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On the Hilbert series of the tangent cones for some 4-generated pseudosymmetric monomial curves

Nil ŞAHİN*
Department of Industrial Engineering, Faculty of Engineering, Bilkent University, Ankara, Turkey

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Abstract: In this article, we study Hilbert series of non-Cohen-Maculay tangent cones for some 4-generated pseudosymmetric monomial curves. We show that the Hilbert function is nondecreasing by explicitly computing it. We also compute standard bases of these toric ideals.

Key words: Hilbert function, tangent cone, monomial curve, numerical semigroup, standard bases

1. Introduction

Hilbert function of the tangent cone of a projective variety carries a lot of information about the discrete invariants of the variety and of its embedding such as dimension, the degree, and the arithmetic genus of the variety. Though the Hilbert function of a Cohen-Macaulay graded ring is well known with the help of Macaulay's theorem, the same thing does not hold for the Hilbert function of its tangent cone [6]. Some partial results are obtained in special cases about the growth of the Hilbert function of the tangent cone, see [2–4, 14–16].

In this paper, we will work with one dimensional Cohen-Macaulay local rings. It is known that when the tangent cone is also Cohen-Macaulay, the Hilbert function of the local ring is nondecreasing. Rossi's conjecture states that "The Hilbert function of a Gorenstein Local ring of dimension one is nondecreasing". This conjecture is still open in embedding dimension 4 even for monomial curves. Nondecreasingness of the Hilbert function of the tangent cone for some 4 generated symmetric monomial curves is studied by Arslan and Mete in [2] and Katsabekis in [11]. Arslan and Mete put an extra condition on the genarators, namely $\alpha_2 \leq \alpha_{21} + \alpha_{24}$ and showed that the tangent cone is Cohen Macaulay and, as a result, they showed that Hilbert function is nondecreasing without the need of explicit Hilbert function computation. Katsabekis also studied Cohen-Macaulay tangent cone case and computed Hilbert function explicitly. For 4 generated symmetric monomial curves, the case $\alpha_2 > \alpha_{21} + \alpha_{24}$ (i.e. not Cohen-Macaulay Tangent cone) is still open. We studied 4-generated pseudosymmetric monomial curves with Cohen Macaulay tangent cones ($\alpha_2 \leq \alpha_{21} + 1$) in [19] and showed nondecreasingness of the Hilbert function. Without Cohen-Macaulayness of the tangent cone ($\alpha_2 > \alpha_{21} + 1$), nondecreasingness of the Hilbert function of the local ring is not guaranteed and hence requires an explicit Hilbert function computation. With an observation that the number of generators in the standard basis increase when α_4 increase, we studied the simplest case $\alpha_4 = 2$ in [18]. We showed that the number of elements in the standard basis depends on a parameter k which is defined as the smallest positive integer such

*Correspondence: nilsahin@bilkent.edu.tr
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that $k(\alpha_2+1)<(k-1)\alpha_1+(k+1)\alpha_{21}+\alpha_3$ and the Hilbert function is nondecreasing when this parameter k is 1. In this paper, we focus on the next case, $\alpha_4=3$. As the standard basis computation requires adding the normal forms of s-polynomials of all the elements in the ideal, it is important to see the polynomials to start with. Understanding this case will be a step towards understanding the general case using induction. This case differs from the case $\alpha_4=2$ as can be seen in the next two examples. Though the parameter $k=2$ in both of these examples, the number of elements and their forms are much different. The following examples are done using SINGULAR. *

Example 1.1 *For $\alpha_{21}=4$, $\alpha_1=22, \alpha_2=13$, $\alpha_3=5$, $\alpha_4=2$, we have $k=2$ and the corresponding standard basis is* $\{X_1^{22}-X_3X_4, X_2^{13}-X_1^4X_4, X_3^5-X_1^{17}X_2, X_4^2-X_1X_2^{12}X_3^4, X_1^5X_3^4-X_2X_4, X_1^{26}-X_2^{13}X_3, X_1^9X_3^4-X_2^{14}, X_1^{35}X_3^3-X_2^{27}\}$.

Example 1.2 *For $\alpha_{21}=101$, $\alpha_1=501, \alpha_2=340$, $\alpha_3=18$, $\alpha_4=3$, we have $k=2$ and the corresponding standard basis is* $\{X_2X_4^2-X_1^{102}X_3^{17}, X_3X_4^2-X_1^{501}, X_4^3-X_1X_2^{339}X_3^{17}, X_3^{18}-X_1^{399}X_2, X_1^{101}X_4-X_2^{340}, X_1^{203}X_3^{17}-X_2^{341}X_4, X_2^{340}X_3X_4-X_1^{602}, X_2^{680}X_3-X_1^{703}, X_1^{906}X_3^{16}-X_2^{1021}X_4, X_1^{1609}X_3^{15}-X_2^{1701}X_4, X_1^{2312}X_3^{14}-X_2^{2381}X_4, X_1^{3015}X_3^{13}-X_2^{3061}X_4, X_1^{3718}X_3^{12}-X_2^{3741}X_4, X_2^{4421}X_4-X_1^{4421}X_3^{11}, X_1^{4522}X_3^{11}-X_2^{4761}, X_1^{5225}X_3^{10}-X_2^{5441}, X_1^{5928}X_3^{9}-X_2^{6121}, X_1^{6631}X_3^{8}-X_2^{6801}, X_1^{7334}X_3^{7}-X_2^{7481}, X_1^{8037}X_3^{6}-X_2^{8161}, X_1^{8740}X_3^{5}-X_2^{8841}, X_1^{9443}X_3^{4}-X_2^{9521}, X_1^{10146}X_3^{3}-X_2^{10201}, X_1^{10849}X_3^{2}-X_2^{10881}, X_1^{11552}X_3-X_2^{11561}, X_2^{12241}-X_1^{12255}\}$

This shows that the standard basis do not only depend on k like in $\alpha_4=2$ case.

2. Basic definitions

Using the same notation with [18], let S be the numerical semigroup $S=\langle n_1,\dots,n_k\rangle=\{\sum_{i=1}^{k}u_in_i|u_i\in\mathbb{N}\}$, where $n_1<n_2<\cdots<n_k$ are positive integers with $\gcd(n_1,\dots,n_k)=1$. Let $K[S]=K[t^{n_1},t^{n_2},\dots,t^{n_k}]$ be the semigroup ring of S, where K is algebraically closed field, and $A=K[X_1,X_2,\dots,X_k]$. If $\varphi: A\longrightarrow K[S]$ with $\varphi(X_i)=t^{n_i}$ and $\ker\varphi=I_S$, then $K[S]\simeq A/I_S$. C_S is the affine curve with parametrization

$$X_1=t^{n_1},\ \ X_2=t^{n_2},\ \dots,\ X_k=t^{n_k}$$

corresponding to S , and I_S is the defining ideal of C_S. n_1 is the multiplicity of C_S. $R_S=K[[t^{n_1},\dots,t^{n_k}]]$ is the local ring with the maximal ideal $\mathfrak{m}=\langle t^{n_1},\dots,t^{n_k}\rangle$. Then $gr_{\mathfrak{m}}(R_S)=\bigoplus_{i=0}^{\infty}\mathfrak{m}^i/\mathfrak{m}^{i+1}\cong A/I_S^*$ is the associated graded ring where $I_S^*=\langle f^*|f\in I_S\rangle$ with f^* denoting the least homogeneous summand of f.

We mean the Hilbert function of the associated graded ring $gr_{\mathfrak{m}}(R_S)=\bigoplus_{i=0}^{\infty}\mathfrak{m}^i/\mathfrak{m}^{i+1}$ by the Hilbert function $H_{R_S}(n)$ of the local ring R_S. That is,

$$H_{R_S}(n)=H_{gr_{\mathfrak{m}}(R_S)}(n)=dim_{R_S/\mathfrak{m}}(\mathfrak{m}^n/\mathfrak{m}^{n+1})\ \ n\geq 0.$$

The Hilbert series of R_S is defined to be the generating function

$$HS_{R_S}(t)=\sum_{n\in\mathbb{N}}H_{R_S}(n)t^n.$$

*Singular 2.0. A Computer Algebra System for Polynomial Computations. Available at http://www.singular.uni-kl.de.

By the Hilbert-Serre theorem, it can also be written as: $HS_{R_S}(t) = \frac{P(t)}{(1-t)^k} = \frac{Q(t)}{(1-t)^d}$, where $P(t)$ and $Q(t)$ are polynomials with coefficients in $\mathbb{Z}$ and d is the Krull dimension of R_S. $P(t)$ is called first Hilbert series and $Q(t)$ is called second Hilbert series [8, 17]. It is also known that there is a polynomial $P_{R_S}(n) \in \mathbb{Q}[n]$ called the Hilbert polynomial of R_S such that $H_{R_S}(n) = P_{R_S}(n)$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$. The smallest n_0 satisfying this condition is the regularity index of the Hilbert function of R_S.

A 4-generated semigroup $S = \langle n_1, n_2, n_3, n_4 \rangle$ is pseudosymmetric if and only if there are integers $\alpha_i > 1$, for $1 \leq i \leq 4$, and $\alpha_{21} > 0$ with $0 < \alpha_{21} < \alpha_1 - 1$, such that

$$\begin{aligned} n_1 &= \alpha_2\alpha_3(\alpha_4 - 1) + 1, \\ n_2 &= \alpha_{21}\alpha_3\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3, \\ n_3 &= \alpha_1\alpha_4 + (\alpha_1 - \alpha_{21} - 1)(\alpha_2 - 1)(\alpha_4 - 1) - \alpha_4 + 1, \\ n_4 &= \alpha_1\alpha_2(\alpha_3 - 1) + \alpha_{21}(\alpha_2 - 1) + \alpha_2. \end{aligned}$$

Then the toric ideal is $I_S = \langle f_1, f_2, f_3, f_4, f_5 \rangle$ with

$$\begin{aligned} f_1 &= X_1^{\alpha_1} - X_3X_4^{\alpha_4 - 1}, \qquad f_2 = X_2^{\alpha_2} - X_1^{\alpha_{21}}X_4, \quad f_3 = X_3^{\alpha_3} - X_1^{\alpha_1 - \alpha_{21} - 1}X_2, \\ f_4 &= X_4^{\alpha_4} - X_1X_2^{\alpha_2 - 1}X_3^{\alpha_3 - 1}, \quad f_5 = X_1^{\alpha_{21}+1}X_3^{\alpha_3 - 1} - X_2X_4^{\alpha_4 - 1}. \end{aligned}$$

See [12] for the details.

