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Curves and stick figures not contained in a hypersurface of a given degree

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Abstract: A stick figure $X \subset \mathbb{P}^r$ is a nodal curve whose irreducible components are lines. For fixed integers $r \ge 3$, $s \ge 2$ and d we study the maximal arithmetic genus of a connected stick figure (or any reduced and connected curve) $X \subset \mathbb{P}^r$ such that $\deg(X) = d$ and $h^0(\mathcal{I}_X(s-1)) = 0$. We consider Halphen's problem of obtaining all arithmetic genera below the maximal one.

Key words: Curves in projective spaces, stick figures, reducible curves, arithmetic genus

1. Introduction

We recall that a stick figure is a reduced nodal curve $X \subset \mathbb{P}^r$ whose irreducible components are lines ([1, 10, 11, 16]); sometimes a stick figure is also assumed to be connected. Following the classical case of smooth space curves considered by Halphen we introduce the following players. Fix integers $d \ge r \ge 3$ and $s \ge 2$. Let $\Gamma(r,d,s)$ (resp. $\Gamma'(r,d,s)$, resp. $\Gamma''(r,d,s)$) denote the set of all smooth and connected (resp. integral, resp. reduced and connected) curves $X \subset \mathbb{P}^r$ such that $h^0(\mathcal{I}_X(s-1)) = 0$. Since $s \ge 2$, all $X \in \Gamma''(r,d,s)$ spans $\mathbb{P}^r$. Let $\Gamma_1(r,d,s)$ denote the set of all connected degree d stick figures X such that $h^0(\mathcal{I}_X(s-1)) = 0$. Let $\gamma(r,d,s)$ (resp. $\gamma'(r,d,s)$, resp. $\gamma''(r,d,s)$, resp. $\gamma_1(r,d,s)$) denote the maximum of all integers $p_a(X)$ for some element X of $\Gamma(r,d,s)$ (resp. $\Gamma'(r,d,s)$, resp. $\Gamma''(r,d,s)$, resp. $\Gamma_1(r,d,s)$). We only consider reducible connected curves (in particular connected stick figure), because for all $r \ge 3$ and $s \ge 2$ for large d we land in the range of arithmetic genera $>$ Castelnuovo's upper bound of the arithmetic genus of degree d nondegenerate curves $X \subset \mathbb{P}^r$ ([14, Ch. III]). Thus we do not consider $\Gamma(r,d,s)$ and $\gamma(r,d,s)$. We only mention that for $r = 3$ and $d > s^2$ the integer $\gamma(3,d,s)$ is known. Gruson and Peskine proved that for $d > s^2$ the integer $\gamma(3,d,s)$ is the integer $G_C(d,s)$ defined in the following way ([12]). Fix integers $s \ge 4$ and $d > s^2$. Write $d = sk - e$ with k and e integers such that $k > s$ and $0 \le e < s$. Set

$$G_C(d,s) := \frac{d^2}{2k} + \frac{(k-4)d}{2} + 1 - \frac{e}{2}(k - e + \frac{e}{k}).$$

Moreover, $\gamma'(3,d,s) = G_C(d,s)$, i.e. allowing singular, but integral, curves does not increase the maximal arithmetic genus. Not all genera below the maximal one are obtained ([9]). The standard conjecture for smooth space curves with $d := \deg(C) > s^2$ and $h^0(\mathcal{I}_C(s-1)) = 0$ asks if all genera g between 0 and $G_C(d,s+1)$ are obtained by some curve. In the few gaps known in the range $G_C(d,s+1) < g < G_C(d,s)$ there is no difference

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if we allow the arithmetic genera of integral curves. Just a glance to [6–8] shows that the situation is very messy for $\mathbb{P}^r$, $r > 3$.

We also consider the same sets adding the superscript ^{00} (so for instance $\Gamma(r,d,s)^{00}$, $\Gamma_1(r,d,s)^{00}$, $\Gamma'(r,d,s)^{00}$) if we drop the assumption that the curve is connected and call $\gamma(r,d,s)^{00}$, $\gamma_1(r,d,s)^{00}$, $\gamma'(r,d,s)^{00}$ the maximum of the arithmetic genera of their elements. In this case there are 2 different definitions of arithmetic genus and both are useful. Let X be a reduced projective curve with c connected components. The h^1-arithmetic genus $h^1(X)$ of X is the integer $h^1(\mathcal{O}_X)$. The χ-arithmetic genus $\chi(X)$ of X is the integer $-\chi(\mathcal{O}_X) - 1$. Obviously $h^1(X) = \chi(X) + c - 1$. The χ-arithmetic genus is a more natural measure of the complexity of X from the point of view of extremal properties, like the maximal genus of curves with some properties. The h^1-arithmetic genus has the very useful property that $h^1(\mathcal{O}_Y) \le h^1(\mathcal{O}_X)$ for any subcurve Y of X and $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y) + h^1(\mathcal{O}_W)$ for any subcurves Y, W of X with $Y \cap W = \emptyset$ and $Y \cup W = X$ (additivity for connected components). For this reason the h^1-genus is used for instance in [1] to construct smooth connected curves using at intermediate steps disconnected curves.

We ask the following questions:

1. Compute (or give upper/lower bounds for them) the integers $\gamma(r,d,s)$, $\gamma'(r,d,s)$, $\gamma''(r,d,s)$ and $\gamma_1(r,d,s)$.
2. Describe large parts of the arithmetic genera of elements of $\Gamma(r,d,s)$, $\Gamma(r,d,s)'$, $\Gamma(r,d,s)''$ and $\Gamma_1(r,d,s)$. In particular prove that except for the top strip of the arithmetic genera, all other genera occur with no intermediate gap.

Theorem 1.1 *Fix integers $r \ge 3$, $s \ge 2$ and a real number $\epsilon > 0$. Then*

$$\lim_{d\to+\infty} \gamma_1(r,d,s)/d^2 = \lim_{d\to+\infty} \gamma(r,d,s)^{00}/d^2 = \frac{1}{2}.$$

Moreover, there is an integer $d(r,s,\epsilon)$ such that for all integers d, q such that $d \ge d(r,s,\epsilon)$ and $0 \le q \le \frac{d^2}{2+\epsilon}$ there are a connected stick figure $X \subset \mathbb{P}^r$ such that $\deg(X) = d$, $p_a(X) = q$ and $h^0(\mathcal{I}_X(s-1)) = 0$ and a smooth disconnected curve $Y \subset \mathbb{P}^r$ such that it has at most $\lceil \binom{r+s-1}{r}/s \rceil + 2$ connected components, $\deg(Y) = d$, $h^1(\mathcal{O}_Y) = q$ and $h^0(\mathcal{I}_Y(s-1)) = 0$.

The bounds are very different for connected stick figures in $\mathbb{P}^3$ if we impose that they are contained in an integral surface of prescribed degree. In Section 3 we consider space curves contained in a degree k integral surface and prove the following results.

Theorem 1.2 *Fix integers $d > k^2 \ge 4$. Let $Y \subset \mathbb{P}^3$ be a reduced and connected curve contained in an integral degree k surface S. Then*

$$p_a(Y) \le \frac{d^2}{2k} + \frac{(k-4)d}{2} + 1 - \frac{e}{2}(k - e + \frac{e}{k}), \tag{1.1}$$

where $d = kx - e$, x an integer and $0 \le e \le k-1$.

