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Clairaut Riemannian maps

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Abstract: In this paper, first we define Clairaut Riemannian map between Riemannian manifolds by using a geodesic curve on the base space and find necessary and sufficient conditions for a Riemannian map to be Clairaut with a nontrivial example. We also obtain necessary and sufficient condition for a Clairaut Riemannian map to be harmonic. Thereafter, we study Clairaut Riemannian map from Riemannian manifold to Ricci soliton with a nontrivial example. We obtain scalar curvatures of $rangeF_*$ and $(rangeF_*)^\perp$ by using Ricci soliton. Further, we obtain necessary conditions for the leaves of $rangeF_*$ to be almost Ricci soliton and Einstein. We also obtain necessary condition for the vector field $\dot{\beta}$ to be conformal on $rangeF_*$ and necessary and sufficient condition for the vector field $\dot{\beta}$ to be Killing on $(rangeF_*)^\perp$, where β is a geodesic curve on the base space of Clairaut Riemannian map. Also, we obtain necessary condition for the mean curvature vector field of $rangeF_*$ to be constant. Finally, we introduce Clairaut antiinvariant Riemannian map from Riemannian manifold to Kähler manifold, and obtain necessary and sufficient condition for an antiinvariant Riemannian map to be Clairaut with a nontrivial example. Further, we find necessary condition for $rangeF_*$ to be minimal and totally geodesic. We also obtain necessary and sufficient condition for Clairaut antiinvariant Riemannian maps to be harmonic.

Key words: Riemannian manifold, Kähler manifold, Riemannian map, Clairaut Riemannian map, antiinvariant Riemannian map, Ricci soliton

1. Introduction

The geometry of Riemannian submersions has been discussed widely in [8]. In 1992, Fischer introduced Riemannian map between Riemannian manifolds as a generalization of an isometric immersion and Riemannian submersion that satisfies the well-known generalized eikonal equation $\|F_*\|^2 = rankF$, which is a bridge between geometric optics and physical optics [9]. Further, the geometry of Riemannian maps was investigated in [2, 3, 20–26].

An important Clairaut's relation states that $\tilde{r} sin\theta$ is constant, where θ is the angle between the velocity vector of a geodesic and a meridian, and $\tilde{r}$ is the distance to the axis of a surface of revolution. In 1972, Bishop defined Clairaut Riemannian submersion with connected fibers and gave a necessary and sufficient condition for a Riemannian submersion to be Clairaut Riemannian submersion [5]. Further, Clairaut submersions were studied in [1, 12, 14]. In [25], Şahin introduced Clairaut Riemannian map by using a geodesic curve on the total space and obtained necessary and sufficient conditions for Riemannian map to be Clairaut Riemannian map.

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Further, Şahin gave an open problem to find characterizations for Clairaut Riemannian maps (see [26], page 165, open problem 2). In Section 3, we introduce a new type of Clairaut Riemannian map by using a geodesic curve on the base space and obtain necessary and sufficient conditions for a Riemannian map to be Clairaut Riemannian map.

A Riemannian manifold (N, g_2) is called a Ricci soliton [11] if there exists a smooth vector field Z_1 (called potential vector field) on N such that $\frac{1}{2}(L_{Z_1}g_2)(X_1, Y_1) + Ric(X_1, Y_1) + \lambda g_2(X_1, Y_1) = 0$, where L_{Z_1} is the Lie derivative of the metric tensor of g_2 with respect to Z_1, Ric is the Ricci tensor of (N, g_2), λ is a constant function and X_1, Y_1 are arbitrary vector fields on N. We shall denote a Ricci soliton by (N, g_2, Z_1, λ). The Ricci soliton (N, g_2, Z_1, λ) is said to be shrinking, steady or expanding accordingly as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold [4] with Z_1 zero or Killing (Lie derivative of metric tensor g_2 with respect to Z_1 is vanishes). Ricci soliton can be used to solve the Poincaré conjecture [17]. A Ricci soliton (N, g_2, Z_1, λ) becomes an almost Ricci soliton [18] if the function λ is a variable. The Ricci soliton (N, g_2, Z_1, λ) is said to be a gradient Ricci soliton if the potential vector field Z_1 is the gradient of some smooth function f on N, which is denoted by (N, g_2, f, λ). Moreover, a non-Killing tangent vector field Z_1 on a Riemannian manifold (N, g_2) is called conformal [7] if it satisfies $L_{Z_1}g_2 = 2fg_2$, where f is called the potential function of Z_1. The submersions and Riemannian maps from a Ricci soliton to a Riemannian manifold were studied in [10, 13, 15, 29, 30]. In [32], present authors introduced Riemannian map from a Riemannian manifold to a Ricci soliton. In Section 4, we introduce Clairaut Riemannian map from a Riemannian manifold to a Ricci soliton.

In [28], Watson studied almost Hermitian submersions. In [23], Şahin introduced holomorphic Riemannian map as generalization of holomorphic submersion and holomorphic submanifold. In [2, 3, 20, 22] invariant, antiinvariant and semiinvariant Riemannian maps were studied from a Riemannian manifold to a Kähler manifold. Recently, present authors introduced Clairaut invariant Riemannian map from a Riemannian manifold to a Kähler manifold in [31]. In Section 5, we introduce Clairaut antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold.

2. Preliminaries

In this section, we recall the notion of Riemannian map between Riemannian manifolds and give a brief review of basic facts.

Let $F : (M^m, g_1) \to (N^n, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF \leq min\{m, n\}$, where $dim(M) = m$ and $dim(N) = n$. We denote the kernel space of F_* by $\nu_p = kerF_{*p}$ at $p \in M$ and consider the orthogonal complementary space $\mathcal{H}_p = (kerF_{*p})^\perp$ to $kerF_{*p}$ in T_pM. Then the tangent space T_pM of M at p has the decomposition $T_pM = (kerF_{*p}) \oplus (kerF_{*p})^\perp = \nu_p \oplus \mathcal{H}_p$. We denote the range of F_* by $rangeF_*$ at $p \in M$ and consider the orthogonal complementary space $(rangeF_{*p})^\perp$ to $rangeF_{*p}$ in the tangent space $T_{F(p)}N$ of N at $F(p) \in N$. Since $rankF \leq min\{m, n\}$, we have $(rangeF_*)^\perp \neq \{0\}$. Thus the tangent space $T_{F(p)}N$ of N at $F(p) \in N$ has the decomposition $T_{F(p)}N = (rangeF_{*p}) \oplus (rangeF_{*p})^\perp$. Then F is called Riemannian map at $p \in M$ if the horizontal restriction $F^h_{*p} : (kerF_{*p})^\perp \to (rangeF_{*p})$ is a linear isometry between the spaces $((kerF_{*p})^\perp, g_{1(p)}|_{(kerF_{*p})^\perp})$ and $(rangeF_{*p}, g_{2(p_1)}|_{(rangeF_{*p})})$, where $F(p) = p_1$. In other words, F_* satisfies

$$g_2(F_*X, F_*Y) = g_1(X, Y), \tag{2.1}$$

for all X, Y vector field tangent to $\Gamma(kerF_{*p})^{\perp}$. It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^{\perp} = \{0\}$, respectively. The differential map F_* of F can be viewed as a section of bundle $Hom(TM, F^{-1}TN) \to M$, where $F^{-1}TN$ is the pullback bundle whose fibers at $p \in M$ is $(F^{-1}TN)_p = T_{F(p)}N$, $p \in M$. The bundle $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection $\overset{N}{\nabla^F}$. Then the second fundamental form of F is given by [16]

$$(\nabla F_*)(X, Y) = \overset{N}{\nabla^F_X} F_* Y - F_*(\nabla^M_X Y), \tag{2.2}$$

for all $X, Y \in \Gamma(TM)$, where $\overset{N}{\nabla^F_X} F_* Y \circ F = \nabla^N_{F_* X} F_* Y$. It is known that the second fundamental form is symmetric. In [20] Şahin proved that $(\nabla F_*)(X, Y)$ has no component in $rangeF_*$, for all $X, Y \in \Gamma(kerF_*)^{\perp}$. More precisely, we have

$$(\nabla F_*)(X, Y) \in \Gamma(rangeF_*)^{\perp}. \tag{2.3}$$

The tension field of F is defined to be the trace of the second fundamental form of F, i.e. $\tau(F) = trace(\nabla F_*) = \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i)$, where $m = dim(M)$ and $\{e_1, e_2, ..., e_m\}$ is the orthonormal frame on M. Moreover, a map $F : (M^m, g_1) \to (N^n, g_2)$ between Riemannian manifolds is harmonic if and only if the tension field of F vanishes at each point $p \in M$.

Lemma 2.1 *[21] Let $F : (M^m, g_1) \to (N^n, g_2)$ be a Riemannian map between Riemannian manifolds. Then the tension field of F is given by $\tau(F) = -rF_*(H) + (m - r)H_2$, where $r = dim(kerF_*)$, $(m - r) = rankF$, H and H_2 are the mean curvature vector fields of the distribution $kerF_*$ and $rangeF_*$, respectively.*

Lemma 2.2 *[22] Let $F : (M, g_1) \to (N, g_2)$ be a Riemannian map between Riemannian manifolds. Then F is umbilical Riemannian map if and only if*

$$(\nabla F_*)(X, Y) = g_1(X, Y)H_2,$$

for $X, Y \in \Gamma(kerF_*)^{\perp}$ and H_2 is the mean curvature vector field of $rangeF_*$.

