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The normalized Miller-Ross function and its geometric properties

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Abstract: The main objective of this paper is to study certain geometric properties (like univalence, starlikeness, convexity, close-to-convexity) for the normalized Miller-Ross function. The various results, which we have established in the present investigation, are believed to be new, and their importance is illustrated by several interesting consequences and examples. Furthermore, some of the main results improve the corresponding results available in the literature [15].

Key words: Miller-Ross function, analytic function, univalent function, starlike function, close-to-convex function

1. Introduction

Geometric Function Theory is an important branch of complex analysis, it deals with the geometric properties of analytic functions. This branch of complex analysis is widely used in a variety of fields of mathematics, namely in pure and applied mathematics. In the literature, several researchers have studied certain geometric properties for some special classes of univalent functions such as problems for studying the geometric properties (including univalency, starlikeness, or convexity) of some classes of analytic functions (in the unit disk) associated with some special functions have always been attracted by several researchers. Regarding treatises on this investigation, we refer, e.g., to [20] for the Fox-Wright function, to [12, 19, 24, 25] for the Mittag-Leffler, to [18, 19, 26] for the Wright function, to [11, 13] to Dini functions, to [21] for the modified Bessel function, to [15] for the Miller-Ross functions and to [3, 4, 7–9, 23, 29, 31] for some class of functions related to the Bessel function and its q-analog. In a series of papers [32], the authors have determined sufficient conditions on the parameters of some other special functions to belong to a certain class of univalent functions, such as convex, starlike, close-to-convex, etc. Let us now recall some known definitions and results in Geometric Function Theory.

Let $\mathcal{H}$ denote the class of all analytic functions inside the unit disk

$$\mathcal{D}=\Big\{z: z\in\mathbb{C} \text{ and } |z|<1\Big\}.$$

Assume that $\mathcal{A}$ denoted the collection of all functions $f\in\mathcal{H}$, satisfying the normalization $f(0)=f'(0)-1=0$ such that

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \qquad (\forall z\in\mathcal{D}).$$

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A function $f \in \mathcal{A}$ is said to be a starlike function (with respect to the origin 0) in $\mathcal{D}$, if f is univalent in $\mathcal{D}$ and $f(\mathcal{D})$ is a starlike domain with respect to 0 in $\mathbb{C}$. This class of starlike functions is denoted by $\mathcal{S}^*$. The analytic characterization of $\mathcal{S}^*$ is given [14] below:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (\forall z \in \mathcal{D}) \quad \Longleftrightarrow \quad f \in \mathcal{S}^*.$$

If $f(z)$ is a univalent function in $\mathcal{D}$ and $f(\mathcal{D})$ is a convex domain in $\mathbb{C}$, then $f \in \mathcal{A}$ is said to be a convex function in $\mathcal{D}$. We denote this class of convex functions by $\mathcal{K}$, which can also be described as follows:

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > 0, \; (\forall z \in \mathcal{D}) \quad \Longleftrightarrow \quad f \in \mathcal{K}.$$

Then, an analytic function f is convex, if and only if, the function zf' is starlike. An analytic function f in $\mathcal{A}$ is called close-to-convex in the open unit disk $\mathcal{D}$ if there exists a function $g(z)$, which is starlike in $\mathcal{D}$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0 \quad (\forall z \in \mathcal{D}).$$

The class of all close-to-convex functions is denoted by $\mathcal{C}$. It can be easily verified that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C}$. It can be noted that every close-to-convex function in $\mathcal{D}$ is also univalent in $\mathcal{D}$.