We focus on the case $\alpha_4 = 3$.

3. Standard bases

Remark 3.1 *If* $n_1 < n_2$, *then* $2\alpha_2 + 1 < 2\alpha_{21} + \alpha_1$.

Proof

$$\begin{array}{llll} & n_1 & < & n_2 \\ \Longrightarrow & 2\alpha_2\alpha_3 + 1 & < & 3\alpha_{21}\alpha_3 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3 \\ \Longrightarrow & 2\alpha_2\alpha_3 + 1 & < & 2\alpha_{21}\alpha_3 + \alpha_1\alpha_3 - \alpha_1 + \alpha_{21} + 1 \\ \Longrightarrow & \alpha_3(2\alpha_2 - 2\alpha_{21} - \alpha_1) & < & \alpha_{21} - \alpha_1 \\ \Longrightarrow & \alpha_3(2\alpha_2 - 2\alpha_{21} - \alpha_1 + 1) & < & \alpha_3 + \alpha_{21} - \alpha_1 < 0 \quad \text{by}(2) \\ \Longrightarrow & 2\alpha_2 - 2\alpha_{21} - \alpha_1 + 1 & < & 0 \\ \Longrightarrow & 2\alpha_2 + 1 & < & 2\alpha_{21} + \alpha_1 \end{array}$$

□

If $n_1 < n_2 < n_3 < n_4$, then it is known from [19] that

(1) $\alpha_1 > \alpha_4$

(2) $\alpha_3 < \alpha_1 - \alpha_{21}$

(3) $\alpha_4 < \alpha_2 + \alpha_3 - 1$

and these conditions completely determine the leading monomials of f_1, f_3 and f_4. Indeed, $\mathrm{LM}(f_1) = X_3X_4^{\alpha_4 - 1}$ by (1), $\mathrm{LM}(f_3) = X_3^{\alpha_3}$ by (2), $\mathrm{LM}(f_4) = X_4^{\alpha_4}$ by (3). If we also let

(4) $\alpha_2 > \alpha_{21} + 1$,

then $\mathrm{LM}(f_2) = X_1^{\alpha_{21}} X_4$ by (4). Then if $n_1 < n_2 < n_3 < n_4$, using the remarks in [18],

(5) $\alpha_{21} + \alpha_3 > \alpha_4$

(6) $\alpha_1 + \alpha_{21} + 1 \geq \alpha_2 + \alpha_4$

Now using remark 3.1,

(7) $2\alpha_2 + 1 < 2\alpha_{21} + \alpha_1$

These determine $\mathrm{LM}(f_5)$ and will determine $\mathrm{LM}(f_6)$ and $\mathrm{LM}(f_7)$. We know that the standard basis when $\alpha_4 = 2$ depends on the parameter k. We will show that the standard basis when $\alpha_4 = 3$ depends on three parameters k, s, and t defined as follows:

Definition 3.2 *Define k, l and s as the smallest integers satisfying*

$$(k-1)\alpha_1 + (k+1)\alpha_{21} + \alpha_3 > k\alpha_2 + (k+1)$$

$$s\alpha_1 + (2s+2)\alpha_{21} + \alpha_3 > (2s+1)\alpha_2 + (s+2)$$

$$l\alpha_1 + (2l+3)\alpha_{21} + \alpha_3 > (2l+2)\alpha_2 + (l+1)$$

respectively.

Remark 3.3 *s, k, and l always exist.*

Proof Assume to the contrary that s does not exist. Then for any integer i, $i\alpha_1 + (2i+2)\alpha_{21} + \alpha_3 \leq (2i+1)\alpha_2 + i + 2 \implies i(\alpha_1 + 2\alpha_{21} - \alpha_2 - 1) \leq \alpha_2 + 2 - \alpha_3 - 2\alpha_{21}$ which gives a contradiction since the right hand side of this inequality is fixed and $\alpha_1 + 2\alpha_{21} - \alpha_2 - 1$ is positive by (6).

Using a similar argument, k and l always exist. □

Remark 3.4 *If $n_1 < n_2 < n_3 < n_4$, then k is at most 2.*

Proof Assume to the contrary that $k > 2$. Then

$$\begin{array}{rcll} 2(\alpha_2+1) & \geq & \alpha_1 + 3\alpha_{21} + \alpha_3 - 1 & \\ 2\alpha_2 & \geq & \alpha_1 + 3\alpha_{21} + \alpha_3 - 3 & > \alpha_1 + \alpha_{21} \end{array}$$

Then

$$\alpha_1 + \alpha_{21} - 2\alpha_2 < 0 \tag{3.1}$$

On the other hand, $n_1 < n_2$ implies

$$\begin{array}{rcll} 2\alpha_2\alpha_3 + 1 & < & 2\alpha_{21}\alpha_3 + (\alpha_1 - \alpha_{21} - 1)(\alpha_3 - 1) + \alpha_3 & \\ \alpha_1 - \alpha_{21} & < & \alpha_3(\alpha_1 + \alpha_{21} - 2\alpha_2) & < \quad 0 \\ \alpha_1 - \alpha_{21} & < & 0 & \end{array}$$

which is a contradiction. Hence, k cannot exceed 2. □

Remark 3.5 $s \leq l$

Proof We know by (4) that $\alpha_2 - \alpha_{21} - 1 > 0$. If l satisfies $l\alpha_1 + (2l+3)\alpha_{21} + \alpha_3 \geq (2l+2)\alpha_2 + (l+1)$, then $l\alpha_1 + (2l+2)\alpha_{21} + \alpha_3 \geq (2l+1)\alpha_2 + (l+2) + \alpha_2 - \alpha_{21} - 1 > (2l+1)\alpha_2 + (l+2)$. Since s is the smallest integer satisfying this inequality, we must have $s \leq l$. □

Remark 3.6 *If $k = 1$, then $s = 0$.*

Proof If $k = 1$, then $\alpha_2 + 1 < 2\alpha_{21} + \alpha_3 - 1$ which is $s\alpha_1 + (2s+2)\alpha_{21} + \alpha_3 > (2s+1)\alpha_2 + s + 2$ for $s = 0$. Hence, if $k = 1$, then $s = 0$. □

Theorem 3.7 *Let $S = \langle n_1, n_2, n_3, n_4 \rangle$ be a 4-generated pseudosymmetric numerical semigroup with $n_1 < n_2 < n_3 < n_4$ and $\alpha_2 > \alpha_{21} + 1$ with $\alpha_4 = 3$. If k, s, and l are defined as above, then the standard basis for I_S is*

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, g_0, g_1, g_2, ..., g_s, h_s, h_{s+1}..., h_l\},$$

where $f_6 = X_1^{\alpha_1+\alpha_{21}} - X_2^{\alpha_2} X_3 X_4$, $f_7 = X_1^{\alpha_1+2\alpha_{21}} - X_2^{2\alpha_2} X_3$, $g_i = X_2^{(2i+1)\alpha_2+1} X_4 - X_1^{i\alpha_1+(2i+2)\alpha_{21}+1} X_3^{\alpha_3-(i+1)}$, *and* $h_j = X_2^{(2j+2)\alpha_2+1} - X_1^{j\alpha_1+(2j+3)\alpha_{21}+1} X_3^{\alpha_3-(j+1)}$.

Before we prove the theorem, let us state and prove the next lemma:

Lemma 3.8 $\mathrm{NF}(g_j|G) = 0$ *for $j > s$.*

Proof $T_{g_j} = \{g_s\}$ and $\mathrm{spoly}(g_j, g_s) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1} X_2^{2(j-s)\alpha_2} X_3^{\alpha_3-(s+1)} - X_1^{j\alpha_1+(2j+2)\alpha_{21}+1} X_3^{\alpha_3-(j+1)} = r_1$. $T_{r_1} = \{f_7\}$ and $\mathrm{spoly}(r_1, f_7) = X_1^{(s+1)\alpha_1+(2s+4)\alpha_{21}+1} X_2^{2(j-s-1)\alpha_2} X_3^{\alpha_3-(s+2)} - X_1^{j\alpha_1+(2j+2)\alpha_{21}+1} X_3^{\alpha_3-(j+1)} = r_2$. $T_{r_2} = \{f_7\}$ and continuing inductively $r_{j-s} = \mathrm{spoly}(r_{j-s-1}, f_7) = X_1^{(j-1)\alpha_1+2j\alpha_{21}+1} X_3^{\alpha_3-(j+1)} f_7$. Hence, $\mathrm{NF}(g_j|G) = 0$. □

Now we are ready for the proof of Theorem 3.7.

Proof We will prove the theorem by applying standard basis algorithm with NFMora as the normal form algorithm, see [8]. Here $G = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, g_0, g_1, g_2, ..., g_s, h_s, h_{s+1}..., h_t\}$ and T_h denotes the set $\{g \in G : \mathrm{LM}(g) \mid \mathrm{LM}(h)\}$ and ecart(h) is $deg(h) - deg(\mathrm{LM}(h))$. Note that $\mathrm{LM}(f_6) = X_2^{\alpha_2} X_3 X_4$ by (6), $\mathrm{LM}(g_i) = X_1^{i\alpha_1+(2i+2)\alpha_{21}+1} X_3^{\alpha_3-(i+1)}$ for all $i < s$, $\mathrm{LM}(g_s) = X_2^{(2s+1)\alpha_2+1} X_4$, $\mathrm{LM}(h_j) = X_1^{j\alpha_1+(2j+3)\alpha_{21}+1} X_3^{\alpha_3-(j+1)}$ for all $s \leq j < l$ and $\mathrm{LM}(h_l) = X_2^{(2l+2)\alpha_2+1}$ by the definitions of s and l.