For any vector field X on M and any section V of $(rangeF_*)^{\perp}$, we have $\nabla^{F\perp}_X V$, which is the orthogonal projection of $\nabla^N_X V$ on $(rangeF_*)^{\perp}$, where $\nabla^{F\perp}$ is linear connection on $(rangeF_*)^{\perp}$ such that $\nabla^{F\perp} g_2 = 0$. Now, for a Riemannian map F we define $\mathcal{S}_V$ as ([24], p. 188)

$$\nabla^N_{F_* X} V = -\mathcal{S}_V F_* X + \nabla^{F\perp}_X V, \tag{2.4}$$

where ∇^N is Levi-Civita connection on N, $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla^N_{F_* X} V$. Thus at $p \in M$, we have $\nabla^N_{F_* X} V(p) \in T_{F(p)}N$, $\mathcal{S}_V F_* X \in F_{*p}(T_p M)$ and $\nabla^{F\perp}_X V(p) \in (F_{*p}(T_p M))^{\perp}$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V, and $F_* X$ at p depends only on V_p and $F_{*p} X_p$. Hence from (2.2) and (2.4), we obtain

$$g_2(\mathcal{S}_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)), \tag{2.5}$$

for $X, Y \in \Gamma(kerF_*)^{\perp}$ and $V \in \Gamma(rangeF_*)^{\perp}$, where $\mathcal{S}_V$ is self-adjoint operator.

3. Clairaut Riemannian map between Riemannian manifolds

In this section, we define Clairaut Riemannian map between Riemannian manifolds by using a geodesic curve [6] on the base space and investigate geometry.

The notion of Clairaut Riemannian map was defined by Şahin in [25]. According to the definition, a Riemannian map $F:(M,g_1)\to(N,g_2)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $\tilde{r}:M\to\mathbb{R}^+$ such that for every geodesic α on M, the function $(\tilde{r}\circ\alpha)sin\theta$ is constant, where, for all t, $\theta(t)$ is the angle between $\dot{\alpha}(t)$ and the horizontal space at $\alpha(t)$.

Thus, the notion of Clairaut Riemannian map comes from a geodesic curve on a surface of revolution. Therefore, we are going to give a definition of Clairaut Riemannian map by using geodesic curve on the base space.

Definition 3.1 *A Riemannian map $F:(M,g_1)\to(N,g_2)$ between Riemannian manifolds is called Clairaut Riemannian map if there is a function $\tilde{s}:N\to\mathbb{R}^+$ such that for every geodesic β on N, the function $(\tilde{s}\circ\beta)sin\omega(t)$ is constant, where, $F_*X\in\Gamma(rangeF_*)$ for $X\in\Gamma(kerF_*)^\perp$ and $V\in\Gamma(rangeF_*)^\perp$ are components of $\dot{\beta}(t)$, and $\omega(t)$ is the angle between $\dot{\beta}(t)$ and V for all t.*

Note: For all $U,V\in\Gamma(rangeF_*)^\perp$ we define

$$\nabla^N_U V=\mathcal{R}(\nabla^N_U V)+\nabla^{F\perp}_U V,$$

where $\mathcal{R}(\nabla^N_U V)$ and $\nabla^{F\perp}_U V$ denote $rangeF_*$ and $(rangeF_*)^\perp$ part of $\nabla^N_U V$, respectively. Therefore $(rangeF_*)^\perp$ is totally geodesic if and only if

$$\nabla^N_U V=\nabla^{F\perp}_U V.$$

Note that from now, throughout the paper, we are assuming $(rangeF_*)^\perp$ is totally geodesic.

Lemma 3.2 *Let $F:(M,g_1)\to(N,g_2)$ be a Riemannian map between Riemannian manifolds and $\alpha:I\to M$ be a geodesic curve on M. Then the curve $\beta=F\circ\alpha$ is geodesic curve on N if and only if*

$$(\nabla F_*)(X,X)+\nabla^{F\perp}_X V+\nabla^{F\perp}_V V=0, \tag{3.1}$$

$$-\mathcal{S}_V F_*X+F_*(\nabla^M_X X)+\nabla^N_V F_*X=0, \tag{3.2}$$

where $F_*X\in\Gamma(rangeF_*),V\in\Gamma(rangeF_*)^\perp$ are components of $\dot{\beta}(t)$ and ∇^N is Levi-Civita connection on N and $\nabla^{F\perp}$ is a linear connection on $(rangeF_*)^\perp$.

Proof Let $\alpha:I\to M$ be a geodesic on M with $U(t)=\nu\dot{\alpha}(t)$ and $X(t)=\mathcal{H}\dot{\alpha}(t)$. Let $\beta=F\circ\alpha$ be a geodesic on N with $F_*X\in\Gamma(rangeF_*)$ and $V\in\Gamma(rangeF_*)^\perp$ are components of $\dot{\beta}(t)$.
Now,

$$\nabla^N_{\dot{\beta}}\dot{\beta}=\nabla^N_{F_*X+V}(F_*X+V),$$

which implies

$$\nabla^N_{\dot{\beta}}\dot{\beta}=\nabla^N_{F_*X}F_*X+\nabla^N_{F_*X}V+\nabla^N_V F_*X+\nabla^N_V V.$$

Using (2.4) in above equation, we get

$$\nabla^N_{\dot\beta}\dot\beta = \overset{N}{\nabla^F_X} F_*X \circ F + (-\mathcal{S}_V F_*X + \nabla^{F\perp}_X V) + \nabla^N_V F_*X + \nabla^N_V V.$$

Using (2.2) in above equation, we get

$$\nabla^N_{\dot\beta}\dot\beta = (\nabla F_*)(X,X) + F_*(\nabla^M_X X) - \mathcal{S}_V F_*X + \nabla^{F\perp}_X V + \nabla^N_V F_*X + \nabla^N_V V. \tag{3.3}$$

Since $(rangeF_*)^\perp$ is totally geodesic, (3.3) can be written as

$$\nabla^N_{\dot\beta}\dot\beta = (\nabla F_*)(X,X) + F_*(\nabla^M_X X) - \mathcal{S}_V F_*X + \nabla^{F\perp}_X V + \nabla^N_V F_*X + \nabla^{F\perp}_V V. \tag{3.4}$$

Now β is geodesic on N if and only if $\nabla^N_{\dot\beta}\dot\beta = 0$. Then (3.4) implies $(\nabla F_*)(X,X) + F_*(\nabla^M_X X) - \mathcal{S}_V F_*X + \nabla^{F\perp}_X V + \nabla^N_V F_*X + \nabla^{F\perp}_V V = 0$, which completes the proof. □

Theorem 3.3 *Let $F : (M, g_1) \to (N, g_2)$ be a Riemannian map between Riemannian manifolds such that $rangeF_*$ is connected and α, $\beta = F \circ \alpha$ are geodesic curves on M and N, respectively. Then F is Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if any one of the following conditions holds:*

(i) $\mathcal{S}_V F_*X = -V(g)F_*X$, where $F_*X \in \Gamma(rangeF_*), V \in \Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$.

(ii) F is umbilical map, and has $H_2 = -\nabla^N g$, where g is a smooth function on N and H_2 is the mean curvature vector field of $rangeF_*$.

Proof First we prove F is a Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if for any geodesic $\beta : I \to N$ with tangential components $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^\perp$, $t \in I$ the equation

$$g_{2\beta(t)}(F_*X(t), F_*X(t)) g_2(\dot\beta(t), (\nabla^N g)) + g_2(\mathcal{S}_V F_*X(t), F_*X(t)) = 0, \tag{3.5}$$

is satisfied. To prove this, let β be a geodesic on N with $\dot\beta(t) = F_*X(t) + V(t)$ and let $\omega(t) \in [0,\pi]$ denote the angle between $\dot\beta(t)$ and $V(t)$. If $\dot\beta(t) \in \Gamma(rangeF_*)^\perp$, then we have $F_*X(t_0) = 0$ (i.e. (3.5) is satisfied), which implies $\sin\omega(t) = 0$ at point $\beta(t_0)$. Thus for any function $\tilde{s} = e^g$ on M, $(\tilde{s}(\beta(t)))\sin\omega(t)$ identically vanishes. Therefore, the statement holds trivially in this case. Now, we consider the case $\sin\omega(t) \neq 0$, i.e. $\dot\beta(t)$ does not belongs only in $\Gamma(rangeF_*)^\perp$. Since β is geodesic, its speed is constant $b = \|\dot\beta\|^2$ (say). Then

$$g_{2\beta(t)}(V,V) = bcos^2\omega(t), \tag{3.6}$$

$$g_{2\beta(t)}(F_*X, F_*X) = bsin^2\omega(t). \tag{3.7}$$

Now differentiating (3.7) along β, we get

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2bsin\omega cos\omega \frac{d\omega}{dt}. \tag{3.8}$$

On the other hand,

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\nabla^N_{\dot\beta} F_*X, F_*X).$$

By putting $\dot{\beta} = F_*X + V$ in above equation, we get

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\nabla^N_{F_*X} F_*X + \nabla^N_V F_*X, F_*X),$$

which implies

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\overset{N}{\nabla^F_X} F_*X \circ F + \nabla^N_V F_*X, F_*X). \tag{3.9}$$

Using (2.2) and (3.2) in (3.9), we get

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2((\nabla F_*)(X, X) + F_*(\nabla^M_X X) + \mathcal{S}_V F_*X - F_*(\nabla^M_X X), F_*X).$$

Using (2.3) in above equation, we get

$$\frac{d}{dt} g_2(F_*X, F_*X) = 2g_2(\mathcal{S}_V F_*X, F_*X). \tag{3.10}$$

Now from (3.8) and (3.10), we get

$$g_2(\mathcal{S}_V F_*X, F_*X) = b sin\omega cos\omega \frac{d\omega}{dt}. \tag{3.11}$$

Moreover, F is a Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if $\frac{d}{dt}(e^{g\circ\beta} sin\omega) = 0$, that is, $e^{g\circ\beta} sin\omega \frac{d(g\circ\beta)}{dt} + e^{g\circ\beta} cos\omega \frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $b sin\omega$ and using (3.7), we get

$$g_2(F_*X, F_*X)\frac{d(g\circ\beta)}{dt} = -b sin\omega cos\omega \frac{d\omega}{dt}. \tag{3.12}$$

