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Locally product-like statistical submersions

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Abstract: In this paper, the main identities on locally product-like statistical submersions are obtained with the aid of statistical structures and their Riemannian curvature tensors. Some examples of locally product-like statistical submersions are presented. Some results on F-invariant, F^*-invariant and antiinvariant locally product-like statistical submersions are given.

Key words: Product manifold, statistical submersion, Riemannian curvature tensor

1. Introduction

Immersions and submersions are the most commonly used smooth maps to reveal the relationship between two Riemannian manifolds. Although not as much as isometric immersions, the concept of submersions is widely studied today and it is an interesting field of study in differential geometry. Riemannian submersions between Riemannian manifolds were initially introduced by O'Neill [20] and Gray [16], independently. The notion of antiinvariant Riemannian submersion in Riemannian submersion theory was firstly introduced by Şahin in [24]. Besides the mathematical applications, there exist some applications of these mappings in the Kaluza-Klein theory [12, 18, 33], the statical machine learning process [36], medical imaging [22], the statical analysis [10] and the robotic theory [4, 25, 26].

In addition to these facts, statistical manifolds were introduced by Amari [2] in 1985. Later, these structures drew the attention of several authors. Some basic properties of hypersurfaces of statistical manifolds were presented by Furuhata in [14, 15]. Various characterizations on submanifolds of statistical manifolds admitting almost contact, complex and product structures were obtained in [5–8, 19, 27, 28], etc. Furthermore, Riemannian submersions between statistical manifolds were studied in [17, 29, 30, 32], etc.

In [30, 31], the author developed a new perspective on statistical complex manifolds, which can be considered as a generalization of Hermitian manifolds as follows.

Let $(\widetilde{M}, \widetilde{g})$ be a semi-Riemannian manifold endowed with two almost complex structures J and J^* satisfying

$$\widetilde{g}(JX, Y) = \widetilde{g}(X, J^*Y) \tag{1.1}$$

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for any tangent vector fields on $\widetilde{M}$. Then the triple $(\widetilde{M}, \widetilde{g}, J)$ is called a Hermite-like manifold. It is clear that if $J = J^*$, then $(\widetilde{M}, \widetilde{g}, J)$ becomes a Hermitian manifold. Inspiring the definition of Hermite-like manifolds, almost product-like Riemannian manifolds were introduced by the authors in [11] (see equations (2.6) and (2.7)).

Motivated by the above facts, locally product-like statistical submersions are investigated in this paper. Various characterizations on F-invariant, F^*-invariant and antiinvariant locally product-like statistical submersions are obtained.

2. Preliminaries

An n-dimensional semi-Riemannian manifold is a smooth manifold M^n furnished with a metric tensor g, where g is a symmetric, nondegenerate tensor field on M of constant index. The common value ν of index g on M is called the index of M $(0 \leq \nu \leq n)$ and we denote a semi-Riemannian manifold by M^n_ν. If $\nu = 0$, then M is a Riemannian manifold. For each $p \in M$, a tangent vector E to M is spacelike (resp. null, timelike) if $g(E, E) > 0$ or $E = 0$ (resp. $g(E, E) = 0$ and $E \neq 0$, $g(E, E) < 0$). Let $\mathbb{R}^n_\nu$ be an n-dimensional real vector space with an inner product of signature $(\nu, n - \nu)$ given by

$$\langle x, x\rangle = -\sum_{i=1}^{\nu} x_i^2 + \sum_{i=\nu+1}^{n} x_i^2, \tag{2.1}$$

where $x = (x_1, \ldots, x_n)$ is the natural coordinate of $\mathbb{R}^n_\nu$. $\mathbb{R}^n_\nu$ is called an n-dimensional semi-Euclidean space. If $\nu = 0$ (resp. $\nu = 1$), then $\mathbb{R}^n$ (resp. $\mathbb{R}^n_1$) is an Euclidean space (resp. a Lorentzian space).

