Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2023) 47: 687 – 693
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doi:10.55730/1300-0098.3387

Boundedness of Bergman projections acting on weighted mixed norm spaces

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Received: 11.12.2022 • **Accepted/Published Online:** 20.01.2023 • **Final Version:** 09.03.2023

Abstract: We prove that Bergman projections on weighted mixed norm spaces on smoothly bounded domains in $\mathbb{R}^n$ are bounded for a certain range of parameters of such spaces and assuming certain conditions on weights. The proof relies on estimates of integral means of $M_p(P_\gamma f, r)$ in terms of integral means of f. This result complements earlier result on boundedness of P_γ on a closely related space $L^{p,q}_\alpha(\Omega)$.

Key words: Bergman projections, mixed norm spaces

1. Introduction and preliminary notations

The boundedness of Bergman projection was investigated in [1] using equivalence of certain norms on weighted mixed norm space of harmonic functions. This result on the boundedness of the Bergman projection was applied to describe the dual space of harmonic mixed norm space, for a certain range of its parameters. In this paper, we consider the boundedness of Bergman projections on weighted mixed norm space without using this equivalence. We use a similar approach to that in [5] and [4]. Notice that these papers considered Bergman projections on mixed norm spaces in the unit ball of $\mathbb{R}^n$. In this paper, we consider more general domains, actually bounded domains with smooth boundaries and we need estimates of the Bergman kernels on this type of domains which are formulated in Proposition 2.3 of [6] as a special case of delicate estimates obtained by Engliš in [2]. We use these estimates to obtain estimates of integral means of the Bergman projections in terms of integral means of functions.

We are going to use terminology and notation from [1]. Namely, throughout the paper, Ω denotes a bounded domain in $\mathbb{R}^n$ (i.e. open and connected) with C^∞ boundary, and $h(\Omega)$ denotes the vector space of all real-valued harmonic functions in Ω. Let $\rho(x)$ be a defining function for Ω. This means ρ is a real-valued continuous function on $\mathbb{R}^n$ which is C^∞ in a neighborhood of the boundary $\partial\Omega$ of Ω such that $\Omega = \{x \in \mathbb{R}^n : \rho(x) > 0\}$ is bounded and $|\nabla\rho(x)| \neq 0$ on $\partial\Omega$. Throughout this paper such a domain Ω is fixed. It is convenient to work with a particular defining function, namely the distance function $r(x)$ defined by $r(x) = d(x, \partial\Omega)$ for $x \in \overline{\Omega}$ and $r(x) = -d(x, \partial\Omega)$ for $x \notin \overline{\Omega}$. Indeed, there is an $\epsilon > 0$ such that for all $0 < r \leq \epsilon$ the set $\Omega_r = \{x \in \mathbb{R}^n : r(x) > r\}$ is a smoothly bounded subdomain of Ω with defining function $r(x) - r$. We fix such $\epsilon > 0$. We denote, for $0 < r \leq \epsilon$, by Γ_r the boundary $\partial\Omega_r = \{x \in \mathbb{R}^n : r(x) = r\}$. For $x, y \in \Omega$, we introduce a quasi distance $D(x, y) = r(x) + r(y) + |x - y|$ on Ω, which is useful in estimates of the Bergman kernel.

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2010 *AMS Mathematics Subject Classification:* 42B35, 31B05, 31B10

We denote by $d\sigma_r$ the induced surface measure on $\partial\Omega_r$. dm denotes the Lebesgue volume measure on $\mathbb{R}^n$. We also work with weighted measures $dm_\gamma(x) = r(x)^\gamma dm(x)$ on Ω, where $\gamma \in \mathbb{R}$ and set $L^p_\gamma(\Omega) = L^p(\Omega, dm_\gamma)$. The exponent conjugate to $1 \leq p \leq +\infty$ is denoted by p'.

The weighted Bergman spaces are $b^p_\gamma(\Omega) = L^p(\Omega, dm_\gamma) \cap h(\Omega)$ where $0 < p \leq +\infty$ and $\gamma > -1$.