Now from (3.11) and (3.12), we get

$$g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X)\frac{d(g\circ\beta)}{dt},$$

which means

$$g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X) g_2(\nabla^N g, \dot{\beta}). \tag{3.13}$$

Indeed assuming (3.5) and considering any geodesic β on N with initial tangent vector which belongs in $\Gamma(rangeF_*)$, then by using $V(t_0) = 0$ in (3.13), we get g is constant on $rangeF_*$ and since $rangeF_*$ is connected, $\nabla^N g \in \Gamma(rangeF_*)^\perp$. Then by (3.13), we get

$$g_2(\mathcal{S}_V F_*X, F_*X) = -g_2(F_*X, F_*X) g_2(\nabla^N g, V). \tag{3.14}$$

Thus $\mathcal{S}_V F_*X = -V(g) F_*X$, where $V(g)$ is a smooth function on N, which implies the proof of (i). Now, by using (2.5) in (3.14), we get

$$g_2(V, (\nabla F_*)(X, X)) = -g_2(F_*X, F_*X) g_2(\nabla^N g, V), \tag{3.15}$$

for $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^\perp$. Now using (2.2) in (3.15), we get

$$g_2(V, \overset{N}{\nabla}{}^F_X F_*X) = -g_2(\nabla^N g, V) g_2(F_*X, F_*X).$$

Thus by comparing, we get

$$\overset{N}{\nabla}{}^F_X F_*X = -(\nabla^N g) g_2(F_*X, F_*X). \tag{3.16}$$

Taking trace of (3.16), we get

$$\sum_{j=r+1}^{m} \overset{N}{\nabla}{}^F_{X_j} F_*X_j = -(\nabla^N g)(m-r), \tag{3.17}$$

where $\{X_{r+1}, X_{r+2}, ..., X_m\}$ and $\{F_*X_{r+1}, F_*X_{r+2}, ..., F_*X_m\}$ are orthonormal bases of $(kerF_*)^\perp$ and $rangeF_*$, respectively.
Moreover, the mean curvature vector field of $rangeF_*$ is defined by ([21], [24] page 199)

$$H_2 = \frac{1}{m-r} \sum_{j=r+1}^{m} \overset{N}{\nabla}{}^F_{X_j} F_*X_j, \tag{3.18}$$

where $\{X_j\}_{r+1 \leq j \leq m}$ is an orthonormal basis of $(kerF_*)^\perp$. Then from (3.17) and (3.18), we get

$$H_2 = -\nabla^N g. \tag{3.19}$$

Also, by (3.15), we get

$$(\nabla F_*)(X, X) = -g_2(F_*X, F_*X)(\nabla^N g). \tag{3.20}$$

Since F is Riemannian map, using (2.1) in (3.20), we get

$$(\nabla F_*)(X, X) = -g_1(X, X)(\nabla^N g). \tag{3.21}$$

From (3.19) and (3.21), we get

$$(\nabla F_*)(X, X) = g_1(X, X) H_2.$$

Thus by Lemma 2.2 F is umbilical map, which completes the proof. □

Remark 3.4 *In [25], Şahin considered geodesic curve on the total manifold of a Riemannian map F, then by using Clairaut relation fibers of F are totally umbilical. On the other hand, in Definition 3.1, we considered geodesic curve on the base manifold of F, then by using Clairaut's relation F becomes totally umbilical.*

Theorem 3.5 *Let $F : (M^m, g_1) \to (N^n, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds such that $kerF_*$ is minimal. Then F is harmonic if and only if g is constant function on N.*

Proof Since $H = 0$, then by Lemma 2.1 F is harmonic if and only if $H_2 = 0$ if and only if $\nabla^N g = 0$, which completes the proof. □

Theorem 3.6 *Let $F : (M^m, g_1) \to (N^n, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then $N = N_{(rangeF_*)^\perp} \times_f N_{rangeF_*}$ is a twisted product manifold.*

Proof By (3.20), (3.21) and Theorem 3.3, we have $\overset{N}{\nabla}{}^F_X F_*Y = g_1(X,Y)H_2$ for $X, Y \in \Gamma(kerF_*)^\perp$, which implies $rangeF_*$ is totally umbilical. Then proof follows by [19]. □

Example 3.7 *Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0\}$ be a Riemannian manifold with Riemannian metric $g_1 = e^{2x_2}dx_1^2 + dx_2^2$ on M. Let $N = \{(y_1, y_2) \in \mathbb{R}^2\}$ be a Riemannian manifold with Riemannian metric $g_2 = e^{2x_2}dy_1^2 + dy_2^2$ on N. Consider a map $F : (M, g_1) \to (N, g_2)$ defined by*

$$F(x_1, x_2) = (x_1, 0).$$

Then, we get

$$kerF_* = span\{U = e_2\} \text{ and } (kerF_*)^\perp = span\{X = e_1\},$$

where $\left\{e_1 = e^{-x_2}\frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}\right\}$ and $\left\{e_1' = e^{-x_2}\frac{\partial}{\partial y_1}, e_2' = \frac{\partial}{\partial y_2}\right\}$ are bases on T_pM and $T_{F(p)}N$, respectively, for all $p \in M$. By easy computations, we see that $F_*(X) = e_1'$ and $g_1(X,X) = g_2(F_*X, F_*X)$ for $X \in \Gamma(kerF_*)^\perp$. Thus F is Riemannian map with $rangeF_* = span\{F_*(X) = e_1'\}$ and $(rangeF_*)^\perp = span\{e_2'\}$. Now to show F is Clairaut Riemannian map we will verify Theorem 3.3, for this we will verify (3.14). Since V and $(\nabla F_*)(X,X) \in \Gamma(rangeF_*)^\perp$,e here we can write $V = ae_2'$ and $(\nabla F_*)(X,X) = be_2'$ for some $a, b \in \mathbb{R}$. Then we get

$$g_2(V, (\nabla F_*)(X,X)) = g_2(ae_2', be_2') = ab, \tag{3.22}$$

and

$$g_2(F_*X, F_*X) = g_2(e_1', e_1') = 1. \tag{3.23}$$

Since $\nabla^N g = \sum_{i,j=1}^{2} g_2^{ij}\frac{\partial g}{\partial y_i}\frac{\partial}{\partial y_j}$. Therefore for the function $g = -by_2$

$$g_2(\nabla^N g, V) = -ab. \tag{3.24}$$

Thus by using (2.5), (3.22), (3.23) and (3.24) we see that (3.14) holds. Thus F is a Clairaut Riemannian map.

4. Clairaut Riemannian map from Riemannian manifold to Ricci soliton

In this section, we study Clairaut Riemannian map $F : (M, g_1) \to (N, g_2)$ from a Riemannian manifold to a Ricci soliton and give some characterizations.

Lemma 4.1 *[32] Let $F : (M^m, g_1) \to (N^n, g_2)$ be a Riemannian map between Riemannian manifolds. Then the Ricci tensor on (N, g_2) given by*

$$\begin{aligned} Ric(F_*X, F_*Y) &= Ric^{rangeF_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1}\Big\{g_2(\mathcal{S}_{\nabla^{F\perp}_{e_k}e_k}F_*X, F_*Y) \\ &-g_2(\nabla^N_{e_k}\mathcal{S}_{e_k}F_*X, F_*Y) + g_2(\mathcal{S}_{e_k}F_*X, \mathcal{S}_{e_k}F_*Y) + g_2(\nabla^N_{e_k}F_*X, \mathcal{S}_{e_k}F_*Y)\Big\}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} Ric(V, W) &= Ric^{(rangeF_*)^\perp}(V, W) - \sum_{j=r+1}^{m}\Big\{g_2(\mathcal{S}_{\nabla^{F\perp}_V W}F_*X_j, F_*X_j) \\ &+g_2(\mathcal{S}_V F_*X_j, \mathcal{S}_W F_*X_j) - \nabla^N_V(g_2(\mathcal{S}_W F_*X_j, F_*X_j)) + 2g_2(\mathcal{S}_W F_*X_j, \nabla^N_V F_*X_j)\Big\}, \end{aligned} \tag{4.2}$$

$$Ric(F_*X,V) = \sum_{j=r+1}^{m}\Big\{g_2((\tilde\nabla_X\mathcal{S})_V F_*X_j,F_*X_j)-g_2((\tilde\nabla_{X_j}\mathcal{S})_V F_*X,F_*X_j)\Big\}-\sum_{k=1}^{n_1}g_2(R^{F\perp}(F_*X,e_k)V,e_k), \quad (4.3)$$

for $X,Y\in\Gamma(kerF_*)^\perp$, $V,W\in\Gamma(rangeF_*)^\perp$ *and* $F_*X,F_*Y\in\Gamma(rangeF_*)$, *where* $\{F_*X_j\}_{r+1\le j\le m}$ *and* $\{e_k\}_{1\le k\le n_1}$ *are orthonormal bases of* $rangeF_*$ *and* $(rangeF_*)^\perp$, *respectively.*