Moreover, equality holds in (1.1) if and only if Y is arithmetically Cohen-Macaulay and either the complete intersection of S and a degree d/k surface (case $e = 0$) or linked by the complete intersection of S and a surface of degree $\lceil d/k \rceil$ to a plane curve of degree e (case $e > 0$).

Theorem 1.3 *Fix integers $d > k^2 \geq 4$. Let $\alpha(3,d,k)$ denote the maximal arithmetic genus of a degree d connected stick figure contained in an integral degree k surface.*

(a) We have

$$\alpha(3,k,d) \leq \frac{d^2}{2k} + \frac{(k-4)d}{2} + 1 - \frac{e}{2}(k - e + \frac{e}{k}), \tag{1.2}$$

where $0 \leq e \leq k-1$ and $d = k\lceil d/k\rceil - e$. All stick figures achieving equality are either complete intersection of S and a degree d/k surface (case $e = 0$) or linked to a plane curve of degree e by a complete intersection of S and a degree $\lceil d/k\rceil$ surface (case $e > 0$).

(b) Assume $k \geq 3$. There is an integer $d_1(k)$ such that

$$\alpha(3,k,d) < \frac{d^2}{2k} + \frac{(k-4)d}{2} + 1 - \frac{e}{2}(k - e + \frac{e}{k}) \tag{1.3}$$

for all integers $d \geq d_1(k)$.

We construct elements of $\Gamma_1(r,d,s)$, $r \geq 4$, with very high arithmetic genus (Theorem 5.1). The same construction gives elements of $\Gamma(r,d,s)^{00}$ with almost the same arithmetic genus for a larger set of degrees d and a controlled number of connected components (Remark 5.8). For $r = 3$ we have slightly higher arithmetic genera (see Theorem 4.10).

2. Preliminaries

Let M be a projective variety and D an effective Cartier divisor of M. For any closed subscheme E of M the residual scheme $\mathrm{Res}_D(E)$ of E is the closed subscheme of M with $\mathcal{I}_E : \mathcal{I}_D$ as its ideal sheaf. For any line bundle $\mathcal{L}$ on M there is an exact sequence (called the residual exact sequence of D):

$$0 \to \mathcal{I}_{\mathrm{Res}_D(E)} \otimes \mathcal{L}(-D) \to \mathcal{I}_E \otimes \mathcal{L} \to \mathcal{I}_{E\cap D,D} \otimes \mathcal{L}_{|D} \to 0\ . \tag{2.1}$$

Thus

$$h^i(M, \mathcal{I}_E \otimes \mathcal{L}) \leq h^i(M, \mathcal{I}_{\mathrm{Res}_D(E)} \otimes \mathcal{L}(-D)) + h^i(D, \mathcal{I}_{E\cap D,D} \otimes \mathcal{L}_{|D})$$

for all $i \in \mathbb{N}$.

Remark 2.1 *Fix a a reducible conic $D \subset \mathbb{P}^r$, $r \geq 3$, and a 3-dimensional linear subspace U of $\mathbb{P}^r$ containing the plane M spanned by D. Let o be the singular point of D. Let A be the union of D and a degree 2 connected zero-dimensional scheme v with $v_{\mathrm{red}} = \{o\}$, $v \subset U$ and $v \nsubseteq M$. The scheme A is called a sundial in [3]. The scheme A is a flat limit of a family of pairs of disjoint lines contained in U ([17, example 2.1] or [3]). Let $H \subset \mathbb{P}^r$ be a hyperplane such that $H \cap U = M$. We have $A \cap H = D$ (as schemes) and $\mathrm{Res}_H(A) = \{o\}$.*

Let $V \subseteq \mathbb{P}^r$ be a linear subspace. For each $o \in V$ let $(2o, V)$ denote the closed subscheme of V with $(\mathcal{I}_{o,V})^2$ as its ideal sheaf. The scheme $(2o,V)$ is zero-dimensional, $(2o,V)_{\mathrm{red}} = \{o\}$ and $\deg(2o,V) = \dim V + 1$. If $\dim V = 1$, then $(2o,V)$ is called a tangent vector. If $\dim V = 2$ we will say that $(2o,V)$ is a planar 2-point.

Remark 2.2 *Fix an integral projective variety Y, a line bundle L on Y and any linear subspace $V \subseteq H^0(L)$. For each zero-dimensional scheme $Z \subset Y$ set $V(-Z) := V \cap H^0(\mathcal{I}_Z \otimes L)$. If Z is a general tangent vector of Y, then $\dim V(-Z) = \max\{0, \dim V - 2\}$ ([2, lemma 1.4], [5]). Since V is not assumed to be a complete linear system, we also get $\dim V(-Z) = \max\{0, \dim V - 2x\}$ if Z is a general union of x tangent vectors of Y.*

3. Space curves contained in an integral surface

Proof [Proof of Theorem 1.2:] The inequality (1.1) is obtained using verbatim the proof given by Harris in [13, §1], i.e. the section of his paper which does not use the uniform position principle, or even the (weaker) linear general position of a general plane section of Y. Now assume that equality holds. First of all, by the proofs in [13, §1] we get that a bunch of inequalities are equalities and, as shown explicitly in [13, §1], this implies that Y is arithmetically Cohen-Macaulay. Then it is shown that $\lfloor (d-1)/k \rfloor + 1 = \lceil d/k \rceil$ is the minimal degree of a surface $S' \supset Y$ with $S' \nsubseteq S$. Since S is integral, $T := S \cap S'$ is a complete intersection curve of degree $k\lceil d/k \rceil$ containing Y. If $e = 0$ we get $Y = T$. Now assume $e > 0$. The complete intersection curve T links Y to a locally Cohen-Macaulay curve $E \subset \mathbb{P}^3$ with $\deg(E) = e$. Since Y is arithmetically Cohen-Macaulay, E is arithmetically Cohen-Macaulay. We have $p_a(Y) - p_a(E) = (k + \lceil d/k \rceil)(d-e)/2$ ([19, part (vi) of proposition 3.1]). We get that the highest value of $p_a(Y)$ is achieved when $p_a(E)$ is highest among the locally Cohen-Macaulay curves E of degree e. It is easy to check that this is the case if and only if E is a plane curve ([15]). Taking $p_a(E) = (e-1)(e-2)/2$ in the formula of [19, part (vi) of proposition 3.1] we get that equality holds in (1.1). □

Proof [Proof of Theorem 1.3:] Part (a) is a particular case of Theorem 1.2 and hence it is sufficient to prove part (b).

