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Numerical solutions of differential equations having cubic nonlinearity using Boole collocation method

Kübra ERDEM BİÇER*, **Hale Gül DAĞ**
Department of Mathematics, Manisa Celal Bayar University, Manisa, Turkey

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Abstract: The aim of the study is to develop a numerical method for the solution of cubic nonlinear differential equations in which the numerical solution is based on Boole polynomials. That solution is in the form of the truncated series and gives approximate solution for nonlinear equations of cubic type. In this method, firstly, the matrix form of the serial solution is set and the nonlinear differential equation is converted into a matrix equation system. By adding the effect of both the conditions of the problem and the collocation points to this system of equations, we obtain the new system of equations. The coefficients of Boole-based serial solution are obtained from the solution of the resulting system of equations. The theoretical part is reinforced by considering three test problems. Numerical data for Boole solutions of test problems and absolute error functions are given in tables and figures.

Key words: Boole polynomials, numerical methods, the cubic nonlinear differential equations, the error analysis

1. Introduction

Mathematical modeling is important because it appears even in the simplest equations in science and engineering. Among these models, the models of nonlinear differential equations have a very important place. Problems such as heat conduction and transfer, diffusion problems, financial mathematics, and nuclear physics are modeled by the nonlinear differential equations in many papers [5, 15, 17, 19, 20, 31].

The methods that are developed on the solutions of these models are as important as the modeling of nonlinear differential equations. Some of those methods are well-known and applied in many science contexts can be summarized as follows. The approximate solutions of the quadratic nonlinear differential equations are obtained by using Bernoulli matrix-collocation method [6]. The numerical solutions of Lane-Emden type and Abel-type nonlinear differential equations are gained with Taylor matrix-collocation method [2, 4]. The Abel equation is also solved by the collocation method based on Chebyshev polynomials [8]. A collocation method based on the Berstein polynomials is improved for the numerical solutions of fractional Riccati type differential equations [32]. The solutions of second-order nonlinear Lane-Emden type pantograph differential equations are calculated by Pell-Lucas collocation method [35]. The approximate solutions of the Riccati differential-difference equations and a class of high-order nonlinear differential equations are obtained by the collocation method based on the Bessel functions of the first kind [33, 36]. The Legendre wavelet method is used in the solutions of high-order nonlinear ordinary differential equations with variable and proportional delays [12]. Besides, the improved

*Correspondence: kubra.erdem@cbu.edu.tr
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Bessel collocation method is applied to the class of Lane-Emden differential equations [37]. Another collocation method in which the Mott polynomials are considered, is developed for the approximate solution of the model differential equations involving specific nonlinearities of quartic type [20]. Additionally, the numerical methods such as the modified differential transform method [22], Homotopy Perturbation method [1], Simplest equation method [18], Euler matrix method [21], Wavelet Galerkin method [26], Haar wavelet quasi linearization method [28], Bessel collocation method [34], Chelyshkov matrix method [14], Taylor wavelet method [11], cubic Hermite collocation method [7], Lucas matrix method [9, 10], and Newton-Product method [3] have been developed to obtain the solutions of nonlinear equations.

In this study, the numerical method based on the Boole polynomial is improved to obtain the approximate and exact solutions of the cubic nonlinear differential equations. The Boole polynomial and the collocation points are used in this numerical method. In Section 2, the cubic nonlinear differential equation, the Boole polynomial, and its matrix relations are presented. The Boole collocation method is improved in Section 3. In Section 4, the error function is given. In Section 5, the approximate solutions in terms of Boole polynomial are calculated, and the results are compared in the table and the figure.

2. The main problem, Boole polynomials and their matrix forms

A class of the differential equation having cubic nonlinearity can be written as

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x)y^{(p)}(x)y^{(q)}(x)y^{(r)}(x) = g(x), \tag{2.1}$$

where $P_k(x)$, $Q_{pqr}(x)$, and $g(x)$ are functions in the interval $-\infty < a \le x, t \le b < \infty$. The solution of the equation (2.1) is found with initial-boundary conditions:

$$\sum_{k=0}^{m-1} a_{kj}y^{(k)}(a) + b_{kj}y^{(k)}(b) + c_{kj}y^{(k)}(c) = \lambda_j, \quad j = 0, 1, 2, \dots, m-1. \tag{2.2}$$

The generated function of Boole polynomials is defined as

$$\sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n = \frac{2(1+t)^x}{2+t} \tag{2.3}$$

so that the general form of the Boole polynomials is

$$R_n(x) = \sum_{m=0}^{n} \frac{(-1)^m}{2^m} \binom{x}{n-m} \tag{2.4}$$

[16, 27]. For $n = 3$, the Boole polynomials is obtained as follows:

$$R_0(x) = 1$$

$$R_1(x) = x - \frac{1}{2}$$

$$R_2(x) = x^2 - 2x + \frac{1}{2}$$

$$R_3(x) = x^3 - \frac{9}{2}x^2 + 5x - \frac{3}{4}$$

According to the base functions in which Boole polynomials are considered in this study, the solution of the equation (2.1) is considered in the truncated Boole series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n R_n(x) \tag{2.5}$$