A function $f \in \mathcal{A}$ is said to be uniformly convex (starlike) if for every circular arc γ contained in $\mathcal{D}$ with center $\zeta \in \mathcal{D}$ the image arc $f(\gamma)$ is convex (starlike w.r.t. the image $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by UCV (or UST) (see, for details,[28]). In addition, Ronning [28] considered a newly-defined class of starlike functions $\mathcal{S}_p$ as follows:

$$\mathcal{S}_p := \{f : f(z) = zg'(z) \;\; (g \in UCF)\}.$$

Here, and in what follows, we use $E_{\nu,c}(z)$ to denote the Miller-Ross function which are defined by [22, p. 88]:

$$E_{\nu,c}(z) = z^{\nu}\sum_{k=0}^{\infty}\frac{(cz)^k}{\Gamma(\nu+k+1)}, \;\; (\nu > -1, c, z \in \mathbb{C}). \tag{1.1}$$

The main purpose of this paper is to investigate certain geometric properties for the normalized form of the Miller-Ross function defined by [15, Eq. (2)]

$$\mathbb{E}_{\nu,c}(z) = z + \sum_{k=2}^{\infty}\frac{\Gamma(\nu+1)c^{k-1}z^k}{\Gamma(\nu+k)}, \;\; (\nu > -1, c, z \in \mathbb{C}). \tag{1.2}$$

Although formula (1.2) holds true for $c, z \in \mathbb{C}$ and $\nu > -1$, yet in this article, we will restrict our attention to the case involving positive real-valued parameter $c > 0, \nu > -1$ and the argument $z \in \mathcal{D}$.

In order to prove our results the following lemmas will be helpful. The first and second Lemmas is due to T. H. MacGregor [16, 17].

Lemma 1.1 **([16])** *Let $f \in \mathcal{A}$ and $|f(z)/z - 1| < 1$ for each $z \in \mathcal{D}$, then f is univalent and starlike in*

$$\mathcal{D}_{\frac{1}{2}} = \left\{z : z \in \mathbb{C} \text{ and } |z| < \frac{1}{2}\right\}.$$

Lemma 1.2 (**[17]**) *Let $f \in \mathcal{A}$ and $|f'(z)-1|<1$ for each $z \in \mathcal{D}$, then f is convex in $\mathcal{D}_{\frac{1}{2}}$.*

The well-known analytical characterization of the class UCV and $\mathcal{S}_p$ is given by the following Lemma:

Lemma 1.3 [27] *Assume that $f \in \mathcal{A}$. Then the following results hold true:*

1. *If* $\left|\dfrac{zf''(z)}{f'(z)}\right| < \dfrac{1}{2}$, *then* $f \in UCV$.

2. *If* $\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2}$, *then* $f \in \mathcal{S}_p$.

2. Main results

The following Theorem asserts our first major finding.

Theorem 2.1 *Let $c > \frac{1}{2}$ and $\nu > -1$. If $\nu > \max(2c-1+\sqrt{2c^2+1}, 1+\sqrt{2})$, then the function $\mathbb{E}_{\nu,c}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}_{\nu,1}(z)$ in $\mathcal{D}$.*

Proof By means of [15, Theorem 3], the function $\mathbb{E}_{\nu,1}(z)$ is starlike in $\mathcal{D}$ under the given condition $\nu > 1+\sqrt{2}$. By the definition, to prove that the function $\mathbb{E}_{\nu,c}(z)$ is close-to-convex with respect to the starlike function $\mathbb{E}_{\nu,1}(z)$ in $\mathcal{D}$, it is enough to show that

$$\Re\left(\frac{z\mathbb{E}'_{\nu,c}(z)}{\mathbb{E}_{\nu,1}(z)}\right) > 0, \; z \in \mathcal{D},$$

which can be proved by showing

$$\left|\frac{z\mathbb{E}'_{\nu,c}(z)}{\mathbb{E}_{\nu,1}(z)} - 1\right| < 1, \; z \in \mathcal{D}.$$

For all $z \in \mathcal{D}$, we have

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z) - \frac{\mathbb{E}_{\nu,1}(z)}{z}\right| &= \left|\sum_{k=2}^{\infty} \frac{\Gamma(\nu+1)[kc^{k-1}-1]z^{k-1}}{\Gamma(\nu+k)}\right| \\ &< \left|\sum_{k=2}^{\infty} \frac{\Gamma(\nu+1)[kc^{k-1}-1]}{\Gamma(\nu+k)}\right|. \end{aligned} \tag{2.1}$$

