1(\rho)[g(\dot{x},V)\dot{V} - g(\dot{x},\dot{V})V] = 0; \\ \ddot{V} + [B_2(\rho)\|\dot{x}\|^2 + F_2(\rho)\|\dot{V}\|^2]V \quad = 0. \end{cases} \tag{3.31}$$

By Proposition 3.9, $\|\dot{x}\|$ is constant. Making the scalar product of the second equation of (3.31) by $\dot{V}$ and V, respectively, we obtain $g(\ddot{V},\dot{V}) = 0$, i.e. $\|\dot{V}\|$ is constant, and (3.29).

On the other hand, using (3.29), it is easy to check that $B_2(\rho)\|\dot{x}\|^2 + F_2(\rho)\|\dot{V}\|^2 = \frac{1}{\rho}\|\dot{V}\|^2$, so that the second equation of (3.31) becomes:

$$\ddot{V} + \frac{1}{\rho}\|\dot{V}\|^2 V = 0. \tag{3.32}$$

Making the scalar product of the first equation of (3.31) by V and using (3.32) and the fact that $g(\ddot{x},V) = \frac{d}{dt}(g(\dot{x},V)) - g(\dot{x},\dot{V})$, we obtain

$$\frac{d}{dt}(g(\dot{x},V)) - [1 + 2c\rho C_1(\rho)]g(\dot{x},\dot{V}) = 0. \tag{3.33}$$

Making the scalar product of the first equation of (3.31) by $\dot{V}$ and using (3.33) and $g(\ddot{x},\dot{V}) = \frac{d}{dt}(g(\dot{x},\dot{V})) - g(\dot{x},\ddot{V})$, we get:

$$\frac{d}{dt}(g(\dot{x},\dot{V})) + \frac{1}{\rho}[1 + 2c\rho C_1(\rho)]\|\dot{V}\|^2 g(\dot{x},V) = 0. \tag{3.34}$$

Putting $\lambda := 1 - \frac{c\rho\alpha_1(\rho)}{(\alpha_1+\alpha_3)(\rho)}$ and $g(\dot{x},V) = \|\dot{x}\|\|V\|\cos\theta$, $\theta \in [0,\pi]$, we have two cases:

- If $\lambda = 0$, then equations (3.33) and (3.33) imply that $g(\dot{x},V)$ and $g(\dot{x},\dot{V})$ are constant. In particular, θ is constant. If we denote κ_1 the first curvature of x, so that $\|\ddot{x}\| = \kappa_1^2$, then we have, by the first equation of (3.31), $\kappa_1^2 = 4c^2.C_1(\rho)^2[g(\dot{x},V)^2\|\dot{V}\|^2 + \rho g(\dot{x},\dot{V})^2]$, which is constant. Hence, x is a Riemannian circle on (M,g).
- If $\lambda \neq 0$, then substituting from (3.33) into (3.34), we get

$$\frac{d^2}{dt^2}(g(\dot{x},V)) + \frac{\lambda^2\|\dot{V}\|^2}{\rho}g(\dot{x},V) = 0,$$

whose solution gives (3.30).

□

Remarks 3.14 *1. When $\alpha_1 + t\alpha_1' \neq 0$ and $(\alpha_1+\alpha_3)' = 0$ everywhere (The Sasaki metric and the Cheeger-Gromoll metric are examples of such metrics), then equation (3.29) yields $\|\dot{V}\| = 0$, which contradicts the fact that γ is an oblique curve. Hence, for such metric types, there is no oblique geodesic curve on TM belonging to a tangent sphere bundle.*

2. An oblique geodesic curve $\gamma = (x, V)$ on (TM, G), such that $\rho := \|V\|^2$ is constant, is lightlike if and only if

$$\rho\alpha_1(\rho)(\alpha_1+\alpha_3)'(\rho) - (\alpha_1+\alpha_3)(\rho)[\alpha_1(\rho) + \rho\alpha_1'(\rho)] = 0. \tag{3.35}$$

Indeed, substituting from $G(\dot{\gamma}, \dot{\gamma}) = \rho(\alpha_1+\alpha_3)(\rho) + \kappa^2\alpha_1(\rho) = 0$ into (3.29), we obtain (3.35). For example, the metrics defined by (3.28) satisfy equation (3.29), for any $\rho \in I$. We deduce that for such metrics, any oblique geodesic curve $\gamma = (x, V)$ with $\rho := \|V\|^2$ a constant in I is lightlike.

3.3.3. Oblique geodesics on $(TM^2(c), G)$

In this section, we will give a classification of oblique geodesics on the tangent bundle $TM^2(c)$ of a Riemannian 2-dimensional manifold $(M^2(c), g)$ of constant gaussian curvature c, equipped with a pseudo-Riemannian Kaluza-Klein g-natural metric G. Let $\gamma = (x, V)$ be a nonvanishing oblique curve on $TM^2(c)$, i.e. $\dot{x}$ does not vanish on the domain I of γ. Denote by

- $\sigma := \|\dot{x}\|^2$;
- $\rho := \|\dot{V}\|^2$ and $r := \|V\| = \sqrt{\rho}$;
- κ the curvature of the curve x of $M^2(c)$;
- ξ_1 the unit tangent vector field along x;
- ξ_2 the principal normal vector field along x;

so that we have

$$\dot{x} = \sqrt{\sigma}\xi_1, \quad \dot{\xi}_1 = \kappa\xi_2 \quad \text{and} \quad \dot{\xi}_2 = -\kappa\xi_1. \tag{3.36}$$

We deduce that

$$\ddot{x} = (\sqrt{\sigma})'\xi_1 + \sqrt{\sigma}\kappa\xi_2. \tag{3.37}$$

If we put $W = \frac{1}{r}V$, then $\|W\| = 1$ and we can write

$$W = f_1\xi_1 + f_2\xi_2, \tag{3.38}$$

where f_1 and f_2 are real functions on I such that

$$f_1^2 + f_2^2 = 1. \tag{3.39}$$

It follows then that

$$f_1'f_2 + f_2'f_1 = 0. \tag{3.40}$$

Deriving (3.38) twice and using (3.36), we have

$$\dot{W} = (f_1' - \kappa f_2)\xi_1 + (f_2' + \kappa f_1)\xi_2, \tag{3.41}$$

$$\ddot{W} = (f_1'' - \kappa' f_2 + \kappa^2 f_1)\xi_1 + (f_2'' + \kappa' f_1 + \kappa^2 f_2)\xi_2. \tag{3.42}$$