where $a_n, n = 0, 1, 2, , N$ are unknown Boole coefficients. The Boole polynomial (2.4) is written in terms of matrix form as

$$\mathbf{R}(x) = \mathbf{X}(x)\mathbf{H}^{\mathbf{T}} \tag{2.6}$$

where

$$\mathbf{R}(x) = \begin{bmatrix} 1 & x - \frac{1}{2}, & \dots & R_N(t) \end{bmatrix}, \mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \dots & x^N \end{bmatrix}$$

and

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \dots \\ -\frac{1}{2} & 1 & 0 & \dots \\ \frac{1}{2} & -2 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The matrix form of the solution (2.5) is

$$y(x) \cong y_N(x) = \mathbf{R}(x)\mathbf{A} \tag{2.7}$$

and kth derivative is given as

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{R}^{(k)}(x)\mathbf{A}, \quad \text{for} \quad k = 0, 1 \dots, m, \tag{2.8}$$

where

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}.$$

The relation (2.6) is written in the relation (2.8) and the matrix relation is obtained as follows:

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{X}^{(k)}(x)\mathbf{H}^{\mathbf{T}}\mathbf{A} = \mathbf{X}(x)\mathbf{E}^k\mathbf{H}^{\mathbf{T}}\mathbf{A}, \tag{2.9}$$

where the matrix $\mathbf{E}$ is used to get the connection between Taylor polynomials and their derivatives. Using the matrix (2.6) in the matrix (2.9) yields

$$y^{(k)}(x) \cong y_N^{(k)}(x) = \mathbf{R}(x)\mathbf{D}^k\mathbf{A}, \tag{2.10}$$

where $\mathbf{D}$ is derivative transition matrix of Boole polynomials and is expressed as

$$\mathbf{D}^k = (\mathbf{H})^{(-1)}\mathbf{E}^k\mathbf{H}^T. \tag{2.11}$$

Now, we write the matrix forms of the nonlinear parts of equation (2.1) for different situations that vary according to the equation that can be handled. Therefore, the following matrix equations are constructed as

$$[y(x)]^3 = \mathbf{R}(x)\overline{\mathbf{R}}(x)\overline{\overline{\mathbf{R}}}(x)\overline{\overline{\mathbf{A}}}, \tag{2.12}$$

$$[y(x)]^2 y^{(1)}(x) = \mathbf{R}(x)\overline{\mathbf{R}}(x)\overline{\overline{\mathbf{R}}}(x)\overline{\overline{\mathbf{D}}}\,\overline{\overline{\mathbf{A}}}, \tag{2.13}$$

$$[y^{(1)}(x)]^2 y(x) = \mathbf{R}(x)\mathbf{D}\overline{\mathbf{R}}(x)\overline{\mathbf{D}}\,\overline{\overline{\mathbf{R}}}(x)\overline{\overline{\mathbf{A}}}, \tag{2.14}$$

$$[y^{(1)}(x)]^3 = \mathbf{R}(x)\mathbf{D}\overline{\mathbf{R}}(x)\overline{\mathbf{D}}\,\overline{\overline{\mathbf{R}}}(x)\overline{\overline{\mathbf{D}}}\,\overline{\overline{\mathbf{A}}}, \tag{2.15}$$

where

$$\mathbf{R}(x) = \begin{bmatrix} R_0(x) & R_1(x) & \cdots & R_N(x) \end{bmatrix}_{(N+1)x(N+1)},$$

$$\overline{\mathbf{R}}(x) = diag\begin{bmatrix} \mathbf{R}(x) & \mathbf{R}(x) & \cdots & \mathbf{R}(x) \end{bmatrix}_{(N+1)x(N+1)^2},$$

$$\overline{\overline{\mathbf{R}}}(x) = diag\begin{bmatrix} \overline{\mathbf{R}}(x) & \overline{\mathbf{R}}(x) & \cdots & \overline{\mathbf{R}}(x) \end{bmatrix}_{(N+1)^2x(N+1)^3},$$

$$\overline{\mathbf{D}}(x) = diag\begin{bmatrix} \mathbf{D} & \mathbf{D} & \cdots & \mathbf{D} \end{bmatrix}_{(N+1)^2x(N+1)^2},$$

$$\overline{\overline{\mathbf{D}}}(x) = diag\begin{bmatrix} \overline{\mathbf{D}} & \overline{\mathbf{D}} & \cdots & \overline{\mathbf{D}} \end{bmatrix}_{(N+1)^3x(N+1)^3},$$

and

$$\overline{\overline{\mathbf{A}}}(x) = diag\begin{bmatrix} a_0\overline{\mathbf{A}} & a_1\overline{\mathbf{A}} & \cdots & a_N\overline{\mathbf{A}} \end{bmatrix}^T_{(N+1)x(N+1)^3}.$$