For $k = 1$:

In this case, $g_0 = X_1^{2\alpha_{21}+1} X_3^{\alpha_3-1} - X_2^{\alpha_2+1} X_4$ and $\alpha_2 + 1 < 2\alpha_{21} + \alpha_3$ which implies that $\mathrm{LM}(g_0) = X_2^{\alpha_2+1} X_4$. We need to show that $NF(\mathrm{spoly}(f_m, f_n)|G) = 0$ for all m, n with $1 \leq m < n \leq 6$.

- $\mathrm{spoly}(f_1, f_2) = f_6$; hence, $NF(\mathrm{spoly}(f_1, f_2)|G) = 0$
- $\mathrm{spoly}(f_1, f_3) = X_1^{\alpha_1} X_3^{\alpha_3-1} - X_1^{\alpha_1-\alpha_{21}-1} X_2 X_4^2$ and $\mathrm{LM}(\mathrm{spoly}(f_1, f_3)) = X_1^{\alpha_1-\alpha_{21}-1} X_2 X_4^2$ by (5). Let $r_1 = \mathrm{spoly}(f_1, f_3)$. If $\alpha_1 < 2\alpha_{21}+1$ then $T_{r_1} = \{f_5\}$ and since $\mathrm{spoly}(r_1, g) = 0$, $NF(\mathrm{spoly}(f_1, f_3)|G) = 0$. Otherwise $T_{r_1} = \{f_2\}$ and $\mathrm{spoly}(r_1, f_2) = X_1^{\alpha_1-2\alpha_{21}-1} X_2^{\alpha_2+1} X_4 - X_1^{\alpha_1} X_3^{\alpha_3-1}$. Set $r_2 = \mathrm{spoly}(r_1, f_2)$, $\mathrm{LM}(r_2) = X_1^{\alpha_1-2\alpha_{21}-1} X_2^{\alpha_2+1} X_4$ and $T_{r_2} = \{f_7\}$ and $\mathrm{spoly}(r_2, f_7) = 0$; hence, $NF(\mathrm{spoly}(f_1, f_3)|G) = 0$.

- $\mathrm{spoly}(f_1,f_4)=X_1^{\alpha_1}X_4-X_1X_2^{\alpha_2-1}X_3^{\alpha_3}$. Set $r_1=\mathrm{spoly}(f_1,f_4)$. If $\mathrm{LM}(r_1)=X_1^{\alpha_1}X_4$ then $T_{r_1}=\{f_2\}$ and $\mathrm{spoly}(r_1,f_2)=X_1X_2^{\alpha_2-1}f_3$. If $\mathrm{LM}(r_1)=X_1X_2^{\alpha_2-1}X_3^{\alpha_3}$ then $T_{r_1}=\{f_3\}$ and $\mathrm{spoly}(r_1,f_3)=X_1^{\alpha_1-\alpha_{21}}f_2$. Hence, in both cases, $NF(\mathrm{spoly}(f_1,f_4)|G)=0$

- $\mathrm{spoly}(f_1,f_5)=X_1^{\alpha_{21}+1}X_3^{\alpha_3}-X_1^{\alpha_1}X_2=X_1^{\alpha_{21}+1}f_3$; hence, $NF(\mathrm{spoly}(f_1,f_5)|G)=0$

- $\mathrm{spoly}(f_1,f_6)=X_1^{\alpha_1}f_2$; hence, $NF(\mathrm{spoly}(f_1,f_6)|G)=0$

- $\mathrm{spoly}(f_1,f_7)=X_1^{\alpha_1+2\alpha_{21}}X_4^2-X_1^{\alpha_1}X_2^{2\alpha_2}=r_1$ then $\mathrm{LM}(r_1)=X_1^{\alpha_1+2\alpha_{21}}X_4^2$ by (4) and $T_{r_1}=\{f_2\}$. Then $\mathrm{spoly}(r_1,f_2)=X_1^{\alpha_1}X_2^{\alpha_2}f_2$. Hence, $NF(\mathrm{spoly}(f_1,f_7)|G)=0$.

- $NF(\mathrm{spoly}(f_2,f_3)|G)=0$ as $\mathrm{LM}(f_2)$ and $\mathrm{LM}(f_3)$ are relatively prime.

- $\mathrm{spoly}(f_2,f_4)=X_2^{\alpha_2-1}f_5$; hence, $NF(\mathrm{spoly}(f_2,f_4)|G)=0$

- $\mathrm{spoly}(f_2,f_5)=g_0$; hence, $NF(\mathrm{spoly}(f_2,f_5)|G)=0$

- $\mathrm{spoly}(f_2,f_6)=f_7$; hence, $NF(\mathrm{spoly}(f_2,f_6)|G)=0$.

- $NF(\mathrm{spoly}(f_2,f_7)|G)=0$ as $\mathrm{LM}(f_2)$ and $\mathrm{LM}(f_7)$ are relatively prime.

- $NF(\mathrm{spoly}(f_3,f_4)|G)=0$ as $\mathrm{LM}(f_3)$ and $\mathrm{LM}(f_4)$ are relatively prime.

- $NF(\mathrm{spoly}(f_3,f_5)|G)=0$ as $\mathrm{LM}(f_3)$ and $\mathrm{LM}(f_4)$ are relatively prime.

- $\mathrm{spoly}(f_3,f_6)=X_1^{\alpha_1-\alpha_{21}-1}g_0$; hence, $NF(\mathrm{spoly}(f_3,f_6)|G)=0$

- $\mathrm{spoly}(f_3,f_7)=X_1^{\alpha_1-\alpha_{21}-1}h_0$ (since $k=1$, $s=0$ and $h_0\in G$). Hence, $NF(\mathrm{spoly}(f_3,f_7)|G)=0$

- $\mathrm{spoly}(f_4,f_5)=X_1X_3^{\alpha_3-1}f_2$; hence, $NF(\mathrm{spoly}(f_4,f_5)|G)=0$

- $\mathrm{spoly}(f_4,f_6)=X_1X_2^{2\alpha_2-1}X_3^{\alpha_3}-X_1^{\alpha_1+\alpha_{21}}X_4^2=r_1$. If $\mathrm{LM}(r_1)=X_1X_2^{2\alpha_2-1}X_3^{\alpha_3}$; $T_{r_1}=\{f_3\}$ and $\mathrm{spoly}(f_3,r_1)=X_1^{\alpha_1+\alpha_{21}}X_4^2-X_1^{\alpha_1-\alpha_{21}}X_2^{2\alpha_2}=r_2$. $\mathrm{LM}(r_2)=X_1^{\alpha_1+\alpha_{21}}X_4^2$ and $T_{r_2}=\{f_2\}$. Then $\mathrm{spoly}(f_2,r_2)=0$. If $\mathrm{LM}(r_1)=X_1^{\alpha_1+\alpha_{21}}X_4^2$;$T_{r_1}=\{f_2\}$ and $\mathrm{spoly}(f_2,r_1)=X_1^{\alpha_1}X_2^{\alpha_2}X_4-X_1X_2^{2\alpha_2-1}X_3^{\alpha_3}=r_2$. $\mathrm{LM}(r_2)=X_1^{\alpha_1}X_2^{\alpha_2}X_4$ and $T_{r_2}=\{f_3\}$. Then $\mathrm{spoly}(f_3,r_2)=0$. Hence, in both cases $NF(\mathrm{spoly}(f_4,f_6)|G)$ $=0$

- $NF(\mathrm{spoly}(f_4,f_7)|G)=0$ as $\mathrm{LM}(f_4)$ and $\mathrm{LM}(f_7)$ are relatively prime.

- $\mathrm{spoly}(f_5,f_6)=X_1^{\alpha_{21}+1}X_2^{\alpha_2-1}X_3^{\alpha_3}-X_1^{\alpha_1+\alpha_{21}}X_4$ and let $r_1=\mathrm{spoly}(f_5,f_6)$. If $\mathrm{LM}(r_1)=X_1^{\alpha_{21}+1}X_2^{\alpha_2-1}X_3^{\alpha_3}$ then $T_{r_1}=\{f_3\}$ and $\mathrm{spoly}(r_1,f_3)=X_1^{\alpha_1-\alpha_{21}-1}f_2$; hence, $NF(\mathrm{spoly}(f_5,f_6)|G)=0$. If $\mathrm{LM}(r_1)=X_1^{\alpha_1+\alpha_{21}}X_4$ then $T_{r_1}=\{f_2\}$ and $\mathrm{spoly}(r_1,f_2)=X_2^{\alpha_2-1}f_3$; hence, $NF(\mathrm{spoly}(f_5,f_6)|G)=0$

- $\mathrm{spoly}(f_5,f_7)=X_1^{\alpha_1+2\alpha_{21}}X_4^2-X_1^{\alpha_{21}+1}X_2^{2\alpha_2-1}X_3^{\alpha_3}=r_1$.
  If $\mathrm{LM}(r_1)=X_1^{\alpha_1+2\alpha_{21}}X_4^2$, then $T_{r_1}=f_2$ and $\mathrm{spoly}(r_1,f_2)=X_1^{\alpha_1+\alpha_{21}}X_2^{\alpha_2}X_4-X_1^{\alpha_{21}+1}X_2^{2\alpha_2-1}X_3^{\alpha_3}=r_2$. Depending on the leading monomial of r_2, T_{r_2} is either f_2 or f_3. If it is f_2, $\mathrm{spoly}(r_2,f_2)=X_1^{\alpha_{21}+1}X_2^{2\alpha_2-1}f_3$. If it is f_3, $\mathrm{spoly}(r_2,f_3)=X_1^{\alpha_1}X_2^{\alpha_2}f_2$; hence, $NF(\mathrm{spoly}(f_5,f_7)|G)=0$.

