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Lyapunov-type inequalities for $(\mathtt{n},\mathtt{p})$-type nonlinear fractional boundary value problems

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Abstract: This paper establishes Lyapunov-type inequalities for a family of two-point $(\mathtt{n},\mathtt{p})$-type boundary value problems for Riemann–Liouville fractional differential equations. To demonstrate how the findings can be applied, we provide a few examples, one of which is a fractional differential equation with delay.

Key words: Fractional derivative, boundary value problem, Green's function, Lyapunov-type inequality

1. Introduction

Fractional differential equations (FDEqs) with boundary conditions (BCs) have gained in popularity and now play a vital role in the growth of applied mathematics. Fractional calculus (FC) has sparked a lot of attention among mathematicians and modelers in recent years. FC has proven to be incredibly beneficial in a wide range of fields, including mechanics, chemistry, control structures, dynamic procedures and viscoelasticity, for example; see [2, 7, 13, 15, 18, 22].

Different types of integral inequalities are well known to play significant roles in the investigation of the qualitative properties of solutions of differential and integral equations. One such inequality is the Lyapunov-type inequality, which has been shown to be crucial in analysing the zeros of solutions of differential equations [1, 16]. The famous Lyapunov theorem [14] is as follows: If the boundary value problem (BVP)

$$\left.\begin{aligned} \mathtt{w}''(\mathtt{z}) + \ell(\mathtt{z})\mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{z} \in (\mathtt{a},\mathtt{b}),\\ \mathtt{w}(\mathtt{a}) = 0 = \mathtt{w}(\mathtt{b}), \end{aligned}\right\} \tag{1.1}$$

has a nontrivial solution, where $\ell \in \mathtt{C}([\mathtt{a},\mathtt{b}];\mathbb{R})$, then

$$\int_{\mathtt{a}}^{\mathtt{b}} |\ell(\mathtt{s})| d\mathtt{s} > \frac{4}{\mathtt{b}-\mathtt{a}}. \tag{1.2}$$

The Lyapunov inequality (1.2) is useful in various problems. It has been generalized in various ways due to its significance. In recent years, researchers have derived Lyapunov-type inequalities for a variety of fractional

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boundary value problems (FBVPs). In [9], Ferreira investigated a Lyapunov-type inequality for the FBVP

$$\left.\begin{aligned} {}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \ell(\mathtt{z})\mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{z} \in (\mathtt{a},\mathtt{b}),\\ \mathtt{w}(\mathtt{a}) = 0 = \mathtt{w}(\mathtt{b}). \end{aligned}\right\} \tag{1.3}$$

If ℓ is a real continuous function on $[\mathtt{a},\mathtt{b}]$, $\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}$ denotes Riemann–Liouville (RL) derivative, where $\mathtt{r} \in (1,2]$, then if the FBVP (1.3) has a nontrivial solution, then

$$\int_{\mathtt{a}}^{\mathtt{b}} \big|\ell(\mathtt{s})\big| d\mathtt{s} > \Gamma(\mathtt{r})\left(\frac{4}{\mathtt{b}-\mathtt{a}}\right)^{\mathtt{r}-1}.$$

Further, in [10], the same author developed a Lyapunov-type inequality for a Caputo FBVP. In [9] and [10], the author has presented some interesting applications for identifying the real zeros of certain Mittag–Leffler functions.

In [21], eigenvalue intervals are obtained for which the iterative system of $(\mathtt{n},\mathtt{p})$-type FBVP has a positive solution. In 2019, Ntouyas et al. [16] published survey results on Lyapunov-type inequalities for FDEqs with numerous different BCs. For more details, see boundary value problems for ordinary differential equations [4, 5, 17], or boundary value problems for FDEqs [11, 12, 23]. By applying the Leggett–Williams fixed point theorem to the coupled system of FBVPs, sufficient conditions for the existence of multiple positive solutions are obtained in [19]. Those authors later extended these results to an iterative system of FBVPs in [20]. Inspired and motivated by the aforementioned work, we consider the $(\mathtt{n},\mathtt{p})$-type nonlinear fractional BVP

$${}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \ell(\mathtt{z})\mathtt{F}\mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{z} \in (\mathtt{a},\mathtt{b}), \tag{1.4}$$

$$\left.\begin{aligned} &\mathtt{w}^{(\mathtt{i})}(\mathtt{a}) = 0, \quad \mathtt{i} = 0,1,\ldots,\mathtt{n}-2,\\ &{}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{p}}_{\mathtt{a}+}\mathtt{w}(\mathtt{b}) = 0, \end{aligned}\right\} \tag{1.5}$$

where $\mathtt{b} > \mathtt{a}$, $\mathtt{r} \in (\mathtt{n}-1,\mathtt{n}]$, $\mathtt{n} \geq 2$ is an integer, $\mathtt{p} \in [0,\mathtt{r}-1]$ and ${}^{\mathtt{RL}}\mathfrak{D}^{\bullet}_{\mathtt{a}+}$ denotes the RL derivative.

It has recently become known to us that Dhar and Neugebauer [3] obtained the types of results that we seek in the case $r \in (1,2]$. For completeness in this article, we shall state without proof the results already obtained in [3].

Throughout the paper, we consider the following assumptions:

$(\mathbf{A}_1)$ $\mathtt{F} : \mathtt{C}[\mathtt{a},\mathtt{b}] \to \mathtt{C}[\mathtt{a},\mathtt{b}]$ and there exists $\eta > 0$, independent of $\mathtt{w}$ such that if $\mathtt{w} \in \mathtt{C}[\mathtt{a},\mathtt{b}]$, then

$$\|\mathtt{F}\mathtt{w}\|_\infty \leq \eta \|\mathtt{w}\|_\infty.$$

$(\mathbf{A}_2)$ $\ell \in \mathtt{C}\big([\mathtt{a},\mathtt{b}],\mathbb{R}\big)$.

This article is organized as follows. Section 2 contains some supplementary results. Section 3 contains estimates related to an associated Green's function. The main theorems are presented in Section 4, and three examples are presented in Section 5 to demonstrate how the results can be applied. One of the examples considers a fractional differential equation with delay.