By taking into account the obvious inequality:

$$\frac{\Gamma(\nu+1)}{\Gamma(\nu+k)} \leq \frac{1}{(\nu+1)^{k-1}}, \; (k \in \mathbb{N}, \nu > -1), \tag{2.2}$$

we thus get

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z) - \frac{\mathbb{E}_{\nu,1}(z)}{z}\right| &< \sum_{k=2}^{\infty} \frac{[kc^{k-1}-1]}{(\nu+1)^{k-1}} \\ &= \sum_{k=2}^{\infty} k\left(\frac{c}{\nu+1}\right)^{k-1} - \sum_{k=2}^{\infty} \frac{1}{(\nu+1)^{k-1}} \\ &= \frac{(2c-1)\nu^2 - (c^2-4c+2)\nu - (c-1)^2}{\nu(\nu+1-c)^2}. \end{aligned} \tag{2.3}$$

The above expression is positive if $\nu > \frac{2c^2-4c+2}{2(2c-1)}$. However, with the help of inequality [15, Eq. (8)], we have

$$\left|\frac{\mathbb{E}_{\nu,1}(z)}{z}\right| > \frac{\nu-1}{\nu}. \tag{2.4}$$

Having (2.3) and (2.4) in mind, for any $z \in \mathcal{D}$, we get

$$\left|\frac{z\mathbb{E}'_{\nu,c}(z)}{\mathbb{E}_{\nu,1}(z)} - 1\right| < \frac{(\nu-1)[(2c-1)\nu^2 - (c^2-4c+2)\nu - (c-1)^2]}{(\nu+1-c)^2}. \tag{2.5}$$

So, the above expression is less than 1, if and only if

$$\nu^2 + 2(1-2c)\nu + 2c(c-2) > 0,$$

which holds true if $\nu > 2c - 1 + \sqrt{2c^2+1}$. Therefore,

$$\nu > \max\left(2c-1+\sqrt{2c^2+1}, \frac{2c^2-4c+2}{2(2c-1)}, c-1\right) = 2c-1+\sqrt{2c^2+1}.$$

Hence, the proof of Theorem 2.1 is complete. □

Theorem 2.2 *Let $c > 0$ and $\nu > -1$. If $\nu > 2(ce^c - 1)$, then the function $\mathbb{E}_{\nu,c}(z)$ is starlike in $\mathcal{D}$.*

Proof To show that the function $\mathbb{E}_{\nu,c}(z)$ is starlike in $\mathcal{D}$, it is enough to prove that

$$\left|\frac{z\mathbb{E}'_{\nu,c}(z)}{\mathbb{E}_{\nu,c}(z)} - 1\right| < 1, \; z \in \mathcal{D}.$$

By (1.2) and used the functional relation $\Gamma(z+1) = z\Gamma(z)$, for any $z \in \mathcal{D}$, we obtain

$$\begin{aligned}\left|\mathbb{E}'_{\nu,c}(z) - \frac{\mathbb{E}_{\nu,c}(z)}{z}\right| &< \sum_{k=2}^{\infty} \frac{(k-1)\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)} \\ &= \sum_{k=2}^{\infty} \frac{\Gamma(\nu+1)\Gamma(k)c^{k-1}}{\Gamma(\nu+k)(k-2)!}.\end{aligned} \tag{2.6}$$

Due to the log-convexity property of the Gamma function $\Gamma(z)$, the ratio $z \mapsto \frac{\Gamma(z)}{\Gamma(z+a)}$ is decreasing on $(0,\infty)$ for each $a > 0$. This in turn implies that the following inequality

$$\frac{\Gamma(\nu+1)\Gamma(k)}{\Gamma(\nu+k)} \leq \frac{1}{\nu+1}, \tag{2.7}$$

holds true for all $\nu \geq 0$ and $k \geq 2$. Bearing in mind the above formula and (2.6) we find that

$$\left|\mathbb{E}'_{\nu,c}(z) - \frac{\mathbb{E}_{\nu,c}(z)}{z}\right| < \frac{ce^c}{\nu+1}, \; z \in \mathcal{D}. \tag{2.8}$$