Assume $k \geq 3$. The existence of the Chow scheme gives the existence of an integer $d_0(k)$ such that all integral degree k surfaces with only finitely many lines have at most $d_0(k)$ lines. Thus for $d > d_0(k)$ S must contain infinitely many lines, i.e. it must be either a ruled surface or a cone. S is not a cone, because $d > 2$ and a stick figure by assumption is nodal. Thus S is a ruled surface. Since $k > 2$, S has a unique ruling. Thus S has finitely many lines not contained in the ruling of S. The existence of the Chow scheme gives the existence of integers t_1 and c_1 such that all degree k ruled surfaces have at most t_1 lines not contained in their ruling and any line of the ruling meets at most c_1 other lines of the ruling. For any positive integer x let q_x be the maximal arithmetic genus of a degree x connected stick figure contained in an integral ruled degree k surface S. Let $U \subset S$ be a connected nodal union of lines of S. Take any line L in the ruling of S, but not contained in U. If $U \cup L$ is nodal, then $p_a(U \cup L) \leq p_a(U) + c_1 + t_1 - 1$. For large z we get $q_{z+y} \leq q_z + y(t_1 + c_1 - 1)$ for all $y > 0$. Since $c_1 + t_1$ only depends on k, we get that $\limsup q_x/x \leq c_1 + t_1 - 1$. Hence equality in (1.2) cannot be achieved for large d. □

Remark 3.1 *Take integers k, d and e as in Theorems 1.2 and 1.3. Take any $k + \lceil d/k \rceil$ general planes $H_1, \dots, H_k, M_1, \dots, M_{\lceil d/k \rceil}$ and set $S := H_1 \cup \cdots \cup H_k$, $S' := M_1 \cup \cdots \cup M_{\lceil d/k \rceil}$ and $T := S \cap S'$. Note that S is not integral. If $e = 0$ the curve T is a degree d stick figure contained in the degree k surface S and with $p_a(T)$ achieving equality in (1.2). Now assume $e > 0$ and set $E := M_1 \cap (H_1 \cup \cdots \cup H_e)$. Let Y be the union of the lines of T not contained in E. Y is a connected stick figure. Since Y is linked to the plane curve E by the complete intersection T, the linking formula for the arithmetic genus ([19, part (vi) of proposition 3.1]) gives that $p_a(Y)$ achieves equality in (1.2).*

We ask the following question. Fix integers $r \geq 3$, $k \geq r-1$ and $d > 0$. What is the maximal arithmetic genus $\alpha(r,k,d)$ of degree d connected stick figure contained in a degree k integral and nondegenerate surface? What is its leading term of $\alpha(r,k,d)$ for $d \to +\infty$ and fixed (r,k)?

Remark 3.2 *Since we do not require that the surface is smooth, only that the union of the lines is nodal, a general linear projection gives* $\alpha(r+1,k,d) \le \alpha(r,d,k)$ *for all* $r \ge 3$. *We expect that strict inequality holds, but we do not have a guess for the leading term of* $\alpha(r,d,k)$ *for fixed* (r,k) *and* d *large. For* $k-r+1$ *very small one can use the classification of low degree surfaces in* $\mathbb{P}^r$.

4. Stick figures in $\mathbb{P}^3$

Let $M \subset \mathbb{P}^3$ be a plane. For all integers $m \ge 1$ set $a_m := \lceil \binom{m+3}{3}/(m+1) \rceil$, $b_m := (m+1)a_m - \binom{m+3}{3}$ and $h_m := a_m - 2b_m$. We have $a_m = h_m = (m+3)(m+2)/6$ and $b_m = 0$ if $m \equiv 0, 1 \pmod 3$, $a_m = (m+1)(m+4)/6$ and $b_m = (m+1)/3$ if $m \equiv 2 \pmod 3$.

Note that

$$(m+1)h_m + (2m+1)b_m = \binom{m+3}{m}. \tag{4.1}$$

For all integers $m \ge 1$ we define the following Assertion A_m:

Assertion A_m**:** There is a pair (W, Q) with the following properties:

(1) W is a connected curve with $h_m + b_m$ connected components, h_m of them being lines, b_m of them being reducible conics; write $W = A \cup B$ with A the union of the degree 1 connected components;

(2) W is transversal to M;

(3) $h^i(\mathcal{I}_W(m)) = 0$, $i = 0, 1$;

(4) Q is a smooth quadric;

(5) $\mathrm{Sing}(W) \subset Q$, $\dim W \cap Q = 0$ and Q is transversal to A.

Remark 4.1 *Let* $W \subset \mathbb{P}^3$ *be any disjoint union of* h_m *lines and* b_m *reducible conics. Note that* (4.1) *implies* $h^0(\mathcal{I}_W(m)) = h^1(\mathcal{I}_W(m))$.

Remark 4.2 *We have* $(h_1, b_1) = (2, 0)$, $(h_2, b_2) = (0, 2)$, $(h_3, b_3) = (5, 0)$, $(h_4, b_4) = (7, 0)$, $(h_5, b_5) = (2, 4)$, $(h_6, b_6) = (12, 0)$, $(h_7, b_7) = (13, 0)$, $(h_8, b_8) = (7, 6)$, $(h_9, b_9) = (22, 0)$, $(h_{10}, b_{10}) = (22, 0)$.

Remark 4.3 A_1 *is true, because any* 2 *disjoint lines span* $\mathbb{P}^3$. A_2 *is false, because any union of* 2 *conics is contained in a reducible quadric.*

Lemma 4.4 A_3 *is true.*

Proof Let $Y \subset \mathbb{P}^3$ be a general union of 5 lines. By [17, page 173] $h^i(\mathcal{I}_Y(3)) = 0$, $i = 0, 1$. Any $W \subset \mathbb{P}^3$ projectively equivalent to Y satisfies $h^i(\mathcal{I}_W(3)) = 0$, $i = 0, 1$. Since Y is general, $Y \cap M$ is a general union of 5 points. A general union of 5 points lies on a plane stick figure of degree 3. Up to a projective transformation we may assume that $Y \cap M \subset T_4$. Thus A_3 is true. □

Lemma 4.5 A_4 *is true.*

Proof Recall that $h_4 = 7$ and $b_4 = 0$. Let $Y = R_1 \cup R_2 \cup R_3 \subset \mathbb{P}^3$ be a general union of 3 lines with the only restriction that R_i intersects L_i and let U be a general quadric. Thus Y is transversal to U. Fix $p \in R_1 \cap U$. Since R_1 is general with the only restriction that $R_1 \cap L_1 \neq \emptyset$, p is a general point of U. Let R_4 be the element of $|\mathcal{O}_U(1,0)|$ containing p. Since p is general in U, the point $R_4 \cap M$ is a general point of L_4. Thus $(Y \cap R_4) \cap M$ is formed by 4 points such that no 3 of them are collinear. Let v be the general tangent vector of $\mathbb{P}^3$ with $\{p\}$ as its reduction. Thus $v \nsubseteq R_3$. By [2, lemma 1.4] or [5] and the generality of v, we have $h^0(\mathcal{I}_{Y\cup v}(2)) = 0$ and hence $h^1(\mathcal{I}_{Y\cup v}(2)) = 0$. Let R_5, R_6, R_7 be elements of $|\mathcal{O}_U(1,0)|$ such that the point $R_i \cap M \subset D$ is a point of $L_i \cap D$. Since D is transverse to T_m, we may assume $R_i \neq R_j$ for all $i \neq j$. Set $X := Y \cup v \cup R_4 \cup R_5 \cup R_6 \cup R_7$. By Remark 2.1 (and the quoted explicit proof in [17, example 2.1.1]) X is a flat limit of a family of unions of 7 lines whose intersection with M is contained in T_8. Thus the semicontinuity theorem for cohomology shows that it is sufficient to prove $h^i(\mathcal{I}_X(4)) = 0$, $i = 0, 1$. Note that $\mathrm{Res}_U(X) = Y \cup v$. Thus $h^i(\mathcal{I}_{\mathrm{Res}_U(X)}(2)) = 0$, $i = 0, 1$. $X \cap U$ is the union of 4 elements of $|\mathcal{O}_U(1,0)|$ and the set $S := Y \cap U \setminus \{p\}$. Thus to conclude the proof it is sufficient to prove that $h^i(Q, \mathcal{I}_{S,Q}(0,4)) = 0$. This is true, because $\#S = 5$ and (moving R_2 and R_3) we may assume that no two points of S have the same image by any of the two projections $U = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. □