If $\mathrm{LM}(r_1) = X_1^{\alpha_{21}+1}X_2^{2\alpha_2-1}X_3^{\alpha_3}$, then $T_{r_1} = f_3$ and $\mathrm{spoly}(r_1, f_3) = X_1^{\alpha_1}X_2^{2\alpha_2} - X_1^{\alpha_1+2\alpha_{21}}X_4^2 = r_2$. $\mathrm{LM}(r_2) = X_1^{\alpha_1+2\alpha_{21}}X_4^2$ and $T_{r_2} = \{f_2\}$. Then $\mathrm{spoly}(r_2, f_2) = X_1^{\alpha_1}X_2^{\alpha_2}f_2$. Hence, $NF(\mathrm{spoly}(f_5, f_7)|G) = 0$

- $\mathrm{spoly}(f_6, f_7) = X_1^{\alpha_1+\alpha_{21}}f_2$ and hence, $NF(\mathrm{spoly}(f_6, f_7)|G) = 0$
- $\mathrm{spoly}(f_1, g_0) = X_1^{\alpha_1}X_2^{\alpha_2+1} - X_1^{2\alpha_{21}+1}X_3^{\alpha_3}X_4 = r_1$. Using (2) and (4), $\mathrm{LM}(r_1) = X_1^{2\alpha_{21}+1}X_3^{\alpha_3}X_4$ and $T_{r_1} = \{f_2, f_3\}$. If $\mathrm{ecart}(f_2)$ is minimal, $\mathrm{spoly}(r_1, f_2) = X_1^{\alpha_{21}+1}X_2^{\alpha_2}f_3$. If $\mathrm{ecart}(f_3)$ is minimal, $\mathrm{spoly}(r_1, f_3) = X_1^{\alpha_1}X_2f_2$. Hence, in both cases $NF(\mathrm{spoly}(f_1, g_0)|G) = 0$
- $NF(\mathrm{spoly}(f_2, g_0)|G) = 0$ as $\mathrm{spoly}(f_2, g_0) = h_0$ and $h_0 \in G$.
- $NF(\mathrm{spoly}(f_3, g_0)|G) = 0$ as $\mathrm{LM}(f_3)$ and $\mathrm{LM}(g_0)$ are relatively prime.
- $\mathrm{spoly}(f_4, g_0) = X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}X_4^2 - X_1X_2^{2\alpha_2}X_3^{\alpha_3-1} = r_1$. $\mathrm{LM}(r_1) = X_1^{2\alpha_{21}+1}X_3^{\alpha_3-1}X_4^2$ and $T_{r_1} = \{f_1, f_2\}$, but $\mathrm{ecart}(f_2)$ is minimal. $\mathrm{spoly}(r_1, f_2) = X_1X_2^{\alpha_2}X_3^{\alpha_3-1}f_2$; hence, $NF(\mathrm{spoly}(f_4, g_0)|G) = 0$
- $\mathrm{spoly}(f_5, g_0) = X_1^{\alpha_{21}+1}X_3^{\alpha_3-1}f_2$; hence, $NF(\mathrm{spoly}(f_5, g_0)|G) = 0$
- $\mathrm{spoly}(f_6, g_0) = X_1^{2\alpha_{21}+1}f_3$; hence, $NF(\mathrm{spoly}(f_6, g_0)|G) = 0$
- $\mathrm{spoly}(f_7, g_0) = X_1^{2\alpha_{21}+1}X_2^{\alpha_2-1}X_3^{\alpha_3} - X_1^{\alpha_1+2\alpha_{21}}X_4 = r_1$. If $\mathrm{LM}(r_1) = X_1^{2\alpha_{21}+1}X_2^{\alpha_2-1}X_3^{\alpha_3}$, then $T_{r_1} = \{f_3\}$ and $\mathrm{spoly}(r_1, f_3) = X_1^{\alpha_1+\alpha_{21}}f_2$
  If $\mathrm{LM}(r_1) = X_1^{\alpha_1+2\alpha_{21}}X_4$, then $T_{r_1} = \{f_2\}$ and $\mathrm{spoly}(r_1, f_2) = X_1^{2\alpha_{21}+1}X_2^{\alpha_2-1}f_3$ so in both cases, $NF(\mathrm{spoly}(f_7, g_0)|G) = 0$.
- $NF(\mathrm{spoly}(g_0, h_j)|G) = 0$ as $\mathrm{LM}(g_0)$ and $\mathrm{LM}(h_j)$ are relatively prime for all $0 \le j < l$.
- $\mathrm{spoly}(g_0, h_l) = X_1^{t\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}X_4 - X_1^{2\alpha_{21}+1}X_2^{(2l+1)\alpha_2}X_3^{\alpha_3-1} = r_1$ and $\mathrm{LM}(r_1) = X_1^{2\alpha_{21}+1}X_2^{(2l+1)\alpha_2}X_3^{\alpha_3-1}$. $T_{r_1} = \{f_7\}$ and $\mathrm{spoly}(r_1, f_7) = X_1^{\alpha_1+4\alpha_{21}+1}X_3^{\alpha_3-(l+1)}\left[X_1^{(l-1)\alpha_1+(2l-1)\alpha_{21}}X_4 - X_2^{(2l-1)\alpha_2}X_3^{l-1}\right] = r_2$. $T_{r_2} = \{f_7\}$ and $\mathrm{spoly}(r_2, f_7) = X_1^{2\alpha_1+6\alpha_{21}+1}X_3^{\alpha_3-(l+1)}\left[X_1^{(l-2)\alpha_1+(2l-3)\alpha_{21}}X_4 - X_2^{(2l-3)\alpha_2}X_3^{l-2}\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, we obtain $r_{l+1} = \mathrm{spoly}(r_l, f_7) = X_1^{l\alpha_1+(2l+2)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}f_2$; hence, $NF(\mathrm{spoly}(g_0, h_l)|G) = 0$.
- $NF(\mathrm{spoly}(h_j, h_l)|G) = 0$ as the leading monomials are relatively prime.
- $\mathrm{spoly}(f_1, h_j) = X_1^{(j+1)\alpha_1+(2j+3)\alpha_{21}+1}X_3^{\alpha_3-(j+2)} - X_2^{(2j+2)\alpha_2+1}X_4^2 = r_1$. (6) and $j > s$ implies $\mathrm{LM}(r_1) = X_2^{(2j+2)\alpha_2+1}X_4^2$ and $T_{r_1} = \{f_5, g_s\}$. Since $\mathrm{ecart}(f_5)$ is minimal by (7), $\mathrm{spoly}(r_1, f_5) = X_1^{\alpha_{21}+1}X_2^{(2j+2)\alpha_2}X_3^{\alpha_3-1} - X_1^{(j+1)\alpha_2+(2j+3)\alpha_{21}+1}X_3^{\alpha_3-(j+2)} = r_2$. $T_{r_2} = \{f_7\}$. Then $\mathrm{spoly}(r_2, f_7) = X_1^{\alpha_1+3\alpha_{21}+1}X_3^{\alpha_3-(j+2)}[X_1^{j\alpha_1+2j\alpha_{21}} - X_2^{2j\alpha_2}X_3^{j}] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, we obtain $r_{j+2} = \mathrm{spoly}(r_{j+1}, f_7) = X_1^{(j+1)\alpha_1+(2j+1)\alpha_{21}+1}X_3^{\alpha_3-(j+2)}f_7$; hence, $NF(\mathrm{spoly}(f_1, h_j)|G) = 0$.
- $NF(\mathrm{spoly}(f_1, h_l)|G) = 0$ as the leading monomials are relatively prime.

- $\text{spoly}(f_2,h_j)=X_2^{\alpha_2}g_j$. Hence, by lemma 3.8, $NF(\text{spoly}(f_2,h_j)|G)=0$.

- $NF(\text{spoly}(f_2,h_l)|G)=0$ as the leading monomials are relatively prime.

- $\text{spoly}(f_3,h_j)=X_1^{(j+1)\alpha_1+(2j+2)\alpha_{21}}X_2-X_2^{(2j+2)\alpha_2+1}X_3^{j+1}$. Set this as r_1. Then $\text{LM}(r_1)=X_2^{(2j+2)\alpha_2+1}X_3^{j+1}$ and $T_{r_1}=\{f_7\}$. $\text{spoly}(r_1,f_7)=X_1^{\alpha_1+2\alpha_{21}}X_2\left[X_2^{2j\alpha_2}X_3^{j}-X_1^{(j)\alpha_1+2j\alpha_{21}}\right]=r_2$. $T_{r_2}=\{f_7\}$ and continuing inductively, we obtain $r_{j+1}=\text{spoly}(r_j,f_7)=X_1^{j\alpha_1+2j\alpha_{21}}X_2f_7$; hence, $NF(\text{spoly}(f_3,h_j)|G)=0$.

- $NF(\text{spoly}(f_3,h_l)|G)=0$ as the leading monomials are relatively prime.

- $NF(\text{spoly}(f_4,h_j)|G)=0$ as the leading monomials are relatively prime.

- $NF(\text{spoly}(f_4,h_l)|G)=0$ as the leading monomials are relatively prime.

- $NF(\text{spoly}(f_5,h_j)|G)=0$ as the leading monomials are relatively prime.