However, by using the triangle inequality $|a+b|>||a|-|b||$ and the inequality (2.7), for all $z\in\mathcal{D}$, we get

$$\begin{aligned}\left|\frac{\mathbb{E}_{\nu,c}(z)}{z}\right| &> 1-\sum_{k=2}^{\infty}\frac{\Gamma(\nu+1)\Gamma(k)c^{k-1}}{\Gamma(\nu+k)(k-1)!}\\ &> 1-\sum_{k=2}^{\infty}\frac{c^{k-1}}{(\nu+1)(k-1)!}\\ &= \frac{\nu+2-ce^{c}}{\nu+1}.\end{aligned} \tag{2.9}$$

Furthermore, with the help of the above inequality and (2.8), for any $z\in\mathcal{D}$, we obtain

$$\left|\frac{z\mathbb{E}'_{\nu,c}(z)}{\mathbb{E}_{\nu,c}(z)}-1\right|<\frac{ce^{c}}{\nu+2-ce^{c}}. \tag{2.10}$$

The above inequality needs to be less than 1, which is equivalent to condition $\nu>2ce^{c}-2$. Thus, the proof is complete. □

Specifying $c=\frac{1}{2}$ in Theorem 2.2, we conclude that the following result reads as follows.

Corollary 2.3 *If $\nu>e^{\frac{1}{2}}-2\approx-0.351278$, the function $\mathbb{E}_{\nu,\frac{1}{2}}(z)$ is starlike in $\mathcal{D}$.*

Example 2.4 *The function $\mathbb{E}_{-\frac{1}{3},\frac{1}{2}}(z)$ is starlike in $\mathcal{D}$.*

Remark 2.5 *In* [15, Theorem 3]*, it was proved that the function $\mathbb{E}_{\nu,c}(z)$ is starlike in $\mathcal{D}$, if $\nu>(2+\sqrt{2})c-1$. In particular, the function $\mathbb{E}_{\nu,\frac{1}{2}}(z)$ is starlike in $\mathcal{D}$, if $\nu>\frac{\sqrt{2}}{2}\approx0.707106\cdots$. In view of the Corollary 2.3, Theorem 2.2 improves the corresponding result available in the literature* [15, Theorem 3].

Upon setting $c=\frac{2}{5}$ in Theorem 2.2, we compute the following result.

Corollary 2.6 *If $\nu>\frac{4e^{\frac{2}{5}}-10}{5}\approx-0.806540\cdots$, the function $\mathbb{E}_{\nu,\frac{2}{5}}(z)$ is starlike in $\mathcal{D}$.*

Example 2.7 *The function $\mathbb{E}_{-\frac{8}{10},\frac{2}{5}}(z)$ is starlike in $\mathcal{D}$.*

Remark 2.8 *The following are the graphs of the functions $\mathbb{E}_{-\frac{1}{3},\frac{1}{2}}(z)$ and $\mathbb{E}_{-\frac{8}{10},\frac{2}{5}}(z)$ over $\mathcal{D}$. The Figure 1 and Figure 2 depict the validity of our results.*

Remark 2.9 *In* [15, Theorem 2]*, it is established that the function $\mathbb{E}_{\nu,c}(z)$ is starlike in $\mathcal{D}$, if $\nu>\frac{4c-3+\sqrt{4c^2+8c+1}}{2}$. In particular, the function $\mathbb{E}_{\nu,\frac{2}{5}}(z)$ is starlike in $\mathcal{D}$, if $\nu>0.4$. In view of the Corollary 2.6, we conclude that the result asserted by Theorem 2.2 improves the corresponding results available in* [15, Theorem 2].

Proceeding in a similar way and using part (b) of Lemma 1.3, we obtain the following result.