Suppose now that γ is an oblique geodesic. Then, by Proposition 3.9, we have

$$\sqrt{\sigma} = \frac{K_1}{(\alpha_1 + \alpha_3) \circ \rho} \quad \text{and} \tag{3.43}$$

$$\|\dot{W}\| = \frac{K_2}{\rho.\alpha_1 \circ \rho}, \tag{3.44}$$

where K_1 and K_2 are real constants with the same signs as $(\alpha_1 + \alpha_3) \circ \rho$ and $\alpha_1 \circ \rho$, respectively.

Let us investigate the geodesic equations (3.15) on $(TM^2(c), G)$ using the preceding data.

Investigation of the first equation of (3.15):

The first equation of (3.15) is equivalent, by virtue of (3.37), to the equation

$$\kappa \xi_2 + 2c\rho C_1(\rho)[g(\xi_1, W)\dot{W} - g(\xi_2, \dot{W})W] = 0. \tag{3.45}$$

Substituting from (3.36), (3.38), (3.39), and (3.41) into (3.45), we find that the first equation of (3.15) is equivalent to

$$\kappa(1 + 2c\rho C_1(\rho)) + 2c(f_1 f_2' - f_2 f_1')\rho C_1(\rho) = 0. \tag{3.46}$$

Here, we distinguish two cases:

Case 1: $c = 0$. In this case, we have $\kappa = 0$, i.e. x is a geodesic.

Case 2: $c \neq 0$. By (3.39), either $f_1 \neq 0$ or $f_2 \neq 0$, locally. Suppose for instance that $f_2 \neq 0$ on some interval $J \subset I$, the treatment of the other case being the same. Then we have, by virtue of (3.40),

$$f_2' = -\frac{f_1}{f_2} f_1' \quad \text{on} \quad J. \tag{3.47}$$

Substituting from (3.47) into (3.46) and using (3.39), we find

$$\frac{f_1'}{\sqrt{1 - f_1^2}} = \varepsilon_2 \kappa \Big[1 - \frac{(\alpha_1 + \alpha_3) \circ \rho}{c\rho.\alpha_1 \circ \rho}\Big], \tag{3.48}$$

where ε_2 is the sign of f_2. Integrating (3.48), we obtain:

$$f_1 = \varepsilon \sin \circ \lambda, \tag{3.49}$$

where $\varepsilon = \pm 1$, $\lambda : J \to \mathbb{R}$ being the function defined by

$$\lambda(t) := \int_{t_0}^{t} \kappa(s) \Big[1 - \frac{(\alpha_1 + \alpha_3)(\rho(s))}{c\rho(s)\alpha_1(\rho(s))}\Big] ds + \lambda_0, \tag{3.50}$$

for all $t \in J$, $t_0 \in J$ and $\lambda_0 \in \mathbb{R}$. Using (3.39) and (3.40) again, we get

$$f_2 = \varepsilon \cos \circ \lambda. \tag{3.51}$$

We deduce that, on J, we have:

$$V = \varepsilon \sqrt{\rho}[\sin \circ \lambda.\xi_1 + \cos \circ \lambda.\xi_2]. \tag{3.52}$$

Investigation of the second equation of (3.15):

From the expression $V = rW$, we have

$$\dot{V} = r'W + r\dot{W} \quad \text{and} \quad \ddot{V} = r''W + 2r'\dot{V} + r\ddot{W}. \tag{3.53}$$

Using (3.53), the second equation of (3.15) is equivalent to the following equation:

$$\begin{aligned} &r\ddot{W} + 2r'[1 + r^2F_1(r^2)]\dot{W} + \{r'' + r(r')^2[2F_1(r^2) + F_2(r^2) \\ &+ r^2F_3(r^2)] + r^3F_2(r^2)\|\dot{W}\|^2 + rB_2(r^2)\|\dot{x}\|^2\}W = 0. \end{aligned} \tag{3.54}$$

Using (3.41) and (3.42), we deduce that equation (3.54) is equivalent to the system

$$\begin{cases} 0 = & r(f_1'' - \kappa' f_2 + \kappa^2 f_1) + 2r'(1 + r^2F_1(r^2))(f_1' - \kappa f_2) \\ & + f_1\{r'' + r(r')^2\big[2F_1(r^2) + F_2(r^2) + r^2F_3(r^2)\big] \\ & + r^3F_2(r^2)\big[(f_1' - \kappa f_2)^2 + (f_2' + \kappa f_1)^2\big] + r\sigma B_2(r^2)\}, \\ \\ 0 = & r(f_2'' + \kappa' f_1 + \kappa^2 f_2) + 2r'(1 + r^2F_1(r^2))(f_2' + \kappa f_1) \\ & + f_2\{r'' + r(r')^2\big[2F_1(r^2) + F_2(r^2) + r^2F_3(r^2)\big] \\ & + r^3F_2(r^2)\big[(f_1' - \kappa f_2)^2 + (f_2' + \kappa f_1)^2\big] + r\sigma B_2(r^2)\}, \end{cases} \tag{3.55}$$

which is equivalent to the system

$$\begin{cases} 0 = & r[f_2f_1'' - f_1f_2'' - \kappa'] + 2r'(1 + r^2F_1(r^2))[f_2f_1' - f_1f_2' - \kappa], \\ \\ 0 = & r'' + r(r')^2\big[2F_1(r^2) + F_2(r^2) + r^2F_3(r^2)\big] \\ & + r^3F_2(r^2)\big[(f_1' - \kappa f_2)^2 + (f_2' + \kappa f_1)^2\big] + r\sigma B_2(r^2). \end{cases} \tag{3.56}$$

Note that the first equation of (3.56) is obtained by multiplying the first and the second equations of (3.55) by f_2 and $-f_1$, respectively, and summing up, and that the second equation of (3.56) is obtained by multiplying the first and the second equations of (3.55) by f_1 and f_2, respectively, and summing up.