- $\text{spoly}(f_5,h_l)=X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}X_4^2-X_1^{\alpha_{21}+1}X_2^{(2l+2)\alpha_2}X_3^{\alpha_3-1}$. Set this as r_1. If $\text{LM}(r_1)=X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}X_4^2$, then $T_{r_1}=\{f_1,f_2,f_7\}$, but since $\text{ecart}(f_1)$ is minimal among these, $\text{spoly}(r_1,f_1)=X_1^{\alpha_{21}+1}X_3^{\alpha_3-(l+2)}\left[X_1^{(l+1)\alpha_1+(2l+2)\alpha_{21}}-X_2^{2(l+1)\alpha_2}X_3^{(l+1)}\right]=r_2$ and $T_{r_2}=\{f_7\}$. $\text{spoly}(r_2,f_7)=X_1^{\alpha_{21}+1}X_2^{2\alpha_2}X_3^{\alpha_3-(l+1)}\left[X_1^{(l)\alpha_1+(2l)\alpha_{21}}-X_2^{2(l)\alpha_2}X_3^{(l)}\right]=r_3$ and $T_{r_3}=\{f_7\}$. Continuing inductively we obtain, $\text{spoly}(r_{l+1},f_7)=X_1^{\alpha_{21}+1}X_2^{2l\alpha_2}X_3^{\alpha_3-2}f_7$ which implies $NF(\text{spoly}(f_5,h_l)|G)=0$. If $\text{LM}(r_1)=X_1^{\alpha_{21}+1}X_2^{(2l+2)\alpha_2}X_3^{\alpha_3-1}$ then $T_{r_1}=\{f_7\}$ and
$\text{spoly}(r_1,f_7)=X_1^{\alpha_1+3\alpha_{21}+1}X_3^{\alpha_3-(l+1)}\left[X_2^{2l\alpha_2}X_3^{(l-1)}-X_1^{(l-1)\alpha_1+(2l)\alpha_{21}}X_4^2\right]=r_2$. Then $T_{r_2}=\{f_7\}$ and $\text{spoly}(r_2,f_7)=X_1^{2\alpha_1+5\alpha_{21}+1}X_3^{\alpha_3-(l+1)}\left[X_2^{2(l-1)\alpha_2}X_3^{(l-2)}-X_1^{(l-2)\alpha_1+(2l-2)\alpha_{21}}X_4^2\right]=r_3$ and
$T_{r_3}=\{f_7\}$. Continuing inductively we obtain $r_{l+1}=\text{spoly}(r_l,f_7)=X_1^{l\alpha_1+(2l+1)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}[X_2^{2\alpha_2}-X_1^{2\alpha_{21}}X_4^2]$. $T_{r_{l+1}}=\{f_2\}$ and $\text{spoly}(r_{l+1},f_2)=X_1^{l\alpha_1+(2l+1)\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(l+1)}f_2$. Hence, $NF(\text{spoly}(f_5,h_l)|G)=0$ in this case, too.

- $\text{spoly}(f_6,h_j)=g_{j+1}$ for all $s\leq j<l$. Hence, by lemma 3.8 $NF(\text{spoly}(f_6,h_j)|G)=0$.

- $\text{spoly}(f_6,h_l)=X_1^{\alpha_1+\alpha_{21}}X_2^{(2l+1)\alpha_2+1}-X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-l}X_4=r_1$. $\text{LM}(r_1)=X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-l}X_4$ by the definition of l and (4). Then $T_{r_1}=\{f_2\}$ and $\text{spoly}(r_1,f_2)=X_1^{\alpha_1+\alpha_{21}}X_2^{\alpha_2}h_{j-1}$. Then $NF(\text{spoly}(f_6,h_l)|G)$ $=0$

- $\text{spoly}(f_7,h_j)=h_{j+1}$; hence, $NF(\text{spoly}(f_7,h_j)|G)=0$

- $\text{spoly}(f_7,h_l)=X_1^{\alpha_1+2\alpha_{21}}h_{l-1}$ if $2\alpha_2+1<\alpha_1+2\alpha_{21}$. Otherwise, leading monomials of f_7 and h_l are relatively prime. As a result, in both cases, $NF(\text{spoly}(f_7,h_l)|G)=0$.

<u>For $k=2$:</u>

In this case, since s might be greater than zero, l will be greater than zero and h_0 will not be an element of the standard basis. This means, from the above computations, only spoly(f_3, f_7) must be reconsidered. In addition to the normal forms considered in the case of $k = 1$, we need the following for $k = 2$ to prove the theorem:

- spoly$(f_3, f_7) = X_1^{\alpha_1-\alpha_{21}-1}h_0$. The problem here is that, since $k = 2$, s is not necessarily 0 and we cannot guarantee if $h_0 \in G$. Set spoly(f_3, f_7) as r_1. If $s > 0$, then $l > 0$ and by its definition, $3\alpha_{21}+\alpha_3 < 2\alpha_2+1$ and LM$(r_1) = X_1^{\alpha_1+2\alpha_{21}}X_3^{\alpha_3-1}$ and $T_{r_1} = \{g_0\}$. spoly$(r_1, g_0) = X_1^{\alpha_1-\alpha_{21}-1}X_2^{\alpha_2+1}f_2$. Hence, $NF(\text{spoly}(f_3, f_7)|G) = 0$.

- spoly$(g_i, g_j) = X_1^{(j-i)\alpha_1+2(j-i)\alpha_{21}}X_2^{(2i+1)\alpha_2+1}X_4 - X_2^{(2j+1)\alpha_2+1}X_3^{j-i}X_4$ Set this as r_1. Then LM$(r_1) = X_1^{(j-i)\alpha_1+2(j-i)\alpha_{21}}X_2^{(2i+1)\alpha_2+1}X_4$ and spoly$(r_1, f_7) = X_1^{\alpha_1+2\alpha_{21}}X_2^{(2i+1)\alpha_2+1}X_4\left[X_1^{(j-i-1)\alpha_1+2(j-i-1)\alpha_{21}} - X_2^{2(j-i-1)\alpha_2}X_3^{j-i-1}\right] = r_2$ which implies that $T_{r_2} = \{f_7\}$ and this, continuing inductively, implies that $r_{j-i} = \text{spoly}(r_{j-i-1}, f_7) = X_1^{(j-i-1)\alpha_1+2(j-i-1)\alpha_{21}}X_2^{[2i+1]\alpha_2+1}X_4 f_7$; hence, $NF(\text{spoly}(g_i, g_j)|G) = 0$.

- $NF(\text{spoly}(g_i, g_s)|G) = 0$ as the leading monomials are relatively prime.

- spoly$(g_i, h_j) = X_1^{(j-i)\alpha_1+(2(j-i)+1)\alpha_{21}}X_2^{(2i+1)\alpha_2+1}X_4 - X_2^{(2j+2)\alpha_2+1}X_3^{j-i}$. Set this as r_1. If LM$(r_1) = X_1^{(j-i)\alpha_1+(2(j-i)+1)\alpha_{21}}X_2^{(2i+1)\alpha_2+1}X_4$ then $T_{r_1} = \{f_2\}$ and spoly$(r_1, f_2) = X_2^{(2i+2)\alpha_2+1}\left[X_1^{(j-i)\alpha_1+(2(j-i)+1)\alpha_{21}}X_4 - X_2^{(2(j-i)+1)\alpha_2}X_3^{j-i}\right] = r_2$. LM$(r_2) = X_2^{(2(j-i)+1)\alpha_2}X_3^{j-i}$ and $T_{r_2} = \{f_7\}$. spoly$(r_2, f_7) = X_1^{\alpha_1+2\alpha_{21}}X_2^{2(i+1)\alpha_2+1}\left[X_1^{(j-i-1)\alpha_1+(2(j-i-1))\alpha_{21}} - X_2^{2(j-i-1)\alpha_2}X_3^{j-i-1}\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, finally, we obtain $r_{j-i+1} = \text{spoly}(r_{j-i}, f_7) = X_1^{(j-i-1)\alpha_1+2(j-i-1)\alpha_{21}}X_2^{2(i+1)\alpha_2+1}f_7$; hence, $NF(\text{spoly}(g_i, h_j)|G) = 0$ in this case.
  If LM$(r_1) = X_2^{(2j+2)\alpha_2+1}X_3^{j-i}$, and then $T_{r_1} = \{f_7\}$ and spoly$(r_1, f_7) = X_1^{\alpha_1+2\alpha_{21}}X_2^{(2i+1)\alpha_2+1}\left[X_2^{(2(j-i-1)+1)\alpha_2}X_3^{j-i-1} - X_1^{(j-i-1)\alpha_1+(2(j-i-1)+1)\alpha_{21}}X_4\right] = r_2$. $T_{r_2} = \{f_7\}$ and spoly$(r_2, f_7) = X_1^{2\alpha_1+4\alpha_{21}}X_2^{(2i+1)\alpha_2+1}\left[X_2^{(2(j-i-2)+1)\alpha_2}X_3^{j-i-2} - X_1^{(j-i-2)\alpha_1+(2(j-i-2)+1)\alpha_{21}}X_4\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, finally, we obtain $r_{j-i+1} = \text{spoly}(f_7, r_{j-i}) = X_1^{(j-i)\alpha_1+2(j-i)\alpha_{21}}X_2^{(2i+1)\alpha_2+1}f_2$; hence, $NF(\text{spoly}(g_i, h_j)|G) = 0$ in this case, too.

- $NF(\text{spoly}(g_i, h_l)|G) = 0$ as the leading monomials are relatively prime.

- $NF(\text{spoly}(g_s, h_j)|G) = 0$ as the leading monomials are relatively prime.

- spoly$(g_s, h_l) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_2^{(2(l-s)+1)\alpha_2}X_3^{\alpha_3-(s+1)} - X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}X_4 = r_1$. If LM$(r_1) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_2^{(2(l-s)+1)\alpha_2}X_3^{\alpha_3-(s+1)}$, then $T_{r_1} = \{f_7\}$ and spoly$(r_1, f_7) = X_1^{(s+1)\alpha_1+(2s+4)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}\left[X_2^{(2(l-s-1)+1)\alpha_2}X_3^{l-s-1} - X_1^{(l-s-1)\alpha_1+(2(l-s-1)+1)\alpha_{21}}X_4\right] = r_2$. $T_{r_2} = \{f_7\}$ and continuing inductively, $r_{l-s+1} = \text{spoly}(r_{l-s}, f_7) = X_1^{l\alpha_1+2(l+1)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}f_2$. If LM$(r_1) = X_1^{l\alpha_1+(2l+3)\alpha_{21}+1}X_3^{\alpha_3-(l+1)}X_4$, then $T_{r_1} = \{f_2\}$ and spoly$(r_1, f_2) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(l+1)}\left[X_1^{(l-s)\alpha_1+2(l-s)\alpha_{21}} - X_2^{2(l-s)\alpha_2}X_3^{l-s}\right] = r_2$. $T_{r_2} = \{f_7\}$

and continuing inductively, $T_{r_{l-s}} = \{f_7\}$ and $r_{l-s+1} = \text{spoly}(r_{l-s}, f_7) = X_1^{(l-1)\alpha_1+2(l-1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-(l+1)} f_7$. Hence, in both cases, $NF(\text{spoly}(g_s, h_l)|G) = 0$.