Theorem 4.2 *Let* $F:(M^m,g_1)\to(N^n,g_2)$ *be a Clairaut Riemannian map with* $\tilde s=e^g$ *between Riemannian manifolds. Then the Ricci tensor on* (N,g_2) *given by*

$$\begin{aligned}Ric(F_*X,F_*Y) &= Ric^{rangeF_*}(F_*X,F_*Y)-\sum_{k=1}^{n_1}(e_k(g))^2g_2(F_*X,F_*Y)\\ &+\sum_{k=1}^{n_1}g_2(\nabla^{F\perp}_{e_k}e_k,\nabla^Ng)g_2(F_*X,F_*Y)-\sum_{k=1}^{n_1}(\nabla^N_{e_k}e_k(g))g_2(F_*X,F_*Y),\end{aligned} \quad (4.4)$$

$$\begin{aligned}Ric(V,W) &= Ric^{(rangeF_*)^\perp}(V,W)+(m-r)g_2(\nabla^Ng,\nabla^{F\perp}_VW)\\ &-(m-r)V(g)W(g)-(m-r)\nabla^N_VW(g),\end{aligned} \quad (4.5)$$

$$Ric(F_*X,V)=\sum_{j=r+1}^{m}g_2((\tilde\nabla_X\mathcal{S})_VF_*X_j,F_*X_j)-\sum_{j=r+1}^{m}g_2((\tilde\nabla_{X_j}\mathcal{S})_VF_*X_j,F_*X_j)-\sum_{k=1}^{n_1}g_2(R^{F\perp}(F_*X,e_k)V,e_k), \quad (4.6)$$

for $X,Y\in\Gamma(kerF_*)^\perp$, $V,W\in\Gamma(rangeF_*)^\perp$ *and* $F_*X,F_*Y\in\Gamma(rangeF_*)$, *where* $\{F_*X_j\}_{r+1\le j\le m}$ *and* $\{e_k\}_{1\le k\le n_1}$ *are orthonormal bases of* $rangeF_*$ *and* $(rangeF_*)^\perp$, *respectively.*

Proof Using Theorem 3.3 and (3.14) in (4.1), we get

$$\begin{aligned}Ric(F_*X,F_*Y) &= Ric^{rangeF_*}(F_*X,F_*Y)-\sum_{k=1}^{n_1}(e_k(g))^2g_2(F_*X,F_*Y)+\sum_{k=1}^{n_1}g_2(\nabla^{F\perp}_{e_k}e_k,\nabla^Ng)g_2(F_*X,F_*Y)\\ &-\sum_{k=1}^{n_1}g_2(\nabla^N_{e_k}(e_k(g)F_*X),F_*Y)+\sum_{k=1}^{n_1}g_2(\nabla^N_{e_k}F_*X,e_k(g)F_*Y),\end{aligned}$$

which implies (4.4). Also using Theorem 3.3 and (3.14) in (4.2), we get

$$\begin{aligned}Ric(V,W) &= Ric^{(rangeF_*)^\perp}(V,W)+\sum_{j=r+1}^{m}g_2(\nabla^{F\perp}_VW,\nabla^Ng)g_2(F_*X_j,F_*X_j)-\sum_{j=r+1}^{m}g_2(V(g)F_*X_j,W(g)F_*X_j)\\ &-\sum_{j=r+1}^{m}\nabla^N_V(g_2(W(g)F_*X_j,F_*X_j))+2\sum_{j=r+1}^{m}g_2(W(g)F_*X_j,\nabla^N_VF_*X_j),\end{aligned}$$

which implies (4.5). Also the proof of (4.3) and (4.6) is same. □

Theorem 4.3 *Let* (N,g_2,H_2,λ) *be a Ricci soliton with the potential vector field* $H_2\in\Gamma(rangeF_*)^\perp$ *and* $F:(M,g_1)\to(N,g_2)$ *be a Clairaut Riemannian map with* $\tilde s=e^g$ *between Riemannian manifolds. Then*

$$s^{rangeF_*}=-\lambda(m-r)+(m-r)\Delta g-(m-r)(m-r-2)\|\nabla^Ng\|^2,$$

where s^{rangeF_*} *is the scalar curvature of* $rangeF_*$ *and* $(m-r)=\dim(rangeF_*)$.

Proof Since (N, g_2, H_2, λ) admit Ricci soliton with the potential vector field $H_2 \in \Gamma(rangeF_*)^{\perp}$ then, we have

$$\frac{1}{2}(L_{H_2}g_2)(F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,$$

for $F_*X, F_*Y \in \Gamma(rangeF_*)$, which implies

$$\frac{1}{2}\{g_2(\nabla^N_{F_*X}H_2, F_*Y) + g_2(\nabla^N_{F_*Y}H_2, F_*X)\} + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0.$$

Using (2.4) in above equation, we get

$$\frac{1}{2}\{g_2(-\mathcal{S}_{H_2}F_*X, F_*Y) + g_2(-\mathcal{S}_{H_2}F_*Y, F_*X)\} + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0.$$

Since $\mathcal{S}_{H_2}$ is self-adjoint, above equation can be written as

$$-g_2(\mathcal{S}_{H_2}F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0. \tag{4.7}$$

Using (3.14), (3.19) and (4.4) in (4.7), we get

$$\begin{aligned}&-g_2(\nabla^N g, \nabla^N g)g_2(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1}(e_k(g))^2 g_2(F_*X, F_*Y)\\ &+ \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k}e_k, \nabla^N g)g_2(F_*X, F_*Y) - \sum_{k=1}^{n_1}\nabla^N_{e_k}e_k(g)g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,\end{aligned}$$

where $\{e_k\}_{1\leq k\leq n_1}$ is an orthonormal basis of $(rangeF_*)^{\perp}$. This implies

$$\begin{aligned}&-2\|\nabla^N g\|^2 g_2(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y)\\ &-\sum_{k=1}^{n_1} g_2(e_k, \nabla^N_{e_k}\nabla^N g)g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0.\end{aligned} \tag{4.8}$$

Taking trace of (4.8) for $rangeF_*$, we get

$$s^{rangeF_*} - 2(m-r)\|\nabla^N g\|^2 - (m-r)\sum_{k=1}^{n_1} g_2(\nabla^N_{e_k}\nabla^N g, e_k) + \lambda(m-r) = 0.$$

Using definition of Hessian form of g (i.e. $H^g(X_1, Y_1) = g_2(\nabla^N_{X_1}\nabla^N g, Y_1)$ for all $X_1, Y_1 \in \Gamma(TN)$) from [8] in above equation, we get

$$s^{rangeF_*} + (m-r)\{-2\|\nabla^N g\|^2 - \sum_{k=1}^{n_1} H^g(e_k, e_k) + \lambda\} = 0. \tag{4.9}$$

Since we know that

$$\Delta g = \sum_{j=r+1}^{m} H^g(F_*X_j, F_*X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k), \tag{4.10}$$

where $\{F_*X_j\}_{r+1\leq j\leq m}$ and $\{e_k\}_{1\leq k\leq n_1}$ are orthonormal bases of $rangeF_*$ and $(rangeF_*)^{\perp}$, respectively. Then by using definition of Hessian form of g in (4.10), we get

$$\Delta g = \sum_{j=r+1}^{m} g_2(\nabla^N_{F_*X_j}\nabla^N g, F_*X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k). \tag{4.11}$$

Using (2.4) in (4.11), we get

$$\Delta g = -\sum_{j=r+1}^{m} g_2(\mathcal{S}_{\nabla^N g} F_* X_j, F_* X_j) + \sum_{k=1}^{n_1} H^g(e_k, e_k).$$

Using Theorem 3.3 in above equation, we get

$$\Delta g - (m-r)\|\nabla^N g\|^2 = \sum_{k=1}^{n_1} H^g(e_k, e_k). \tag{4.12}$$

Thus (4.9) and (4.12) implies the proof. □

Theorem 4.4 *Let* (N, g_2, H_2, λ) *be a Ricci soliton with the potential vector field* $H_2 \in \Gamma(rangeF_*)^\perp$ *and* $F : (M^m, g_1) \to (N^n, g_2)$ *be a Clairaut Riemannian map with* $\tilde{s} = e^g$ *between Riemannian manifolds. Then*

$$s^{(rangeF_*)^\perp} = -\lambda n_1 + (m-r+1)\Delta g - (m-r)^2\|\nabla^N g\|^2,$$

where $s^{(rangeF_*)^\perp}$ *denotes the scalar curvature of* $(rangeF_*)^\perp$ *and* $(m-r) = \dim(rangeF_*)$, $n_1 = \dim((rangeF_*)^\perp)$.

Proof Since (N, g_2, H_2, λ) admit Ricci soliton with the potential vector field $H_2 \in \Gamma(rangeF_*)^\perp$ then, we have

$$\frac{1}{2}(L_{H_2} g_2)(V, W) + Ric(V, W) + \lambda g_2(V, W) = 0,$$

for $V, W \in \Gamma(rangeF_*)^\perp$, which implies

$$\frac{1}{2}\{g_2(\nabla^N_V H_2, W) + g_2(\nabla^N_W H_2, V)\} + Ric(V, W) + \lambda g_2(V, W) = 0.$$

Putting $H_2 = -\nabla^N g$ in above equation, we get

$$-\frac{1}{2}\{g_2(\nabla^N_V \nabla^N g, W) + g_2(\nabla^N_W \nabla^N g, V)\} + Ric(V, W) + \lambda g_2(V, W) = 0. \tag{4.13}$$

Using definition of Hessian form of g and (4.5) in (4.13), we get

$$\begin{aligned} &-H^g(V, W) + Ric^{(rangeF_*)^\perp}(V, W) + (m-r)g_2(\nabla^N g, \nabla^{F\perp}_V W) \\ &-(m-r)V(g)W(g) - (m-r)\nabla^N_V W(g) + \lambda g_2(V, W) = 0. \end{aligned} \tag{4.14}$$

Taking trace of (4.14) for $(rangeF_*)^\perp$, we get

$$-\sum_{k=1}^{n_1} H^g(e_k, e_k) + s^{(rangeF_*)^\perp} + \sum_{k=1}^{n_1}(m-r)g_2(\nabla^N g, \nabla^{F\perp}_{e_k} e_k) - (m-r)\sum_{k=1}^{n_1}(e_k(g))^2 - (m-r)\sum_{k=1}^{n_1}\nabla^N_{e_k} e_k(g) + \lambda n_1 = 0,$$

where $\{e_k\}_{1\le k\le n_1}$ is an orthonormal basis of $(rangeF_*)^\perp$, which implies

$$s^{(rangeF_*)^\perp} + \lambda n_1 - (m-r)\sum_{k=1}^{n_1}(e_k(g))^2 - (m-r+1)\sum_{k=1}^{n_1} H^g(e_k, e_k) = 0.$$

Using (4.12) and $(e_k(g))^2 = g_2(\nabla^N g, e_k)^2 = g_2(\nabla^N g, \nabla^N g)$ in above equation, we get the proof. □

Remark 4.5 *Since $rangeF_*$ and $(rangeF_*)^\perp$ are subbundles of TN, they define distributions on N. Then for $F_*X, F_*Y \in \Gamma(rangeF_*)$, we have*

$$\begin{aligned}[F_*X, F_*Y] &= \nabla^N_{F_*X} F_*Y - \nabla^N_{F_*Y} F_*X \\ &= \overset{N}{\nabla}{}^F_X F_*Y \circ F - \overset{N}{\nabla}{}^F_Y F_*X \circ F.\end{aligned}$$

Using (2.2) in above equation, we get

$$[F_*X, F_*Y] = F_*(\nabla_X Y) - F_*(\nabla_Y X) = F_*(\nabla_X Y - \nabla_Y X) \in \Gamma(rangeF_*).$$

Thus $rangeF_*$ is an integrable distribution. Then for any point $F(p) \in N$ there exists maximal integral manifold or a leaf of $rangeF_*$ containing $F(p)$.