However, using (3.49) and (3.51), we get:

$$f_1' = \lambda' f_2, \quad f_2' = -\lambda' f_1, \quad f_1'' = -(\lambda')^2 f_1, \quad f_2'' = -(\lambda')^2 f_2. \tag{3.57}$$

Substituting from (3.57) into (3.56), we find:

$$\begin{cases} 0 = & c\kappa' r^4\alpha_1^2(r^2) + \kappa(\alpha_1 + \alpha_3)(r^2)(r^2\alpha_1(r^2))', \\ \\ 0 = & r'' + r(r')^2\frac{\varphi_1'(r^2)}{\varphi_1(r^2)} + r\left\{\frac{\kappa^2(\alpha_1+\alpha_3)(r^2)}{c^2r^4\varphi_1^2(r^2)\alpha_1^2(r^2)}\big[2c^2r^2\varphi_1(r^2)\alpha_1(r^2)\right. \\ & \left. -(\alpha_1 + \alpha_3)(r^2)(\alpha_1(r^2) + r^2\alpha_1'(r^2))\big] - \sigma\frac{(\alpha_1+\alpha_3)'(r^2)}{\varphi_1(r^2)}\right\}. \end{cases} \tag{3.58}$$

Taking into account the first equation of (3.56), we should consider two cases:

Case 1: κ vanishes identically on some interval $J' \subset J$

Then λ is constant and, by (3.49) and (3.51), f_1 and f_2 are constant on J. We deduce from (3.41) that $\dot{W} = 0$ and hence $K_2 = 0$. It follows from Remark 3.10 that either ρ is constant or r is solution of the differential equation (3.24). Making a derivation of (3.24), standard calculation shows that r is also a solution of the second equation of (3.56).

Case 2: κ does not vanish on some interval $J' \subset J$

In this case, the first equation of (3.58) is equivalent on J' to

$$c\frac{\kappa'}{\kappa} = -\frac{(r^2\alpha_1(r^2))'}{(r^2\alpha_1(r^2))^2}(\alpha_1+\alpha_3)(r^2). \tag{3.59}$$

On the other hand, from (3.41) and (3.57), we get:

$$\|\dot{W}\|^2 = (f_1' - \kappa f_2)^2 + (f_2' + \kappa f_1)^2 = (\lambda' - \kappa)^2 = \frac{\kappa^2(\alpha_1+\alpha_3)^2(r^2)}{c^2 r^4 \alpha_1^2(r^2)}. \tag{3.60}$$

Comparing the last formula with (3.17), we obtain

$$\kappa = \frac{\bar{\varepsilon} c K_2}{(\alpha_1+\alpha_3)(r^2)}, \tag{3.61}$$

where $\bar{\varepsilon} = \pm 1$.

Now since γ is a geodesic then $\|\dot{\gamma}\|^2$ is a constant l. A standard calculation using (3.53), (3.43), and (3.44) gives:

$$(r')^2 = \frac{l}{\varphi_1(r^2)} - \frac{K_1^2}{\varphi_1(r^2)(\alpha_1+\alpha_3)(r^2)} - \frac{K_2^2}{r^2\varphi_1(r^2)\alpha_1(r^2)}. \tag{3.62}$$

Deriving (3.62), we find:

$$\begin{aligned} r'r'' = rr'\Big[&- \frac{\tilde{\varepsilon}\varphi_1'(r^2)}{\varphi_1^2(r^2)} + \frac{K_1^2(\varphi_1'(r^2)(\alpha_1+\alpha_3)(r^2) + \varphi_1(r^2)(\alpha_1+\alpha_3)'(r^2))}{\varphi_1^2(r^2)(\alpha_1+\alpha_3)^2(r^2)} \\ &+ \frac{K_2^2(\varphi_1(r^2)\alpha_1(r^2) + r^2\varphi_1'(r^2)\alpha_1(r^2) + r^2\varphi_1(r^2)\alpha_1'(r^2))}{r^4\varphi_1^2(r^2)\alpha_1^2(r^2)}\Big]. \end{aligned} \tag{3.63}$$

Then

either $r' = 0$ on some interval $J'' \subset J'$, i.e. ρ is constant on J'' and, in this case, κ is constant (by (3.61)) and

$$l\rho\alpha(\rho) = K_1^2\rho\alpha_1(\rho) + K_2^2(\alpha_1+\alpha_3)(\rho), \tag{3.64}$$

or $r' \neq 0$ on some interval $J'' \subset J'$. In this case, using (3.64) and (3.64), we get:

$$r'' + r(r')^2\frac{\varphi_1'(r^2)}{\varphi_1(r^2)} = \frac{K_1^2 r(\alpha_1+\alpha_3)'(r^2)}{\varphi_1(r^2)(\alpha_1+\alpha_3)^2(r^2)} + \frac{K_2^2(\alpha_1(r^2) + r^2\alpha_1'(r^2))}{r^3\varphi_1(r^2)\alpha_1^2(r^2)} \tag{3.65}$$

Comparing the last equation with the second equation of (3.58) and taking into account (3.61), we find: $\frac{2\kappa^2(\alpha_+\alpha_3)(r^2)}{r^2\varphi_1(r^2)\alpha_1(r^2)} = 0$, i.e. $\kappa = 0$, which is a contradiction.