Theorem 2.10 *Let $c>0$ and $\nu\geq0$. If $\nu>3ce^{c}-1$, then the function $\mathbb{E}_{\nu,c}(z)$ belongs to the class $\mathcal{S}_p$ in $\mathcal{D}$.*

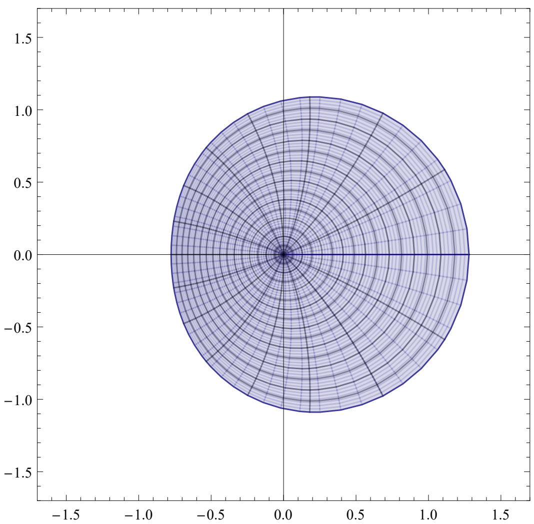

Figure 1. Mapping of $\mathbb{E}_{-\frac{1}{3},\frac{1}{2}}(z)$ over $\mathcal{D}$

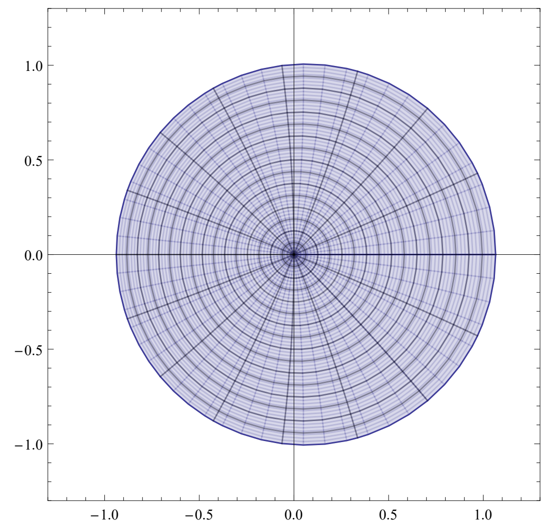

Figure 2. Mapping of $\mathbb{F}_{2,\frac{5}{2}}(z)$ over $\mathcal{D}$

In the next theorem, sufficient conditions are imposed on the parameters of the Miller-Ross function which allow us to conclude that it is convex in $\mathcal{D}$.

Theorem 2.11 *Let $c > 0$. If $\nu > \max(2(c+1)e^c - 3, 1)$, then the function $\mathbb{E}_{\nu,c}(z)$ is convex in $\mathcal{D}$.*

Proof To prove that the function $\mathbb{E}_{\nu,c}(z)$ is convex in $\mathcal{D}$ it suffices to prove that the function $\mathbb{F}_{\nu,c}(z) = z\mathbb{E}'_{\nu,c}(z)$ is starlike in $\mathcal{D}$. Straightforward calculation would yield

$$\left|\mathbb{F}'_{\nu,c}(z) - \frac{\mathbb{F}_{\nu,c}(z)}{z}\right| < \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)(k-2)!},\ z \in \mathcal{D}. \tag{2.11}$$

Once more, by using the monotonicity criterion of the ratio $z \mapsto \frac{\Gamma(z+a)}{\Gamma(z+b)}$, where $b \geq a > 0$ we obtain

$$\frac{\Gamma(k+1)\Gamma(\nu+1)}{\Gamma(\nu+k)} \leq \frac{2}{\nu+1},\ \nu \geq 1, k \geq 2. \tag{2.12}$$

Now, with the help of (2.11) and (2.12), for all $z \in \mathcal{D}$, we find

$$\left|\mathbb{F}'_{\nu,c}(z) - \frac{\mathbb{F}_{\nu,c}(z)}{z}\right| < \frac{2ce^c}{\nu+1}. \tag{2.13}$$

Again, by (2.12), for any $z \in \mathcal{D}$, we have

$$\begin{aligned}\left|\frac{\mathbb{F}_{\nu,c}(z)}{z}\right| &> 1 - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)(k-1)!} \\ &> 1 - \frac{2}{\nu+1}\sum_{k=2}^{\infty} \frac{c^{k-1}}{(k-1)!} \\ &= \frac{\nu+3-2e^c}{\nu+1} > 0,\end{aligned} \tag{2.14}$$

under the given hypothesis. Having in mind (2.13) and (2.14), for all $z \in \mathcal{D}$, we obtain

$$\left|\frac{z\mathbb{F}'_{\nu,c}(z)}{\mathbb{F}_{\nu,c}(z)} - 1\right| < \frac{2ce^c}{\nu + 3 - 2e^c},$$

and the last expression is less than 1 by our assumption. The proof of Theorem 2.11 is thus completed. □

Setting $c = \frac{2}{5}$ in Theorem 2.11, we conclude the following result.