2. Auxiliary results

We present here some auxiliary findings that will be used in our main findings.

Definition 2.1 *[13, 18] Let* $\mathtt{g} \in \mathtt{L}^1\big((\mathtt{a},\mathtt{b});\mathbb{R}\big)$*, where* $\big(\mathtt{a},\mathtt{b}\big) \in \mathbb{R}^2, \mathtt{a} < \mathtt{b}$*. The RL fractional integral of order* $\mathtt{r} > 0$ *of* $\mathtt{g}$ *is defined by*

$$\mathtt{I}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \frac{1}{\Gamma(\mathtt{r})}\int_{\mathtt{a}}^{\mathtt{z}} (\mathtt{z}-\mathtt{s})^{\mathtt{r}-1}\mathtt{g}(\mathtt{s})d\mathtt{s}, \quad a.e.\ \mathtt{z} \in [\mathtt{a},\mathtt{b}].$$

If $\mathtt{r} = 0$*, then*

$$\mathtt{I}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \mathtt{g}(\mathtt{z}).$$

Definition 2.2 *[13, 18] Let* $\mathtt{r} > 0$ *and* $\mathtt{m}$ *be the smallest integer greater than or equal to* $\mathtt{r}$*. The RL fractional derivative of order* $\mathtt{r}$ *of a function* $\mathtt{g} : [\mathtt{a},\mathtt{b}] \to \mathbb{R}$*, where* $\big(\mathtt{a},\mathtt{b}\big) \in \mathbb{R}^2, \mathtt{a} < \mathtt{b}$*, is defined by*

$${}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \left(\frac{d}{d\mathtt{z}}\right)^{\mathtt{m}} \mathtt{I}^{\mathtt{m}-\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z})$$

provided that the right-hand side is defined almost everywhere on $[\mathtt{a},\mathtt{b}]$*. If* $\mathtt{r} = 0$*, then*

$${}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \mathtt{g}(\mathtt{z}).$$

The following basic properties follow from Definitions 2.1 and 2.2 (see for example, [6]).

$$\left.\begin{aligned} &\textbf{(i)}\quad \mathtt{I}^{\mathtt{m}_2}_{\mathtt{a}+}(\mathtt{z}-\mathtt{a})^{\mathtt{m}_1} = \frac{\Gamma(\mathtt{m}_1+1)}{\Gamma(\mathtt{m}_1+\mathtt{m}_2+1)}(\mathtt{z}-\mathtt{a})^{\mathtt{m}_1+\mathtt{m}_2},\ \mathtt{m}_2 \geq 0, \mathtt{m}_1+1>0,\\ &\textbf{(ii)}\quad {}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{m}_2}_{\mathtt{a}+}(\mathtt{z}-\mathtt{a})^{\mathtt{m}_1} = \frac{\Gamma(\mathtt{m}_1+1)}{\Gamma(\mathtt{m}_1-\mathtt{m}_2+1)}(\mathtt{z}-\mathtt{a})^{\mathtt{m}_1-\mathtt{m}_2},\ \mathtt{m}_2 \geq 0, \mathtt{m}_1+1>0, \end{aligned}\right\} \tag{2.1}$$

where it is assumed that $\mathtt{m}_2 - \mathtt{m}_1$ is not a positive integer. The right-hand side of **(ii)** vanishes if $\mathtt{m}_2 - \mathtt{m}_1 \in \mathbb{Z}^+$. To see this, appeal to the convention that $\dfrac{1}{\Gamma(\mathtt{m}_1-\mathtt{m}_2+1)} = 0$ if $\mathtt{m}_2 - \mathtt{m}_1 \in \mathbb{Z}^+$. Moreover, if $\mathtt{g} \in \mathtt{L}^1\big((\mathtt{a},\mathtt{b});\mathbb{R}\big)$,

$$\left.\begin{aligned} &\textbf{(i)}\quad \mathtt{I}^{\mathtt{m}_2}_{\mathtt{a}+}\mathtt{I}^{\mathtt{m}_1}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \mathtt{I}^{\mathtt{m}_2+\mathtt{m}_1}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}),\ \mathtt{m}_1,\mathtt{m}_2 \geq 0,\\ &\textbf{(ii)}\quad {}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{m}_2}_{\mathtt{a}+}\mathtt{I}^{\mathtt{m}_1}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \mathtt{I}^{\mathtt{m}_1-\mathtt{m}_2}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}),\ 0 \leq \mathtt{m}_2 \leq \mathtt{m}_1. \end{aligned}\right\} \tag{2.2}$$

Lemma 2.3 *[13, 18] Suppose that* $\mathtt{g} \in \mathtt{C}(\mathtt{a},\mathtt{b}) \cap \mathtt{L}^1(\mathtt{a},\mathtt{b})$ *with fractional derivative of order* $\mathtt{r} > 0$ *belonging to* $\mathtt{C}(\mathtt{a},\mathtt{b}) \cap \mathtt{L}^1(\mathtt{a},\mathtt{b})$*. Let* $\mathtt{n} = \lfloor \mathtt{r} \rfloor + 1$ *where* $\lfloor \mathtt{r} \rfloor$ *is the greatest integer less than or equal to* $\mathtt{r}$*. Then*

$$\mathtt{I}^{\mathtt{r}}_{\mathtt{a}+}\,{}^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{g}(\mathtt{z}) = \mathtt{g}(\mathtt{z}) + \sum_{\mathtt{k}=1}^{\mathtt{n}} \mathtt{c}_{\mathtt{k}}(\mathtt{z}-\mathtt{a})^{\mathtt{r}-\mathtt{k}}, \quad \mathtt{z} \in [\mathtt{a},\mathtt{b}],$$

where $\mathtt{c}_{\mathtt{k}} \in \mathbb{R},\ \mathtt{k} = 1,\ldots,\mathtt{n}.$