- $\text{spoly}(f_1, g_i) = X_2^{(2i+1)\alpha_2+1} X_4^3 - X_1^{(i+1)\alpha_1+(2i+2)\alpha_{21}+1} X_3^{\alpha_3-(i+2)} = r_1$. $\text{LM}(r_1) = X_2^{(2i+1)\alpha_2+1} X_4^3$ by (7) and (3) and $T_{r_1} = \{f_4, f_5\}$, but $\text{ecart}(f_5)$ is minimal. Then $\text{spoly}(r_1, f_5) = X_1^{\alpha_{21}+1} X_2^{(2i+1)\alpha_2} X_3^{\alpha_3-1} X_4 - X_1^{(i+1)\alpha_1+(2i+2)\alpha_{21}+1} X_3^{\alpha_3-(i+2)} = r_2$. $\text{LM}(r_2) = X_1^{\alpha_{21}+1} X_2^{(2i+1)\alpha_2} X_3^{\alpha_3-1} X_4$ and $T_{r_2} = \{f_2, f_6, f_7\}$ and $\text{spoly}(r_2, f_2) = X_1 X_3^{\alpha_3-(i+2)} \left[X_2^{(2i+2)\alpha_2} X_3^{i+1} - X_1^{(i+1)\alpha_1+2(i+1)\alpha_{21}}\right] = r_3$. $\text{LM}(r_3) = X_1 X_2^{(2i+2)\alpha_2} X_3^{\alpha_3-1}$ and $T_{r_3} = \{f_7\}$, $\text{spoly}(r_3, f_7) = X_1^{\alpha_1+2\alpha_{21}+1} X_3^{\alpha_3-(i+2)} \left[X_2^{(2i)\alpha_2} X_3^i - X_1^{i\alpha_1+2i\alpha_{21}}\right] = r_4$. $T_{r_4} = \{f_7\}$ and continuing inductively, we obtain $r_{i+1} = \text{spoly}(r_{i+2}, f_7) = X_1^{i\alpha_1+2i\alpha_{21}+1} X_3^{\alpha_3-(i+2)} f_7$; hence, $NF(\text{spoly}(f_1, g_i)|G) = 0$

- $\text{spoly}(f_2, g_i) = X_2^{(2i+1)\alpha_2+1} X_4^2 - X_1^{i\alpha_1+(2i+1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-(i+1)} = r_1$. $\text{LM}(r_1) = X_2^{(2i+1)\alpha_2+1} X_4^2$ by (5) and (7). $T_{r_1} = \{f_5\}$. $\text{spoly}(r_1, f_5) = X_1^{\alpha_{21}+1} X_2^{\alpha_2} \left[X_2^{2i\alpha_2} X_3^i - X_1^{i\alpha_1+2i\alpha_{21}}\right] = r_2$. $T_{r_2} = \{f_7\}$. $\text{spoly}(r_2, f_7) = X_1^{\alpha_1+3\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-(i+1)} \left[X_2^{2(i-1)\alpha_2} X_3^{i-1} - X_1^{(i-1)\alpha_1+2(i-1)\alpha_{21}}\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, $r_{i+1} = \text{spoly}(r_i, f_7) = X_1^{(i-1)\alpha_1+(2i+1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-(i+1)} f_7$; hence, $NF(\text{spoly}(f_2, g_i)|G) = 0$

- $\text{spoly}(f_3, g_i) = X_2^{(2i+1)\alpha_2+1} X_3^{i+1} X_4 - X_1^{(i+1)\alpha_1+(2i+1)\alpha_{21}} X_2 = r_1$. Using (7) and (4), $\text{LM}(r_1) = X_2^{(2i+1)\alpha_2+1} X_3^{i+1} X_4$ and $T_{r_1} = \{f_6, f_7\}$, but $\text{ecart}(f_7)$ is minimal. Then $\text{spoly}(r_1, f_7) = X_1^{\alpha_1+2\alpha_{21}} X_2 \left[X_2^{(2i-1)\alpha_2} X_3^i X_4 - X_1^{i\alpha_1+(2i-1)\alpha_{21}}\right] = r_2$. $T_{r_2} = \{f_6, f_7\}$. Continuing inductively, we obtain, $r_{i+1} = \text{spoly}(r_i, f_7) = X_1^{i\alpha_1+2i\alpha_{21}} X_2 f_6$; hence, $NF(\text{spoly}(f_3, g_i)|G) = 0$.

- $NF(\text{spoly}(f_4, g_i)|G) = 0$ as the leading monomials are relatively prime.

- $NF(\text{spoly}(f_5, g_i)|G) = 0$ as the leading monomials are relatively prime.

- $\text{spoly}(f_6, g_i) = X_2^{(2i+2)\alpha_2+1} X_4^2 - X_1^{(i+1)\alpha_1+(2i+3)\alpha_{21}+1} X_3^{\alpha_3-(i+2)} = r_1$. $\text{LM}(r_1) = X_2^{(2i+2)\alpha_2+1} X_4^2$ by (5) and (7). Then $T_{r_1} = \{f_5\}$ and $\text{spoly}(r_1, f_5) = X_1^{\alpha_{21}+1} X_2^{(2i+2)\alpha_2} X_3^{\alpha_3-1} - X_1^{(i+1)\alpha_1+(2i+3)\alpha_{21}+1} X_3^{\alpha_3-(i+2)} = r_2$. $\text{LM}(r_2) = X_1^{\alpha_{21}+1} X_2^{(2i+2)\alpha_2} X_3^{\alpha_3-1}$ and $T_{r_2} = \{f_7\}$.
$\text{spoly}(r_2, f_7) = X_1^{\alpha_1+3\alpha_{21}+1} X_3^{\alpha_3-(i+2)} \left[X_1^{i\alpha_1+2i\alpha_{21}} - X_2^{2i\alpha_2} X_3^i\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, we obtain $r_{i+2} = \text{spoly}(r_{i+1}, f_7) = X_1^{i\alpha_1+(2i+1)\alpha_{21}+1} X_3^{\alpha_3-(i+2)} f_7$; hence, $NF(\text{spoly}(f_6, g_i)|G) = 0$.

- $\text{spoly}(f_7, g_i) = g_{i+1}$. Hence, $NF(\text{spoly}(f_7, g_i)|G) = 0$

- $\text{spoly}(f_1, g_s) = X_1^{\alpha_1} X_2^{(2s+1)\alpha_2+1} - X_1^{s\alpha_1+(2s+2)\alpha_{21}+1} X_3^{\alpha_3-s} X_4$. Set this as r_1. Since $s-1 < l$, by the definition of l and (4), $\text{LM}(r_1) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1} X_3^{\alpha_3-s} X_4$. $T_{r_1} = \{f_2\}$ and $\text{spoly}(r_1, f_2) = X_1^{s\alpha_1+(2s+1)\alpha_{21}+1} X_2^{\alpha_2} X_3^{\alpha_3-s} - X_1^{\alpha_1} X_2^{(2s+1)\alpha_2+1} = r_2$. $T_{r_2} = \{g_{s-1}\}$ and $\text{spoly}(r_2, g_{s-1}) = X_1^{\alpha_1} X_2^{2s\alpha_2+1} f_2$. Hence, $NF(\text{spoly}(f_1, g_s)|G) = 0$

- $\text{spoly}(f_2, g_s) = h_s$. Since $h_s \in G$, $NF(\text{spoly}(f_2, g_s)|G) = 0$

- $NF(\text{spoly}(f_3, g_s)|G) = 0$ as the leading monomials are relatively prime.

- $\text{spoly}(f_4, g_s) = X_1X_2^{(2s+2)\alpha_2}X_3^{\alpha_3-1} - X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_3^{\alpha_3-(s+1)}X_4^2$. Set this as r_1. If $\text{LM}(r_1) = X_1X_2^{(2s+2)\alpha_2}X_3^{\alpha_3-1}$, then $T_{r_1} = \{f_7\}$ and
  $\text{spoly}(r_1, f_7) = X_1^{\alpha_1+2\alpha_{21}+1}X_3^{\alpha_3-(s+1)}\left[X_2^{(2s)\alpha_2}X_3^{s-1} - X_1^{(s-1)\alpha_1+(2s)\alpha_{21}}X_4^2\right] = r_2$. $T_{r_2} = \{f_2\}$ and
  $\text{spoly}(r_2, f_2) = X_1X_2X_3\left[X_2^{(2s-1)\alpha_2}X_3^{s-1} - X_1^{(s-1)\alpha_1+(2s-1)\alpha_{21}}X_4\right] = r_3$. $T_{r_3} = \{f_2\}$ and $\text{spoly}(r_3, f_2) = X_1^{\alpha_1+2\alpha_{21}+1}X_2^{2\alpha_2}X_3^{\alpha_3-(s+1)}\left[X_1^{(s-1)\alpha_1+(2s-2)\alpha_{21}} - X_2^{(2s-2)\alpha_2}X_3^{s-1}\right] = r_4$. $T_{r_4} = \{f_7\}$ and continuing inductively, $r_{s+2} = \text{spoly}(r_s, f_7) = X_1^{s\alpha_1+2\alpha_{21}+1}X_2^{(s)\alpha_2}X_3^{\alpha_3-(s+1)}f_7$. If $\text{LM}(r_1) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_3^{\alpha_3-(s+1)}$ X_4^2, then $T_{r_1} = \{f_1, f_2\}$, but $\text{ecart}(f_2)$ is minimal.
  $\text{spoly}(r_1, f_2) = X_1X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}\left[X_1^{s\alpha_1+(2s+1)\alpha_{21}}X_4 - X_2^{(2s+1)\alpha_2}X_3^s\right]$. Set this as r_2. Observe that $X_2^{\alpha_2}\text{spoly}(f_5, g_s) - X_1^{\alpha_{21}}r_2 = 0$ and $NF(\text{spoly}(f_5, g_s)|G) = 0$ (see below).
  Hence, in both of the cases, $NF(\text{spoly}(f_4, g_s)|G) = 0$