Theorem 4.6 *Let (N, g_2, F_*Z, λ) be a Ricci soliton with the potential vector field $F_*Z \in \Gamma(rangeF_*)$ and $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then a leaf of $rangeF_*$ is an almost Ricci soliton.*

Proof Since (N, g_2, F_*Z, λ) admit Ricci soliton with the potential vector field $F_*Z \in \Gamma(rangeF_*)$ then, we have

$$\frac{1}{2}(L_{F_*Z} g_2)(F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0, \tag{4.15}$$

for $F_*X, F_*Y, F_*Z \in \Gamma(rangeF_*)$. Using (4.4) in (4.15), we get

$$\begin{aligned}&\tfrac{1}{2}(L_{F_*Z} g_2)(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1} (e_k(g))^2 g_2(F_*X, F_*Y) \\ &+ \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k} e_k, \nabla^N g) g_2(F_*X, F_*Y) - \sum_{k=1}^{n_1} \nabla^N_{e_k} e_k(g) g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,\end{aligned}$$

where $\{e_k\}_{1 \leq k \leq n_1}$ is an orthonormal basis of $(rangeF_*)^\perp$, which implies

$$\tfrac{1}{2}(L_{F_*Z} g_2)(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y) + \tilde{\lambda} g_2(F_*X, F_*Y) = 0,$$

where $\tilde{\lambda} = -\sum_{k=1}^{n_1} (e_k(g))^2 + \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k} e_k, \nabla^N g) - \sum_{k=1}^{n_1} e_k(e_k(g)) + \lambda$ is a smooth function on N. Thus a leaf of $rangeF_*$ is an almost Ricci soliton, which completes the proof. □

Theorem 4.7 *Let (N, g_2, V, λ) be a Ricci soliton with the potential vector field $V \in \Gamma(rangeF_*)^\perp$ and $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\tilde{s} = e^g$ between Riemannian manifolds. Then a leaf of $rangeF_*$ is an Einstein.*

Proof Since (N, g_2, F_*Z, λ) admit Ricci soliton with the potential vector field $F_*Z \in \Gamma(rangeF_*)$ then, we have

$$\frac{1}{2}(L_V g_2)(F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0,$$

for $F_*X, F_*Y \in \Gamma(rangeF_*)$, which implies

$$\frac{1}{2}\{g_2(\nabla^N_{F_*X}V, F_*Y) + g_2(\nabla^N_{F_*Y}V, F_*X)\} + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0.$$

Using (2.4) in above equation, we get

$$\frac{1}{2}\{g_2(-\mathcal{S}_V F_*X, F_*Y) + g_2(-\mathcal{S}_V F_*Y, F_*X)\} + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0.$$

Since $\mathcal{S}_V$ is self-adjoint, above equation can be written as

$$-g_2(\mathcal{S}_V F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0. \tag{4.16}$$

Since F is Clairaut Riemannian map, using $\mathcal{S}_V F_*X = -V(g)F_*X$ and (4.4) in (4.16), we get

$$\begin{aligned} &V(g)g_2(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1}(e_k(g))^2 g_2(F_*X, F_*Y) \\ &+ \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k}e_k, \nabla^N g)g_2(F_*X, F_*Y) - \sum_{k=1}^{n_1}\nabla^N_{e_k}e_k(g)g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0, \end{aligned}$$

where $\{e_k\}_{1\le k\le n_1}$ is an orthonormal basis of $(rangeF_*)^\perp$, which implies

$$Ric^{rangeF_*}(F_*X, F_*Y) = \lambda' g_2(F_*X, F_*Y),$$

where $\lambda' = \sum_{k=1}^{n_1}(e_k(g))^2 - \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k}e_k, \nabla^N g) + \sum_{k=1}^{n_1} e_k(e_k(g)) - \lambda - V(g)$ is a smooth function on N. Thus a leaf of $rangeF_*$ is an Einstein, which completes the proof. □

Theorem 4.8 *Let β be a geodesic curve on N and $(N, g_2, \dot\beta, \lambda)$ be a Ricci soliton with the potential vector field $\dot\beta \in \Gamma(TN)$. Let $F : (M, g_1) \to (N, g_2)$ be a Clairaut Riemannian map with $\tilde s = e^g$ from a Riemannian manifold M to an Einstein manifold N. Then the following statements are true:*

(i) $\dot\beta$ is a conformal vector field on $rangeF_*$.

(ii) $\dot\beta$ is Killing vector field on $(rangeF_*)^\perp$ if and only if $V(g)W(g) = -H^g(V, W)$ for all $V, W \in \Gamma(rangeF_*)^\perp$.

Proof Since $(N, g_2, \dot\beta, \lambda)$ is a Ricci soliton then, we have

$$\frac{1}{2}(L_{\dot\beta}g_2)(F_*X, F_*Y) + Ric(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0, \tag{4.17}$$

for $F_*X, F_*Y \in \Gamma(rangeF_*)$. Using (4.4) in (4.17), we get

$$\begin{aligned} &\tfrac{1}{2}(L_{\dot\beta}g_2)(F_*X, F_*Y) + Ric^{rangeF_*}(F_*X, F_*Y) - \sum_{k=1}^{n_1}(e_k(g))^2 g_2(F_*X, F_*Y) \\ &+ \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k}e_k, \nabla^N g)g_2(F_*X, F_*Y) - \sum_{k=1}^{n_1}\nabla^N_{e_k}e_k(g)g_2(F_*X, F_*Y) + \lambda g_2(F_*X, F_*Y) = 0, \end{aligned} \tag{4.18}$$

where $\{e_k\}_{1\leq k\leq n_1}$ is an orthonormal basis of $(rangeF_*)^\perp$. Since N is Einstein, putting $Ric^{rangeF_*}(F_*X, F_*Y) = -\lambda g_2(F_*X, F_*Y)$ in (4.18), we get

$$\tfrac{1}{2}(L_{\dot\beta} g_2)(F_*X, F_*Y) + \mu g_2(F_*X, F_*Y) = 0,$$

where $\mu = -\sum_{k=1}^{n_1}(e_k(g))^2 + \sum_{k=1}^{n_1} g_2(\nabla^{F\perp}_{e_k} e_k, \nabla^N g) - \sum_{k=1}^{n_1} e_k(e_k(g))$ is a smooth function on N. Thus $\dot\beta$ is a conformal vector field on $rangeF_*$. On the other hand, since $(N, g_2, \dot\beta, \lambda)$ is a Ricci soliton then, we have

$$\frac{1}{2}(L_{\dot\beta} g_2)(V, W) + Ric(V, W) + \lambda g_2(V, W) = 0, \tag{4.19}$$

for any $V, W \in \Gamma(rangeF_*)^\perp$. Using (4.5) in (4.19), we get

$$\begin{aligned} &\tfrac{1}{2}(L_{\dot\beta} g_2)(V, W) + Ric^{(rangeF_*)^\perp}(V, W) + (m-r)g_2(\nabla^N g, \nabla^{F\perp}_V W) \\ &-(m-r)V(g)W(g) - (m-r)\nabla^N_V W(g) + \lambda g_2(V, W) = 0. \end{aligned} \tag{4.20}$$

Since N is Einstein, putting $Ric^{(rangeF_*)^\perp}(V, W) = -\lambda g_2(V, W)$ in (4.20), we get

$$\tfrac{1}{2}(L_{\dot\beta} g_2)(V, W) + \{g_2(\nabla^N g, \nabla^{F\perp}_V W) - V(g)W(g) - \nabla^N_V W(g)\}(m-r) = 0.$$

Then by using $\nabla^N_V W(g) = \nabla^N_V(g_2(W, \nabla^N g)) = g_2(\nabla^N_V W, \nabla^N g) + H^g(V, W) = g_2(\nabla^{F\perp}_V W, \nabla^N g) + H^g(V, W)$ in above equation, we get $\frac{1}{2}(L_{\dot\beta} g_2)(V, W) = 0$ if and only if $V(g)W(g) = -H^g(V, W)$. This completes the proof. □

Lemma 4.9 *Let* (N, g_2, X_1, λ) *be a Ricci soliton with the potential vector field* $X_1 \in \Gamma(TN)$ *and* $F : (M^m, g_1) \to (N^n, g_2)$ *be a Clairaut Riemannian map with* $\tilde s = e^g$ *between Riemannian manifolds. Then*

$$s = -\lambda n, \tag{4.21}$$

where s *denotes the scalar curvature of* N.