3. The collocation method

In order to find the unknown coefficients in the approximate series solution of equation (2.1) in terms of Boole polynomials, we create new matrix forms by substituting the collocation points in that equation. The collocation points

$$x_i = a + \frac{b-a}{N}i, \qquad i = 0, 1, \ldots, N \tag{3.1}$$

are written in equation (2.1) to get

$$\sum_{k=0}^{m} P_k(x_i)y^{(k)}(x_i) + \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x_i)y^{(p)}(x_i)y^{(q)}(x_i)y^{(r)}(x_i) = g(x_i). \tag{3.2}$$

Here to get the matrix forms we use the following assignment:

$$y^{(k)}(x_i) \cong y_N^{(k)}(x_i) = \mathbf{R}(x_i)\mathbf{D}^k\mathbf{A} \tag{3.3}$$

and

$$y^{(p)}(x_i)y^{(q)}(x_i)y^{(r)}(x_i) = \mathbf{Y}^{(p,q,r)} = \begin{bmatrix} y^{(p)}(x_0)y^{(q)}(x_0)y^{(r)}(x_0) \\ y^{(p)}(x_1)y^{(q)}(x_1)y^{(r)}(x_1) \\ \vdots \\ y^{(p)}(x_N)y^{(q)}(x_N)y^{(r)}(x_N) \end{bmatrix}.$$

For the different values of p, q, and r, the following equalities are obtained:

$$\mathbf{Y}^{(0,0,0)} = \mathbf{R}^*_{0,0,0}\overline{\overline{\mathbf{A}}}, \mathbf{Y}^{(0,0,1)} = \mathbf{R}^*_{0,0,1}\overline{\overline{\mathbf{A}}}, \mathbf{Y}^{(1,1,0)} = \mathbf{R}^*_{1,1,0}\overline{\overline{\mathbf{A}}}, \mathbf{Y}^{(1,1,1)} = \mathbf{R}^*_{1,1,1}\overline{\overline{\mathbf{A}}}, \tag{3.4}$$

where

$$\mathbf{R}^*_{0,0,0} = \begin{bmatrix} \mathbf{R}(x_0)\overline{\mathbf{R}}(x_0)\overline{\overline{\mathbf{R}}}(x_0) \\ \mathbf{R}(x_1)\overline{\mathbf{R}}(x_1)\overline{\overline{\mathbf{R}}}(x_1) \\ \vdots \\ \mathbf{R}(x_N)\overline{\mathbf{R}}(x_N)\overline{\overline{\mathbf{R}}}(x_N) \end{bmatrix}, \quad \mathbf{R}^*_{0,0,1} = \begin{bmatrix} \mathbf{R}(x_0)\overline{\mathbf{R}}(x_0)\overline{\overline{\mathbf{R}}}(x_0)\overline{\overline{\mathbf{D}}} \\ \mathbf{R}(x_1)\overline{\mathbf{R}}(x_1)\overline{\overline{\mathbf{R}}}(x_1)\overline{\overline{\mathbf{D}}} \\ \vdots \\ \mathbf{R}(x_N)\overline{\mathbf{R}}(x_N)\overline{\overline{\mathbf{R}}}(x_N)\overline{\overline{\mathbf{D}}} \end{bmatrix},$$

$$\mathbf{R}^*_{1,1,0} = \begin{bmatrix} \mathbf{R}(x_0)\mathbf{D}\overline{\mathbf{R}}(x_0)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_0) \\ \mathbf{R}(x_1)\mathbf{D}\overline{\mathbf{R}}(x_1)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_1) \\ \vdots \\ \mathbf{R}(x_N)\mathbf{D}\overline{\mathbf{R}}(x_N)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_N) \end{bmatrix}, \quad \mathbf{R}^*_{1,1,1} = \begin{bmatrix} \mathbf{R}(x_0)\mathbf{D}\overline{\mathbf{R}}(x_0)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_0)\overline{\overline{\mathbf{D}}} \\ \mathbf{R}(x_1)\mathbf{D}\overline{\mathbf{R}}(x_1)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_1)\overline{\overline{\mathbf{D}}} \\ \vdots \\ \mathbf{R}(x_N)\mathbf{D}\overline{\mathbf{R}}(x_N)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(x_N)\overline{\overline{\mathbf{D}}} \end{bmatrix},$$

Then, the equation (3.2) becomes as follows:

$$\sum_{k=0}^{m} \mathbf{P}_k\mathbf{R}\mathbf{D}^k\mathbf{A} + \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} \mathbf{Q}_{pqr}\mathbf{R}^*_{p,q,r}\overline{\overline{\mathbf{A}}} = \mathbf{G}, \tag{3.5}$$

where

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}(x_0) \\ \mathbf{R}(x_1) \\ \vdots \\ \mathbf{R}(x_1) \end{bmatrix} = \begin{bmatrix} R_0(x_0) & R_1(x_0) & \cdots & R_N(x_0) \\ R_0(x_1) & R_1(x_1) & \cdots & R_N(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ R_0(x_N) & R_1(x_N) & \cdots & R_N(x_N) \end{bmatrix}_{(N+1)x(N+1)}, \mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \dots & 0 \\ 0 & P_k(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_k(x_N) \end{bmatrix}_{(N+1)x(N+1)},$$

$$\mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \mathbf{Q}_{pqr} = \begin{bmatrix} Q_{pqr}(x_0) & 0 & \dots & 0 \\ 0 & Q_{pqr}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{pqr}(x_N) \end{bmatrix}_{(N+1)x(N+1)}.$$