For $0 < p < \infty$ and $0 < r \leq \epsilon$, we set

$$M_p(f, r) = \left\{ \int_{\Gamma_r} |f(\zeta)|^p d\sigma_r(\zeta) \right\}^{\frac{1}{p}}, \tag{1.1}$$

with obvious modification for the case $p = +\infty$. Now let $0 < p \leq +\infty$, $0 < q < +\infty$ and $\alpha > 0$. We define a mixed norm space $B^{p,q}_\alpha(\Omega)$ as the space of all $f \in h(\Omega)$ such that the (quasi) norm

$$\|f\|_{B^{p,q}_\alpha} = \left\{ \int_0^\epsilon r^{\alpha q - 1} M_p^q(f, r) dr \right\}^{1/q} \tag{1.2}$$

is finite, again with obvious modification to include the case $q = +\infty$. The space $B^{p,q}_\alpha(\Omega)$ is a Banach space for $1 \leq p \leq +\infty$ and $1 \leq q \leq +\infty$. This scale of spaces includes weighted Bergman spaces: $b^p_\gamma(\Omega) = B^{p,p}_{(\gamma+1)/p}(\Omega)$, $\gamma > -1$, $0 < p < \infty$.

Also, we define a mixed norm space $\tilde{L}^{p,q}_\alpha(\Omega)$ as the space of all Lebesgue measurable functions on Ω, which vanish on Ω_ϵ, such that the integral in (1.2) is finite i.e. we have a (quasi) norm on this space

$$\|f\|_{\tilde{L}^{p,q}_\alpha} = \left\{ \int_0^\epsilon r^{\alpha q - 1} M_p^q(f, r) dr \right\}^{1/q}. \tag{1.3}$$

This space does not include $B^{p,q}_\alpha(\Omega)$, therefore we will consider a direct sum of that space with another one. For $0 < s < +\infty$, we denote by $\tilde{L}^s(\Omega)$ a space of Lebesgue measurable functions on Ω which vanish on $\Omega \setminus \Omega_\epsilon$, such that $\int_{\Omega_\epsilon} |f(x)|^s dm(x) < +\infty$. We set $\tilde{L}^{p,q,s}_\alpha(\Omega) = \tilde{L}^{p,q}_\alpha(\Omega) \oplus \tilde{L}^s(\Omega)$ which means that for every function $f \in \tilde{L}^{p,q,s}_\alpha(\Omega)$, we have $\|f\|_{\tilde{L}^{p,q,s}_\alpha} = \|f\chi_{\Omega\setminus\Omega_\epsilon}\|_{\tilde{L}^{p,q}_\alpha} + \|f\chi_{\Omega_\epsilon}\|_{\tilde{L}^s}$. Now we have $B^{p,q}_\alpha(\Omega) \subset \tilde{L}^{p,q,s}_\alpha(\Omega)$.

For $\gamma > -1$, let $R_\gamma(x, y)$ be the reproducing kernel of the harmonic Bergman space $b^2_\gamma(\Omega)$. For every function $f \in b^2_\gamma(\Omega)$ we have a reproducing formula

$$f(x) = \int_\Omega R_\gamma(x, y) f(y) dm_\gamma(y), \qquad x \in \Omega.$$

The kernel $R_\gamma(x, y)$ is symmetric and real-valued. The (weighted) Bergman projection P_γ is the orthogonal projection from $L^2_\gamma(\Omega)$ onto its subspace $b^2_\gamma(\Omega)$; it is given by the following integral formula

$$P_\gamma f(x) = \int_\Omega R_\gamma(x, y) f(y) dm_\gamma(y), \qquad x \in \Omega. \tag{1.4}$$

We will see in Section 3 that the Bergman projection is a bounded operator from $\tilde{L}^{p,q}_\alpha(\Omega)$ to $B^{p,q}_\alpha(\Omega)$, under certain conditions on parameters, and also it is a compact operator from $\tilde{L}^s(\Omega)$ to $B^{p,q}_\alpha(\Omega)$.