Proof The proof is similar to remark 9 of [30]; therefore, we are omitting it. □

Theorem 4.10 *Let* $(N, g_2, -H_2, \lambda)$ *be a Ricci soliton with the potential vector field* $-H_2 \in \Gamma(rangeF_*)^\perp$ *and* $F : (M, g_1) \to (N, g_2)$ *be a Clairaut Riemannian map with* $\tilde s = e^g$ *between Riemannian manifolds. Then following statements are true:*
(i) N *admits a gradient Ricci soliton.*
(ii) The mean curvature vector field of $rangeF_*$ *is constant.*

Proof By similar proof as theorem 10 of [30], we get

$$\Delta g = 0.$$

Hence $\nabla^N(\nabla^N g) = 0$, i.e. $\nabla^N H_2 = 0$, which means H_2 is constant. This completes the proof. □

Example 4.11 *The map $F: M \to N$ given in Example 3.7 is Clairaut Riemannian map. Now, we will show that N admits a Ricci soliton, i.e.*

$$\frac{1}{2}(L_{Z_1}g_2)(X_1,Y_1) + Ric(X_1,Y_1) + \lambda g_2(X_1,Y_1) = 0, \tag{4.22}$$

for any $X_1, Y_1, Z_1 \in \Gamma(TN)$. By similar computations as example 6.1 of [32], we get

$$\frac{1}{2}(L_{Z_1}g_2)(X_1,Y_1) = 0, \tag{4.23}$$

$$g_2(X_1,Y_1) = (a_1a_3 + a_2a_4), \tag{4.24}$$

and

$$Ric(X_1,Y_1) = a_1a_3Ric(e_1',e_1') + (a_1a_4 + a_2a_3)Ric(e_1',e_2') + a_2a_4Ric(e_2',e_2'). \tag{4.25}$$

By (4.4), we get

$$Ric(e_1',e_1') = Ric^{rangeF_*}(e_1',e_1') - (g_2(\nabla^N g, e_2'))^2 + g_2(\nabla^{F\perp}_{e_2'}e_2', \nabla^N g) - \nabla^N_{e_2'}(g_2(e_2', \nabla^N g)).$$

Since dimension of $rangeF_*$ is one, $Ric^{rangeF_*}(e_1',e_1') = 0$ and we have $\nabla^N g = -be_2'$ for some $b \in \mathbb{R}$. So

$$Ric(e_1',e_1') = -b^2, \tag{4.26}$$

By (4.5), we get

$$Ric(e_2',e_2') = Ric^{(rangeF_*)^\perp}(e_2',e_2') + g_2(\nabla^N g, \nabla^{F\perp}_{e_2'}e_2') - e_2'(g)e_2'(g) - \nabla^N_{e_2'}(e_2'(g)).$$

Since dimension of $(rangeF_*)^\perp$ is one, $Ric^{(rangeF_*)^\perp}(e_2',e_2') = 0$ and putting $\nabla^N g = -be_2'$ for some $b \in \mathbb{R}$, we get

$$Ric(e_2',e_2') = -b^2. \tag{4.27}$$

And by similar computation as example 6.1 of [32], we get

$$Ric(e_1',e_2') = 0. \tag{4.28}$$

Using (4.26), (4.27) and (4.28) in (4.25), we get

$$Ric(X_1,Y_1) = -(a_1a_3 + a_2a_4)b^2. \tag{4.29}$$

Now, using (4.23), (4.24) and (4.29) in (4.22), we obtain that metric g_2 admits Ricci soliton for

$$\lambda = b^2.$$

Since $b \in \mathbb{R}$, for some choices of b Ricci soliton (N, g_2) will be expanding or steady according to $\lambda > 0$ or $\lambda = 0$.

5. Clairaut antiinvariant Riemannian map from Riemannian manifold to Kähler manifold

In this section, we introduce Clairaut antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold and investigate the geometry with a nontrivial example.

Let (N, g_2) be an almost Hermitian manifold [33], then N admits a tensor J of type (1, 1) on N such that $J^2 = -I$ and

$$g_2(JX_1, JY_1) = g_2(X_1, Y_1), \tag{5.1}$$

for all $X_1, Y_1 \in \Gamma(TN)$. An almost Hermitian manifold N is called Kähler manifold if

$$(\nabla^N_{X_1} J)Y_1 = 0,$$

for all $X_1, Y_1 \in \Gamma(TN)$, where ∇^N is the Levi-Civita connection on N.

Definition 5.1 *[20] Let $F : (M, g_1) \to (N, g_2)$ be a proper Riemannian map from a Riemannian manifold M to an almost Hermitian manifold N with almost complex structure J. We say that F is an antiinvariant Riemannian map at $p \in M$ if $J(rangeF_{*p}) \subset (rangeF_{*p})^\perp$. If F is an antiinvariant Riemannian map for every $p \in M$ then F is called an antiinvariant Riemannian map.*

In this case we denote the orthogonal subbundle to $J(rangeF_*)$ in $(rangeF_*)^\perp$ by μ, i.e. $(rangeF_*)^\perp = J(rangeF_*) \oplus \mu$. For any $V \in \Gamma(rangeF_*)^\perp$, we have

$$JV = BV + CV, \tag{5.2}$$

where $BV \in \Gamma(rangeF_*)$ and $CV \in \mu$. Note that if $\mu = 0$ then F is called Lagrangian Riemannian map [27].

Lemma 5.2 *Let $F : (M, g_1) \to (N, g_2, J)$ be an antiinvariant Riemannian map from a Riemannian manifold M to a Kähler manifold N and $\alpha : I \to M$ be a geodesic curve on M. Then the curve $\beta = F \circ \alpha$ is geodesic on N if and only if*

$$-\mathcal{S}_{JF_*X}F_*X - \mathcal{S}_{CV}F_*X + \nabla^N_V BV + F_*(\nabla^{M}_X {}^*F_*BV) = 0, \tag{5.3}$$

$$(\nabla F_*)(X, {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X + \nabla^{F\perp}_X CV + \nabla^{F\perp}_V CV = 0, \tag{5.4}$$

where $F_*X \in \Gamma(rangeF_*), V \in \Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$ and ${}^*F_*$ is the adjoint map of F_*, and ∇^N is the Levi-Civita connection on N, and $\nabla^{F\perp}$ is a linear connection on $(rangeF_*)^\perp$.

Proof Let $\alpha : I \to M$ be a geodesic on M and let $\beta = F \circ \alpha$ be a geodesic on N with $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$. Since N is Kähler manifold, $\nabla^N_{\dot\beta}\dot\beta = -J\nabla^N_{\dot\beta}J\dot\beta$. Thus

$$\nabla^N_{\dot\beta}\dot\beta = -J\nabla^N_{\dot\beta}J\dot\beta = -J\nabla^N_{F_*X+V}J(F_*X + V),$$

which implies

$$\nabla^N_{\dot\beta}\dot\beta = -J(\nabla^N_{F_*X}JF_*X + \nabla^N_{F_*X}JV + \nabla^N_V JF_*X + \nabla^N_V JV). \tag{5.5}$$

Using (2.4) and (5.2) in (5.5), we get

$$\begin{aligned} \nabla^N_{\dot\beta}\dot\beta &= -J(-\mathcal{S}_{JF_*X}F_*X - \mathcal{S}_{CV}F_*X + \nabla^N_V BV + \nabla^N_{F_*X}BV \\ &\quad + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X + \nabla^{F\perp}_X CV + \nabla^{F\perp}_V CV). \end{aligned} \tag{5.6}$$

Since ∇^N is Levi-Civita connection on N and $g_2(\nabla^N_V BV, U) = 0$ for any $U \in \Gamma(rangeF_*)^\perp$, $\nabla^N_V BV \in \Gamma(rangeF_*)$ and using (2.2), we get $\nabla^N_{F_*X} BV = \overset{N}{\nabla}{}^F_X BV \circ F = (\nabla F_*)(X, {}^*F_*BV) + F_*(\nabla^M_X {}^*F_*BV)$. Then by (5.6), we get

$$\begin{aligned}\nabla^N_{\dot\beta}\dot\beta &= -J(-\mathcal{S}_{JF_*X}F_*X - \mathcal{S}_{CV}F_*X + \nabla^N_V BV + (\nabla F_*)(X, {}^*F_*BV) \\ &\quad + F_*(\nabla^M_X {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X + \nabla^{F\perp}_X CV + \nabla^{F\perp}_V CV).\end{aligned}$$

Now β is geodesic on N $\iff$ $\nabla^N_{\dot\beta}\dot\beta = 0$ $\iff$ $-\mathcal{S}_{JF_*X}F_*X - \mathcal{S}_{CV}F_*X + \nabla^N_V BV + (\nabla F_*)(X, {}^*F_*BV) + F_*(\nabla^M_X {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X + \nabla^{F\perp}_X CV + \nabla^{F\perp}_V CV = 0$, which completes the proof. □

Definition 5.3 *An antiinvariant Riemannian map from a Riemannian manifold to a Kähler manifold is called Clairaut antiinvariant Riemannian map if it satisfies the condition of Clairaut Riemannian map.*

Theorem 5.4 *Let $F : (M, g_1) \to (N, g_2, J)$ be an antiinvariant Riemannian map from a Riemannian manifold M to a Kähler manifold N and α, $\beta = F \circ \alpha$ are geodesic curves on M and N, respectively. Then F is Clairaut antiinvariant Riemannian map with $\tilde{s} = e^g$ if and only if $g_2(\mathcal{S}_{JF_*X}F_*X + \mathcal{S}_{CV}F_*X, BV) - g_2((\nabla F_*)(X, {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X, CV) - g_2(F_*X, F_*X)\frac{d(g\circ\beta)}{dt} = 0$, where g is a smooth function on N and $F_*X \in \Gamma(rangeF_*)$, $V \in \Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$.*