Let (M, g) be a semi-Riemannian manifold. Denote a torsion-free affine connection by ∇. The triple (M, ∇, g) is called a statistical manifold if ∇g is symmetric. For the statistical manifold (M, ∇, g), we define another affine connection ∇^* by

$$Eg(G, H) = g(\nabla_E G, H) + g(G, \nabla^*_E H) \tag{2.2}$$

for vector fields E, G and H on M. The affine connection ∇^* is called conjugate (or dual) to ∇ with respect to g. The affine connection ∇^* is torsion-free, $\nabla^* g$ is symmetric and satisfies $(\nabla^*)^* = \nabla$. Clearly, the triple (M, ∇^*, g) is a statistical manifold. We denote the curvature tensors on M with respect to the affine connection ∇ and ∇^* by R and R^*, respectively. Then we find

$$g\Big(R(E, G)H, E'\Big) = -g\Big(H, R^*(E, G)E'\Big) \tag{2.3}$$

for vector fields E, G, H, E' on M, where $R(E, G)H = [\nabla_E, \nabla_G]H - \nabla_{[E,G]}H$. Therefore R vanishes identically if and only if so is R^*. We call that (M, ∇, g) is flat if R vanishes identically.

Let M be a smooth manifold with a tensor field F of type $(1, 1)$ on M such that

$$F^2 = I, \tag{2.4}$$

where I stands for the identity transformation. Then we say that M is an almost product manifold with almost product structure F. We consider the semi-Riemannian manifold on the almost product manifold M. If F preserves the metric g, that is,

$$g(FE, FG) = g(E, G) \tag{2.5}$$

for any vector fields E and G on M, then (M, g, F) is called an almost product semi-Riemannian manifold. Now, we consider the semi-Riemannian manifold (M, g) with the almost product structure F which has another tensor field F^* of type $(1,1)$ satisfying

$$g(FE, G) = g(E, F^*G) \tag{2.6}$$

for any vector fields E and G. Then the triple (M, g, F) is called an almost product-like semi-Riemannian manifold. We see that $(F^*)^* = F$, $(F^*)^2 = I$ and

$$g(FE, F^*G) = g(E, G). \tag{2.7}$$

Lemma 2.1 *([11]) The triple (M, g, F) is an almost product-like semi-Riemannian manifold if and only if so is (M, g, F^*).*

Let (N, g^N) be a submanifold of (M, g, F). From (2.7), it can be obtained

(1) $F(T_pN) \subset T_pN$ if and only if $F^*(T_pN)^\perp \subset (T_pN)^\perp$,

(2) $F(T_pN) \subset (T_pN)^\perp$ if and only if $F^*(T_pN) \subset (T_pN)^\perp$,

(3) $F(T_pN)^\perp \subset T_pN$ if and only if $F^*(T_pN)^\perp \subset T_pN$,

(4) $F(T_pN)^\perp \subset (T_pN)^\perp$ if and only if $F^*(T_pN) \subset T_pN$.

If $F(T_pN) \subset T_pN$ (resp. $F^*(T_pN) \subset T_pN$) for each $p \in N$, then N is said to be F-invariant (resp. F^*-invariant) in M. For more details on submanifolds of almost product-like Riemannian manifolds, we refer to [11].

Example 2.2 *Let*

$$M_+ = \{ (x_1, x_2, x_3, x_4) \,|\, x_1 > 0 \}, \qquad M_- = \{ (x_1, x_2, x_3, x_4) \,|\, x_1 < 0 \}$$

be smooth manifolds which admit the following almost product structure F:

$$F = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus the pair (M_+, F) and (M_-, F) are almost product manifolds. If we put

$$g = \begin{pmatrix} -1 + e^{x_1} & 0 & 0 & 0 \\ 0 & e^{-x_1} & 0 & 0 \\ 0 & 0 & e^{x_1} & 0 \\ 0 & 0 & 0 & e^{-x_1} \end{pmatrix}, \qquad F^* = \begin{pmatrix} 0 & 0 & (1 - e^{-x_1})^{-1} & 0 \\ 0 & 0 & 0 & 1 \\ 1 - e^{-x_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then (M_+, g, F) (resp. (M_-, g, F)) is an almost product-like Riemannian (resp. semi-Riemannian) manifold and so is (M_+, g, F^*) (resp. (M_-, g, F^*)).