3. Green's function and bounds

Lemma 3.1 *Let* $x \in C([a,b],\mathbb{R}), r \in (n-1,n]$, *where* $n \geq 2$ *denotes an integer, and* $p \in [0, r-1]$. *Then* $w \in C([a,b],\mathbb{R})$ *is a solution of the FDEq*

$$ {}^{RL}\mathfrak{D}^{r}_{a+}w(z) + x(z) = 0, \quad z \in (a,b), \tag{3.1}$$

satisfying the boundary conditions (1.5) *if and only if*

$$w(z) = \int_a^b \mathcal{G}_r(z,s)x(s)ds,$$

where

$$\mathcal{G}_r(z,s) = \begin{cases} \dfrac{(b-s)^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)}, & a \leq z \leq s \leq b, \\ \dfrac{(b-s)^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)} - \dfrac{(z-s)^{r-1}}{\Gamma(r)}, & a \leq s \leq z \leq b, \end{cases} \tag{3.2}$$

if $p < r-1$, *or*

$$\mathcal{G}_r(z,s) = \begin{cases} \dfrac{(z-a)^{r-1}}{\Gamma(r)}, & a \leq z \leq s \leq b, \\ \dfrac{(z-a)^{r-1}}{\Gamma(r)} - \dfrac{(z-s)^{r-1}}{\Gamma(r)}, & a \leq s \leq z \leq b, \end{cases} \tag{3.3}$$

if $p = r-1$.

Proof Let $w \in C[a,b]$ be the solution to the FBVP (3.1), (1.5). Then we have by Lemma 2.3,

$$w(z) = \sum_{k=1}^{n} c_k(z-a)^{r-k} - I^{r}_{a+}x(s)ds,$$

where $c_k \in \mathbb{R}$ for $k = 1, \ldots, n$. Therefore,

$$w(z) = \sum_{k=1}^{n} c_k(z-a)^{r-k} - \frac{1}{\Gamma(r)}\int_a^z (z-s)^{r-1}x(s)ds.$$

Using $w^{(i)}(a) = 0$, $i = 0, \ldots, n-2$, we obtain $c_n = c_{n-1} = \cdots = c_2 = 0$. Therefore,

$$w(z) = c_1\big(z-a\big)^{r-1} - \frac{1}{\Gamma(r)}\int_a^z (z-s)^{r-1}x(s)ds. \tag{3.4}$$

Apply ${}^{RL}\mathfrak{D}^{p}_{a+}$ to both sides of (3.4) and employ (2.1) and (2.2) to obtain

$${}^{RL}\mathfrak{D}^{p}_{a+}w(z) = \frac{\Gamma(r)(z-a)^{r-p-1}}{\Gamma(r-p)}c_1 - \frac{1}{\Gamma(r-p)}\int_a^z (z-s)^{r-p-1}x(s)ds, \tag{3.5}$$

if $p < r-1$, or

$${}^{RL}\mathfrak{D}^{r-1}_{a+}w(z) = \Gamma(r)c_1 - \int_a^z x(s)ds.$$

Since ${}^{RL}\mathfrak{D}^{p}_{a+}w(b) = 0$, $c_1 = \int_a^b \left[\frac{(b-s)^{r-p-1}}{\Gamma(r)(b-a)^{r-p-1}}\right]x(s)ds$ if $p < r-1$, or $c_1 = \int_a^b \left[\frac{x(s)}{\Gamma(r)}\right]ds$ if $p = r-1$.
Thus, the unique solution of FBVP (3.1), (1.5) is given by

$$w(z) = \int_a^b \mathcal{G}_r(z,s)x(s)ds.$$

The converse follows by direct computation and the proof is complete. □

Remark 3.2 *The Green's function given by* (3.2) *has been constructed and studied by others. For example, in [12], the authors assume* $p = 0$ *and maximize* $\mathcal{G}_r(z,s)$ *over* $[a,b]\times[a,b]$. *In [8], the authors obtain comparison results on* $\mathcal{G}_r(z,s)$ *as a function of the parameter,* p, *and obtain sufficient conditions for a unique limiting Green's function as* $b \to +\infty$.

In each of the following three cases,

(*i*) $1 < r \le 2$, and $0 \le p \le r-1$,

(*ii*) $2 < r$, and $0 \le p < 1$,

(*iii*) $2 < r$, and $1 \le p \le r-1$,

we shall obtain

$$\max_{(z,s)\in[a,b]\times[a,b]} |\mathcal{G}_r(z,s)| = \max_{(z,s)\in[a,b]\times[a,b]} \mathcal{G}_r(z,s)$$

where the kernel $\mathcal{G}_r$ is defined by (3.2).

3.1. The case $1 < r \le 2$ *and* $0 \le p \le r-1$

The following lemma has been stated and proven in [3, lemma 3.2]. We state, without proof, the following lemma for the sake of completeness of this article.

Lemma 3.3 *Assume* $1 < r \le 2$ *and* $0 \le p \le r-1$. *The kernel* $\mathcal{G}_r(z,s)$ *given by* (3.2) *or* (3.3) *has the properties:*

(i) $\mathcal{G}_r(z,s) \ge 0, \ \forall\, z,s \in [a,b]$.

(ii) $$\max_{z\in[a,b]} \mathcal{G}_r(z,s) = \mathcal{G}_r(s,s) = \begin{cases} \dfrac{(b-s)^{r-p-1}(s-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)}, & \forall\, s\in[a,b], \quad \textit{if } p < r-1, \\ \dfrac{(s-a)^{r-1}}{\Gamma(r)} & \forall\, s\in[a,b], \quad \textit{if } p = r-1. \end{cases}$$

(iii) $$\max_{s\in[a,b]} \mathcal{G}_r(s,s) = \begin{cases} \dfrac{1}{\Gamma(r)}\left[\dfrac{r-p-1}{2(r-1)-p}\right]^{r-p-1}\left[\dfrac{r-1}{2(r-1)-p}\right]^{r-1}(b-a)^{r-1}, & \textit{if } p < r-1, \\ \dfrac{(b-a)^{r-1}}{\Gamma(r)}, & \textit{if } p = r-1. \end{cases}$$

Note that in the case, $r = 2$, $p = 0$, $\left[\frac{r-p-1}{2(r-1)-p}\right]^{r-p-1}\left[\frac{r-1}{2(r-1)-p}\right]^{r-1}$ reduces to $\frac{1}{4}$ as in the original case studied by Lyapunov [14].