Remark 4.6 *Fix integers $b \geq a > 0$ and $t > 0$. Let $Z \subset Q$ be a general union of t 2-points of Q, i.e. planar 2-points of $\mathbb{P}^3$ scheme-theoretically contained in Q. By [18] or [4, corollary 2.3] we obtain $h^0(Q, \mathcal{I}_{Z,Q}(a,b)) = \max\{0, (a+1)(b+1) - 3t\}$ and $h^1(Q, \mathcal{I}_{Z,Q}(a,b)) = \max\{0, 3t - (a+1)(b+1)\}$, unless $(a,b,t) = (2, t-1, t)$. In the latter case $h^0(Q, \mathcal{I}_{Z,Q}(2,t-1)) = h^1(Q, \mathcal{I}_{Z,Q}(2,t-1)) = 1$.*

Lemma 4.7 *Fix an integer $m \geq 5$ and assume that A_{m-2} is true. Then A_m is true.*

Proof Take (W, Q) satisfying A_{m-2} and write $W = A \cup B$ with B the union of the reducible conics.

(a) Assume $m \equiv 2 \pmod 3$. Thus $B = \emptyset$, $h_{m-2} = (m+1)m/6$, $b_m = (m+1)/3$ and $h_m - h_{m-2} = 2(m+1)/3$. Thus we may take as Q a general quadric, so that no line of Q contains 2 points of an irreducible component of A.

Fix $2(m+1)/3$ lines $R_1, \dots, R_{2(m+1)/3}$ of A and call p_i one of the points of $R_i \cap Q$ for $1 \leq i \leq (m+1)/3$. Call D_i, $1 \leq i \leq (m+1)/3$, the line of $|\mathcal{O}_Q(1,0)|$ containing p_i. For $(m+1)/3 < i \leq 2(m+1)/3$ take as D_i a general element of $|\mathcal{O}_Q(1,0)|$. Deforming A if necessary keeping $A \cap M$ fixed, we may assume that $\{p_1, \dots, p_{(m+1)/3}\}$ is a general subset of Q with cardinality $(m+1)/3$ (here we are silently using the semicontinuity theorem for cohomology). Let E be the union of W and all lines D_i. E has lines and reducible conics with vertex contained in Q as its connected components and exactly $(m+1)/3$ of them are conics. Moreover, E is transversal to M. Since $\mathrm{Res}_Q(E) = W$, $h^i(\mathcal{I}_{\mathrm{Res}_Q(W)}(m-2)) = 0$. Thus by the residual exact sequence of U to prove that $h^i(\mathcal{I}_E(m)) = 0$, $i = 0, 1$, (and hence to prove A_m for $m \equiv 2 \mod 3$) it is sufficient to prove that $h^i(Q, \mathcal{I}_{E\cap Q,Q}(m,m)) = 0$, $i = 0, 1$, i.e. to prove that $h^i(Q, \mathcal{I}_{S,Q}((m-2)/3, m)) = 0$, $i = 0, 1$, where $S = W \cap Q \setminus \{p_1, \dots, p_{(m+1)/3}\}$. Use $m-2$ times Remark 4.9. Then we deform E to a curve E' isomorphic to E as an abstract scheme so that $\mathrm{Sing}(E') \subset Q$ and $\dim E' \cap Q = 0$.

(b) Assume $m \equiv 0 \pmod 3$. Since $b_{m-2} = b_m = 0$, we may repeat the proof of step (a) taking all D_i's as general elements of $|\mathcal{O}_Q(1,0)|$.

(c) Assume $m \equiv 1 \pmod 3$. Take (W,Q) satisfying A_{m-2}. By semicontinuity we may deform W so that no irreducible component of W is contained in Q. Set $S := \mathrm{Sing}(W)$ and $Z := \cup_{p\in S}(2p,\mathbb{P}^3)$. The scheme $W \cup Z$ is a flat limit of a family of unions of h_{m-2} disjoint lines. To get A_m is sufficient to take the union of $W\cup Z$ and $h_m - h_{m-2}$ general elements of $|\mathcal{O}_Q(1,0)|$ and then apply Remark 2.1 and the semicontinuity theorem. □

Remark 4.8 *By Remark 4.3 and Lemmas 4.4, 4.5, and 4.7* A_m *is true for all* $m \neq 2$*.*

Remark 4.9 *Let* $Q \subset \mathbb{P}^3$ *be a smooth quadric and* $L \subset \mathbb{P}^3$ *be a line such that* $L \nsubseteq Q$*. Fix* $(a,b) \in \mathbb{N}^2$ *and a zero-dimensional scheme* $Z \subset Q$ *such that* $h^1(Q,\mathcal{I}_{Z,Q}(a,b)) = 0$ *and* $h^0(Q,\mathcal{I}_{Z,Q}(a,b)) \geq 2$*. Fix a general* $p \in Q$ *and call* H *the plane spanned by* L *and* p*. For a general* p *the scheme* $D := H \cap Q$ *is a smooth conic. Since* p *is general in* Q*,* $p \notin Z_{\mathrm{reg}}$ *and* $Z \cap H \subseteq L \cap Q$*. A general line* R *though* p *meeting* L *intersects* Q *in* p *and a general point* q_R *of* D*. Since* p *is general,* $h^1(Q,\mathcal{I}_{Z\cup\{p\},Q}(a,b)) = 0$ *and* $h^0(Q,\mathcal{I}_{Z,Q\cup\{p\}}(a,b)) = h^0(Q,\mathcal{I}_{Z,Q}(a,b)) - 1 > 0$*. Assume* $h^1(\mathcal{I}_{Z\cup\{p,q_R\},Q}(a,b)) > 0$*. Since* q_R *is general in* D*, this means that* D *is in the base locus of* $|\mathcal{I}_{Z\cup\{p\},Q}(a,b)|$*, i.e. we have*

$$h^0(Q,\mathcal{I}_{\mathrm{Res}_D(Z)}(a-1,b-1)) = h^0(U,\mathcal{I}_{Z\cup\{p\},Q}(a,b)) - 1.$$

Since $h^1(Q,\mathcal{I}_{Z,Q}(a,b)) = 0$*,* $h^0(Q,\mathcal{O}_Q(a,b)) = h^0(Q,\mathcal{O}_Q(a-1,b-1)) + a + b + 1$ *and* $Z \cap D = Z \cap L$*, we get* $a + b \leq \deg(Z \cap L) \leq 2$*.*

Theorem 4.10 *Fix integers* $s \geq 4$ *and* $d \geq s + a_{s-1}$*. Then there is a connected stick figure* $X \subset \mathbb{P}^3$ *such that* $\deg(X) = d$*,* $p_a(X) = b_{s-1} + (d - a_{s-1} - 1)(d - a_{s-2} - 2)/2$ *and* $h^0(\mathcal{I}_X(s-1)) = 0$*.*

Proof Fix a plane $M \subset \mathbb{P}^n$ and take a general union T_s of s lines. Let $E \subset M$ be a general union of $d - s - a_{s-1}$ lines. Take (W,Q) satisfying A_{s-1} with respect to T_s and set $X := T_s \cup E \cup W$. Note that X is a connected stick figure, $\deg(X) = d$ and $p_a(X) = p_a(T_s \cup E) + b_{s-1}$. Assume $h^0(\mathcal{I}_X(s-1)) \neq 0$ and take $U \in |\mathcal{I}_X(s-1)|$. Since $\deg(E \cup T_s) \geq \deg(T_s) = s$, M is a component of U. Write $U = M + V$ with $V \in |\mathcal{O}_{\mathbb{P}^3}(s-2)|$. Note that $V \in |\mathcal{I}_W(s-1)|$, contradicting the assumption that W satisfies A_{s-1}. □