Proof Let $\alpha : I \to M$ be a geodesic on M and let $\beta = F \circ \alpha$ be a geodesic on N with $F_*X \in \Gamma(rangeF_*)$ and $V \in \Gamma(rangeF_*)^\perp$ are components of $\dot\beta(t)$ and $\omega(t)$ denote the angle in $[0, \pi]$ between $\dot\beta$ and V. Assuming $b = \|\dot\beta(t)\|^2$, then we get

$$g_{2\beta(t)}(V, V) = bcos^2\omega(t), \tag{5.7}$$

$$g_{2\beta(t)}(F_*X, F_*X) = bsin^2\omega(t). \tag{5.8}$$

Now differentiating (5.7) along β, we get

$$\frac{d}{dt}g_2(V, V) = -2bsin\omega(t)cos\omega(t)\frac{d\omega}{dt}. \tag{5.9}$$

On the other hand by (5.1), we get

$$\frac{d}{dt}g_2(V, V) = \frac{d}{dt}g_2(JV, JV).$$

Using (5.2) in above equation, we get

$$\frac{d}{dt}g_2(V, V) = \frac{d}{dt}\Big(g_2(BV, BV) + g_2(CV, CV)\Big),$$

which implies

$$\frac{d}{dt}g_2(V, V) = 2g_2(\nabla^N_{\dot\beta} BV, BV) + 2g_2(\nabla^N_{\dot\beta} CV, CV). \tag{5.10}$$

Putting $\dot{\beta} = F_*X + V$ in (5.10), we get

$$\frac{d}{dt}g_2(V,V) = 2g_2(\nabla^N_{F_*X}BV, BV) + 2g_2(\nabla^N_{F_*X}CV, CV) + 2g_2(\nabla^N_V BV, BV) + 2g_2(\nabla^N_V CV, CV).$$

Since $(rangeF_*)^\perp$ is totally geodesic, above equation can be written as

$$\frac{d}{dt}g_2(V,V) = 2g_2(\overset{N}{\nabla}{}^F_X BV \circ F, BV) + 2g_2(\nabla^N_{F_*X}CV, CV) + 2g_2(\nabla^N_V BV, BV) + 2g_2(\nabla^{F\perp}_V CV, CV). \tag{5.11}$$

Using (2.2), (2.3) and (2.4) in (5.11), we get

$$\frac{d}{dt}g_2(V,V) = 2g_2(F_*(\nabla^{M*}_X F_*BV) + \nabla^N_V BV, BV) + 2g_2(\nabla^{F\perp}_X CV + \nabla^{F\perp}_V CV, CV). \tag{5.12}$$

Using (5.3) and (5.4) in (5.12), we get

$$\frac{d}{dt}g_2(V,V) = 2g_2(\mathcal{S}_{JF_*X}F_*X + \mathcal{S}_{CV}F_*X, BV) - 2g_2\Big((\nabla F_*)(X, {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X, CV\Big). \tag{5.13}$$

Now from (5.9) and (5.13), we get

$$g_2(\mathcal{S}_{JF_*X}F_*X + \mathcal{S}_{CV}F_*X, BV) - g_2\Big((\nabla F_*)(X, {}^*F_*BV) + \nabla^{F\perp}_X JF_*X + \nabla^{F\perp}_V JF_*X, CV\Big) = -bsin\omega cos\omega\frac{d\omega}{dt}. \tag{5.14}$$

Moreover, F is a Clairaut Riemannian map with $\tilde{s} = e^g$ if and only if $\frac{d}{dt}(e^{g\circ\beta}sin\omega) = 0$, that is, $e^{g\circ\beta}sin\omega\frac{d(g\circ\beta)}{dt} + e^{g\circ\beta}cos\omega\frac{d\omega}{dt} = 0$. By multiplying this with nonzero factor $bsin\omega$ and using (5.8), we get

$$g_2(F_*X, F_*X)\frac{d(g\circ\beta)}{dt} = -bsin\omega cos\omega\frac{d\omega}{dt}. \tag{5.15}$$

Thus (5.14) and (5.15) complete the proof. □

Theorem 5.5 *Let $F : (M^m, g_1) \to (N^n, g_2, J)$ be a Clairaut antiinvariant Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold M to a Kähler manifold N. Then at least one of the following statement is true:*
(i) $\dim(rangeF_*) = 1$*,*
(ii) g is constant on $J(rangeF_*)$, where g is a smooth function on N.

Proof Since F is Clairaut Riemannian map with $\tilde{s} = e^g$ then using (2.2) in (3.21), we get

$$\overset{N}{\nabla}{}^F_X F_*Y - F_*(\nabla^M_X Y) = -g_1(X,Y)\nabla^N g, \tag{5.16}$$

for $F_*Y \in \Gamma(rangeF_*)$ and $X, Y \in \Gamma(kerF_*)^\perp$. Taking inner product of (5.16) with $JF_*Z \in \Gamma(rangeF_*)^\perp$, we get

$$g_2(\overset{N}{\nabla}{}^F_X F_*Y - F_*(\nabla^M_X Y), JF_*Z) = -g_1(X,Y)g_2(\nabla^N g, JF_*Z). \tag{5.17}$$

Since $\overset{N}{\nabla}{}^F$ is pullback connection of the Levi-Civita connection ∇^N. Therefore $\overset{N}{\nabla}{}^F$ is also Levi-Civita connection. Then using metric compatibility condition in (5.17), we get

$$-g_2(\overset{N}{\nabla}{}^F_X JF_*Z, F_*Y) = -g_1(X,Y)g_2(\nabla^N g, JF_*Z),$$

which implies

$$g_2(J\overset{N}{\nabla}{}^F_X F_*Z, F_*Y) = g_1(X,Y)g_2(\nabla^N g, JF_*Z). \tag{5.18}$$

Using (5.1) in (5.18), we get

$$-g_2(\overset{N}{\nabla}{}^F_X F_*Z, JF_*Y) = g_1(X,Y)g_2(\nabla^N g, JF_*Z).$$

Using (5.16) in above equation, we get

$$g_1(X,Z)g_2(\nabla^N g, JF_*Y) = g_1(X,Y)g_2(\nabla^N g, JF_*Z). \tag{5.19}$$

Now putting $X = Y$ in (5.19), we get

$$g_1(X,Z)g_2(\nabla^N g, JF_*X) = g_1(X,X)g_2(\nabla^N g, JF_*Z). \tag{5.20}$$

Now interchanging X and Z in (5.20), we get

$$g_1(X,Z)g_2(\nabla^N g, JF_*Z) = g_1(Z,Z)g_2(\nabla^N g, JF_*X). \tag{5.21}$$

From (5.20) and (5.21), we get

$$g_2(\nabla^N g, JF_*X)\left(1 - \frac{g_1(X,X)g_1(Z,Z)}{g_1(X,Z)g_1(X,Z)}\right) = 0,$$

which implies either $\dim((kerF_*)^\perp) = 1$ or $g_2(\nabla^N g, JF_*X) = 0$, which means $(JF_*X)(g) = 0$, which completes the proof. □

Theorem 5.6 *Let $F : (M^m, g_1) \to (N^n, g_2, J)$ be a Clairaut Lagrangian Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold M to a Kähler manifold N such that $\dim(rangeF_*) > 1$. Then following statements are true:*
(i) $rangeF_*$ is minimal.
(ii) $rangeF_*$ is totally geodesic.

Proof Since F is Clairaut Riemannian map then from (3.21) and Theorem 3.3, we have

$$(\nabla F_*)(X,X) = g_1(X,X)H_2,$$

for $X \in \Gamma(kerF_*)^\perp$ and H_2 is the mean curvature vector field of $rangeF_*$. Now multiply above equation by $U \in \Gamma(rangeF_*)^\perp$, we get

$$g_2((\nabla F_*)(X,X), U) = g_1(X,X)g_2(H_2, U). \tag{5.22}$$

Using (2.2) in (5.22), we get

$$g_2(\overset{N}{\nabla}{}^F_X F_*X, U) = g_1(X,X)g_2(H_2, U). \tag{5.23}$$

Since N is Kähler manifold, using (5.1) in (5.23), we get

$$g_2(\overset{N}{\nabla}{}^F_X JF_*X, JU) = g_1(X,X)g_2(H_2, U). \tag{5.24}$$

Since ∇^N is Levi-Civita connection on N, using metric compatibility condition in (5.24), we get

$$-g_2(JF_*X, \overset{N}{\nabla}{}^F_X JU) = g_1(X,X)g_2(H_2,U). \tag{5.25}$$

Using (5.23) in (5.25), we get

$$-g_2(JF_*X, g_1(X, {}^*F_*JU)H_2) = g_1(X,X)g_2(H_2,U), \tag{5.26}$$

where ${}^*F_*$ is the adjoint map of F_*. Now using $H_2 = -\nabla^N g$ in (5.26), we get

$$g_1(X, {}^*F_*JU)g_2(JF_*X, \nabla^N g) = g_1(X,X)g_2(H_2,U),$$

which implies

$$g_1(X, {}^*F_*JU)JF_*X(g) = g_1(X,X)g_2(H_2,U). \tag{5.27}$$

Since $\dim(rangeF_*) > 1$ then by Theorem 5.5, g is constant on $J(rangeF_*)$, which means $JF_*X(g) = 0$. Then (5.27) implies $g_2(H_2,U) = 0$. Thus

$$H_2 = 0, \tag{5.28}$$

which implies (i).