3.2. The case $2 < r$ and $0 \le p < 1$

Lemma 3.4 *Let $2 < r$ and $0 \le p < 1$. The kernel $\mathcal{G}_r(z,s)$ given by (3.2) has the properties:*

(i) $\mathcal{G}_r(z,s) \ge 0, \ \forall \ z,s \in [a,b]$,

(ii) $\max_{z\in[a,b]} \mathcal{G}_r(z,s) = \mathcal{G}_r(s^*,s) = \dfrac{(b-s)^{r-p-1}(s-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)\big(1-D\big)^{r-2}}$, *where*

$$s^* = \frac{s-aD}{1-D}, \ \text{and } D = \left(\frac{b-s}{b-a}\right)^{\frac{r-p-1}{r-2}}.$$

(iii) $\max_{s\in[a,b]} \mathcal{G}_r(s^*,s) = \dfrac{(b-a)^{r-p-1}\varpi_{r,p}^{r-p-1}(1-\varpi_{r,p})^{r-1}}{\Gamma(r)\left(1-\varpi_{r,p}^{\frac{r-p-1}{r-2}}\right)^{r-2}}$, *where $\varpi_{r,p}$ denotes the unique root of the nonlinear equation*

$$\varpi^{\frac{2r-p-3}{r-2}} - \left(2-\frac{p}{r-1}\right)\varpi + \frac{r-p-1}{r-1} = 0$$

in the interval $\left(0, \left[\left(2-\dfrac{p}{r-1}\right)\left(\dfrac{r-2}{2r-p-3}\right)\right]^{\frac{r-2}{r-p-1}}\right) \subset \Big(0,1\Big)$.

Proof The kernel $\mathcal{G}_r(z,s)$ provided in (3.2) is clearly continuous on $[a,b]\times[a,b]$. Let $a < z \le s \le b$. Then

$$\mathcal{G}_r(z,s) = \frac{(b-s)^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)} \ge 0.$$

Let $a < s \le z \le b$. Then

$$\begin{aligned}
\mathcal{G}_r(z,s) &= \frac{(b-s)^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)} - \frac{(z-s)^{r-1}}{\Gamma(r)} \\
&= \frac{((b-a)-(s-a))^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)} - \frac{((z-a)-(s-a))^{r-1}}{\Gamma(r)} \\
&= \left[\frac{(z-a)^{r-1}}{\Gamma(r)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-p-1} - \left[1-\left(\frac{s-a}{z-a}\right)\right]^{r-1}\right\} \\
&\ge \left[\frac{(z-a)^{r-1}}{\Gamma(r)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-p-1} - \left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-1}\right\} \\
&= \left[\frac{(z-a)^{r-1}}{\Gamma(r)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{-p} - 1\right\}\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-1}.
\end{aligned}$$

For $a < s < b$, each term in the product is positive since $0 < \dfrac{s-a}{b-a} < 1$. Thus, $\mathcal{G}_r(z,s) \ge 0$ and (i) holds.

We now verify **(ii)**. Let $s \in (a,b)$ be fixed. As in Lemma (3.3), $\mathcal{G}_r(z,s)$ is increasing in z for $a \le z \le s$. For $s \le z \le b$, we have

$$\frac{\partial}{\partial z}\mathcal{G}_r(z,s) = \frac{(r-1)(b-s)^{r-p-1}(z-a)^{r-2}}{(b-a)^{r-p-1}\Gamma(r)} - \frac{(r-1)(z-s)^{r-2}}{\Gamma(r)}$$
$$= \left\{\left(\frac{b-s}{b-a}\right)^{r-p-1} - \left(\frac{z-s}{z-a}\right)^{r-2}\right\}\left[\frac{(z-a)^{r-2}}{\Gamma(r-1)}\right].$$

Note that $\frac{\partial}{\partial z}\mathcal{G}_r(s,s) > 0$ and $\frac{\partial}{\partial z}\mathcal{G}_r(b,s) < 0$ since $r-p-1 > r-2$. Moreover, it is clear that, for $s < z < b$, $\frac{\partial}{\partial z}\mathcal{G}_r(z,s)$ has a unique root at $z = s^* = \dfrac{s-aD}{1-D}$, where $D = \left(\dfrac{b-s}{b-a}\right)^{\frac{r-p-1}{r-2}}$.

We now calculate $\mathcal{G}_r(s^*,s)$. Write

$$\mathcal{G}_r(s^*,s) = \frac{D^{r-2}(s^*-a)^{r-1}}{\Gamma(r)} - \frac{(s^*-s)^{r-1}}{\Gamma(r)},$$

and note that

$$s^*-a = \frac{s-a}{1-D}, \quad s^*-s = \frac{D}{1-D}(s-a).$$

Thus,

$$\mathcal{G}_r(s^*,s) = \frac{1}{\Gamma(r)}\left[D^{r-2}\left(\frac{s-a}{1-D}\right)^{r-1} - \left(\frac{D}{1-D}\right)^{r-1}(s-a)^{r-1}\right]$$
$$= \frac{(b-s)^{r-p-1}(s-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)\big(1-D\big)^{r-2}}.$$

To verify **(iii)**, we introduce the notation employed in the proof of lemma 3.5 in [12]. Define $h(s) = \mathcal{G}_r(s^*,s)$ and note that h is continuous on $[a,b]$ and differentiable on (a,b). Moreover,

$$\max_{s\in[a,b]} \mathcal{G}_r(s^*,s) = \max_{s\in[a,b]} h(s).$$

As in [12], set $\varpi = \dfrac{b-s}{b-a}$ and define

$$\varphi(\varpi) = \frac{(b-a)^{r-1}\varpi^{r-p-1}(1-\varpi)^{r-1}}{\Gamma(r)\left(1-\varpi^{\frac{r-p-1}{r-2}}\right)^{r-2}}.$$