Next, if F is parallel with respect to the affine connection ∇, then (M, ∇, g, F) is called a locally product-like statistical manifold. From (2.6), we get

$$g((\nabla_D F)E, G) = g(E, (\nabla^*_D F^*)G). \tag{2.8}$$

Hence we have ([11]).

Lemma 2.3 *(M, ∇, g, F) is a locally product-like statistical manifold if and only if so is (M, ∇^*, g, F^*).*

We put for any vector fields E, G, H on the almost product-like statistical manifold

$$R(E,G)H = c\,[\,g(G,H)E - g(E,H)G + g(G,FH)FE - g(E,FH)FG + \{g(FE,G) - g(E,FG)\}FH\,], \tag{2.9}$$

where c is a constant. Then the tensor R satisfies the 1st and 2nd Bianchi identities and $FR(E,G) = R(E,G)F$. Moreover, we have

$$\begin{aligned} R^*(E,G)H = c[\,&g(G,H)E - g(E,H)G + g(G,F^*H)F^*E - g(E,F^*H)F^*G \\ &+ \{\,g(F^*E,G) - g(E,F^*G)\}F^*H\,]. \end{aligned} \tag{2.10}$$

Example 2.4 *Let (M_+, g, F) (resp. (M_-, g, F)) be an almost product-like Riemannian (resp. semi-Riemannian) manifold of Example 2.2. We put the affine connection ∇ as follows:*

$$\nabla_{\partial_1}\partial_1 = \nabla_{\partial_3}\partial_3 = e^{x_1}\partial_1,$$

$$\nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \nabla_{\partial_3}\partial_4 = \nabla_{\partial_4}\partial_3 = -e^{x_1}\partial_2,$$

$$\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = e^{x_1}\partial_3,$$

$$\nabla_{\partial_1}\partial_4 = \nabla_{\partial_4}\partial_1 = \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = -e^{x_1}\partial_4,$$

$$\nabla_{\partial_2}\partial_2 = \nabla_{\partial_4}\partial_4 = -e^{-x_1}\partial_1 + e^{x_1}\partial_4,$$

$$\nabla_{\partial_2}\partial_4 = \nabla_{\partial_4}\partial_2 = e^{x_1}\partial_2 - e^{-x_1}\partial_3,$$

where $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1,2,3,4)$. Therefore ∂_1 is timelike and $\partial_2, \partial_3, \partial_4$ are spacelike on M_-. Then we find

$$\nabla^*_{\partial_1}\partial_1 = -e^{x_1}(e^{x_1} - 2)(e^{x_1} - 1)^{-1}\partial_1,$$

$$\nabla^*_{\partial_1}\partial_2 = \nabla^*_{\partial_2}\partial_1 = (e^{x_1} - 1)\,\partial_2,$$

$$\nabla^*_{\partial_1}\partial_3 = \nabla^*_{\partial_3}\partial_1 = -(e^{x_1} - 1)\,\partial_3,$$

$$\nabla^*_{\partial_1}\partial_4 = \nabla^*_{\partial_4}\partial_1 = (e^{x_1} - 1)\,\partial_4,$$

$$\nabla^*_{\partial_2}\partial_2 = \nabla^*_{\partial_4}\partial_4 = (e^{x_1} - 1)^{-1}\partial_1 - e^{x_1}\partial_4,$$

$$\nabla^*_{\partial_2}\partial_3 = \nabla^*_{\partial_3}\partial_2 = e^{x_1}\partial_4,$$

$$\nabla^*_{\partial_2}\partial_4 = \nabla^*_{\partial_4}\partial_2 = -e^{x_1}\partial_2 + e^{-x_1}\partial_3,$$

$$\nabla^*_{\partial_3}\partial_3 = -e^{2x_1}(e^{x_1} - 1)^{-1}\partial_1,$$

$$\nabla^*_{\partial_3}\partial_4 = \nabla^*_{\partial_4}\partial_3 = e^{x_1}\partial_2.$$