Summarizing the previous discussion, we have

Theorem 3.15 *Let $(M^2(c), g)$ a Riemannian 2-dimensional manifold of constant gaussian curvature $c \neq 0$ and let its tangent bundle $TM^2(c)$ be equipped with a pseudo-Riemannian Kaluza-Klein g-natural metric G. Let $\gamma = (x, V)$ be a nonvanishing oblique curve on $TM^2(c)$, i.e. $\dot{x}$ does not vanish on the domain I of γ. Then γ is a geodesic on (TM, G) if and only if one of the following conditions holds locally* *(i)* x *is a geodesic on* $(M^2(c), g)$*,* V *makes a constant angle with* $\dot{x}$ *and the length* r *of* V *satisfies the ODE*

$$(r')^2 = \frac{l}{\varphi_1(r^2)} - \frac{K_1^2}{\varphi_1(r^2)(\alpha_1 + \alpha_3)(r^2)}, \tag{3.66}$$

where $l := \|\dot{\gamma}\|^2$*.*

(ii) x *is a Riemannian circle (with constant curvature* κ*),* $\rho := \|V\|^2$ *is constant and* V *makes an affine angle* θ *with* $\dot{x}$ *defined by*

$$\theta(t) = \kappa \left(1 - \frac{(\alpha_1 + \alpha_3)(\rho)}{c\rho\alpha_1(\rho)}\right) t + \theta_0,$$

where θ_0 *is a constant.*

Remarks 3.16 *1. For a flat 2-dimensional manifold* $(M^2(0), g)$*, any oblique geodesic on* $(TM^2(0), G)$ *is a vector field along a geodesic.*

2. If, in Theorem 3.15, $c = \frac{(\alpha_1+\alpha_3)(\rho)}{\rho\alpha_1(\rho)}$*, then for any oblique geodesic* (x, V)*,* V *makes constant angle with* $\dot{x}$*.*

3. If G *is the Sasaki metric or the Cheeger-Gromoll metric on* $T^2M(c)$*, then* $\varphi_1 = \alpha_1 + \alpha_3 = 1$ *and condition (3.66) is equivalent to the fact that* r *is an affine function. Another nonclassical example can be given by the family of Kaluza-Klein* g*-natural metrics defined by* $\varphi_1 = 1$ *and* $(\alpha_1 + \alpha_3)(t) = ae^{b\sqrt{t}}$*, for* $t \in [t_0, +\infty[$*,* $t_0 > 0$*,* $a \neq 0$*,* $b \neq 0$*. Then the solution of (3.24) is the function defined by* $r(t) = -\frac{1}{b}\left\{\ln \frac{a}{K_1^2}\big(l - \frac{b^2K_1^2}{4a^2}(\varepsilon t + d\big)^2\right\}$*, where* $\varepsilon = \pm 1$ *is fixed. The domain of* r *depends of the sign of* a*, which is the sign is* $\alpha_1 + \alpha_3$*:*

- *If* $a > 0$*, then* $l > 0$ *(by the expression of* r*), i.e. oblique geodesics of type (ii) in the theorem are all spacelike. In this case,* $K_1 > 0$ *and the domain of* r *should be in the set* $\{t \geq t_0, -\frac{2a\sqrt{l}}{|b|K_1} - d < \varepsilon t < \frac{2a\sqrt{l}}{|b|K_1} - d\}$*.*
- *If* $a < 0$*, then we have*

  either $l \leq 0$ *and* r *is defined on* $[t_0, +\infty[$*.*

  or $l > 0$ *and* r *is defined on* $\{t \geq t_0, -\frac{2a\sqrt{l}}{|b|K_1} - d < \varepsilon t < \frac{2a\sqrt{l}}{|b|K_1} - d\}$*.*

3.3.4. Example

Now, we give an example of oblique geodesics of type (ii) (in Theorem 3.15), when the base manifold is the hyperbolic plane $M = \mathbb{H}^2 = \{(u, v) \in \mathbb{R}^2; v > 0\}$, equipped with the metric $g = \frac{1}{4v^2}(du^2 + dv^2)$.

We consider the Riemannian circle in $\mathbb{H}^2(-4)$ given by:

$$x(s) = (2\sin(\mu(s)), 2 - 2\cos(\mu(s))), \tag{3.67}$$

with $\mu'(s) + 2cos(\mu(s)) = 2$, whose solution is given by:

$$\mu(s) = -2\,\text{arccot}(2s). \tag{3.68}$$

Then we have $\dot{x}(s) = (2\mu'(s)\cos(\mu(s)), 2\mu'(s)\sin(\mu(s))$ and $\|\dot{x}(s)\| = 1$. A standard calculation using the previous identities yields

$$\nabla_{\dot{x}}\dot{x} = (-4\mu'(s)\sin(\mu(s), 4\mu'(s)\cos(\mu(s)).$$

By Theorem 3.15, a nonvanishing oblique geodesic on $(T\mathbb{H}^2(-4), G)$ based on the Riemannian circle x is a vector field V along x which makes an affine angle θ with $\dot{x}$ defined by:

$$\theta(t) = \kappa\left(1 - \frac{(\alpha_1+\alpha_3)(\rho)}{c\rho\alpha_1(\rho)}\right)t + \theta_0,$$

where θ_0 is a constant and $\kappa^2 = g(\nabla_{\dot{x}}\dot{x}, \nabla_{\dot{x}}\dot{x}) = 4$.

Putting $V(s) = (V_1(s), V_2(s))$, with $\|V(s)\|^2 = \rho = 1$, we have:

$$1 = g_{x(s)}(V(s), V(s)) = \frac{1}{4\mu'^2(s)}[V_1^2(s) + V_2^2(s)] \text{ and}$$

$$g_{x(s)}(V(s), \dot{x}(s)) = \|\dot{x}\|\|V\|\cos(\theta(s)) = \frac{1}{4\mu'^2(s)}[2\mu'\cos(\mu)V_1 + \mu'\sin(\mu)V_2],$$

so that we obtain:

$$\begin{cases} V_1^2(s) + V_2^2(s) = & 4\mu'^2(s), \\ V_1\cos(\mu) + V_2\sin(\mu) = & 2\mu'(s)\cos(\theta(s)). \end{cases}$$

i.e

$$\begin{cases} \pm\sqrt{4\mu'^2(s) - V_1^2(s)} = & V_2, \\ V_1\cos(\mu) + V_2\sin(\mu) = & 2\mu'(s)\cos(\theta(s)). \end{cases} \tag{3.69}$$

with $4\mu'^2(s) - V_1^2 \geq 0$. Supposing that $V_2(s) = \sqrt{4\mu'^2(s) - V_1^2(s)}$ and replacing in the second equation of (3.69) we obtain:

$$0 = V_1^2 - 4\mu'(s)\cos(\theta(s))\cos(\mu(s))V_1 + 4\mu'^2(s)\cos(\theta(s)) - 4\mu'^2\sin^2(\mu(s)), \tag{3.70}$$

whose discriminant is $\Delta = 16\mu'^2\sin^2(\mu(s))\sin^2(\theta(s))$, so that

$$V_1(s) = 2\mu'[\cos(\theta(s))\cos(\mu(s)) \pm \sin(\theta(s))\sin(\mu(s))] = 2\mu'\cos(\theta(s) \pm \mu).$$