So,

$$\max_{s\in[a,b]} h(s) = \max_{\varpi\in[0,1]} \varphi(\varpi).$$

Differentiate φ with respect to ϖ to obtain

$$\varphi'(\varpi) = \frac{(b-a)^{r-1}}{\Gamma(r)}\varpi^{r-p-1}(1-\varpi)^{r-1}\left(1-\varpi^{\frac{r-p-1}{r-2}}\right)^{2-r}\mu(\varpi)$$

where

$$\mu(\varpi) = \frac{(\mathtt{r}-1)\varpi^{\frac{2\mathtt{r}-\mathtt{p}-3}{\mathtt{r}-2}} - \Big(2(\mathtt{r}-1)-\mathtt{p}\Big)\varpi + (\mathtt{r}-\mathtt{p}-1)}{\varpi(1-\varpi)\Big(1-\varpi^{\frac{\mathtt{r}-\mathtt{p}-1}{\mathtt{r}-2}}\Big)}.$$

Define $\mathtt{P}(\varpi) = \varpi^{\frac{2\mathtt{r}-\mathtt{p}-3}{\mathtt{r}-2}} - \Big(2-\frac{\mathtt{p}}{\mathtt{r}-1}\Big)\varpi + \frac{\mathtt{r}-\mathtt{p}-1}{\mathtt{r}-1}$, so that

$$\mu(\varpi) = \frac{(\mathtt{r}-1)\mathtt{P}(\varpi)}{\varpi(1-\varpi)\Big(1-\varpi^{\frac{\mathtt{r}-\mathtt{p}-1}{\mathtt{r}-2}}\Big)}.$$

Note that $\dfrac{d\mathtt{P}}{d\varpi} = \left(\dfrac{2\mathtt{r}-\mathtt{p}-3}{\mathtt{r}-2}\right)\varpi^{\frac{\mathtt{r}-\mathtt{p}-1}{\mathtt{r}-2}} - \left(2-\dfrac{\mathtt{p}}{\mathtt{r}-1}\right)$ has a unique root at

$$\begin{aligned}\varpi^* &= \left[\left(2-\frac{\mathtt{p}}{\mathtt{r}-1}\right)\left(\frac{\mathtt{r}-2}{2\mathtt{r}-\mathtt{p}-3}\right)\right]^{\frac{\mathtt{r}-2}{\mathtt{r}-\mathtt{p}-1}}\\ &= \left[\left(\frac{2\mathtt{r}-4}{2\mathtt{r}-\mathtt{p}-3}\right) - \left(\frac{\mathtt{p}}{\mathtt{r}-1}\right)\left(\frac{\mathtt{r}-2}{2\mathtt{r}-\mathtt{p}-3}\right)\right]^{\frac{\mathtt{r}-2}{\mathtt{r}-\mathtt{p}-1}}.\end{aligned}$$

The first representation of ϖ^* implies $0 < \varpi^*$ and the second representation of ϖ^* implies $\varpi^* < 1$. Thus, $(0,\varpi^*) \subset (0,1)$. Moreover, $\mathtt{P}'(\varpi) < 0$ for $\varpi \in (0,\varpi^*)$, $\mathtt{P}'(\varpi) > 0$ for $\varpi \in (\varpi^*,1)$, $\mathtt{P}(0) > 0$, and $\mathtt{P}(1) = 0$. That implies that there exists a unique $\varpi_{\mathtt{r},\mathtt{p}} \in (0,\varpi^*)$ such that $\mathtt{P}(\varpi_{\mathtt{r},\mathtt{p}}) = 0$. In particular,

$$\max_{\varpi\in[0,1]} \varphi(\varpi) = \varphi(\varpi_{\mathtt{r},\mathtt{p}})$$

and **(iii)** is verified. □

3.3. The case $2 < \mathtt{r}$ *and* $1 \le \mathtt{p} \le \mathtt{r}-1$

Lemma 3.5 *Assume* $2 < \mathtt{r}$ *and* $1 \le \mathtt{p} \le \mathtt{r}-1$. *The kernel* $\mathcal{G}_{\mathtt{r}}(\mathtt{z},\mathtt{s})$ *given by* (3.2) *has the properties:*

(i) $\mathcal{G}_{\mathtt{r}}(\mathtt{z},\mathtt{s}) \ge 0, \ \forall\ \mathtt{z},\mathtt{s} \in [\mathtt{a},\mathtt{b}]$,

(ii) $\mathcal{G}_{\mathtt{r}}(\mathtt{z},\mathtt{s}) \le \mathcal{G}_{\mathtt{r}}(\mathtt{b},\mathtt{s}), \quad$ *for* $\mathtt{z},\mathtt{s} \in [\mathtt{a},\mathtt{b}]$,

(iii) $\displaystyle\max_{\mathtt{s}\in[\mathtt{a},\mathtt{b}]} \mathcal{G}_{\mathtt{r}}(\mathtt{b},\mathtt{s}) = \frac{(\mathtt{b}-\mathtt{a})^{\mathtt{r}-1}}{\Gamma(\mathtt{r})}$.

It is of interest to note that $\max\limits_{\mathtt{s}\in[\mathtt{a},\mathtt{b}]} \mathcal{G}_{\mathtt{r}}(\mathtt{b},\mathtt{s})$ is independent of $\mathtt{p}$.

Proof The kernel $\mathcal{G}_{\mathtt{r}}(\mathtt{z},\mathtt{s})$ provided in (3.2) is clearly continuous on $[\mathtt{a},\mathtt{b}]\times[\mathtt{a},\mathtt{b}]$. Then for $\mathtt{a} < \mathtt{z} \le \mathtt{s} \le \mathtt{b}$,

$$\mathcal{G}_{\mathtt{r}}(\mathtt{z},\mathtt{s}) = \frac{(\mathtt{b}-\mathtt{s})^{\mathtt{r}-\mathtt{p}-1}(\mathtt{z}-\mathtt{a})^{\mathtt{r}-1}}{(\mathtt{b}-\mathtt{a})^{\mathtt{r}-\mathtt{p}-1}\Gamma(\mathtt{r})} \ge 0.$$

and for $a < s \le z \le b$, as in the proof of Lemma 3.4, Case (**i**),

$$\begin{aligned}\mathcal{G}_r(z,s) &= \frac{(b-s)^{r-p-1}(z-a)^{r-1}}{(b-a)^{r-p-1}\Gamma(r)} - \frac{(z-s)^{r-1}}{\Gamma(r)}\\ &\ge \left[\frac{(z-a)^{r-1}}{\Gamma(r)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{-p} - 1\right\}\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-1}.\end{aligned}$$

For $a < s < b$, each term in the product is positive since $0 < \frac{s-a}{b-a} < 1$. Thus, $\mathcal{G}_r(z,s) \ge 0$ and (**i**) holds.