- $\text{spoly}(f_5, g_s) = X_1^{\alpha_{21}+1}X_2^{(2s+1)\alpha_2}X_3^{\alpha_3-1} - X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_3^{\alpha_3-(s+1)}X_4$. Set this as r_1. If $\text{LM}(r_1) = X_1^{\alpha_{21}+1}X_2^{(2s+1)\alpha_2}X_3^{\alpha_3-1}$, then $T_{r_1} = \{f_7\}$ and
  $\text{spoly}(r_1, f_7) = X_1^{\alpha_1+3\alpha_{21}+1}X_3^{\alpha_3-(s+1)}\left[X_2^{(2s-1)\alpha_2}X_3^{s-1} - X_1^{(s-1)\alpha_1+(2s-1)\alpha_{21}}X_4\right] = r_2$. $T_{r_2} = \{f_2\}$ and
  $\text{spoly}(r_2, f_2) = X_1^{\alpha_1+3\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}\left[X_1^{(s-1)\alpha_1+(2s-2)\alpha_{21}} - X_2^{(2s-2)\alpha_2}X_3^{s-1}\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, $r_{s+1} = \text{spoly}(r_s, f_7) = X_1^{s\alpha_1+(2s-1)\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}f_7$. If $\text{LM}(r_1) = X_1^{s\alpha_1+(2s+2)\alpha_{21}+1}X_3^{\alpha_3-(s+1)}X_4$, then $T_{r_1} = \{f_2\}$ $\text{spoly}(r_1, f_2) = X_1^{\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}$ $[X_1^{s\alpha_1+2s\alpha_{21}} -$ $X_2^{2s\alpha_2}X_3^s = r_2$. $T_{r_2} = \{f_7\}$ and
  $\text{spoly}(r_2, f_7) = X_1^{\alpha_1+3\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}\left[X_1^{(s-1)\alpha_1+(2s-2)\alpha_{21}} - X_2^{(2s-2)\alpha_2}X_3^{s-1}\right] = r_3$. $T_{r_3} = \{f_7\}$ and continuing inductively, $r_{s+1} = \text{spoly}(r_s, f_7) = X_1^{s\alpha_1+(2s-1)\alpha_{21}+1}X_2^{\alpha_2}X_3^{\alpha_3-(s+1)}f_7$ in this case, too. Hence, in both of the cases $NF(\text{spoly}(f_5, g_s)|G) = 0$

- $\text{spoly}(f_6, g_s) = X_1^{\alpha_1+\alpha_{21}}h_s$; hence, $NF(\text{spoly}(f_6, g_s)|G) = 0$

- $\text{spoly}(f_7, g_s) = X_1^{\alpha_1+2\alpha_{21}}g_{s-1}$; hence, $NF(\text{spoly}(f_7, g_s)|G) = 0$

Since all normal forms reduce to zero, G is a standard basis for I_C □

Corollary 3.9 $\{f_1{}^*, f_2{}^*, ..., f_7{}^*, g_0{}^*, ..., g_s{}^*, h_s{}^*, ..., h_l{}^*\}$ *is a standard basis for* I_C^**, where* $f_1{}^* = X_3X_4^2$*,* $f_2{}^* = X_1^{\alpha_{21}}X_4$*,* $f_3{}^* = X_3^{\alpha_3}$*,* $f_4{}^* = X_4^3$*,* $f_5{}^* = X_2X_4^2$*,* $f_6{}^* = X_2^{\alpha_2}X_3X_4$*,* $f_7^* = X_2^{2\alpha_2}X_3$ *and* $g_i{}^* = X_1^{i\alpha_1+(2i+2)\alpha_{21}+1}X_3^{\alpha_3-(i+1)}$ *for* $i = 1, 2, ..., s-1$*,* $g_s{}^* = X_2^{(2s+1)\alpha_2+1}X_4$*,* $h_j{}^* = X_1^{j\alpha_1+(2j+3)\alpha_{21}+1}X_3^{\alpha_3-(j+1)}$ *for* $j = s, s+1, ..., l-1$*,* $h_l{}^* = X_2^{(2l+2)\alpha_2+1}$*. Since* $X_1|f_2{}^*$*, the tangent cone is not Cohen-Macaulay by the criterion given in [1].*

4. Hilbert function

Let $P(I_S^*)$ denote the numerator of the Hilbert series of A/I_S^*

Theorem 4.1 *The numerator of the Hilbert series of the local ring R_S is*

$P(I_S^*) = 1-3t^3+3t^4-t^5-t^{\alpha_{21}+1}(1-t)(1+t-2t^2+t^3)-t^{\alpha_3}(1-t^{\alpha_{21}+1}-t^2(1-t^{\alpha_{21}}))-t^{\alpha_2+2}(1-t)(1-t^{\alpha_{21}})(1-t^{\alpha_3-1})-t^{2\alpha_2+1}(1-t)(1-t^{\alpha_3-1})-(1-t)^2(1-t^{2\alpha_2})R(t)-(1-t)^2(1-t^{\alpha_{21}})t^{(2s+1)\alpha_2+2}-(1-t)^2t^{(2l+2)\alpha_2+1}$*, where* $R(t) = t^{2\alpha_{21}+\alpha_3}\sum_{j=0}^{s-1} t^{j(\alpha_1+2\alpha_{21}-1)} + t^{s\alpha_1+(2s+3)\alpha_{21}+\alpha_3-s}\sum_{j=0}^{l-s-1} t^{j(\alpha_1+2\alpha_{21}-1)}$ *for any* $s>0$ *and* $R(t)=0$ *if* $s=0$ *and* $l=0$*.*

Proof To compute the Hilbert series, we use Algorithm 2.6 of [5] that is formed by continuous use of the proposition:

"If I is a monomial ideal with $I =< J, w >$, then the numerator of the Hilbert series of A/I is $P(I) = P(J) - t^{\deg w}P(J : w)$ and $P(w) = 1 - t^{\deg w}$, where w is a monomial and $\deg w$ is the total degree of w."

Taking $w_1 = {h_l}^*$, $w_2 = {h_s}^*$, $w_3 = {h_{l-1}}^*$, $w_4 = {h_{l-2}}^*, \cdots, w_{l-s+2} = {h_s}^*, w_{l-s+3} = {g_s}^*, \ldots, w_{l+2} = {g_0}^*$, $w_{l+3} = {f_7}^*$, $w_{l+4} = {f_6}^*$, $w_{l+5} = {f_3}^*$, $w_{l+6} = {f_2}^*$, $w_{l+7} = {f_4}^*$, $w_{l+8} = {f_5}^*$, $w_{l+9} = {f_1}^*$. If we set $J_0 = I_S^*$, $J_{i+1} = J_i - \{w_{i+1}\}$ for $i = 0, \cdots, t+8$ in the Algorithm, we get $P(J_i) = P(J_{i+1}) - t^{\deg w_{i+1}}P(J_i : w_{i+1})$ and we obtain the desired result.

□

Corollary 4.2 *The second Hilbert series of the local ring is* $Q(t) = (1+t+t^2+\cdots+t^{\alpha_{21}-1})(t+2t^2+t^4+t^5+\cdots+t^{\alpha_3}) + (1+t+t^2+\cdots+t^{\alpha_3-2})(1+t+t^2+\cdots+t^{2\alpha_2}-t^{\alpha_2+2}(1+t+\cdots+t^{\alpha_{21}-1})) - (1+t+\cdots+t^{\alpha_{21}-1})t^{(2s+1)\alpha_2+2} + t^{\alpha_3-1}(1+t+\cdots+t^{(2l+2)\alpha_2-\alpha_3+1}) - (1+t+\cdots+t^{2\alpha_2-1})R(t)$

Proof Since $P(I_S^*) = \dfrac{Q(t)}{(1-t)^3}$, the result is a direct consequence of theorem 4.1. □

Clearly, since the krull dimension is one, if there are no negative terms in the second Hilbert series, then the Hilbert function will be nondecreasing. We can state and prove the next theorem.

Theorem 4.3 *The local ring R_S has a nondecreasing Hilbert function if $l = 0$.*

Proof Since $\alpha_2 > \alpha_{21}+1$ by (4), $2\alpha_2 > \alpha_2+\alpha_{21}+1$; hence,

$Q(t) = (1+t+t^2+\cdots+t^{\alpha_{21}-1})(t+2t^2+t^4+t^5+\cdots+t^{\alpha_3}) + (1+t+t^2+\cdots+t^{\alpha_3-2})(1+t+t^2+\cdots+t^{\alpha_2+1}+t^{\alpha_2+\alpha_{21}+2}+\cdots+t^{2\alpha_2}) - (1+t+\cdots+t^{\alpha_{21}-1})t^{(2s+1)\alpha_2+2} + t^{\alpha_3-1}(1+t+\cdots+t^{(2l+2)\alpha_2-\alpha_3+1}) - (1+t+\cdots+t^{2\alpha_2-1})R(t)$

When $l = 0$, then $s = 0$ and $R(t) = 0$. There are two cases:

If $\alpha_3 \geq \alpha_2+3$:

$Q(t) = (1+t+t^2+\cdots+t^{\alpha_{21}-1})\left[t+2t^2+t^4+\cdots+t^{\alpha_2+1}+t^{\alpha_2+3}+\cdots+t^{\alpha_3}\right] + (1+t+\cdots+t^{\alpha_3-2})\left[1+t+\cdots+t^{\alpha_2+1} + t^{\alpha_2+\alpha_{21}+2}+\cdots+t^{2\alpha_2}\right] + t^{\alpha_3-1}\left[1+t+\cdots+t^{2\alpha_2-\alpha_3+1}\right]$

If $\alpha_3 < \alpha_2+3$:

$Q(t) = (1+t+t^2+\cdots+t^{\alpha_{21}-1})\left[t+2t^2+t^4+\cdots+t^{\alpha_3}\right] + (1+t+\cdots+t^{\alpha_3-2})\left[1+t+\cdots+t^{\alpha_2+1}+t^{\alpha_2+\alpha_{21}+2}+\cdots+t^{2\alpha_2}\right] + \left[t^{\alpha_3-1}+\cdots+t^{\alpha_2+1}+t^{\alpha_2+\alpha_{21}+2}+\cdots+t^{2\alpha_2}\right]$

In both cases, there are no negative terms in the second Hilbert series; hence, the Hilbert function is nondecreasing.