Throughout this paper, we will use the convention of using C to denote any positive constant which may change from one occurrence to the next. Given two positive quantities A and B, we write $A \asymp B$ if there are constants $0 < c \leq C < +\infty$ such that $cA \leq B \leq CA$.

2. Auxilary results

We begin with a proposition from [6] which is in fact a result from [2], adapted for our situation.

Proposition 2.1 *Let $\gamma > -1$. There is a positive constant $C = C_{\gamma,\Omega}$ such that $|R_\gamma(x,y)| \le C\frac{1}{D(x,y)^{n+\gamma}}$ and $|\frac{\partial R_\gamma(y,x)}{\partial y}| \le C\frac{1}{D(x,y)^{n+\gamma+1}}$. Moreover, for some constant $c > 0$ we have*

$$|R_\gamma(x,x)| \ge c\frac{1}{r(x)^{n+\gamma}}.$$

We also need the next lemma which essentially comes from [3].

Lemma 2.2 *For any $s > n-1$, there exists some C (depending on s) such that for all $x \in \Omega$ and $0 < r \le \epsilon$*

$$\int_{\Gamma_r} \frac{d\sigma_r(y)}{D(x,y)^s} \le \frac{C}{(r(x)+r)^{s-(n-1)}}$$

The next lemma gives an estimate for the integral means of $P_\gamma f$ in terms of integral means of f.

Lemma 2.3 *Let $\gamma > -1$, and $1 \le p \le \infty$ Then, for $f \in \tilde{L}^{p,q}_\alpha(\Omega)$ we have*

$$M_p(P_\gamma f, \rho) \le C\int_0^\epsilon \frac{r^\gamma}{(r+\rho)^{\gamma+1}} M_p(f,r)dr, \qquad 0 < \rho \le \epsilon.$$

Proof For $x \in \Gamma_\rho$ and $y \in \Gamma_r$ we have $r(x) = \rho$ and $r(y) = r$. Using definition of the Bergman projection P_γ and the estimate of the Bergman kernel R_γ, stated in Proposition 2.1, we obtain

$$\begin{aligned} |P_\gamma f(x)| &\le C\int_0^\epsilon \int_{\Gamma_r} |R_\gamma(x,y)||f(y)|d\sigma_r(y) r^\gamma dr \\ &\le C\int_0^\epsilon r^\gamma dr \int_{\Gamma_r} \frac{|f(y)|d\sigma_r(y)}{D(x,y)^{n+\gamma}} = C\int_0^\epsilon \tilde{f}(r,x) r^\gamma dr \end{aligned} \tag{2.1}$$

where $\tilde{f}(r,x) = \int_{\Gamma_r} \frac{|f(y)|d\sigma_r(y)}{D(x,y)^{n+\gamma}}$. If $1 < p < \infty$ using Hölder's inequality with conjugate exponents p and p' and Lemma 2.2 for $s = n+\gamma$, we obtain

$$\begin{aligned} \tilde{f}(r,x) &\le \left(\int_{\Gamma_r} \frac{d\sigma_r(y)}{D(x,y)^{n+\gamma}}\right)^{\frac{1}{p'}} \left(\int_{\Gamma_r} \frac{|f(y)|^p d\sigma_r(y)}{D(x,y)^{n+\gamma}}\right)^{\frac{1}{p}} \\ &\le \frac{C}{(r(x)+r)^{(\gamma+1)/p'}} \left(\int_{\Gamma_r} \frac{|f(y)|^p d\sigma_r(y)}{D(x,y)^{n+\gamma}}\right)^{\frac{1}{p}}. \end{aligned} \tag{2.2}$$