Therefore (M_+, ∇, g, F) *(resp.* (M_-, ∇, g, F)*) is a locally product-like statistical manifold and so is* (M_+, ∇^*, g, F^*) *(resp.* (M_-, ∇^*, g, F^*)*).*

Let M and B be semi-Riemannian manifolds. A surjective mapping $\pi : M \to B$ is called a semi-Riemannian submersion if π has maximal rank and π_* preserves lengths of horizontal vectors. Let $\pi : M \to B$ be a semi-Riemannian submersion. We put $\dim M = m$ and $\dim B = n$. For each point $x \in B$, semi-Riemannian submanifold $\pi^{-1}(x)$ with the induced metric $\overline{g}$ is called a fiber and denoted by $\overline{M}_x$ or $\overline{M}$ simply. We notice that the dimension of each fiber is always $m-n$. A vector field on M is vertical if it is always tangent to fibers, horizontal if always orthogonal to fibers. We denote the vertical and horizontal subspace in the tangent space T_pM of the total space M by $\mathcal{V}_p(M)$ and $\mathcal{H}_p(M)$ for each point $p \in M$, and the vertical and horizontal distributions in the tangent bundle TM of M by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then TM is the direct sum of $\mathcal{V}(M)$ and $\mathcal{H}(M)$. The projection mappings are denoted $\mathcal{V} : TM \to \mathcal{V}(M)$ and $\mathcal{H} : TM \to \mathcal{H}(M)$, respectively. We call a vector field X on M projectable if there exists a vector field X_* on B such that $\pi_*(X_p) = X_{*\pi(p)}$ for each $p \in M$, and say that X and X_* are π-related. Also, a vector field X on M is called basic if it is projectable and horizontal. Then we have ([9, 16, 20]).

Lemma 2.5 *If* X *and* Y *are basic vector fields on* M *which are* π*-related to* X_* *and* Y_* *on* B*, then*

- *(1)* $g(X, Y) = g_B(X_*, Y_*) \circ \pi$*, where* g *is the metric on* M *and* g_B *the metric on* B*,*
- *(2)* $\mathcal{H}[X, Y]$ *is basic and* π*-related to* $[X_*, Y_*]$*,*
- *(3)* $\mathcal{H}\nabla'_X Y$ *is basic and* π*-related to* $\widehat{\nabla}'_{X_*} Y_*$*, where* ∇' *and* $\widehat{\nabla}'$ *are the Levi-Civita connections of* M *and* B*, respectively.*

Let (M, ∇, g) be a statistical manifold and $\pi : M \to B$ be a semi-Riemannian submersion. We denote the affine connections of $\overline{M}$ by $\overline{\nabla}$ and $\overline{\nabla}^*$. Notice that $\overline{\nabla}_U V$ and $\overline{\nabla}^*_U V$ are well-defined vertical vector fields on M for vertical vector fields U and V on M, more precisely $\overline{\nabla}_U V = \mathcal{V}\nabla_U V$ and $\overline{\nabla}^*_U V = \mathcal{V}\nabla^*_U V$. Moreover, $\overline{\nabla}$ and $\overline{\nabla}^*$ are torsion-free and conjugate to each other with respect to $\overline{g}$. The triple $(\overline{M}, \overline{\nabla}, \overline{g})$ is a statistical manifold and so is $(\overline{M}, \overline{\nabla}^*, \overline{g})$.