Note that condition $4\mu'^2 - V_1^2 \geq 0$ is always satisfied because $4\mu'^2 - V_1^2 = 4\mu'^2[1 - \cos(\theta(s) \pm \mu)] \geq 0$, that is

$$V(s) = (V_1(s), \sqrt{4\rho\mu'^2(s) - V_1^2(s)}), \tag{3.71}$$

with $V_1 = 2\mu'\cos(\theta(s) \pm \mu)$

In the following, we draw some pictures representing some different situations showing the curve x on $\mathbb{H}^2(-4)$, together with the vector field $V = 2\mu'\cos(\theta(s) - \mu(s))$ along x, where μ is given by (3.68):

- Figure 3 illustrates the case when $a = 1$ and $\theta_0 = 0$;

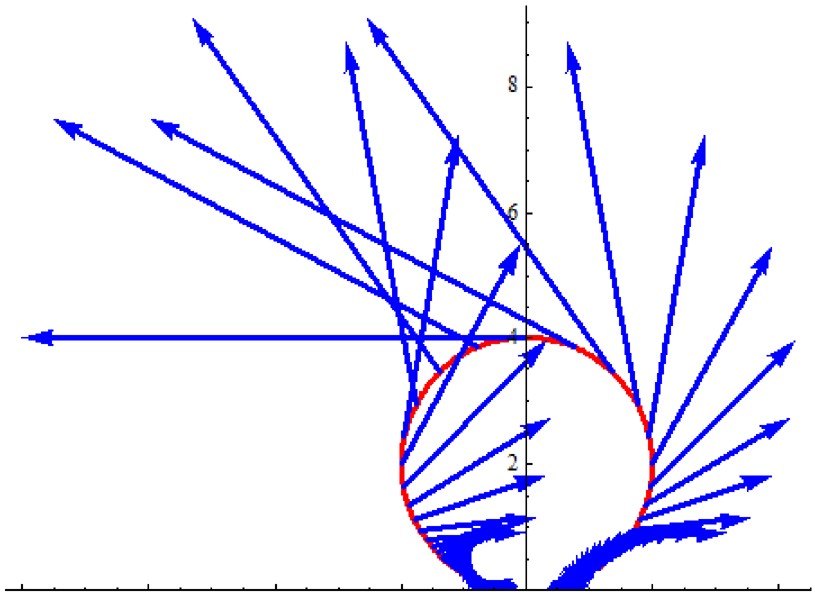

Figure 3. $a = 1, \; \theta_0 = 0$.

- Figure 4 gives the case when $a = 1/2$ and $\theta_0 = 1$;

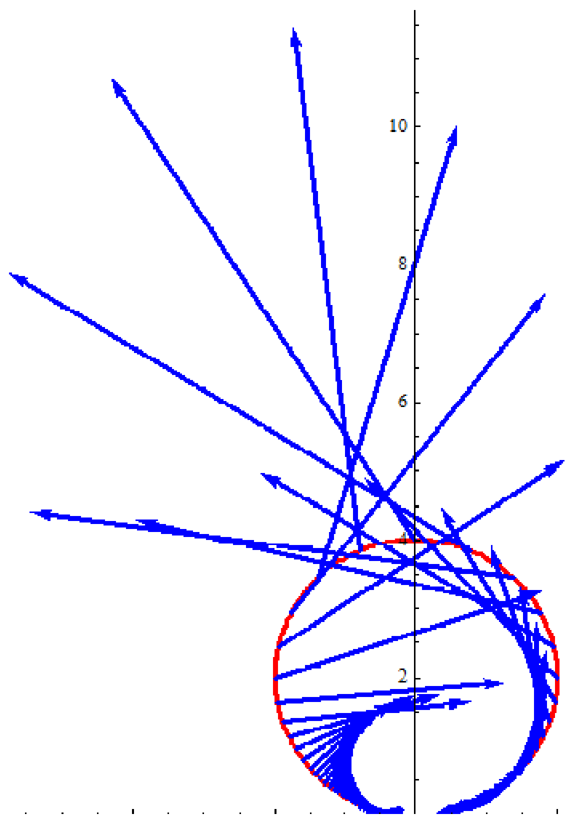

Figure 4. $a = 1/2, \; \theta_0 = 1$.

4. Natural complex magnetic curves on the tangent bundle of a space form

In this section, we suppose that (M,g) is a space of constant sectional curvature c and we endow its tangent bundle with a pseudo-Riemannian g-natural metric of Kaluza-Klein type G. We consider on TM an almost Kaëhlerian g-natural almost complex structure J given by (2.7). Under the previous assumptions, we have $\alpha_2 = \beta_2 = 0$ and, by virtue of (2.8) and Proposition 2.6, we have:

$$a_1 = \lambda\alpha_1, \qquad a_2 = 0, \qquad a_3 = \lambda\alpha_3, \tag{4.1}$$

$$A_1 = \lambda\varphi_1, \qquad A_2 = 0, \qquad A_3 = \lambda\varphi_3, \tag{4.2}$$

$$\lambda^2 = \alpha = \alpha_1(\alpha_1+\alpha_3), \qquad \mu^2 = \varphi = \varphi_1(\varphi_1+\varphi_3), \tag{4.3}$$

$$\mu = (t\lambda)', \quad \text{i.e.} \quad \sqrt{\alpha\varphi} = \pm(t\alpha_1)'. \tag{4.4}$$

The fundamental 2-form Ω of the almost Kaëhlerian structure (TM,G,J) defined by $\Omega(X,Y) := G(JX,Y)$, for all $X,Y \in \mathfrak{X}(TM)$, is closed and then defines a magnetic field on TM, which we call g-natural complex magnetic field, whose Lorentz force is exactly J. The corresponding Lorentz equation is given by:

$$\nabla_{\dot\gamma}\dot\gamma = qJ(\dot\gamma),$$

where q is a constant. We call the associated magnetic trajectories (curves) g-natural complex magnetic trajectories (curves).