Remark 4.11 *Fix an integer* $s \geq 2$*. Note that*

$$\lim_{d\to+\infty}(\frac{(d-h_{s-1})^2}{2} + b(s-1))/\frac{d^2}{2} = 1$$

while

$$\lim_{d\to+\infty}\frac{G_C(d,s)}{d^2/2s} = 1$$

. Thus $\frac{(d-h_{s-1})^2}{2} \sim \frac{d^2}{2}$*, while* $G_C(d,s) \sim \frac{d^2}{2s}$ *for* $d \gg 0$*. By the definition of Range C and [12] we have* $G_C(d,s) = \gamma(3,d,s)$ *for all* $d > s^2$*. Hence we may use* $\gamma(3,d,s)$ *instead of* $G_C(d,s)$*.*

5. Inside $\mathbb{P}^r$, $r > 3$

The case of curves in $\mathbb{P}^r$, $r > 3$, is far more complicated. See [6–8] for a partial extensions in various directions of Halphen theory. These papers look at maximal genera, but do not consider lower genera, i.e. the existence or nonexistence of some genus intermediate between two genera obtained by two curves. The case $s = 1$ (nondegeneracy of the curve) shows that the case of integral curves and the case of connected, but reducible, curves is very different. For integral curves there is Castelnuovo's theorem which gives an upper bound for the arithmetic genus (for fixed r and degree d), classify the degree d nondegenerate curves with genus g and go a little bit to lower genera ([14, chapter III]). Gaps near the top of the genus exists, even for $r = 3$, and are described in [14, chapter III].

For all integers $r \geq 3$ and $m \geq 1$ set $a_{r,m} := \lceil \frac{\binom{r+m}{r}}{m+1} \rceil$ and $b_{r,m} := (m+1)a_{r,m} - \binom{r+m}{m}$. Note that

$$(m+1)a_{r,m} - b_{r,m} = \binom{r+m}{r}, \ 0 \leq b_{r,m} \leq m \tag{5.1}$$

and that the integers $a_{r,m}$ and $b_{r,m}$ are uniquely determined by (5.1). Taking the difference between the equation in (5.1) and the same equation for $(r, m-1)$ we get

$$a_{r,m} + m(a_{r,m} - a_{r,m-1}) + b_{r,m-1} - b_{r,m} = \binom{r+m-1}{r-1}. \tag{5.2}$$

Our aim is to prove the following result, which we also use in the proof of Theorem 1.1.

Theorem 5.1 *Fix integers $r \geq 4$, $s \geq 3$, $d \geq a_{r,s-1} + \lceil a_{r,s-1}/2 \rceil$. Then there is a connected stick figure $X \subset \mathbb{P}^r$ such that $\deg(X) = d$, $p_a(X) = b_{r,s-1} + (d - a_{r,s-1} - 1)(d - a_{r,s-1} - 2)/2$ and $h^0(\mathcal{I}_X(s-1)) = 0$.*

Proof [Proof of Theorem 1.1:] Set $\alpha := \lceil \binom{r+s-1}{r}/s \rceil$. It is easy to check that any reduced curve $X \subset \mathbb{P}^r$ with $\deg(X) = d \geq 6$ has arithmetic genus $\leq (d-1)(d-2)/2$ and that equality holds if and only X is a plane curve ([15]). Thus the limits in the statement (if they exist) are upper bounded by $1/2$. Theorem 5.1 shows that the $\liminf_{d \to +\infty}$ for connected stick figures is at least $1/2$. Thus the limits exist and $1/2$ is their value, except perhaps for smooth disconnected curves. For them instead of Theorem 5.1 use the union of s smooth plane curve of degree $d - \alpha$ and α general lines and use [17, Theoreme 1.1].

Fix a real number $\epsilon > 0$.

(a) In this step we prove the existence of the connected stick figure. Take a positive integer d and an integer q such that $0 \leq q \leq \frac{d^2}{2+\epsilon}$. Define the positive integer h and the integers $x_1 > \cdots > x_h > 0$ and $e_i \in \mathbb{N}$, $i = 1, \dots, h$, in the following way. If $q \leq 2$ set $h := 1$, $x_1 := 1$ and $e_1 := q$. Assume $q \geq 3$. Let x_1 be the maximal positive integer such that $(x_1 - 1)(x_1 - 2)/2 \leq q$. Set $e_1 := q - (x_1 - 1)(x_1 - 2)/2$. If $e_1 \leq 2$ set $h = 1$. Assume $e_1 \geq 3$. Let x_2 be the maximal integer such that $e_2 := e_1 - (x_2 - 1) - (x_2 - 1)(x_2 - 2)/2 \geq 0$. If $e_2 \leq 2$, set $h := 2$. Assume defined the integers x_i, e_i for some integer $m \geq 2$. If $e_m = 0$, set $h := m$. If $e_m = 1$, set $h := m + 1$, $x_h := 2$ and $e_h := 0$. If $e_m \geq 2$, let x_{m+1} be the maximal integer such that $(x_{m+1} - 1) + (x_{m+1} - 1)(x_{m+1} - 2)/2 \leq e_m$. Set $e_{m+1} := e_m - (x_{m+1} - 1) - (x_{m+1} - 1)(x_{m+1} - 2)/2$. After finitely many steps we get $x_1, e_1, \dots, x_h, e_h$ with $e_h = 0$, $e_i > 0$ for all $i < h$ and e_i decreasing. Note that $x - 1 + (x-1)(x-2)/2 = x(x-1)/2$. Hence if $e_i \geq 3$, then x_{i+1} is the largest integer $t \geq 2$ such that

$t(t-1)/2 \le e_i - 1$, i.e. $t(t-1) \le 2e_i - 2$. Note that $x_i \le 2\sqrt{x_{i-1}+2}$ for all $i > 1$. Since $\epsilon > 0$, there is a real number $\eta(\epsilon)$ such that $x_1 \le \frac{d}{1+\eta(\epsilon)}$.

Let $E \subset \mathbb{P}^r$ be a general union of α lines. By [17, Theoreme 1.1] $h^0(\mathcal{I}_E(s-1)) = 0$. Thus to prove the theorem it is sufficient to find a connected stick figure of degree d (for large d) containing E and with arithmetic genus q. Assume for the moment to have constructed a connected stick figure $Z \subset \mathbb{P}^r$ with $d' := \deg(Z) \le d - 2\alpha$ and $p_a(Y) = q$. Moving if necessary the components of E we may assume $E \cap Z = \emptyset$. Let $L_1, \dots, L_\alpha$ be the connected components of E. Fix an irreducible component L of Z. Let R_i, $1 \le i \le \alpha$, be a general line intersecting both L and L_i. Set $G := W \cup Z \cup R_1 \cup \cdots \cup R_\alpha \cup E$. By construction G is connected. Since $W \cup Z$ is a connected stick figure, E and $R_1, \dots, R_\alpha$ are general and $r \ge 3$, G is a connected stick figure with $p_a(G) = p_a(Z \cup W) = q$.