We establish the inequality (**ii**). Let $a \le z \le s \le b$. Then

$$\frac{\partial}{\partial z}\mathcal{G}_r(z,s) = \frac{(r-1)(z-a)^{r-2}(b-s)^{r-1-p}}{(b-a)^{r-p-1}\Gamma(r)} \ge 0.$$

So $\mathcal{G}_r(z,s)$ is increasing with respect to z yielding $\mathcal{G}_r(z,s) \le \mathcal{G}_r(s,s)$, $\forall\, z \in [a,s]$. Let $a \le s \le z \le b$. Then, similar to the calculation performed in the previous paragraph,

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{G}_r(z,s) =& \frac{(r-1)(b-s)^{r-p-1}(z-a)^{r-2}}{(b-a)^{r-p-1}\Gamma(r)} - \frac{(r-1)(z-s)^{r-2}}{\Gamma(r)}\\ &= \left[\frac{(z-a)^{r-2}}{\Gamma(r-1)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-p-1} - \left[1-\left(\frac{s-a}{z-a}\right)\right]^{r-2}\right\}\\ &\ge \left[\frac{(z-a)^{r-2}}{\Gamma(r-1)}\right]\left\{\left[1-\left(\frac{s-a}{b-a}\right)\right]^{-p+1} - 1\right\}\left[1-\left(\frac{s-a}{b-a}\right)\right]^{r-2}.\end{aligned}$$

Since $1 \le p$, it is again the case that each term in the product is nonnegative. Thus, for $a \le z \le b$,

$$\frac{\partial}{\partial z}\mathcal{G}_r(z,s) \ge 0,$$

implying $\mathcal{G}_r(z,s) \le \mathcal{G}_r(b,s)$, $\forall\, z \in [a,b]$. Hence, the inequality (**ii**) is proven.

Now, we prove the inequality (**iii**). Let us define:

$$\mathcal{G}_r(b,s) = \frac{(b-s)^{r-p-1}}{\Gamma(r)(b-a)^{-p}}, \quad a \le s \le b.$$

Note that

$$\frac{d}{ds}\Big\{\mathcal{G}_r(b,s)\Big\} = \frac{(r-p-1)(b-s)^{r-p-2}}{\Gamma(r)(b-a)^{-p}}(-1) \le 0.$$

As a result, $\mathcal{G}_r(b,s)$ is decreasing with respect to s. Thus, $\mathcal{G}_r(b,s)$ has a maximum at $s = a \in [a,b]$ given by $\max\limits_{s\in[a,b]} \mathcal{G}_r(b,s) = \dfrac{(b-a)^{r-1}}{\Gamma(r)}$. Hence, (**iii**) holds. □

4. Lyapunov-type inequalities

For (n,p)-type RL FBVP (1.4)–(1.5), we can now obtain Lyapunov-type inequalities. As we are focused on continuous solutions, we consider the Banach space $E = \Big\{w : \ w \in C[a,b]\Big\}$ with the norm

$$\|w\|_\infty = \max_{z\in[a,b]} \big|w(z)\big|.$$

We continue to consider the three cases,

(i) $2 < \mathtt{r}$, and $0 \leq \mathtt{p} \leq \mathtt{r}-1$, and $1 < \mathtt{r} \leq 2$,

(ii) $2 < \mathtt{r}$, and $0 \leq \mathtt{p} < 1$,

(iii) $2 < \mathtt{r}$,and $1 \leq \mathtt{p} \leq \mathtt{r}-1$.

4.1. The case $1 < \mathtt{r} \leq 2$ *and* $0 \leq \mathtt{p} \leq \mathtt{r}-1$

The following theorem has been stated and proven in [3, theorem 3.1]. We state, without proof, the following theorem for the sake of completeness of this article. To state Theorem 4.1, we assume $\mathtt{r} < 2$.

Theorem 4.1 *Assume* $1 < \mathtt{r} < 2$ *and* $0 \leq \mathtt{p} \leq \mathtt{r}-1$*. Assume that* $(\mathbf{A}_1)$*,* $(\mathbf{A}_2)$ *hold. Then the estimate*

$$\int_{\mathtt{a}}^{\mathtt{b}} \left|\ell(\mathtt{s})\right| d\mathtt{s} < \left[\frac{\Gamma(\mathtt{r})}{\eta}\right] \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-\mathtt{p}-1}\right]^{\mathtt{r}-\mathtt{p}-1} \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-1}\right]^{\mathtt{r}-1} \left(\frac{1}{\mathtt{b}-\mathtt{a}}\right)^{\mathtt{r}-1}$$

implies that the FBVP (1.4)–(1.5) has only the trivial continuous solution $\mathtt{w}(\mathtt{z}) \equiv 0$*.*

Corollary 4.2 *Let* $1 < \mathtt{r} < 2$ *and* $0 \leq \mathtt{p} \leq \mathtt{r}-1$*. Then any eigenvalue* μ *of the FDEq*

$$^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \mu \mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{a} < \mathtt{z} < \mathtt{b}, \tag{4.1}$$

coupled with BCs (1.5), satisfies

$$|\mu| \geq \Gamma(\mathtt{r}) \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-\mathtt{p}-1}\right]^{\mathtt{r}-\mathtt{p}-1} \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-1}\right]^{\mathtt{r}-1} \frac{1}{(\mathtt{b}-\mathtt{a})^{\mathtt{r}}}.$$