We call that $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a statistical submersion if $\pi : M \to B$ satisfies $\pi_*(\nabla_X Y)_p = (\widehat{\nabla}_{X_*} Y_*)_{\pi(p)}$ for basic vector fields X, Y and $p \in M$. The tensor fields T and A of type (1,2) are defined by

$$T_E G = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}G + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}G, \qquad A_E G = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}G + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}G$$

for any vector fields E and G on M. Changing ∇ to ∇^* in the above equations, we set T^* and A^*, respectively. Then we find $T^{**} = T$ and $A^{**} = A$. For vertical vector fields, T and T^* have the symmetry property. For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$, we obtain

$$g(T_U V, X) = -g(V, T^*_U X), \qquad g(A_X Y, U) = -g(Y, A^*_X U). \tag{2.11}$$

Thus, T (resp. A) vanishes identically if and only if T^* (resp. A^*) vanishes identically. Since A is related to the integrability of $\mathcal{H}(M)$, it is identically zero if and only if $\mathcal{H}(M)$ is integrable with respect to ∇. Moreover, if A and T vanish identically, then the total space is a locally product space of the base space and the fiber. If T vanishes identically, then we said to be with isometric fiber. It is known that ([1]).

Theorem 2.6 *Let $\pi : M \to B$ be a semi-Riemannian submersion. Then (M, ∇, g) is a statistical manifold if and only if the following conditions hold:*

(1) $\mathcal{H}S_V X = A_X V - A^*_X V$,

(2) $\mathcal{V}S_X V = T_V X - T^*_V X$,

(3) $(\overline{M}, \overline{\nabla}, \overline{g})$ *is a statistical manifold for each* $x \in B$,

(4) $(B, \widehat{\nabla}, g_B)$ *is a statistical manifold,*

where $S_E G = \nabla_E G - \nabla^*_E G$ *for* $E, G \in TM$.

For the statistical submersion $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$, we have the following lemmas ([30]):

Lemma 2.7 *If X and Y are horizontal vector fields, then $A_X Y = -A^*_Y X$.*

Lemma 2.8 *For $X, Y \in \mathcal{H}(M)$ and $U, V \in \mathcal{V}(M)$ we have*

$$\begin{aligned} \nabla_U V &= T_U V + \overline{\nabla}_U V, & \nabla^*_U V &= T^*_U V + \overline{\nabla}^*_U V, \\ \nabla_U X &= \mathcal{H}\nabla_U X + T_U X, & \nabla^*_U X &= \mathcal{H}\nabla^*_U X + T^*_U X, \\ \nabla_X U &= A_X U + \mathcal{V}\nabla_X U, & \nabla^*_X U &= A^*_X U + \mathcal{V}\nabla^*_X U, \\ \nabla_X Y &= \mathcal{H}\nabla_X Y + A_X Y, & \nabla^*_X Y &= \mathcal{H}\nabla^*_X Y + A^*_X Y. \end{aligned}$$

Furthermore, if X is basic, then $\mathcal{H}\nabla_U X = A_X U$ and $\mathcal{H}\nabla^*_U X = A^*_X U$.

We define the covariant derivatives ∇T and ∇A by

$$(\nabla_E T)_G H = \nabla_E (T_G H) - T_{\nabla_E G} H - T_G(\nabla_E H),$$
$$(\nabla_E A)_G H = \nabla_E (A_G H) - A_{\nabla_E G} H - A_G(\nabla_E H)$$

for $E, G, H \in TM$. We change ∇ to ∇^*, then the covariant derivatives $\nabla^* T^*$ and $\nabla^* A^*$ are defined similarly. We consider the curvature tensor on the statistical submersion. Let $\overline{R}$ (resp. $\overline{R}^*$) be the curvature tensor with respect to the induced affine connection $\overline{\nabla}$ (resp. $\overline{\nabla}^*$) of each fiber. Also, let $\widehat{R}(X,Y)Z$ (resp. $\widehat{R}^*(X,Y)Z$) be a horizontal vector field such that $\pi_*(\widehat{R}(X,Y)Z) = \widehat{R}(\pi_* X, \pi_* Y)\pi_* Z$ (resp. $\pi_*(\widehat{R}^*(X,Y)Z) = \widehat{R}^*(\pi_* X, \pi_* Y)\pi_* Z$) at each $p \in M$, where $\widehat{R}$ (resp. $\widehat{R}^*$) is the curvature tensor on B of the affine connection $\widehat{\nabla}$ (resp. $\widehat{\nabla}^*$). Then we have ([30]).