Assume for the moment $h \ge 2$. Since $x_1 \le \frac{d}{1+\eta(\epsilon)}$ with $\eta(\epsilon) > 0$, for large d (depending on r, s and ϵ) we have $\sum_{i=1}^{h} x_i + x_h + 3 \le d - 2\alpha$. Take a plane M_1 and a degree x_1 stick figure $E_1 \subset M_1$. Fix a smooth point $p_1 \in E_1$ and let $M_2 \subset \mathbb{P}^r$ be a general plane containing p_1. Let $E_2 \subset M_2$ be a general degree x_2 stick figure with the only restriction that p_1 is a smooth point of E_2. Fix a smooth point $p_2 \in E_2$ such that $p_2 \neq p_1$. If $h > 2$ we continue with stick figures $E_3, \dots, E_h$ contained in planes $M_3, \dots, M_h$ and points $p_i \in E_i$ with $\#\{p_1, \dots, p_h\} = h$ so that $T := E_1 \cup \cdots \cup \cdots \cup E_h$ is a connected stick figure of degree $x_1 + \cdots + x_h$ with arithmetic genus $\sum_{i=1}^{h}(x_i-1)(x_i-2)/2$ with $0 \le e_h \le 2$ and $e_h = q - p_a(T)$. There is a connected stick figure of degree 5 and arithmetic genus 2. Thus we may find a stick figure with at most degree 5 such that its union with T is a connected stick figure Z with arithmetic genus q and degree $\le d - 2\alpha$. If its degree is smaller, we may add general lines meeting E_1. Now assume $h = 1$. Instead of $\cup_{i=1}^{h} E_i$ we take E_1.

(b) In this step we modify the construction of part (a) to get a smooth curve Y with at most $\alpha + 2$ connected component. Instead of W take a smooth and connected W' with $W' \cap Z = \emptyset$, $\deg(W') = \deg(W) + \alpha$ and set $Y := W' \cup Z \cup E$. We only use M_1 and M_2 (case $h \ge 2$) or E_1 (case $h = 1$), because we want to avoid the dependence from h (and hence on q) on the number of connected components of the smooth curve Y. The curve W of step (a) requires at most another connected component. Since we do not use $R_1, \dots, R_\alpha$, we only need to construct a smooth curve Γ with degree $d - \alpha$, $h^1(\mathcal{O}_\Gamma) = q$ and at most 3 connected components. If $h = 1$, then it is easier. Thus we assume $h \ge 2$. It is sufficient to have $e_2 := q - (x_1-1)(x_1-2)/2 - (x_2-1)(x_2-2)/2 \le d - \alpha - 3$. We have $e_2 \le x_2 - 1 \le 4\sqrt{x_1+4}$. Since $x_1 \le \frac{d}{1+\eta(\epsilon)}$ with $\eta(\epsilon) > 0$, we conclude for large d only depending on r, s and ϵ. $\square$

Now we start the preliminary remarks and notation used in the proof of Theorem 5.1.

We fix a plane $M \subset \mathbb{P}^r$, $r \ge 4$, and a hyperplane $H \supset M$ of $\mathbb{P}^r$. Consider the following Assertion $H_{r,m}$:

Assertion $H_{r,m}$, $r \ge 4$, $m \ge 1$: There is a disjoint union $X \subset \mathbb{P}^r$ of $a_{r,m} - 2b_{r,m}$ lines and $b_{r,m}$ singular conics such that:

1. $\mathrm{Sing}(X) \subset H$;

2. $e := \min\{\binom{m+2}{2}, a_{r,m} - b_{r,m}\}$ of the degree one components of X intersects M; if $e < \binom{m+2}{2}$, then $\min\{\binom{m+2}{2} - e, 2b_{m,r}\}$ of the irreducible components of the conics of X meet M and $M \cap \mathrm{Sing}(X) = \emptyset$;

3. $h^i(\mathcal{I}_X(m)) = 0$, $i = 0, 1$.

The assertion $H_{r,m}$ without the mention of the plane M is called $H'_{m,r}$ in [17].

Remark 5.2 *We have* $a_{r,1} = (r+1)/3$ *and* $b_{r,1} = 0$ *if* r *is odd and* $a_{r,1} = r/2$ *and* $b_{r,1} = 1$ *if* r *is even. We have* $(a_{r,2}, b_{r,2}) = (\frac{(r+2)(r+1)}{6}, 0)$ *if* $r \equiv 1, 2 \pmod 3$, $(a_{r,2}, b_{r,2}) = (\frac{r^2+3r+6}{6}, 2)$ *if* $r \equiv 0 \pmod 3$, $(a_{4,3}, b_{4,3}) = (9,1)$, $(a_{4,4}, b_{4,4}) = (14,0)$, $(a_{4,5}, b_{4,5}) = (21,0)$, $(a_{4,6}, b_{4,6}) = (30,0)$, $(a_{4,7}, b_{4,7}) = (42,6)$, $(a_{4,8}, b_{4,8}) = (55,0)$, $(a_{4,9}, b_{4,9}) = (72,5)$, $(a_{4,10}, b_{4,10}) = (91,0)$, $(a_{4,11}, b_{4,11}) = (114,3)$, $(a_{4,12}, b_{4,12}) = (140,0)$, $(a_{5,3}, b_{5,3}) = (14,0)$, $(a_{5,4}, b_{5,4}) = (26,4)$, $(a_{5,5}, b_{5,5}) = (38,0)$, $(a_{6,3}, b_{6,3}) = (21,0)$.

Remark 5.3 *Since any* 2 *points of* $\mathbb{P}^r$ *are contained in a plane,* $H_{r,1}$ *is true. Since any* 3 *points of* $\mathbb{P}^r$ *are contained in a plane,* $H_{r,2}$ *is true if and only if the assertion called* $H'_{r,2}$ *or* $H_{r,2}$ *in [17] are true. Thus* $H_{r,2}$ *is true if and only if* $r \geq 4$ *(see part (b) of Lemma 5.4 for the case* r *even).*

Lemma 5.4 *Fix a plane* $M \subset \mathbb{P}^r$, $r \geq 3$. *Let* $A_x \subset \mathbb{P}^r$ *be a general union of* x *lines intersecting* M. *Then* $h^0(\mathcal{I}_{M\cup A_x}(2)) = \max\{\binom{r+2}{2} - 6 - 2x, 0\}$ *and* $h^0(\mathcal{I}_{A_x}(2)) = h^0(\mathcal{I}_{M\cup A_x}(2)) + \max\{0, 6-x\}$.