For the simplicity, when we denote $\mathbf{W}, \mathbf{V}$ for the linear and the nonlinear part of the equation (3.5) respectively, we get

$$\mathbf{W}\mathbf{A} + \mathbf{V}\overline{\overline{\mathbf{A}}} = \mathbf{G}, \tag{3.6}$$

where

$$\mathbf{W} = [w_{ij}] = \sum_{k=0}^{m} \mathbf{P}_k\mathbf{R}\mathbf{D}^k, \quad \text{for} \quad i, j = 0, 1, \dots, N \tag{3.7}$$

and

$$\mathbf{V} = [v_{mn}] = \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} \mathbf{Q}_{pqr}\mathbf{R}^*_{p,q,r} \tag{3.8}$$

for $m = 0, 1, , N,\ n = 0, 1, , (N+1)^3 - 1$. The augmented matrix form of the relation (3.6) is written as

$$[\mathbf{W}; \mathbf{V}; \mathbf{G}] = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & v_{00} & v_{01} & \dots & v_{0N} & : & g(x_0) \\ w_{10} & w_{11} & \dots & w_{1N} & ; & v_{10} & v_{11} & \dots & v_{1N} & : & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N0} & w_{N1} & \dots & w_{NN} & ; & v_{N0} & v_{N1} & \dots & v_{NN} & : & g(x_N) \end{bmatrix} \tag{3.9}$$

Using the relation (2.10), the matrix form of the conditions (2.2) is obtained as

$$\sum_{k=0}^{m-1} [a_{kj}\mathbf{R}(a) + b_{kj}\mathbf{R}(b) + c_{kj}\mathbf{R}(c)]\mathbf{D}^k\mathbf{A} = \lambda_j, \tag{3.10}$$

i.e.

$$\mathbf{UA} + \mathbf{0}^*\overline{\overline{\mathbf{A}}} = \lambda \tag{3.11}$$

or

$$[\mathbf{U}; \mathbf{0}^* : \overline{\overline{\mathbf{A}}}] = \begin{bmatrix} u_{00} & u_{01} & \dots & u_{0N} & ; & 0 & 0 & \dots & 0 & : & \lambda_0 \\ u_{10} & u_{11} & \dots & u_{1N} & ; & 0 & 0 & \dots & 0 & : & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \dots & u_{m-1,N} & ; & 0 & 0 & \dots & 0 & : & \lambda_{m-1} \end{bmatrix} \tag{3.12}$$

in which the matrices are indicated as

$$\mathbf{U} = \begin{bmatrix} u_{j0} & u_{j1} & \dots & u_{jN} \end{bmatrix}, \quad j = 0, 1, 2, \dots, m-1$$

$$\lambda = \begin{bmatrix} \lambda_0 & \lambda_1 & \dots & \lambda_N \end{bmatrix} \text{ and } \mathbf{0}^* = \begin{bmatrix} 0 & 0 & \dots & 0 \end{bmatrix}.$$

In the final step, the m rows of the matrix (3.9) are deleted, and by replacing the matrix forms of conditions (3.12), the new augmented matrix is obtained. From the solution of that matrix, the unknown Boole coefficients are calculated. Therefore, by writing the obtained coefficients in the solution (2.5), the semianalytical solution as in the form of truncated Boole series is found.

4. Accuracy of solution

In this section, the accuracy of the solutions is examined, and it is written as

$$E(x_r) = \left| \sum_{k=0}^{m} P_k(x_r)y^{(k)}(x_r) + \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x_r)y^{(p)}(x_r)y^{(q)}(x_r)y^{(r)}(x_r) - g(x_r) \right| \cong 0 \tag{4.1}$$

or

$$E(x_r) \leq 10^{-k_r}(k_r \text{is any positive integer}) \quad \text{for} \quad x_r \in [a, b], r = 0, 1, 2, \dots \tag{4.2}$$

The Boole solution must be satisfied by the equation (2.1) approximately. If max $10^{-k_r} = 10^{-k}$ (k_r is any positive integer) is defined, the truncation limit N is increased until the variation of $E(x_r)$ at each point becomes smaller than the defined max 10^{-k} [6, 23]. Since the exact solution is $y(x)$ and the approximate solution is $y_N(x)$, the absolute error function is expressed as $e_N(x) = |y(x) - y_N(x)|$. Also, the residual function is defined as follows [36, 38–40]:

$$E_N(x) = \sum_{k=0}^{m} P_k(x)y_N^{(k)}(x) + \sum_{p=0}^{1}\sum_{q=0}^{p}\sum_{r=0}^{q} Q_{pqr}(x)y_N^{(p)}(x)y_N^{(q)}(x)y_N^{(r)}(x) - g(x). \tag{4.3}$$