Integral Minkowski's inequality, above inequality, and Lemma 2.2 give us

$$
\begin{aligned}
M_p(P_\gamma f,\rho) &\le C\left(\int_{\Gamma_\rho}\left(\int_0^\epsilon \tilde{f}(r,x)r^\gamma dr\right)^p d\sigma_\rho(x)\right)^{\frac{1}{p}}\\
&\le C\int_0^\epsilon r^\gamma\left(\int_{\Gamma_\rho}\tilde{f}(r,x)^p d\sigma_\rho(x)\right)^{\frac{1}{p}} dr\\
&\le C\int_0^\epsilon r^\gamma\left(\int_{\Gamma_\rho}\left(\frac{1}{(\rho+r)^{(\gamma+1)\frac{p}{p'}}}\int_{\Gamma_r}\frac{|f(y)|^p d\sigma_r(y)}{D(x,y)^{n+\gamma}}\right)d\sigma_\rho(x)\right)^{\frac{1}{p}} dr\\
&= C\int_0^\epsilon \frac{r^\gamma}{(\rho+r)^{\frac{\gamma+1}{p'}}}\left(\int_{\Gamma_r}|f(y)|^p\int_{\Gamma_\rho}\frac{d\sigma_\rho(x)}{D(x,y)^{n+\gamma}}d\sigma_r(y)\right)^{1/p} dr\\
&\le C\int_0^\epsilon \frac{r^\gamma}{(\rho+r)^{\frac{\gamma+1}{p'}}}\left(\int_{\Gamma_r}\frac{|f(y)|^p}{(\rho+r)^{\gamma+1}}d\sigma_r(y)\right)^{1/p} dr\\
&= C\int_0^\epsilon \frac{r^\gamma}{(r+\rho)^{\gamma+1}}M_p(f,r)dr.
\end{aligned}
$$

If $p=1$ or $p=\infty$, the lemma follows from inequality (2.1) and Lemma 2.2. □

The following lemma is elementary, therefore the proof is omitted.

Lemma 2.4 *Let $s_1,\ s_2>0$ and $0<r\le\epsilon$. Then*

$$
\int_0^\epsilon \frac{\rho^{s_1-1}d\rho}{(r+\rho)^{s_2}} \le \begin{cases} C, & s_2<s_1\\ C\ln(1+\frac{\epsilon}{r}), & s_2=s_1\\ \frac{C}{r^{s_2-s_1}}, & s_2>s_1\end{cases}
$$

3. Main results

Theorem 3.1 *Let $\alpha>0$, $1\le p\le\infty$, $1\le q<\infty$. If $\gamma=p(\alpha-\frac{1}{q})>\alpha-1$ then the Bergman projection P_γ is a bounded operator from $\tilde{L}^{p,q}_\alpha(\Omega)$ to $B^{p,q}_\alpha(\Omega)$.*

Proof Let $\gamma_1,\gamma_2,\gamma_3$ and γ_4 be positive numbers such that $\gamma+1=\gamma_1+\gamma_2=\gamma_3+\gamma_4$, $\alpha+\gamma_1>\gamma_3>\gamma_1$ and $\gamma_2>\alpha$. For example, taking a sufficiently small $\eta>0$, and assuming $\gamma_1=\gamma+1-\alpha(1+\eta)$, $\gamma_2=(1+\eta)\alpha$, $\gamma_3=\gamma+1-(1-\eta)\alpha$ and $\gamma_4=(1-\eta)\alpha$, we see that $\gamma_1,\gamma_2,\gamma_3$ and γ_4 satisfy the above conditions. Let $1\le p\le\infty$ and $1<q<\infty$. First, we will prove

$$
M_p(P_\gamma f,\rho)\le\frac{C}{\rho^{\gamma_3-\gamma_1}}\left(\int_0^\epsilon\frac{r^{q\gamma_2-1}}{(r+\rho)^{q\gamma_4}}M_p^q(f,r)dr\right)^{\frac{1}{q}}.
$$

Namely, using Lemma 2.3, Hölder's inequality with conjugate exponents q and q' and with respect to measure