Proof Since $A_x \cap M$ is the union of x general points of M and $h^0(\mathcal{O}_M(2)) = 6$, the first equality for all x implies the second one for all x. The first equality holds for $x = 0$. Fix $x \in \mathbb{N}$ such that the first equality holds for all $y \leq x$. Let $L \subset \mathbb{P}^r$ be a general line meeting M. To conclude the proof of the lemma by induction on x it is sufficient to prove that either $h^0(\mathcal{I}_{M\cup A_x\cup L}(2)) = 0$ or $h^0(\mathcal{I}_{M\cup A_x\cup L}(2)) = h^0(\mathcal{I}_{M\cup A_x}(2)) - 2$. Assume $h^0(\mathcal{I}_{M\cup A_x\cup L}(2)) \neq 0$ and $h^0(\mathcal{I}_{M\cup A_x\cup L}(2)) \geq h^0(\mathcal{I}_{M\cup A_x}(2)) - 1$. Since (after fixing $M \cup A_x$) we may take as L a line containing a general point of $\mathbb{P}^r$, $h^0(\mathcal{I}_{M\cup A_x\cup L}(2)) = h^0(\mathcal{I}_{M\cup A_x}(2)) - 1$. Fix a general $Q \in |\mathcal{I}_{M\cup A_x\cup L}(2)|$. There is a Zariski open set U of Q such that $h^0(\mathcal{I}_{M\cup A_x\cup\{o\}}(2)) = h^0(\mathcal{I}_{M\cup A_x}(2)) - 1$. Assuming that the lemma fails for the integer $x+1$ we get $h^0(\mathcal{I}_{M\cup A_x\cup R}(2)) = h^0(\mathcal{I}_{M\cup A_x\cup\{o\}}(2))$ for a general line containing o and in the 3-dimensional space V spanned by o and M. We get $V \subset Q$. Varying $o \in U$ we get a contradiction. □

Lemma 5.5 $H_{r,2}$ *is true for all* $r \geq 4$.

Proof Lemma 5.4 gives the case $b_{r,2} = 0$. Thus we may assume $r \equiv 0 \pmod 3$. In this case $b_{r,2} = 2$ and $a_{r,2} = (r^2 + 3r + 6)/6$. Let $A \subset \mathbb{P}^r$ be a general union of $a_{r,2} - 2$ lines, say $A = L_1 \cup \cdots \cup L_{a_{r,2}-2}$. Lemma 5.4 gives $h^1(\mathcal{I}_A(2)) = 0$, i.e. $h^0(\mathcal{I}_A(2)) = 4$, and that this true even if we assume that 6 of the lines of A, say L_i for $1 \leq i \leq 6$, are only general among the lines meeting M. Fix $7 \leq i < j \leq a_{r,2} - 2$. To prove the lemma it is sufficient to find lines R_i, R_j such that $L_i \cap R_i$ is a point of H, $R_j \cap H$ is a point of H and $h^0(\mathcal{I}_{A\cup R_i\cup R_j}(2)) = 0$. For general L_i, L_j the set $L_i \cap H$ (resp. $L_j \cap H$) is a unique point, o_i (resp. o_j).

Let R_i (resp. R_j) denote a general line containing o_i (resp. o_j). To prove the lemma we need to find i, j such that $h^0(\mathcal{I}_{A\cup R_i}(2)) = 2$ and $h^0(\mathcal{I}_{A\cup R_i\cup R_j}(2)) = 0$.

Suppose $h^0(\mathcal{I}_{A\cup R_i}(2)) > 2$. Since R_i contains a general point of $\mathbb{P}^r$, $h^0(\mathcal{I}_{A\cup R_i}(2)) = h^0(\mathcal{I}_A(2)) - 1$. Fix a general $Q \in |\mathcal{I}_A(2)|$. Since $\dim |\mathcal{I}_A(2)| > 0$, there is a nonempty Zariski open subset U of Q (maybe not dense if Q is reducible) such that $h^0(\mathcal{I}_{A\cup\{p\}}(2)) = h^0(\mathcal{I}_A(2)) - 1$ for all $p \in U$ and $o_i \notin U$. For any $p \in U$ let R_p be the line spanned by $\{o_i, p\}$. Our assumptions give $h^0(\mathcal{I}_{A\cup\{p\}}(2)) = h^0(\mathcal{I}_{A\cup R_p}(2))$. Hence $R_p \subset Q$. Since $\dim U = \dim Q = r - 1$ and this is true for all $p \in U$, Q is a quadric cone with vertex containing o_i. Taking as i another element of $\{7, \dots, a_{r,2} - 2\}$ we get that a general $Q \in |\mathcal{I}_A(2)|$ is a cone with vertex containing all points $L_h \cap H$, $7 \leq h \leq a_{r,2} - 2$, which are $a_{r,2} - 8 \geq 2$ general points of H. Lemma 5.4 applied to a linear projection of one of these points gives a contradiction. Thus $h^0(\mathcal{I}_{A\cup R_i}(2)) = 2$ for some i, say for $i = 7$. Applying the same reasoning to a general element of $|\mathcal{I}_{A\cup R_7}(2)|$ we get the lemma. □

Lemma 5.6 *Fix an integer $r \geq 4$. Then,*

(a) $a_{r,k} \geq 2b_{r,k}$ for all $k > 0$;

(b) $a_{r,k} - 2b_{r,k} \geq \binom{k+2}{2}$, unless $r = 4$ and $2 \leq k \leq 4$;

(c) assume $b_{r,k} \geq b_{r,k-1}$; we have $a_{r,k} - a_{r,k-1} \geq k + 1 + b_{r,k} - b_{r,k-1}$ for all $r \geq 4$, $k \geq 2$.

Proof Statements (a) and (b) are true for $k = 1$. Assume $k \geq 2$. Since $b_{r,k} \leq k$, $a_{r,k} \geq \binom{r+k}{r}/(k+1)$ and $\binom{k+2}{2} + 2k = (k^2 + 11k + 8)/2$, part (a) is true for all $k > 0$.

Now we check part (b). The inequality in part (b) is true for $r = 4$ at least for $k \geq 13$. Use Remark 5.2 or $r = 4$ and $2 \leq k \leq 12$ and for all r for $k = 2$. For $r \geq 5$ and $k \geq 6$ to get (2) use $\binom{r+k}{r}/(k+1) \geq \binom{k+5}{5}/(k+1) = (k+5)(k+4)(k+3)(k+2)/120$. For $r = 5$ and $2 \leq k \leq 5$ use Remark 5.2. For $r \geq 6$ and $k \geq 3$ use the inequality $\binom{r+k}{r} \geq (k+1)(k^2 + 11k + 8)/2$ and the case $r = 6$, $k = 3$ given by Remark 5.2.

Now we check part (c). Assume $a_{r,k} - a_{r,k-1} \leq k + b_{r,k} - b_{r,k-1}$. By (5.2) we get

$$a_{r,k} + k^2 + (k-1)(b_{r,k} - b_{r,k-1}) \geq \binom{r+k-1}{r-1}. \tag{5.3}$$

Multiplying (5.3) by $k + 1$ and using that $(k+1)a_{r,k} \leq \binom{r+k}{r} + k$, that $b_{r,k} \leq k$, that $b_{r,k-1} \geq 0$ and that $k(2k-1) + k + 1 = 2k^2 + 1$, we get

$$\binom{r+k}{r} + 2k^2 + 1 \geq (k+1)\binom{r+k-1}{r-1}. \tag{5.4}$$

Call $\varphi(r,k)$ the difference between the right hand side and the left end side of (5.4). Note that $(k+1)\binom{r+k-1}{r-1} - \binom{r+k}{r} = \frac{(r+k-1)!}{r!k!}(r(k+1) - (r+k))$. Thus $\varphi(r,k) < \varphi(r+1,k)$ for all $r \geq 4$. For $r = 4$ (5.4) is the inequality

$$2k^2 + 1 \geq (k+3)(k+2)(k+1)k/8, \tag{5.5}$$

which is false for all $k \geq 2$. □

Lemma 5.7 *$H_{r,k}$ is true for all $r \geq 4$ and $k \geq 2$.*

Proof By Lemma 5.5 we may assume $k \geq 3$. Any plane and any line of $\mathbb{P}^3$ meet. By Remark 4.8 we may assume that a statement similar to $H_{3,k}$ with the plane M, but no mention of H is true in $\mathbb{P}^3$. Thus we may assume that the lemma is true in H. For the fixed r we may assume that $H_{r,t}$ is true for all $2 \leq t < k$.