5. Numerical examples

Example 1. First, the exact solution of the cubic nonlinear differential equation

$$x^2y^{(1)}(x) - 2y(x) + [y^{(1)}]^2y(x) - x[y^{(1)}(x)]^3 = -4x^4 + 2x^3 - 6x^2 + 2 \qquad 0 \leq x \leq 1 \tag{5.1}$$

will be found with the initial condition $y(0) = -1$, where $P_0(x) = -2, P_1(x) = x^2, Q_{110}(x) = 1, Q_{111}(x) = -x$. For $N = 2$, the solution (2.5) is obtained as

$$y(x) = y_2(x) = \sum_{n=0}^{2} a_nR_n(x) = a_0R_0(x) + a_1R_1(x) + a_2R_2(x) \tag{5.2}$$

and the collocation points (3.1) is $x_0 = 0, x_1 = \frac{1}{2}$ and $x_2 = 1$. According to the fundamental matrix relation (3.5), the problem (5.1) is written as

$$(\mathbf{P}_0\mathbf{R} + \mathbf{P}_1\mathbf{R}\mathbf{D}^1)\mathbf{A} + (\mathbf{Q}_{110}\mathbf{R}^*_{1,1,0} + \mathbf{Q}_{111}\mathbf{R}^*_{1,1,1})\overline{\overline{\mathbf{A}}} = \mathbf{G}, \tag{5.3}$$

where

$$\mathbf{P}_0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{3x3}, \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}, \mathbf{R} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{4} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}_{3x3},$$

$$\mathbf{D}^1 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}, \mathbf{Q}_{110} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3x3}, \mathbf{Q}_{111} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}_{3x3},$$

$$\mathbf{R}^*_{1,1,0} = \begin{bmatrix} \mathbf{R}(0)\mathbf{D}\overline{\mathbf{R}}(0)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(0) \\ \mathbf{R}(\frac{1}{2})\mathbf{D}\overline{R}(\frac{1}{2})\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(\frac{1}{2}) \\ \mathbf{R}(1)\mathbf{D}\overline{\mathbf{R}}(1)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(1) \end{bmatrix}_{3x3}, \mathbf{R}^*_{1,1,1} = \begin{bmatrix} \mathbf{R}(0)\mathbf{D}\overline{\mathbf{R}}(0)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(0)\overline{\overline{\mathbf{D}}} \\ \mathbf{R}(\frac{1}{2})\mathbf{D}\overline{\mathbf{R}}(\frac{1}{2})\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(\frac{1}{2})\overline{\overline{\mathbf{D}}} \\ \mathbf{R}(1)\mathbf{D}\overline{\mathbf{R}}(1)\overline{\mathbf{D}}\ \overline{\overline{\mathbf{R}}}(1)\overline{\overline{\mathbf{D}}} \end{bmatrix}_{3x3}, \mathbf{G} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ -6 \end{bmatrix}.$$

Using the condition (3.10) by decomposing it into two parts, namely, the linear condition and the nonlinear condition of the problem (5.1) are obtained as

$$\mathbf{U} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}_{1x3} \quad \textit{and} \quad \mathbf{0}^* = \begin{bmatrix} 0 & 0 & \ldots & 0 \end{bmatrix}_{1x27}. \tag{5.4}$$

The matrix equation which is obtained by replacing the last row of (5.3) with the condition matrix (5.4), is solved for the unknown Boole coefficients and so

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

is found. Therefore, the approximated solution of the equation (5.1) given in the form of (2.5) is obtained as

$$y(x) = x^2 - 1,$$

which is also the exact solution of the equation (5.1).

Example 2. The Duffing equation

$$y^{(2)}(x) + 2y^{(1)}(x) + y(x) + 8y^3(x) = e^{-3x} \tag{5.5}$$

with the initial conditions $y(0) = \frac{1}{2},\ \ y^{(1)}(0) = -\frac{1}{2}$ is considered [24, 25, 30]. The exact solution of this equation is

$$y(x) = 0.5e^{-x}.$$

In [30], the authors have solved the problem (5.5) using the Adomian decomposition method (ADM) and obtained 8^{th} degree polynomial by taking 7 iteration. Also, in [25], the authors have solved the problem (5.5) using the Laguerre wavelet method (LWM) for $M = 5$ and obtained 5^{th} degree polynomial. For $M = 7$, in [24], the authors have solved with Taylor wavelet method (TWM) and obtained 6^{th} degree polynomial. In this study, for $N = 5, 8$ the problem (5.5) is solved by the Boole collocation method (BCM) and the absolute error functions $|e_N(x)|$ of the problem (5.5) is obtained. These results are compared with the Laguerre wavelet method, the Adomian decomposition method and the Taylor wavelet method, in the Table 1. For $N = 5, 8$, the Boole solutions $y_N(x)$ and the residual functions $E_N(x)$ of the problem (5.5) are calculated. The values of the Boole solutions and the exact solution are compared in Figure 1. The residual functions of the problem (5.5) are given in Figure 2, for $N = 5, 8$.