$\frac{dr}{r}$ and Lemma 2.4 with $s_1 = \gamma_1 q'$ and $s_2 = \gamma_3 q'$, we obtain

$$
\begin{aligned}
M_p(P_\gamma f, \rho) &\le C \int_0^\epsilon \frac{r^{\gamma_1+\gamma_2}}{(r+\rho)^{\gamma_3+\gamma_4}} M_p(f,r) \frac{dr}{r} \\
&\le C \left(\int_0^\epsilon \frac{r^{\gamma_2 q}}{(r+\rho)^{\gamma_4 q}} M_p^q(f,r) \frac{dr}{r} \right)^{\frac{1}{q}} \left(\int_0^\epsilon \frac{r^{\gamma_1 q'}}{(r+\rho)^{\gamma_3 q'}} \frac{dr}{r} \right)^{\frac{1}{q'}} \\
&\le C \left(\frac{1}{\rho^{(\gamma_3-\gamma_1)q'}} \right)^{\frac{1}{q'}} \left(\int_0^\epsilon \frac{r^{\gamma_2 q}}{(r+\rho)^{\gamma_4 q}} M_p^q(f,r) \frac{dr}{r} \right)^{\frac{1}{q}}.
\end{aligned}
$$

Thus, using above inequality, properties of numbers $\gamma_1, \gamma_2, \gamma_3$ and γ_4 , Fubini theorem and Lemma 2.4 with $s_2 = q\gamma_4$ and $s_1 = q(\alpha - \gamma_3 + \gamma_1)$, when $1 \le p \le \infty$ and $1 < q < \infty$, we have

$$
\begin{aligned}
\|P_\gamma f\|^q_{B^{p,q}_\alpha} &= \int_0^\epsilon \rho^{\alpha q - 1} M_p^q(P_\gamma f, \rho) d\rho \\
&\le C \int_0^\epsilon \frac{\rho^{\alpha q-1}}{\rho^{q(\gamma_3-\gamma_1)}} \left(\int_0^\epsilon \frac{r^{q\gamma_2 - 1}}{(r+\rho)^{q\gamma_4}} M_p^q(f,r) \right) d\rho \\
&= C \int_0^\epsilon \left(\int_0^\epsilon \frac{\rho^{q(\alpha-\gamma_3+\gamma_1)-1} d\rho}{(r+\rho)^{q\gamma_4}} \right) r^{q\gamma_2-1} M_p^q(f,r) dr \\
&\le C \int_0^\epsilon \frac{r^{q\gamma_2-1}}{r^{q\gamma_4 - q(\alpha-\gamma_3+\gamma_1)}} M_p^q(f,r) dr \\
&= C \int_0^\epsilon r^{q(\gamma_2-\gamma_4)+q(\gamma_1-\gamma_3)} r^{\alpha q-1} M_p^q(f,r) dr \\
&= C \int_0^\epsilon r^{\alpha q-1} M_p^q(f,r) dr = \|f\|_{\tilde{L}^{p,q}_\alpha}.
\end{aligned}
$$

When $q = 1$, the result follows directly from Lemma 2.3 and Lemma 2.4. □

Notice that in previous theorem operator P_γ is not a projection of space $\tilde{L}^{p,q}_\alpha(\Omega)$ to $B^{p,q}_\alpha(\Omega)$, since the space $B^{p,q}_\alpha(\Omega)$ is not a subspace of $\tilde{L}^{p,q}_\alpha(\Omega)$. Namely, harmonic functions which vanish on open subset Ω_ϵ of Ω, would vanish on Ω, according to the uniqueness theorem for harmonic functions. On the other hand, as we noticed in the introduction, the space $B^{p,q}_\alpha(\Omega)$ is a subspace of $\tilde{L}^{p,q,s}_\alpha(\Omega)$, and our goal is to obtain projection operator to $B^{p,q}_\alpha(\Omega)$. Intending to produce that result, we formulate the next proposition.