We shall investigate g-natural complex magnetic trajectories on (TM,G). Using (3.1), we have:

$$J(\dot\gamma) = [-a_1\dot V - b_1 g(\dot V,V)V]^h + [(a_1+a_3)\dot x + (b_1+b_3)g(\dot x,V)V]^v. \tag{4.5}$$

Substituting from the last identity into the Lorentz equation and using Proposition 2.3, we have the following characterization of g-natural complex magnetic curves on TM:

Theorem 4.1 *γ is a g-natural complex magnetic curve on (TM,G,J) with strength q if and only if*

$$\begin{cases} 0 = \ddot x + 2C(V,\dot x,\dot V) + q[a_1\dot V + b_1 g(\dot V,V)V]; \\ 0 = \ddot V + B(V,\dot x,\dot x) + F(V,\dot V,\dot V) - q[(a_1+a_3)\dot x + (b_1+b_3)g(\dot x,V)V]. \end{cases} \tag{4.6}$$

4.1. Horizontal g-natural complex magnetic curves

Recall that a horizontal curve $\gamma = (x,V)$ is a curves transverse to fibers (i.e. $\dot x \neq 0$ everywhere) such that $\dot V = 0$, i.e. V is parallel along x. In particular, $\rho = \|V\|^2 = cte$. In this case, by virtue of (4.6), γ is a g-natural complex magnetic curve if and only if

$$\begin{cases} 0 = \ddot x; \\ 0 = [2B_1 g(\dot x,V) - q(a_1+a_3)]\,\dot x + \left[B_2\|\dot x\|^2 + B_3 g(\dot x,V)^2 - q(b_1+b_3)\right]V. \end{cases} \tag{4.7}$$

Note that all the quantities a_i, b_i, and B_i are taking at ρ and consequently are constant. We shall study (4.7) in the case where $q \neq 0$, i.e. we shall study horizontal nongeodesic magnetic curves on (TM,G).

Theorem 4.2 *Let (M,g) be a space of constant sectional curvature and G be a pseudo-Riemannian g-natural metric of Kaluza-Klein type on TM. Let $\gamma=(x,V)$ be a (horizontal) curve such that $\|\dot{\gamma}\|^2=\varepsilon$ ($\varepsilon=\pm1$ or 0) and $\dot{V}=0$ (In this case $\rho=\|V\|^2$ is a constant). Then γ is a nongeodesic g-natural complex magnetic curve on (TM,G) if and only if $\rho\neq 0$ and one of the following assertions holds:*

1) *x is a geodesic , $(\beta_1+\beta_3)(\rho)\neq 0$, $g(\dot{x},V)=-\frac{q\alpha_1(a_1+a_3)}{\beta_1+\beta_3}(\rho)$ and*

$$\begin{aligned}4\varepsilon B_1^2(\rho)B_2(\rho)=&q^2(a_1+a_3)(\rho)[2B_1(\rho)(\alpha_1+\alpha_3)(\rho)(b_1+b_3)(\rho)\\&+(B_2(\beta_1+\beta_3)-B_3(\alpha_1+\alpha_3))(\rho)(a_1+a_3)(\rho)].\end{aligned}\tag{4.8}$$

2) *$\varepsilon=sing((\varphi_1+\varphi_3)(\rho))\neq 0$, $\|\dot{x}\|^2=\frac{\varepsilon}{(\varphi_1+\varphi_3)(\rho)}$, $q^2\neq\frac{4\rho B_1^2(\rho)}{(a_1+a_3)^2(\rho)(\varphi_1+\varphi_3)(\rho)}$, $q(b_1+b_3)(\rho)(\varphi_+\varphi_3)(\rho)\neq\varepsilon(B_2+\rho B_3)(\rho)$ and γ is, up to a real factor, the velocity vector field of a geodesic curve on (M,g).*

Proof Suppose that γ is a nongeodesic g-natural complex magnetic curve on (TM,G). We claim that $\rho\neq 0$. Indeed, suppose that $\rho=0$. Then the second equation of (4.7) reduces to $-q(a_1+a_3)\dot{x}=0$, which is a contradiction since $\dot{x}\neq 0$ everywhere, $q\neq 0$ and $a_1+a_3\neq 0$.

By the first equation of (4.7), x is a geodesic. Since $\dot{x}$ and V are parallel, then the function $g(\dot{x},V)$ is constant. On the other hand, since magnetic curves are of constant speed, we can suppose without loss of generality; that $\|\dot{\gamma}\|^2=\varepsilon$ where $\varepsilon=-1;0$ or 1 according to the cases when γ is timelike, lightlike, or spacelike, respectively, i.e. $\varepsilon=\|\dot{\gamma}\|^2=(\alpha_1+\alpha_3)\|\dot{x}\|^2+(\beta_1+\beta_3)g(\dot{x},V)^2$ then

$$\|\dot{x}\|^2=\frac{1}{\alpha_1+\alpha_3}[\varepsilon-(\beta_1+\beta_3)g(\dot{x},V)^2].\tag{4.9}$$

We deduce that $\|\dot{x}\|^2$ is also constant. It follows that the second equation is of the form $\xi_1\dot{x}+\xi_2V=0$, where ξ_1 and ξ_2 are the constants

$$\xi_1:=2B_1g(\dot{x},V)-q(a_1+a_3),\qquad \xi_2:=B_2\|\dot{x}\|^2+B_3g(\dot{x},V)^2-q(b_1+b_3).$$

Since $\dot{x}$ and V do not vanish, then ξ_1 and ξ_2 are zero or not mutual.