When $l > 0$, since $l \geq s$, and $\alpha_2 > \alpha_{21} + 1$, $(2l+2)\alpha_2 > (2s+1)\alpha_2 + \alpha_{21} + 1$ and $\alpha_3 < (2s+1)\alpha_2 + 3$, which means that all of the negative terms $-(1+t+\cdots+t^{\alpha_{21}-1})t^{(2s+1)\alpha_2+2}$ in $Q(t)$ will be cancelled out by the terms in $t^{\alpha_3-1}(1+t+\cdots+t^{(2l+2)\alpha_2-\alpha_3+1})$.

Then it is enough to show that the negative terms $-(1+t+\cdots+t^{2\alpha_2-1})R(t)$ will also be cancelled out. Note that

$(1+t+\cdots+t^{2\alpha_2-1})R(t) = (1+t+\cdots+t^{2\alpha_2-1})(t^{2\alpha_{21}+3} +$ higher degree terms $+t^{(l-1)\alpha_1+(2l+1)\alpha_{21}+\alpha_3-l+1} = t^{2\alpha_{21}+3} +$ some higher degree terms $+\, t^{(l-1)\alpha_1+(2l+1)\alpha_{21}+\alpha_3-l+2\alpha_2}$.

Since l is the smallest integer with $l\alpha_1 + (2l+3)\alpha_{21} + \alpha_3 \geq (2l+2)\alpha_2 + l + 1$, for $l-1$, we have $(l-1)\alpha_1 + (2l+1)\alpha_{21} + \alpha_3 < 2l\alpha_{21} + l \implies (l-1)\alpha_1 + (2l+1)\alpha_{21} + \alpha_3 - l + 2\alpha_2 < (2l+2)\alpha_2$.

Also, since $2\alpha_2 - 1 < \alpha_1 + 2\alpha_{21} - 1$, all of the terms in $(1+t+\cdots+t^{2\alpha_2-1})R(t)$ has coefficient 1.

Hence, all of the negative terms disappear in $Q(t)$ and the Hilbert function is nondecreasing.

□

5. Examples

Example 5.1 *Let* $\alpha_{21} = 12, \alpha_1 = 38, \alpha_2 = 20, \alpha_3 = 8, \alpha_4 = 3$. *Then* $k = 1$, $s = 0$ *and* $l = 0$ *and the corresponding standard basis is* $\{f_1 = X_1^{38} - X_3X_4^2, f_2 = X_2^{20} - X_1^{12}X_4, f_3 = X_3^8 - X_1^{25}X_2, f_4 = X_4^3 - X_1X_2^{19}X_3^7, f_5 = X_1^{13}X_3^7 - X_2X_4^2, f_6 = X_1^{50} - X_2^{20}X_3X_4, f_7 = X_1^{62} - X_2^{40}X_3, g_0 = X_2^{21}X_4 - X_1^{25}X_3^7, h_0 = X_2^{41} - X_1^{37}X_3^7\}$. *The fisrt Hilbert series is* $P(I_S^*) = 1 - 3t^3 + 3t^4 - t^5 - t^8 + t^{10} - t^{13} + 3t^{15} - 3t^{16} + t^{17} + t^{21} - 3t^{22} + 3t^{23} - t^{24} + t^{29} - t^{30} + 2t^{34} - 3t^{35} + t^{36} - 3t^{41} + 4t^{42} - t^{43} + t^{48} - t^{49}$ *and the second Hilbert series is* $Q(t) = 1 + 3t + 6t^2 + 7t^3 + 9t^4 + 11t^5 + 13t^6 + 15t^7 + 16t^8 + 16t^9 + 16t^{10} + 16t^{11} + 16t^{12} + 15t^{13} + 13t^{14} + 13t^{15} + 12t^{16} + 11t^{17} + 10t^{18} + 9t^{19} + 8t^{20} + 8t^{21} + 6t^{22} + 5t^{23} + 4t^{24} + 3t^{25} + 2t^{26} + t^{27} + 2t^{34} + 3t^{35} + 4t^{36} + 5t^{37} + 6t^{38} + 7t^{39} + 8t^{40} + 6t^{41} + 5t^{42} + 4t^{43} + 3t^{44} + 2t^{45} + t^{46}$. *Since there are no negative terms, Hilbert function is nondecreasing.*

Example 5.2 *For* $\alpha_{21} = 11$, $\alpha_1 = 62, \alpha_2 = 40$, $\alpha_3 = 14$, $\alpha_4 = 3$, *we have* $k = 2$, $l = 12$ *and* $s = 3$. *Corresponding standard basis is:*

$\{f_1 = X_1^{62} - X_3X_4^2, f_2 = X_2^{40} - X_1^{11}X_4, f_3 = X_3^{14} - X_1^{50}X_2, f_4 = X_4^3 - X_1X_2^{39}X_3^{13}, f_5 = X_1^{12}X_3^{13} - X_2X_4^2, f_6 = X_1^{73} - X_2^{40}X_3X_4, f_7 = X_1^{84} - X_2^{80}X_3, g_0 = X_2^{41}X_4 - X_1^{23}X_3^{13}, g_1 = X_2^{121}X_4 - X_1^{107}X_3^{12}, g_2 = X_2^{201}X_4 - X_1^{191}X_3^{11}, g_3 = X_2^{281}X_4 - X_1^{275}X_3^{10}, h_4 = X_2^{321} - X_1^{286}X_3^{10}, h_5 = X_2^{401} - X_1^{370}X_3^9, h_6 = X_2^{481} - X_1^{454}X_3^8, h_7 = X_2^{561} - X_1^{538}X_3^7, h_8 = X_2^{641} - X_1^{622}X_3^6, h_9 = X_2^{721} - X_1^{706}X_3^5, h_{10} = X_2^{801} - X_1^{790}X_3^4, h_{11} = X_2^{881} - X_1^{874}X_3^3, h_{12} = X_2^{961} - X_1^{958}X_3^2, h_{13} = X_2^{1041} - X_1^{1042}X_3, \}$. *The first Hilbert series is* $P(I_S^*) = 1 - 3t^3 + 3t^4 - t^5 - t^{12} + 2t^{14} - 3t^{15} + 2t^{16} + t^{26} - t^{27} - t^{36} + 2t^{37} - t^{38} - t^{42} + t^{43} + t^{53} - t^{54} + t^{55} - t^{56} - t^{66} + t^{67} - t^{81} + t^{82} + t^{94} - t^{95} + t^{116} - 2t^{117} + t^{118} - t^{119} + 2t^{120} - t^{121} + t^{199} - 2t^{200} + t^{201} - t^{202} + 2t^{203} - t^{204} + 2t^{293} - 2t^{294} + t^{295} - t^{296} + 2t^{297} - t^{298} + t^{376} - 2t^{377} + t^{378} - t^{379} + 2t^{380} - t^{381} + t^{459} - 2t^{460} + t^{461} - t^{462} + 2t^{463} - t^{464} + t^{542} - 2t^{543} + t^{544} - t^{545} + 2t^{546} - t^{547} + t^{625} - 2t^{626} + t^{627} - t^{628} + 2t^{629} - t^{630} + t^{708} - 2t^{709} + t^{710} - t^{711} + 2t^{712} - t^{713} + t^{791} - t^{792} + t^{793} - t^{794} + 2t^{795} - t^{796} + t^{824} - 2t^{875} + t^{876} - t^{877} + 2t^{878} - t^{879} + t^{957} - 2t^{958} + t^{959} - t^{960} + 2t^{961} - t^{962} + t^{1040} - 3t^{1041} + 3t^{1042} - t^{1043}$ *and the second Hilbert series is* $Q(t) = 1 + 3t + 6t^2 + 7t^3 + 9t^4 + 11t^5 + 13t^6 + 15t^7 + 17t^8 + 19t^9 + 21t^{10} + 23t^{11} + 24t^{12} + 24t^{13} + 25t^{14} + 24t^{15} + 23t^{16} + 22t^{17} + 21t^{18} + 20t^{19} + 19t^{20} + 18t^{21} + 17t^{22} + 16t^{23} + 15t^{24} + 14t^{25} + 14t^{26} + 14t^{27} + 14t^{28} + 14t^{29} + 14t^{30} + 14t^{31} + 14t^{32} + 14t^{33} + 14t^{34} + 14t^{35} + 13t^{36} + 13t^{37} + 13t^{38} + 13t^{39} + 13t^{40} + 13t^{41} +$

$12t^{42}+11t^{43}+10t^{44}+9t^{45}+8t^{46}+7t^{47}+6t^{48}+5t^{49}+4t^{50}+3t^{51}+2t^{52}+2t^{53}+2t^{54}+3t^{55}+4t^{56}+5t^{57}+6t^{58}+7t^{59}+8t^{60}+9t^{61}+10t^{62}+11t^{63}+12t^{64}+13t^{65}+13t^{66}+13t^{67}+13t^{68}+13t^{69}+13t^{70}+13t^{71}+13t^{72}+13t^{73}+13t^{74}+13t^{75}+13t^{76}+13t^{77}+13t^{78}+13t^{79}+13t^{80}+12t^{81}+11t^{82}+10t^{83}+9t^{84}+8t^{85}+7t^{86}+6t^{87}+5t^{88}+4t^{89}+3t^{90}+2t^{91}+t^{92}+t^{116}+t^{117}+t^{118}+t^{199}+t^{200}+t^{201}+t^{293}+t^{294}+t^{295}+t^{376}+t^{377}+t^{378}+t^{459}+t^{460}+t^{461}+t^{542}+t^{543}+t^{544}+t^{625}+t^{626}+t^{627}+t^{708}+t^{709}+t^{710}+t^{791}+t^{792}+t^{793}+t^{874}+t^{875}+t^{876}+t^{957}+t^{958}+t^{959}+t^{1040}$.
Since there are no negative terms, Hilbert function is nondecreasing.

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