Proposition 3.2 *Let* $\alpha > 0$, $1 \le p, s \le +\infty$, $1 \le q < +\infty$. *If* $\gamma = p(\alpha - \frac{1}{q})$ *then the Bergman projection* $P_\gamma : \tilde{L}^s(\Omega) \to B^{p,q}_\alpha(\Omega)$ *is a compact linear operator.*

Proof First, we will prove that $P_\gamma : L^1(\Omega) \to C(\overline{\Omega}) \cap h(\Omega)$ is a compact operator. Let $f \in L^1(\Omega)$ such that

$\|f\|_{L^1(\Omega)} \leq 1$ and $u(x) = P_\gamma f(x)$. Then we have

$$
\begin{aligned}
|u(x)| =& |P_\gamma f(x)| = \left| \int_{\Omega_\epsilon} R_\gamma(x,y) f(y) dm_\gamma(y) \right| \\
&\leq \sup_{(x,y)\in\Omega\times\Omega_\epsilon} |R_\gamma(x,y)| r(y)^\gamma m(\Omega_\epsilon) \\
&\leq C_\gamma \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}_\epsilon} |R_\gamma(x,y)|
\end{aligned}
$$

We have that $R_\gamma(x,y) \in C^\infty(\overline{\Omega}\times\overline{\Omega} \setminus \{(\zeta,\zeta) : \zeta \in \partial\Omega\})$ (see [2]) and $\overline{\Omega}\times\overline{\Omega_\epsilon} \subset \overline{\Omega}\times\overline{\Omega}\setminus\{(\zeta,\zeta) : \zeta\in\partial\Omega\}$, so $\sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}_\epsilon}|R_\gamma(x,y)|$ is a constant depending on ϵ and we denote it by C_ϵ. Hence, we obtain $|u(x)| \leq C_\gamma C_\epsilon$ i.e. the set $K = P_\gamma(B)$ is a bounded subset of $C(\overline{\Omega})$, where $B = \{f \in L^1(\Omega) : \|f\|_{L^1(\Omega)} \leq 1\}$.

Let us prove that the family $K \subset C(\overline{\Omega})$ is equicontinuous. For the gradient of the Bergman projection of function f we have

$$|\nabla u(x)| \leq C_\gamma \sup_{(x,y)\in\Omega\times\Omega_\epsilon} |\nabla_x R_\gamma(x,y)|.$$

Again, using that $R_\gamma(x,y) \in C^\infty(\overline{\Omega}\times\overline{\Omega}\setminus\{(\zeta,\zeta) : \zeta\in\partial\Omega\})$, we obtain the desired equicontinuity.

Hence, by the Arzelà-Ascoli theorem, P_γ is compact operator from $L^1(\Omega)$ to $C(\overline{\Omega}) \cap h(\Omega)$. Since $s \geq 1$, the space $\tilde{L}^s(\Omega)$ is continuously embedded into $L^1(\Omega)$. Beside that, $C(\overline{\Omega}) \cap h(\Omega)$ is continuously embedded into $B^{p,q}_\alpha(\Omega)$, so $P_\gamma : \tilde{L}^s(\Omega) \to B^{p,q}_\alpha(\Omega)$ is a compact operator. □

Remark 3.3 *Note that* $P_\gamma : L^1(\Omega) \to C^k(\overline{\Omega}) \cap h(\Omega)$ *is compact, for any* $k \in \mathbb{N}$, *since* $\nabla_x^{k+1} R_\gamma(x,y) \in C^\infty(\overline{\Omega}\times\overline{\Omega_\epsilon})$. *We need only the case* $k = 0$.

We obtain the desired result by combining Theorem 3.1 and Proposition 3.2, which we formulate in the next theorem.

Theorem 3.4 *Let* $\alpha > 0$, $1 \leq p, s \leq \infty$, $1 \leq q < \infty$. *If* $\gamma = p(\alpha - \frac{1}{q}) > \alpha - 1$ *then the Bergman projection* P_γ *is a bounded projection from* $\tilde{L}^{p,q,s}_\alpha(\Omega)$ *to* $B^{p,q}_\alpha(\Omega)$.

Acknowledgment

The author would like to thank Professor Miloš Arsenović for pointing out the compactness of the operator in Proposition 3.2 which leads to our main goal, and other valuable suggestions during the writing of this paper. The author is also grateful to the anonymous referee for the useful suggestions.

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