Table 1. The comparison of the obtained absolute error functions of the presented method, the LWM, the ADM and the TWM for the problem (5.5)

	$\|e_N(x)\|$ for BCM	$\|e_N(x)\|$ for LWM [25]	$\|e_N(x)\|$ for BCM	$\|e_N(x)\|$ for ADM [30]	$\|e_N(x)\|$ for TWM [24]
x_i	$N = 5$	$M = 5$	$N = 8$	7 iteration	$M = 7$
0	0	0	0	0	0
0.2	2.8618e-07	1.08e-09	2.6700e-11	5.47e-07	1.09e-09
0.4	4.7533e-07	5.13e-09	4.3417e-11	6.29e-05	5.13e-09
0.6	6.6969e-07	7.99e-10	4.9505e-11	9.86e-04	8.00e-10
0.8	1.2269e-06	5.08e-11	4.0439e-11	6.93e-03	5.07e-11
1.0	3.1614e-05	1.58e-06	4.0643e-09	3.18e-02	1.58e-06

Example 3. The Boole solutions of the following Van Der Pol differential equation

$$y^{(2)}(x) - y^{(1)}(x)(1 - y^2(x)) + y(x) = (2 + \sin(x))\cos(x)\sin(x) + 1 \tag{5.6}$$

with the initial conditions $y(0) = y^{(0)} = 1$ are investigated [13]. The exact solution of the equation (5.6) is

$$y(x) = 1 + \sin(x).$$

The Boole solutions $y_N(x)$ and the error functions $E_N(x)$ of the problem (5.6) are calculated by the Boole collocation method (BCM), for $N = 2, 4, 5$. In Figure 3, the exact solution and the Boole solutions are compared. The values of the residual functions are given in Figure 4, for $N = 2, 4, 5$. Also, the authors in reference [13] have solved the problem (5.6) using the Laguerre matrix method (LMM). The absolute error functions $|e_N(x)|$ of the problem (5.6) is obtained by the Boole collocation method (BCM) for $N = 2, 4, 5$. In Table 2, these results are compared with the Laguerre matrix method (LMM).

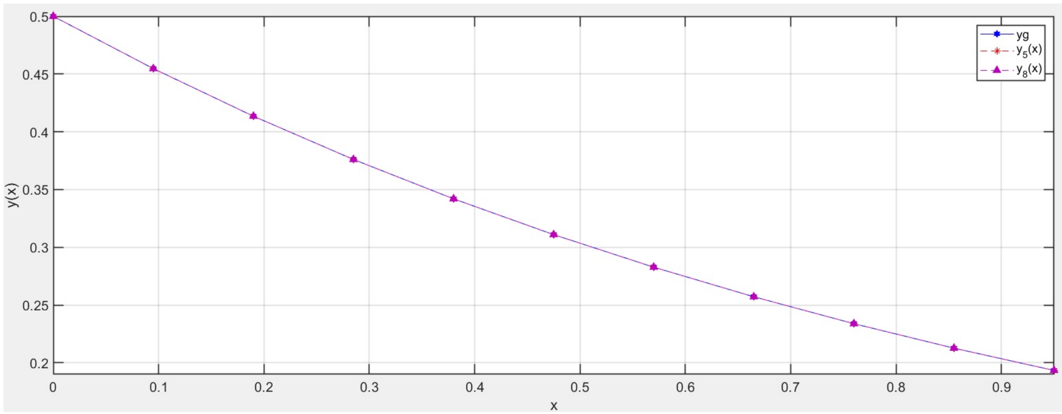

Figure 1. The comparison of the exact solution and the Boole solution of the problem (5.5) for $N = 5, 8$.

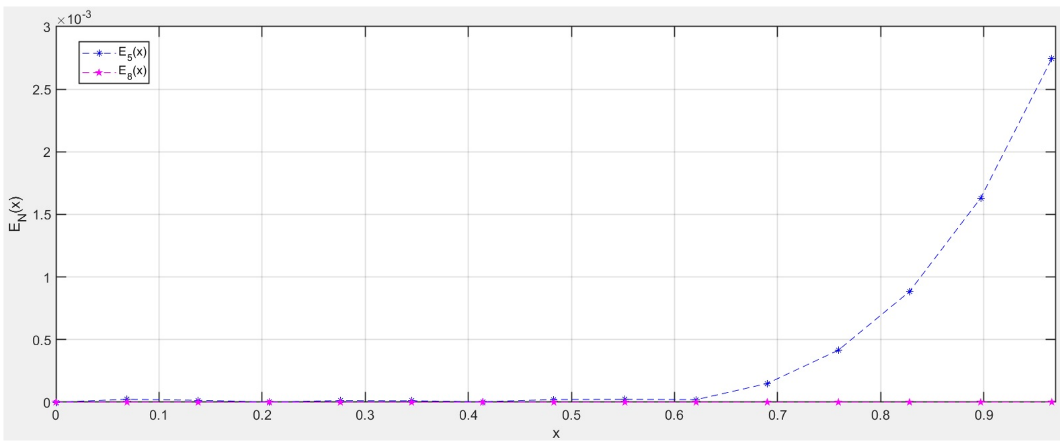

Figure 2. The comparison of the residual functions for the problem (5.5) for $N = 5, 8$.