Corollary 2.12 *If* $\nu > 1.177110$, *then the function* $\mathbb{E}_{\nu,\frac{2}{5}}(z)$ *is convex on* $\mathcal{D}$.

Example 2.13 *The function* $\mathbb{E}_{\frac{6}{5},\frac{2}{5}}(z)$ *is convex on* $\mathcal{D}$.

Theorem 2.14 *Let* $c > 0$. *If* $\nu > e^c - 2$, *then the function* $\mathbb{E}_{\nu,c}(z)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Proof By using (1.2), for any $z \in \mathcal{D}$, it follows that

$$\left|\frac{\mathbb{E}_{\nu,c}(z)}{z} - 1\right| < \sum_{k=2}^{\infty} \frac{\Gamma(\nu+1)\Gamma(k)}{\Gamma(\nu+k)} \frac{c^{k-1!}}{(k-1)!}. \tag{2.15}$$

Since the function $z \mapsto \frac{\Gamma(z)}{\Gamma(z+a)}$ is decreasing on $(0, \infty)$ for all $a > 0$, we have

$$\frac{\Gamma(k)\Gamma(\nu+1)}{\Gamma(\nu+k)} \leq \frac{1}{\nu+1}, \; (k \geq 2, \nu > 0).$$

The above inequality combined with the inequality (2.15) gives

$$\begin{aligned}\left|\frac{\mathbb{E}_{\nu,c}(z)}{z} - 1\right| &< \frac{1}{\nu+1} \sum_{k=2}^{\infty} \frac{c^{k-1}}{(k-1)!} \\ &= \frac{e^c - 1}{\nu+1}.\end{aligned} \tag{2.16}$$

Finally, with the help of Lemma 1.1 and the conditions imposed on the parameters ν and c we obtain the desired result. Hence, the proof is complete. □

Upon setting $\nu = 1$ in Theorem 2.14, we get the following result.

Corollary 2.15 *If* $0 < c < \log(3)$, *then the function* $\mathbb{E}_{1,c}(z)$ *is stralike in* $\mathcal{D}_{\frac{1}{2}}$.

Example 2.16 *The function* $\mathbb{E}_{1,\frac{12}{11}}(z)$ *is starlike in* $\mathcal{D}_{\frac{1}{2}}$.

Remark 2.17 *In* [15, Theorem 5], *it was derived that the function* $\mathbb{E}_{\nu,c}(z)$ *is stralike in* $\mathcal{D}_{\frac{1}{2}}$ *if* $\nu > 2c - 1$. *In particular, the function* $\mathbb{E}_{1,c}(z)$ *is stralike in* $\mathcal{D}_{\frac{1}{2}}$ *if* $0 < c < 1$. *Consequently, in view of Corollary 2.15, Theorem 2.14 improves the corresponding results available in* [15, Theorem 5].

Theorem 2.18 *Let $\nu \geq 1$ and $c > 0$. Assume that one of the following hypotheses holds true:*
(a). The parameters ν and c satisfy $\nu > 2e^c - 3$,
(b). The parameters ν and c satisfy $c < 1$ and $\nu > \frac{2c-1}{1-c}$.
Then the function $\mathbb{E}_{\nu,c}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$.