Proof If μ is an eigenvalue of FBVP (4.1), (1.5), then FBVP (4.1), (1.5) admits at least one nontrivial solution $\mathtt{w}_{\mu}$. Set $\mu = \ell(\mathtt{z})$ and $\eta = 1$. As a result of Theorem 4.1, we have

$$\int_{\mathtt{a}}^{\mathtt{b}} |\mu| d\mathtt{s} \geq \left[\frac{\Gamma(\mathtt{r})}{\eta}\right] \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-\mathtt{p}-1}\right]^{\mathtt{r}-\mathtt{p}-1} \left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-1}\right]^{\mathtt{r}-1} \left(\frac{1}{\mathtt{b}-\mathtt{a}}\right)^{\mathtt{r}-1}.$$

□

We state the following corollary without proof since details for the case $0 \leq \mathtt{p} \leq \mathtt{r}-1$ and $2 = \mathtt{r}$ are completely analogous.

Corollary 4.3 *Let* $\mathtt{r} = 2$ *and* $\mathtt{p} = 0$*. Assume that* $(\mathbf{A}_1)$*,* $(\mathbf{A}_2)$ *hold. Then the estimate*

$$\int_{\mathtt{a}}^{\mathtt{b}} \left|\ell(\mathtt{s})\right| d\mathtt{s} < \frac{1}{\eta}\left(\frac{4}{\mathtt{b}-\mathtt{a}}\right)$$

implies that the FBVP (1.4)–(1.5) has only the trivial continuous solution $\mathtt{w}(\mathtt{z}) \equiv 0$*.*

Corollary 4.4 *Let* $\mathtt{r} = 2$ *and* $\mathtt{p} = 0$*. Assume that condition* $(\mathbf{A}_2)$ *holds. Then any eigenvalue* μ *of the FDEq*

$$^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \mu \mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{a} < \mathtt{z} < \mathtt{b}, \tag{4.2}$$

coupled with BCs (1.5), satisfies

$$|\mu| \geq \left[\frac{4}{(\mathtt{b}-\mathtt{a})^2}\right].$$

4.2. The case $2 < \mathtt{r}$ *and* $0 \leq \mathtt{p} < 1$

Theorem 4.5 *Assume* $2 < \mathtt{r}$ *and* $0 \leq \mathtt{p} < 1$. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *hold. Then the estimate*

$$\int_{\mathtt{a}}^{\mathtt{b}} |\ell(\mathtt{s})| d\mathtt{s} < \frac{\Gamma(\mathtt{r})\left(1 - \varpi_{\mathtt{r,p}}^{\frac{\mathtt{r-p-1}}{\mathtt{r-2}}}\right)^{\mathtt{r}-2}}{\eta(\mathtt{b}-\mathtt{a})^{\mathtt{r-p-1}} \varpi_{\mathtt{r,p}}^{\mathtt{r-p-1}} (1-\varpi_{\mathtt{r,p}})^{\mathtt{r}-1}}$$

implies that the FBVP (1.4)–(1.5) *has only the trivial continuous solution* $\mathtt{w}(\mathtt{z}) \equiv 0$, *where* $\varpi_{\mathtt{r,p}}$ *denotes the unique solution of the nonlinear equation*

$$\varpi^{\frac{2\mathtt{r-p-3}}{\mathtt{r-2}}} + \left(\frac{\mathtt{p}}{\mathtt{r}-1} - 2\right)\varpi + \frac{\mathtt{r-p-1}}{\mathtt{r}-1} = 0$$

in the interval $\left(0, \left[\left(2 - \frac{\mathtt{p}}{\mathtt{r}-1}\right)\left(\frac{\mathtt{r}-2}{2\mathtt{r-p-3}}\right)\right]^{\frac{\mathtt{r}-2}{\mathtt{r}} - \mathtt{p} - 1}\right)$.

As the method of proof is precisely as in the proof of Theorem 4.1, we omit the proof.

Corollary 4.6 *Assume* $2 < \mathtt{r}$ *and* $0 \leq \mathtt{p} < 1$. *Then any eigenvalue* μ *of the FDEq*

$$^{\mathrm{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \mu\mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{a} < \mathtt{z} < \mathtt{b}, \tag{4.3}$$

coupled with BCs (1.5), *satisfies*

$$|\mu| \geq \frac{\Gamma(\mathtt{r})\left(1 - \varpi_{\mathtt{r,p}}^{\frac{\mathtt{r-p-1}}{\mathtt{r-2}}}\right)^{\mathtt{r}-2}}{(\mathtt{b}-\mathtt{a})^{\mathtt{r-p}} \varpi_{\mathtt{r,p}}^{\mathtt{r-p-1}} (1-\varpi_{\mathtt{r,p}})^{\mathtt{r}-1}},$$

where $\varpi_{\mathtt{r,p}}$ *is the unique root of the nonlinear equation*

$$\varpi^{\frac{2\mathtt{r-p-3}}{\mathtt{r-2}}} + \left(-2 + \frac{\mathtt{p}}{\mathtt{r}-1}\right)\varpi + \frac{\mathtt{r-p-1}}{\mathtt{r}-1} = 0$$

in the interval $\left(0, \left[\left(2 - \frac{\mathtt{p}}{\mathtt{r}-1}\right)\left(\frac{\mathtt{r}-2}{2\mathtt{r-p-3}}\right)\right]^{\frac{\mathtt{r}-2}{\mathtt{r}} - \mathtt{p} - 1}\right)$.

4.3. The case $2 < \mathtt{r}$ *and* $1 \leq \mathtt{p} \leq \mathtt{r} - 1$

Theorem 4.7 *Assume* $2 < \mathtt{r}$ *and* $1 \leq \mathtt{p} \leq \mathtt{r} - 1$. *Assume that* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *hold. Then the estimate*

$$\int_{\mathtt{a}}^{\mathtt{b}} |\ell(\mathtt{s})| d\mathtt{s} < \frac{\Gamma(\mathtt{r})}{(\mathtt{b}-\mathtt{a})^{\mathtt{r}-1}\eta}$$

implies that the FBVP (1.4)–(1.5) *has only the trivial continuous solution* $\mathtt{w}(\mathtt{z}) \equiv 0$.