Suppose that $\xi_1=\xi_2=0$. Then $B_1\neq 0$, i.e. $(\beta_1+\beta_3)(\rho)\neq 0$ and $g(\dot{x},V)=\frac{q(a_1+a_3)}{2B_1}(\rho)=-\frac{q\alpha_1(a_1+a_3)}{\beta_1+\beta_3}(\rho)$. Using (4.9) and the expression of $g(\dot{x},V)$, equation $\xi_2=0$ is equivalent to the condition (4.8). With these conditions, the second equation of (4.7) is always satisfied.

For $\xi_1\neq 0$ and $\xi_2\neq 0$, we deduce from the second equation of (4.7) that V and $\dot{x}$ are collinear. In particular, we have $g(\dot{x},V)^2=\|\dot{x}\|^2\|V\|^2$ which gives, by virtue of (4.9), $g(\dot{x},V)^2=\frac{\varepsilon\rho}{\varphi_1+\varphi_3}$. This implies that $\varepsilon=\mathrm{sign}(\varphi_1+\varphi_3)$ and that $\|\dot{x}\|^2=\frac{\varepsilon}{\varphi_1+\varphi_3}$. Then $\mu_1\neq 0$ is equivalent to $q^2\neq\frac{4\rho B_1^2}{(a_1+a_3)^2(\varphi_1+\varphi_3)}$. It follows that $\mu_2\neq 0$ is equivalent to $q(b_1+b_3)(\varphi_+\varphi_3)\neq\varepsilon(B_2+\rho B_3)$. □

Corollary 4.3 *Under the same assumptions of Theorem 4.2, G is a Kaluza-Klein metric, i.e. $\beta_1+\beta_3$ vanishes identically, then γ is a nongeodesic g-natural complex magnetic curve on (TM,G) if and only if $\rho\neq 0$, $\varepsilon=sing((\alpha_1+\alpha_3)(\rho))\neq 0$, $\|\dot{x}\|^2=\frac{\varepsilon}{(\alpha_1+\alpha_3)(\rho)}$, $(\alpha_1+\alpha_3)'(\rho)\neq 0$ and γ is, up to a real factor, the velocity vector field of a geodesic curve on (M,g).*

In particular, if G is the Sasaki metric, then every horizontal g-natural complex magnetic curve on (TM, G) is a geodesic.

4.2. g-natural complex magnetic curves of oblique type

In this section, we restrict ourselves to Kaluza-Klein g-natural metrics on TM, i.e. we assume that $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$. In this case, we have

$$b_1 + b_3 = \lambda'(\alpha_1 + \alpha_3). \tag{4.10}$$

Recall that curve $\gamma = (x, V)$ on TM is slant if V makes a constant angle with $\dot{x}$, i.e. $g(\dot{x}, V) = \|\dot{x}\|\|V\| \cos\theta$, where θ is constant. We are interested in slant g-natural complex magnetic curves of oblique type. We shall investigate the case when γ is the velocity vector field of a normal curve and the case when V is of constant norm along x.

Let $\gamma = (x, V)$ be an oblique curve ($\dot{V} \neq 0$ everywhere) such that $\|\dot{\gamma}\|^2 = \varepsilon$, where $\varepsilon = \pm 1$ or 0, i.e.

$$\varepsilon = (\alpha_1 + \alpha_3)(\rho)\|\dot{x}\|^2 + \alpha_1(\rho)\|\dot{V}\|^2 + \beta_1(\rho) g(\dot{V}, V). \tag{4.11}$$

where $\rho = g(V, V)$. In this case, γ is a g-natural complex magnetic curve in (TM, G) if and only if

$$\begin{cases} 0 = \ddot{x} + 2cC_1[g(\dot{x}, V)\dot{V} - g(\dot{x}, \dot{V})V] + 2g(V, \dot{V})C_3\dot{x} \\ \quad +q[a_1\dot{V} + b_1 g(\dot{V}, V)V]; \\ \\ 0 = \ddot{V} + 2F_1 g(V, \dot{V})]\dot{V} + [B_2\|\dot{x}\|^2 + F_2\|\dot{V}\|^2 + F_3 g(\dot{V}, V)^2 \\ \quad -q(b_1 + b_3)g(\dot{x}, V)]V - q(a_1 + a_3)\dot{x}. \end{cases} \tag{4.12}$$

4.2.1. Velocity vector fields

Proposition 4.4 *Let (M, g) be a space of constant sectional curvature c and G be a pseudo-Riemannian g-natural Kaluza-Klein metric on (TM, G). Let $x(s)$ be a nongeodesic curve of constant speed ($\|\dot{x}\|^2 = \rho$) in (M, g) and $\gamma(s) = (x(s); \dot{x}(s))$ its velocity curve. γ is a g-natural complex magnetic curve on (TM, G) if and only if*

i) *x is a Riemannian circle.*

ii) $c = \frac{(\alpha_1+\alpha_3)(\rho)[1+qa_1(\rho)]}{\rho\alpha_1(\rho)}$

iii) *one of the following assertions holds*

- $\rho(\alpha_1 + \alpha_3)'(\rho) = q[A_1 + A_3](\alpha_1 + \alpha_3)$ *and* $\alpha_1(\rho) + \rho\alpha_1^{'} = 0$
- $[q\varphi_1(A_1 + A_3) + (\alpha_1 + \alpha_3)'][\alpha_1 + \rho\alpha_1^{'}](\rho) < 0$ *and* $\kappa_1 = \pm\sqrt{-\frac{[q\varphi_1(A_1+A_3)+(\alpha_1+\alpha_3)']}{\alpha_1+\rho\alpha_1^{'}}(\rho)}$

Proof Denoting $V(t) = \dot{x}(t)$, we have $\rho = \|V\|^2$ is constant and $g(\dot{x}, \ddot{x}) = g(V, \dot{V}) = g(\dot{x}, \dot{V}) = g(V, \ddot{x}) = 0$. If we denote by ξ_1, ξ_2, and ξ_3 the unit tangent vector, the principal normal vector and the principal binormal vector of x, respectively, and by κ_1 and κ_2, the first and second curvatures of x, then by Frenet Formulas, we have:

$$\dot{x} = \sqrt{\rho}\xi_1 \quad \nabla_{\dot{x}}\dot{x} = \rho\kappa_1\xi_2, \quad \dddot{x} = \rho^{\frac{3}{2}}[\kappa_1\kappa_2\xi_3 - \kappa_1^2\xi_1 + \dot{\kappa}_1\xi_2]. \tag{4.13}$$