Proof (a). Let $z \in \mathcal{D}$. A simple computation leads us to

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z) - 1\right| &< \sum_{k=2}^{\infty} \frac{k\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)} \\ &= \sum_{k=2}^{\infty} \frac{\Gamma(\nu+1)\Gamma(k+1)c^{k-1}}{\Gamma(\nu+k)(k-1)!}. \end{aligned} \tag{2.17}$$

Bearing in mind the above formula and (2.12) we find that

$$\left|\mathbb{E}'_{\nu,c}(z) - 1\right| < \frac{2(e^c-1)}{\nu+1}, \ z \in \mathcal{D}. \tag{2.18}$$

Therefore, under the given conditions imposed on the parameters ν and c and with the aid of Lemma 1.2 we confirm the desired result.
(b). Thanking to the following inequality [30, Lemma 7, Eq. (10)]

$$\frac{k\Gamma(a+1)}{\Gamma(a+k)} \leq \frac{1}{(a+1)^{k-2}}, \ (a \geq 1, k \in \mathbb{N} \setminus \{1\}), \tag{2.19}$$

combined with (2.17), for any $z \in \mathcal{D}$, we get

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z) - 1\right| &< \sum_{k=2}^{\infty} \frac{c^{k-1}}{(\nu+1)^{k-2}} \\ &= \frac{c(\nu+1)}{\nu+1-c}, \end{aligned} \tag{2.20}$$

since $\nu \geq \frac{2c-1}{1-c} > c-1$. Again, by using Lemma 1.2, the desired result can be established. □

Taking $\nu = 1$ in part (a) of Theorem 2.18, we compute the following result.

Corollary 2.19 *If $0 < c < \log(2) \approx 0.693\cdots$, then the function $\mathbb{E}_{1,c}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$*

Example 2.20 *The function $\mathbb{E}_{1,\frac{2}{3}}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$.*

Letting $c = \frac{2}{3}$ in part (b) of Theorem 2.18, we get the following result.

Corollary 2.21 *If $\nu \geq 1$ then the function $\mathbb{E}_{\nu,\frac{2}{3}}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$.*

Remark 2.22 *Let us mention that Eker et al.* [15, Theorem 6]*, derive that the function $\mathbb{E}_{\nu,c}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$ if $\nu > (2+\sqrt{2})c-1$. However, the function $\mathbb{E}_{1,c}(z)$ (resp. $\mathbb{E}_{\nu,\frac{2}{3}}(z)$ is convex in $\mathcal{D}_{\frac{1}{2}}$ if $0 < c < \frac{2}{2+\sqrt{2}} \approx 0.585\cdots$ (resp. $\nu > 1.276142\cdots$). Hence, in view of Corollary 2.19 and Corollary 2.21, Theorem 2.18 improves the corresponding result established in* [15, Theorem 6].

Theorem 2.23 *Consider that one of the following assertions is valid:*
(a). The parameters $c>0$ *and* $\nu \geq 1$ *such that* $\nu > (2+4c)e^c - 3$.
(b). The parameters $c>0$ *and* $\nu > (2+\sqrt{2})c - 1$ *satisfy the following conditions*

$$(\nu+1)^3 - 9c(\nu+1)^2 + 6c^2(\nu+1) - 2c^3 > 0.$$

Then, the function $\mathbb{E}_{\nu,c}(z)$ *is uniformly convex in* $\mathcal{D}$.

Proof (a). For any $z \in \mathcal{D}$, we obtain

$$\begin{aligned} \left|z\mathbb{E}''_{\nu,c}(z)\right| &< \sum_{k=2}^{\infty} \frac{k(k-1)\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)} \\ &= \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(\nu+1)c^{k-1}}{(k-2)!\Gamma(\nu+k)}. \end{aligned} \tag{2.21}$$

Therefore, by combining (2.21) and (2.12), for any $z \in \mathcal{D}$, it follows that

$$\left|z\mathbb{E}''_{\nu,c}(z)\right| < \frac{2ce^c}{\nu+1}, \quad (\nu \geq 1, c > 0). \tag{2.22}$$

Furthermore, with the help of the inequality (2.12), we get

$$\begin{aligned} \left|\mathbb{E}'_{\nu,c}(z)\right| &> 1 - \sum_{k=2}^{\infty} \frac{k\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)} \\ &= 1 - \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(\nu+1)c^{k-1}}{(k-1)!\Gamma(\nu+k)} \\ &> \frac{\nu+3-2e^c}{\nu+1}, \quad (z \in \mathcal{D}). \end{aligned} \tag{2.23}$$