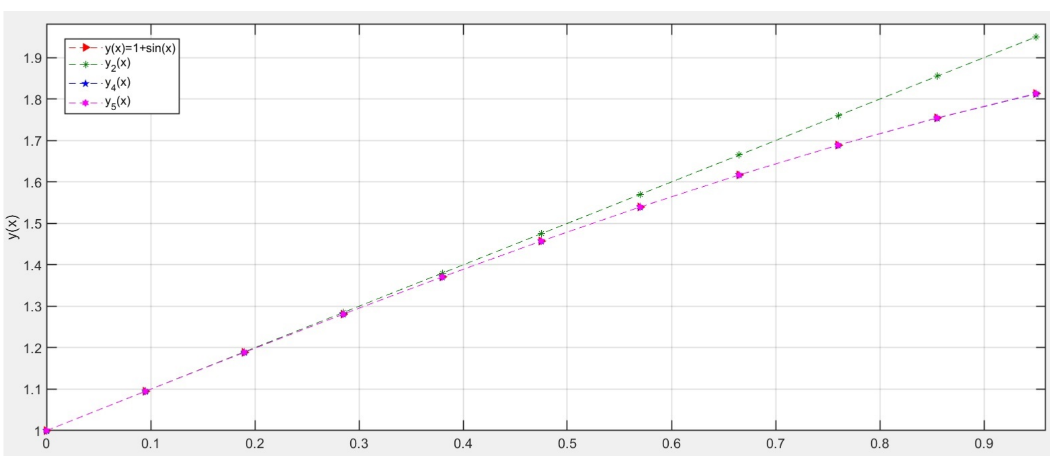

Figure 3. The comparison of the exact solution and the Boole solution of the problem (5.6) for $N = 2, 4, 5$.

Example 4. In the last example, the differential equation is considered

$$y^{(2)}(x) - \mu(1 - y^2(x))y^{(1)}(x) + y(x) = 2\sin^3(x), \qquad 0 < x < 1 \tag{5.7}$$

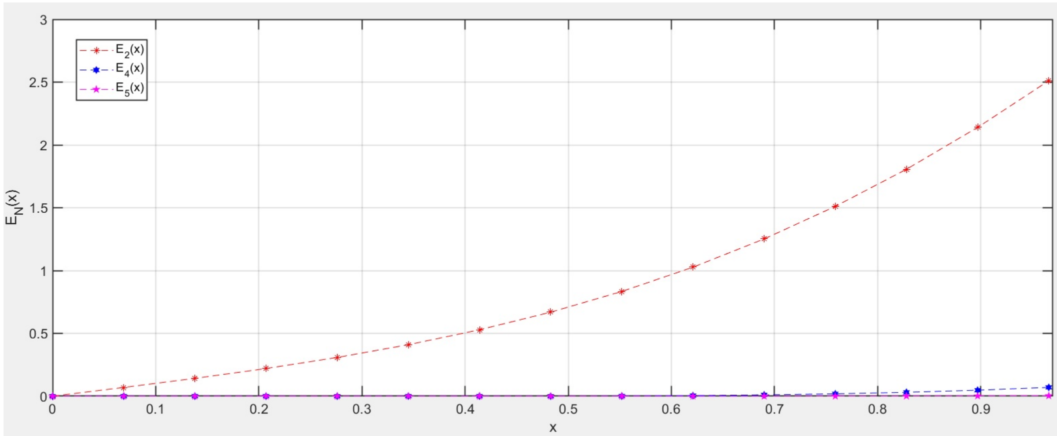

Figure 4. The comparison of the residual functions for the problem (5.6) for $N = 2, 4, 5$.

Table 2. The comparison of the obtained absolute error functions of the presented method and the LMM for the problem (5.6).

	$\lvert e_N(x)\rvert$ for BCM			$\lvert e_N(x)\rvert$ for LMM [13]		
x_i	$N=2$	$N=4$	$N=5$	$N=2$	$N=4$	$N=5$
0	0	1.8367e-40	0	0	0	0
0.2	1.3307e-03	1.2313e-05	2.1628e-07	0.280551e-3	0.281048e-4	0.194988e-5
0.4	1.0582e-02	3.2892e-05	4.2774e-07	0.236493e-3	0.671672e-4	0.339969e-5
0.6	3.5358e-02	3.0153e-05	6.6214e-07	0.842376e-2	0.476701e-4	0.530256e-5
0.8	8.2644e-02	2.0351e-04	1.2793e-06	0.211081e-1	0.617047e-3	0.104534e-4
1.0	1.5853e-01	1.2619e-03	3.5726e-05	0.436563e-1	0.420369e-2	0.282554e-3

with the initial conditions $y(0) = 1, y^{(1)}(0) = 0$, the constant $\mu = 2$ [29]. The exact solution of the equation (5.7) is

$$y(x) = \cos(x).$$

For $N = 4, 7$, the Boole solutions $y_N(x)$ and the error functions $E_N(x)$ of the problem (5.7) are obtained. In Figure 5, the exact solution and the Boole solutions are compared. The values of the residual functions are given in Figure 6, for $N = 4, 7$. The Morgan-Voyce matrix-collocation method (MVMCM) are used by the authors in [29] to solve the problem (5.7). The absolute error functions $|e_N(x)|$ of the problem (5.7) is obtained by the Boole collocation method (BCM) for $N = 4, 7$. In Table 3, these results are compared with the Morgan-Voyce matrix-collocation method are given for $N = 4, 7$.