Again, the proof is omitted.

Corollary 4.8 *Assume* $2 < \mathtt{r}$ *and* $1 \le \mathtt{p} \le \mathtt{r} - 1$. *Then any eigenvalue* μ *of the FDEq*

$$^{\mathtt{RL}}\mathfrak{D}^{\mathtt{r}}_{\mathtt{a}+}\mathtt{w}(\mathtt{z}) + \mu\mathtt{w}(\mathtt{z}) = 0, \quad \mathtt{a} < \mathtt{z} < \mathtt{b}, \tag{4.4}$$

coupled with BCs (1.5), *satisfies* $|\mu| \ge \dfrac{\Gamma(\mathtt{r})}{(\mathtt{b}-\mathtt{a})^{\mathtt{r}}}$.

5. Examples

Example 5.1 *Let* $\mathtt{b} = 2, \mathtt{a} = 1, \mathtt{r} = \dfrac{48}{25}$, $\mathtt{p} = \dfrac{11}{21}$. *Consider the eigenvalue problem:*

$$^{\mathtt{RL}}\mathfrak{D}^{\frac{48}{25}}_{1+}\mathtt{w}(\mathtt{z}) + \mu\mathtt{w}(\mathtt{z}) = 0, \quad 1 < \mathtt{z} < 2, \tag{5.1}$$

$$\mathtt{w}(1) = 0, \ ^{\mathtt{RL}}\mathfrak{D}^{\frac{11}{21}}_{1+}\mathtt{w}(2) = 0. \tag{5.2}$$

Applying Corollary 4.2, if μ *is an eigenvalue of the FBVP* (5.1)-(5.2), *then*

$$\begin{aligned}
|\mu| &\ge \Gamma(\mathtt{r})\left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-\mathtt{p}-1}\right]^{\mathtt{r}-\mathtt{p}-1}\left[\frac{2(\mathtt{r}-1)-\mathtt{p}}{\mathtt{r}-1}\right]^{\mathtt{r}-1}\left(\frac{1}{2-1}\right)^{\mathtt{r}-1} \\
&= \Gamma\left(\frac{48}{25}\right)\left[\frac{2(\frac{48}{25}-1)-\frac{11}{21}}{\frac{48}{25}-\frac{11}{21}-1}\right]^{\frac{48}{25}-\frac{11}{21}-1}\left[\frac{2(\frac{48}{25}-1)-\frac{11}{21}}{\frac{48}{25}-1}\right]^{\frac{48}{25}-1} 1^{\frac{48}{25}-1} \\
&\approx 2.16715731.
\end{aligned}$$

Example 5.2 *Let* $\mathtt{b} = 1, \mathtt{a} = 0, \mathtt{r} = \dfrac{12}{5}$, $\mathtt{p} = \dfrac{6}{5}$. *Consider a nonlinear FBVP:*

$$^{\mathtt{RL}}\mathfrak{D}^{\frac{12}{5}}_{0+}\mathtt{w}(\mathtt{z}) + \mu\mathtt{w}(\mathtt{z}) = 0, \quad 0 < \mathtt{z} < 1, \tag{5.3}$$

$$\mathtt{w}(0) = 0, \ \mathtt{w}'(0) = 0, \ \ ^{\mathtt{RL}}\mathfrak{D}^{\frac{6}{5}}_{0+}\mathtt{w}(1) = 0. \tag{5.4}$$

Based on the information provided, then Corollary 4.8 implies that if $\mathtt{w}$ *is a nontrivial solution of the FBVP* (5.3)-(5.4), *then*

$$|\mu| \ge \left[\frac{\Gamma(2.4)}{\left(1-0\right)^{2.4}}\right] \approx 1.24216934.$$

Example 5.3 *In our final example, we consider a fractional differential equation with delay. Let* $\mathtt{b} = 1, \mathtt{a} = 0, \mathtt{r} = \dfrac{12}{5}$, $\mathtt{p} = 1$, *and* $\mathtt{Fw}(\mathtt{z}) = \mathtt{w}\left(\dfrac{\mathtt{z}}{2}\right)$. *Consider a nonlinear FBVP:*

$$^{RL}\mathfrak{D}^{\frac{12}{5}}_{0+}\mathtt{w}(\mathtt{z}) + \ell(\mathtt{z})\mathtt{Fw}(\mathtt{z}) = 0, \quad 0 < \mathtt{z} < 1, \tag{5.5}$$

$$\mathtt{w}(0) = 0, \ \mathtt{w}'(0) = 0, \ \mathtt{w}'(1) = 0. \tag{5.6}$$

Note that $\eta = 1$. Apply Theorem 4.7; if

$$\int_0^1 \left|\ell(\mathtt{s})\right| d\mathtt{s} < \left[\frac{\Gamma(2.4)}{\left(1-0\right)^{1.4}}\right],$$

then the FBVP (5.5)-(5.6) has only the trivial solution.

Note that the estimates in Examples 5.2 and 5.3 agree due to the independence of $\max_{\mathtt{s}\in[\mathtt{a},\mathtt{b}]} \mathcal{G}_{\mathtt{r}}(\mathtt{b},\mathtt{s})$ on $\mathtt{p}$ in in the case $1 \le \mathtt{p} \le \mathtt{r}-1,\ 2 < \mathtt{r}$.

6. Conclusion

The authors consider a family of two-point $(\mathtt{n},\mathtt{p})$-type boundary value problems for Riemann–Liouville fractional differential equations on an interval $(\mathtt{a},\mathtt{b})$. The boundary conditions allow for a fractional boundary condition at the left. The corresponding family of Green's functions are constructed, shown to be nonnegative, and maximized on $[\mathtt{a},\mathtt{b}]\times[\mathtt{a},\mathtt{b}]$. Then under suitable hypotheses, a Lyapunov-type inequality is obtained for each boundary value problem in the family.

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