Claim 1: Assume $b_{r,k} < b_{r,k-1}$. Let $E \subset H$ be a general union of $a_{r,k} - a_{r,k-1}$ lines, $k+1$ of them meeting M. Let $Z \subset H$ be a general union of $b_{r,k-1} - b_{r,k}$ planar 2-points. Let $S \subset M$ be a general union of $\binom{k+1}{2}$ points. Then $h^1(H, \mathcal{I}_{E\cup Z\cup S,H}(x)) = 0$.

Claim 1: Claim 1 is related to condition $H''_{r-1,k}$ of [17], except for two simplifying reasons. We do not need the tangent vectors (because in characteristic 0 to handle them we just quote [5]) and we do not have collinear points (because our set-up is more efficient). However, contrary to [17] we have the restriction that $k+1$ of the lines of E meets M and we have the set S which is only general in M. These two restrictions are not restrictive if $H = \mathbb{P}^3$, i.e. $r = 4$ (for S we also need to use induction on k). Thus we may assume $r \geq 5$. Set $z := \min\{a_{r,k} - a_{r,k-1}, a_{r-1,k-1} - b_{r-1,k-1}\}$. Let M_1 be a hyperplane of H not containing M. Set $L := M \cap H$. Let $S' \subset M \setminus L$ be a general union of $\binom{k+2}{2}$ points. Let $S'' \subset L$ be a general union of k points. Let $F \subset H$ be a general union of z lines, with the only restriction that k of them meet $M \setminus L$. Let $G \subset M_1$ be a general union of $a_{r,k} - a_{r,k-1}$ lines with the only restriction that one of them meets L. Let $Z' \subset M_1$ be a general union of $b_{r,k-1} - b_{r,k}$ planar 2-points. By $H''_{r-2,k}$ of [17] $h^1(M_1, \mathcal{I}_{G\cup L\cup Z',M_1}(k)) = 0$. Thus $h^1(M_1, \mathcal{I}_{G\cup S''\cup Z',M_1}(k)) = 0$. We specialize $E \cup Z \cup S$ to $F \cup G \cup S' \cup S''$ and use the residual exact sequence of the hypersurface M_1 of H.

(a) Assume $b_{r,k} \geq b_{r,k-1}$. Part (c) of Lemma 5.6 gives $a_{r,k} - a_{r,k-1} \geq k+1+b_{r,k} - b_{r,k-1}$. Let $E \subset H$ be a general union of $a_{r,k} - a_{r,k-1}$ lines with the only restriction that $b_{r,k} - b_{r,k-1}$ of them contain one point of $A'' \cap H$ and $k+1$ of the other connected connected components of E meets M. Set $Y := W \cup E$. To prove the lemma in this case it is sufficient to prove $h^i(\mathcal{I}_Y(k)) = 0$, $i = 0,1$. The residual exact sequence of H shows that it is sufficient to prove $h^i(H, \mathcal{I}_{E\cup((H\setminus E)\cap W),H}(k)) = 0$, $i = 0,1$. The scheme $E \cup ((H \setminus E) \cap W)$ is a general union of lines (with the only restriction that at most $k+1$ of them meet M), at most $\binom{k+1}{2}$ general points of M, some general points of $H \setminus M$ and some general tangent vectors. Let $G \subset E$ the union of the lines and the points in M. By the inductive assumption on r we have $h^1(H, \mathcal{I}_{G,H}(k)) = 0$. Since $E \cup ((H \setminus E) \cap W)$ is obtained from G adding general points and general tangent vectors, [2, lemma 1.4] or [5] gives $h^i(H, \mathcal{I}_{E\cup((H\setminus E)\cap W),H}(k)) = 0$, $i = 0,1$.

(b) Assume $b_{r,k} < b_{r,k-1}$. Let $E \subset H$ be a general union of $a_{r,k} - a_{r,k-1}$ lines, $k+1$ of them meeting M. Fix $S \subset \mathrm{Sing}(B)$ with $\#S = b_{r,k-1} - b_{r,k-1}$. For each $o \in S$ let $v_o \subset H$ be a general tangent vector of H with $v_{o\mathrm{red}} = \{o\}$. Set $Z := \cup_{o\in S} v_o$ and $Y := W \cup E \cup Z$. Remark 2.1 shows that Y is a flat limit of a family of $a_{r,k} - 2b_{r,k-1}$ lines, $\binom{k+2}{2}$ of them intersecting M, and $b_{r,k}$ reducible conics with singular point contained in H. By the semicontinuity theorem for cohomology and the residual exact sequence of H to prove the lemma in this case it is sufficient to prove $h^i(H, \mathcal{I}_{Y\cap H,H}(k)) = 0$, $i = 0,1$. The scheme $Y \cap H$ is a general union of $a_{r,k} - 2b_{r,k} - \binom{k+2}{2}$ lines, $\binom{k+2}{2}$ lines meeting M, $b_{r,k}$ tangent vectors and $b_{r,k-1} - b_{r,k}$ planar 2-points. We first use Claim 1 and then quote [2, lemma 1.4] or [5]. □

Proof [Proof of Theorem 5.1:] We fix a plane $M \subset \mathbb{P}^r$, By Lemma 5.7 there is a nodal union $Y \subset \mathbb{P}^r$ of $a_{r,s-1} - 2b_{r,s-1}$ lines and $b_{r,s-1}$ reducible conics such that $\mathrm{Sing}(Y) \cap M = \emptyset$ and each irreducible component of Y contains exactly one point of M. Set $S := Y \cap M$. Moving Y if necessary, we see that we may assume that S is a general subset of M with cardinality $a_{r,s-1}$. Since $d - a_{r,s-1} \geq \lceil a_{r,s-1}/2 \rceil$ and no 3 of the points of S are collinear, there is a nodal union $W \subset M$ of $d - a_{r,s-1}$ lines containing S in its smooth locus. Use the stick figure $W \cup Y$. □

Remark 5.8 *Let $M \subset \mathbb{P}^r$ be a plane and let $S \subset M$ be a general subset with cardinality $a_{r,s-1}$. Let t be the mininum integer such that $t(t+3) \geq 2a_{r,s-1}$. Since S is general, for every integer $x \geq t$ there is a smooth degree* *x curve $C \subset M$ containing S. Thus the proof of Theorem 5.1 (taking smooth conics instead of reducible conics) shows that to get a connected nodal curve X with* $\deg(X) = d$, $p_a(X) = b_{r,s-1} + (d - a_{r,s-1} - 1)(d - a_{r,s-1} - 2)/2$ *and* $h^0(\mathcal{I}_X(s-1)) = 0$ *it is sufficient to assume* $d \geq t + a_{r,s-1}$. *We may do better to get an element W of $G(r,d,s)^{00}$ with $1 + a_{r,s-1} - b_{r,s}$ connected components and $h^1(\mathcal{O}_W) = (d - \alpha - 1)(d - \alpha - 2)/2$. Let $T \subset \mathbb{P}^r$ be a general union of $a_{r,s-1} - 2b_{r,s-1}$ lines and $b_{r,s-1}$ smooth conics. The semicontinuity theorem for cohomology and $H_{r,s-1}$ give $h^0(\mathcal{I}_T(s-1)) = 0$. Take $W := C \cup T$.*

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