Since $H_2 = trace\ (\overset{N}{\nabla}{}^F_X F_*Y)$. Then by (5.28), we get $\overset{N}{\nabla}{}^F_X F_*Y = 0$, which implies (ii). □

Theorem 5.7 *Let $F : (M^m, g_1) \to (N^n, g_2, J)$ be a Clairaut Lagrangian Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold M to a Kähler manifold N such that $\dim(rangeF_*) > 1$. Then F is harmonic if and only if mean curvature vector field of $kerF_*$ is constant.*

Proof Let $F : (M^m, g_1) \to (N^n, g_2)$ be a smooth map between Riemannian manifolds. Then F is harmonic if and only if the tension field $\tau(F)$ of map F vanishes. Then proof follows by Lemma 2.1 and Theorem 5.6. □

Theorem 5.8 *Let $F : (M^m, g_1) \to (N^n, g_2, J)$ be a Clairaut Lagrangian Riemannian map with $\tilde{s} = e^g$ from a Riemannian manifold M to a Kähler manifold N such that $\dim(rangeF_*) > 1$. Then $N = N_{rangeF_*} \times N_{(rangeF_*)^\perp}$ is a usual product manifold.*

Proof The proof follows by [19] and Theorem 5.6. □

Example 5.9 *Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0\}$ be a Riemannian manifold with Riemannian metric $g_1 = e^{2x_2}dx_1^2 + e^{2x_2}dx_2^2$ on M. Let $N = \{(y_1, y_2) \in \mathbb{R}^2\}$ be a Riemannian manifold with Riemannian metric $g_2 = e^{2x_2}dy_1^2 + dy_2^2$ on N and the complex structure J on N defined as $J(y_1, y_2) = (-y_2, y_1)$. Consider a map $F : (M, g_1) \to (N, g_2, J)$ defined by*

$$F(x_1, x_2) = \Big(\frac{x_1 - x_2}{\sqrt{2}}, 0\Big).$$

Then

$$kerF_* = span\Big\{U = \frac{e_1 + e_2}{\sqrt{2}}\Big\} \text{ and } (kerF_*)^\perp = span\Big\{X = \frac{e_1 - e_2}{\sqrt{2}}\Big\},$$

where $\left\{e_1 = e^{-x_2}\frac{\partial}{\partial x_1}, e_2 = e^{-x_2}\frac{\partial}{\partial x_2}\right\}$ *and* $\left\{e'_1 = e^{-x_2}\frac{\partial}{\partial y_1}, e'_2 = \frac{\partial}{\partial y_2}\right\}$ *are bases on* T_pM *and* $T_{F(p)}N$ *respectively, for* $p \in M$. *By easy computations, we see that* $F_*(X) = e'_1$ *and* $g_1(X,X) = g_2(F_*X, F_*X)$ *for* $X \in \Gamma(kerF_*)^\perp$. *Thus* F *is Riemannian map with* $rangeF_* = span\{F_*(X) = e'_1\}$ *and* $(rangeF_*)^\perp = span\{e'_2\}$. *Moreover it is easy to see that* $JF_*X = Je'_1 = -e'_2$. *Thus* F *is an antiinvariant Riemannian map.*

Now to show F *is Clairaut Riemannian map we will find a smooth function* g *on* N *satisfying* $(\nabla F_*)(X,X) = -g_1(X,X)\nabla^N g$ *for* $X \in \Gamma(kerF_*)^\perp$. *Since* $(\nabla F_*)(X,X) \in \Gamma(rangeF_*)^\perp$ *for any* $X \in \Gamma(kerF_*)^\perp$. *So here we can write* $(\nabla F_*)(X,X) = ae'_2$, *for some* $a \in \mathbb{R}$. *Since* $\nabla^N g = e^{-2x_2}\frac{\partial g}{\partial y_1}\frac{\partial}{\partial y_1} + \frac{\partial g}{\partial y_2}\frac{\partial}{\partial y_2}$. *Hence* $\nabla^N g = -a\frac{\partial}{\partial y_2} = -ae'_2$ *for the function* $g = -ay_2$. *Then it is easy to verify that* $(\nabla F_*)(X,X) = -g_1(X,X)\nabla^N g$, *where* $g_1(X,X) = 1$, *for vector field* $X \in \Gamma(kerF_*)^\perp$ *and we can easily see that* $\nabla^N_{e'_2} e'_2 = 0$. *Thus by Theorem 3.3,* F *is Clairaut antiinvariant Riemannian map.*

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References

[1] Allison D. Lorentzian Clairaut submersions. Geometriae Dedicata 1996; 63: 309-319. https://doi.org/10.1007/BF00181419

[2] Akyol MA, Şahin B. Conformal anti-invariant Riemannian maps to Kähler manifolds. UPB Scientific Bulletin, Series A: Applied Mathematics and Physics 2018; 80 (4): 187-198.

[3] Akyol MA, Şahin B. Conformal semi-invariant Riemannian maps to Kähler manifolds. Revista de la Union Matematica Argentina 2019; 60 (2): 459-468. https://doi.org/10.33044/revuma.v60n2a12

[4] Besse AL. Einstein Manifolds. Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[5] Bishop RL. Clairaut submersions. Differential geometry (in Honor of K-Yano), Kinokuniya, Tokyo, 1972; 21-31.

[6] Carmo MP. Differential geometry of curves and surfaces. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1976.

[7] Deshmukh S, Al-Solamy FR. Conformal vector fields on a Riemannian manifold. Balkan Journal of Geometry and its Applications 2014; 19 (2): 86-93.

[8] Falcitelli M, Ianus S, Pastore AM. Riemannian Submersions and Related Topics. River Edge, NJ: World Scientific, 2004.

[9] Fischer AE. Riemannian maps between Riemannian manifolds. Contemporary Mathematics 1992; 132: 331-366. https://doi.org/10.1090/conm/132/1188447

[10] Gupta G, Sachdeva R, Kumar R, Rani R. On conformal Riemannian maps whose total manifolds admit a Ricci soliton. Journal of Geometry and Physics 2022; 178: 1-19. https://doi.org/10.1016/j.geomphys.2022.104539

[11] Hamilton RS. The Ricci flow on surfaces, mathematics and general relativity. Contemporary Mathematics 1966; 71: 237-262. https://doi.org/10.1090/conm/071/954419

[12] Lee J, Park J, Şahin B, Song D. Einstein conditions for the base space of anti-invariant Riemannian submersions and Clairaut submersions. Taiwanese Journal of Mathematics 2015; 19 (4): 1145-1160. https://doi.org/10.11650/tjm.19.2015.5283

[13] Meena K, Yadav A. Conformal submersions whose total manifolds admit a Ricci soliton. Mediterranean Journal of Mathematics, To appear.

[14] Meena K, Zawadzki T. Clairaut conformal submersions. preprint, 2022, arXiv:2202.00393 [math.DG].

[15] Meriç SE, Kiliç E. Riemannian submersions whose total manifolds admit a Ricci soliton. International Journal of Geometric Methods in Modern Physics 2019; 16: 1950196-1-1950196-12. https://doi.org/10.1142/S0219887819501962

[16] Nore T. Second fundamental form of a map. Annali di Matematica pura ed applicata 1986; 146: 281-310. https://doi.org/10.1007/BF/01762368

[17] Perelman G. The Entropy formula for the Ricci flow and its geometric applications. preprint, 2002, arxiv: math/0211159 [math.DG].

[18] Pigola S, Rigoli M, Rimoldi M, Setti AG. Ricci almost solitons. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze 2011; 10 (4): 757-799.

[19] Ponge R, Reckziegel H. Twisted products in pseudo-Riemannian geometry. Geometriae Dedicata 1993; 48 (1): 15-25. https://doi.org/10.1007/BF01265674

[20] Şahin B. Invariant and anti-invariant Riemannian maps to Kähler manifolds. International Journal of Geometric Methods in Modern Physics 2010; 7 (3): 337-355. https://doi.org/10.1142/S0219887810004324

[21] Şahin B. Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems. Acta Applicandae Mathematicae 2010; 109: 829-847. https://doi.org/10.1007/s10440-008-9348-6

[22] Şahin B. Semi-invariant Riemannian maps to Kähler manifolds. International Journal of Geometric Methods in Modern Physics 2011; 8 (7): 1439-1454. https://doi.org/10.1142/S0219887811005725

[23] Şahin B. Holomorphic Riemannian maps. Journal of Mathematical Physics, Analysis, Geometry 2014; 10 (4): 422-430.

[24] Şahin B. Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications. Academic Press: Elsevier, 2017.

[25] Şahin B. Circles along a Riemannian map and Clairaut Riemannian maps. Bulletin of the Korean Mathematical Society 2017; 54 (1): 253-264. https://doi.org/10.4134/BKMS.b160082

[26] Şahin B. A survey on differential geometry of Riemannian maps between Riemannian manifolds. Analele Stiintifice ale Universitatii Al I Cuza din Iasi - Matematica 2017; 63: 151-167.

[27] Taştan HM. On Lagrangian submersions. Hacettepe Journal of Mathematics and Statistics 2014; 43 (6): 993-1000. https://doi.org/10.15672/HJMS.2014437529

[28] Watson B. Almost Hermitian submersions. Journal of Differential Geometry 1976; 11: 147-165.

[29] Yadav A, Meena K. Riemannian maps whose total manifolds admit a Ricci soliton. Journal of Geometry and Physics 2021; 16: 1-13. https://doi.org/10.1016/j.geomphys.2021.104317

[30] Yadav A, Meena K. Clairaut Riemannian maps whose total manifolds admit a Ricci soliton. International Journal of Geometric Methods in Modern Physics 2022; 19 (2): 2250024-1-2250024-17. doi: 10.1142/S0219887822500244

[31] Yadav A, Meena K. Clairaut invariant Riemannian maps with Kähler structure. Turkish Journal of Mathematics 2022; 46 (3): 1020-1035. doi: 10.55730/1300-0098.3139

[32] Yadav A, Meena K. Riemannian maps whose base manifolds admit a Ricci soliton. Publicationes Mathematicae Debrecen, To appear. https://doi.org/10.5486/PMD.2023.9413

[33] Yano K, Kon M. Structure on Manifolds. Singapore: World Scientific, 1984.