Hence, in view of the above inequality and (2.22), for any $z \in \mathcal{D}$, we have

$$\left|\frac{z\mathbb{E}''_{\nu,c}(z)}{\mathbb{E}'_{\nu,c}(z)}\right| < \frac{2ce^c}{\nu+3-2e^c} < \frac{1}{2},$$

where we have made use of the given hypothesis. Hence, Lemma 1.3 helps us to establish the desired result.
(b). With the help of the inequality (2.2), for any $z \in \mathcal{D}$, we find that

$$\begin{aligned} \left|z\mathbb{E}''_{\nu,c}(z)\right| &< \sum_{k=2}^{\infty} \frac{k(k-1)\Gamma(\nu+1)c^{k-1}}{\Gamma(\nu+k)} \\ &< \sum_{k=2}^{\infty} \frac{k(k-1)c^{k-1}}{(\nu+1)^{k-1}} \\ &= \frac{2c(\nu+1)^2}{(\nu+1-c)^3}. \end{aligned} \tag{2.24}$$

A simple computation leads us to

$$\begin{aligned}\left|\mathbb{E}'_{\nu,c}(z)\right| &> 1-\sum_{k=2}^{\infty} k\left(\frac{c}{\nu+1}\right)^{k-1}\\ &= \frac{(\nu+1-c)^2-2c(\nu+1)+c^2}{(\nu+1-c)^2},\end{aligned}\tag{2.25}$$

and the last expression is positive by our assumption. Keeping (2.24) and (2.25) in mind, for any $z\in\mathcal{D}$, we obtain

$$\left|\frac{z\mathbb{E}''_{\nu,c}(z)}{\mathbb{E}'_{\nu,c}(z)}\right| < \frac{2c(\nu+1)^2}{(\nu+1-c)[(\nu+1-c)^2-2c(\nu+1)+c^2]}.\tag{2.26}$$

Therefore, the last expression is less than $\frac{1}{2}$ if and only if

$$F_c(x) := x^3-9cx^2+6c^2x-2c^3>0,\ (\min(x,c)>0).$$

The proof of Theorem 2.23 is thus completed. □

Putting $c=1$ in Part (b) of Theorem 2.23, we obtain the following result as follows:

Corollary 2.24 *If $\nu\geq 7.306675$, then, the function $\mathbb{E}_{\nu,1}(z)$ is uniformly convex in $\mathcal{D}$.*

Example 2.25 *The function $\mathbb{E}_{\frac{22}{3},1}(z)$ is uniformly convex in $\mathcal{D}$.*

Setting $c=\frac{1}{2}$ in Part (b) of Theorem 2.23, we compute the following result as follows:

Corollary 2.26 *If $\nu\geq 3.153338$, then, the function $\mathbb{E}_{\nu,\frac{1}{2}}(z)$ is uniformly convex in $\mathcal{D}$.*

Example 2.27 *The function $\mathbb{E}_{\frac{19}{6},\frac{1}{2}}(z)$ is uniformly convex in $\mathcal{D}$.*

Taking $\nu=-\frac{1}{2}$ in Part (b) of Theorem 2.23, we derive the following result as follows:

Corollary 2.28 *If $0<c<0.060191$, then, the function $\mathbb{E}_{-\frac{1}{2},c}(z)$ is uniformly convex in $\mathcal{D}$.*

Example 2.29 *The function $\mathbb{E}_{-\frac{1}{2},\frac{1}{17}}(z)$ is uniformly convex in $\mathcal{D}$.*

3. Conclusion

In the present paper, we have derived some sufficient conditions so that the normalized Miller-Ross function defined in (1.2) satisfies several geometric properties such as starlikeness, convexity, close-to-convexity, and uniform convexity inside the unit disk $\mathcal{D}$. The various results, which we have established in this paper, are believed to be new, and their importance is illustrated by several interesting consequences and examples. Some of the main results in the present investigation, improve some results available in the literature [15].

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