Substituting from the last formulas into (4.12), γ is a g-natural complex magnetic curve if and only if

$$\begin{cases} [1+2\rho c C_1(\rho)+qa_1(\rho)]\rho\kappa_1\xi_2=0; \\ \big[\rho B_2+\rho(\rho F_2-1)\kappa_1^2-q(A_1+A_3)\big]\xi_1+\rho[\dot{\kappa}_1\xi_2+\kappa_1\kappa_2\xi_3]=0. \end{cases} \tag{4.14}$$

Since $\kappa_1 \neq 0$ almost everywhere (x is a nongeodesic curve), then the first equation (4.14) yields $1+2\rho c C_1(\rho)+qa_1(\rho)=0$, i.e.

$$c=\frac{(\alpha_1+\alpha_3)(\rho)[1+qa_1(\rho)]}{\rho\alpha_1(\rho)} \tag{4.15}$$

On the other hand, the second equation of (4.14) is equivalent to

$$\begin{cases} \dot{\kappa}_1=0; \\ \kappa_1\kappa_2=0; \\ \rho B_2+\rho(\rho F_2-1)\kappa_1^2-q(A_1+A_3)=0. \end{cases} \tag{4.16}$$

We deduce that κ_1 is constant and $\kappa_2=0$. By the third equation of (4.16), we deduce that

either $1-\rho F_2(\rho)=0$, i.e. $\alpha_1(\rho)+\rho\alpha_1'(\rho)=0$ and, in this case, $\rho B_2(\rho)=q(A_1+A_3)(\rho)=0$;

or $\alpha_1(\rho)+\rho\alpha_1'(\rho)\neq 0$ and, in this case, $\kappa_1^2=\frac{-(\alpha_1+\alpha_3)'(1)}{\alpha_1(1)+\alpha_1'}$. In particular, $\frac{-(\alpha_1+\alpha_3)'(1)}{\alpha_1(1)+\alpha_1'(1)}>0$

□

4.2.2. Vector fields of constant norm along curves

Let $\gamma=(x,V)$ be an oblique curve on TM ($\dot{V}\neq 0$ everywhere) such that $\rho:=\|V\|^2$ is constant. We suppose that $\|\dot{\gamma}\|^2=\varepsilon$, where $\varepsilon=\pm1$ or 0. Then (4.11) reduces to

$$\varepsilon=(\alpha_1+\alpha_3)(\rho)\|\dot{x}\|^2+\alpha_1(\rho)\|\dot{V}\|^2. \tag{4.17}$$

Using (4.12), γ is a g-natural complex magnetic curve in (TM,G) if and only if

$$\begin{cases} 0=\ddot{x}+2cC_1[g(\dot{x},V)\dot{V}-g(\dot{x},\dot{V})V]+qa_1\dot{V}; \\ 0=\ddot{V}+[B_2\|\dot{x}\|^2+F_2\|\dot{V}\|^2-q(b_1+b_3)g(\dot{x},V)]V-q(a_1+a_3)\dot{x}. \end{cases} \tag{4.18}$$

Making the scalar product of the first equation of (4.18) by $\dot{x}$ and then by V, we obtain:

$$g(\ddot{x},\dot{x})+qa_1 g(\dot{x},\dot{V})=0; \tag{4.19}$$

$$g(\ddot{x},V)-2c\rho C_1(\rho)g(\dot{x},\dot{V})=0. \tag{4.20}$$

Using (4.19), we obtain:

$$\frac{d}{dt}g(\dot{x},V)=(1+2c\rho C_1(\rho))g(\dot{x},\dot{V}). \tag{4.21}$$

Theorem 4.5 *Let (M, g) be a space of constant sectional curvature and G be a pseudo-Riemannian Kaluza-Klein g-natural metric on TM. Let $\gamma = (x, V)$ be an oblique nongeodesic g-natural complex magnetic curve on (TM, G) such that $\|\dot{\gamma}\|^2 = \varepsilon$ ($\varepsilon = \pm 1$ or 0) and $\rho = \|V\|^2$ is a constant. Then V makes a constant angle $\theta \neq \pi/2$ with $\dot{x}$ if and only if $\|\dot{x}\|$ and $\|\dot{V}\|$ are constant.*

Proof Let θ be the angle that V makes with $\dot{x}$, so that $g(\dot{x}, V) = \sqrt{\rho}\|\dot{x}\| \cos\theta$. Deriving the squares of the two sides last equality, taking into account that ρ is constant, we obtain:

$$[(1 + 2c\rho C_1(\rho))\|\dot{x}\| + qa_1] \cos^2\theta g(\dot{x}, \dot{V}) = \|\dot{x}\|^2\dot{\theta}\cos\theta\sin\theta. \tag{4.22}$$

Suppose that θ is constant, then locally either $(1 + 2c\rho C_1(\rho))\|\dot{x}\| + qa_1 = 0$ or $g(\dot{x}, \dot{V}) = 0$.

- If $(1 + 2c\rho C_1(\rho))\|\dot{x}\| + qa_1 = 0$, then $1 + 2c\rho C_1(\rho) \neq 0$ and $\|\dot{x}\| = -\frac{qa_1}{1+2c\rho C_1(\rho)}$, which is constant.
- If $g(\dot{x}, \dot{V}) = 0$, then $g(\ddot{x}, \dot{x}) = 0$ by virtue of (4.19) and, consequently, $\|\dot{x}\|$ is constant.

In the two cases, $\|\dot{V}\|$ is also constant by (4.17).

Conversely, if $\|\dot{x}\|$ and $\|\dot{V}\|$ are constant, then $g(\dot{x}, \dot{V}) = 0$ by virtue of (4.19). Using (4.22), either $\sin\theta$ or $\dot{\theta}$ vanish locally. In the two cases, θ is constant. □

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