6. Conclusion

In this paper, a new numerical procedure is developed by using the Boole polynomial and collocation points in the sense of matrix equations. The proposed method is applied successfully to obtain the approximate solution of the nonlinear differential equation having cubic nonlinearity. Four test problems are examined to demonstrate validity and applicability of this method. In addition, the residual function of Boole collocation method is improved in Section 4. The residual function is applied to the examples and the results are compared in Figures

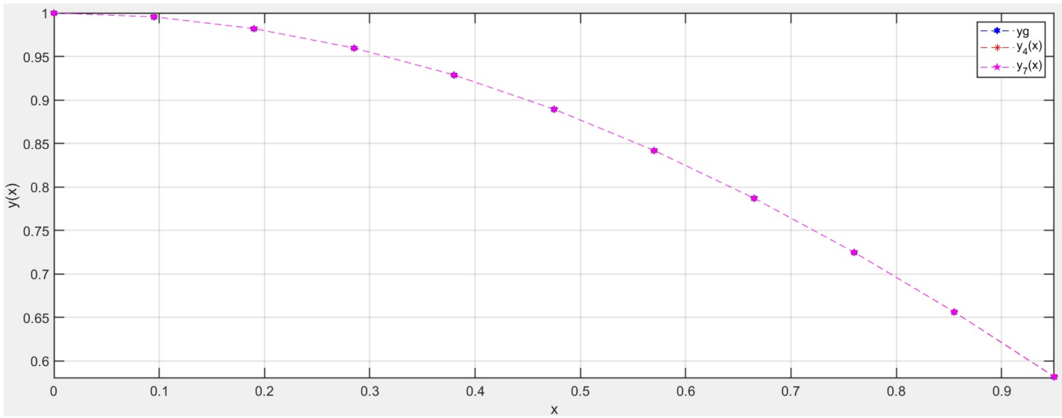

Figure 5. The comparison of the exact solution and the Boole solution of the problem (5.7) for $N = 4, 7$.

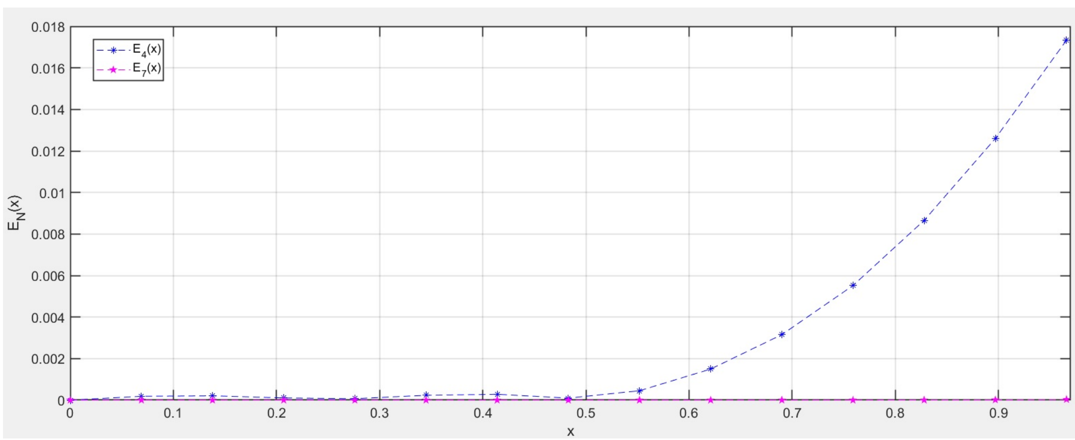

Figure 6. The comparison of the residual functions for the problem (5.7) for $N = 4, 7$.

Table 3. The comparison of the obtained absolute error functions of the presented method and the MVMCM for the problem (5.7).

	$\lvert e_N(x)\rvert$ for BCM		$\lvert e_N(x)\rvert$ for MVMCM [29]	
x_i	$N = 4$	$N = 7$	$N = 4$	$N = 7$
0	3.6734e-40	0	4.44e-16	4.44e-16
0.2	2.8678e-06	2.2602e-09	2.87e-06	2.26e-09
0.4	8.5816e-06	4.8238e-09	8.58e-06	4.82e-09
0.6	8.9878e-06	7.6494e-09	8.99e-06	7.65e-09
0.8	7.4062e-05	1.1216e-08	7.41e-05	1.12e-08
1.0	5.0405e-04	2.1884e-07	5.04e-04	2.19e-07

2, 4, and 6. The exact solution and the approximate solution of the numerical examples are calculated by taking into account the proposed algorithm in MATLAB. Numerical data are compared with the data in the literature and in this way, the obtained results are given by the tables and the figures. In addition, efficient, reliable, and better results have been obtained. Thanks to the study, the Boole collocation method will be applied for the approximate solutions of the partial differential equations, nonlinear delay differential equations, nonlinear integro differential equations and the equation systems.

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