Theorem 2.9 *If $\pi : (M, \nabla, g) \to (B, \widehat{\nabla}, g_B)$ is a statistical submersion, then we get for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$*

$$\begin{aligned}
g(R(U,V)W,W') &= g(\overline{R}(U,V)W,W') + g(T_UW,T_V^*W') - g(T_VW,T_U^*W'),\\
g(R(U,V)W,X) &= g((\nabla_UT)_VW,X) - g((\nabla_VT)_UW,X),\\
g(R(U,V)X,W) &= g((\nabla_UT)_VX,W) - g((\nabla_VT)_UX,W),\\
g(R(U,V)X,Y) &= g((\nabla_UA)_XV,Y) - g((\nabla_VA)_XU,Y) + g(T_UX,T_V^*Y)\\
&\quad -g(T_VX,T_U^*Y) - g(A_XU,A_Y^*V) + g(A_XV,A_Y^*U),\\
g(R(X,U)V,W) &= g([\mathcal{V}\nabla_X,\overline{\nabla}_U]V,W) - g(\nabla_{[X,U]}V,W) - g(T_UV,A_X^*W)\\
&\quad +g(T_U^*W,A_XV),\\
g(R(X,U)V,Y) &= g((\nabla_XT)_UV,Y) - g((\nabla_UA)_XV,Y) + g(A_XU,A_Y^*V)\\
&\quad -g(T_UX,T_V^*Y),\\
g(R(X,U)Y,V) &= g((\nabla_XT)_UY,V) - g((\nabla_UA)_XY,V) + g(T_UX,T_VY)\\
&\quad -g(A_XU,A_YV),\\
g(R(X,U)Y,Z) &= g((\nabla_XA)_YU,Z) - g(T_UX,A_Y^*Z) - g(T_UY,A_X^*Z)\\
&\quad +g(A_XY,T_U^*Z),\\
g(R(X,Y)U,V) &= g([\mathcal{V}\nabla_X,\mathcal{V}\nabla_Y]U,V) - g(\nabla_{[X,Y]}U,V) + g(A_XU,A_Y^*V)\\
&\quad -g(A_YU,A_X^*V),\\
g(R(X,Y)U,Z) &= g((\nabla_XA)_YU,Z) - g((\nabla_YA)_XU,Z) + g(T_U^*Z,\theta_XY),\\
g(R(X,Y)Z,U) &= g((\nabla_XA)_YZ,U) - g((\nabla_YA)_XZ,U) - g(T_UZ,\theta_XY),\\
g(R(X,Y)Z,Z') &= g(\widehat{R}(X,Y)Z,Z') - g(A_YZ,A_X^*Z') + g(A_XZ,A_Y^*Z')\\
&\quad +g(\theta_XY,A_Z^*Z'),
\end{aligned}$$

where we put $\theta_XY = A_XY + A_X^*Y = \mathcal{V}[X,Y]$.

3. Locally product-like statistical submersions

Let (M,∇,g,F) be a locally product-like statistical manifold and $(B,\widehat{\nabla},g_B)$ be a statistical manifold. The statistical submersion $\pi:(M,\nabla,g,F)\to(B,\widehat{\nabla},g_B)$ is called a locally product-like statistical submersion. For $X\in\mathcal{H}(M)$, we put ([35])

$$FX = fX + hX, \qquad F^*X = f^*X + h^*X, \tag{3.1}$$

where $fX,\ f^*X\in\mathcal{H}(M)$ and $hX,\ h^*X\in\mathcal{V}(M)$. For $V\in\mathcal{V}(M)$, we set

$$FV = tV + sV, \qquad F^*V = t^*V + s^*V, \tag{3.2}$$

where $tV,\ t^*V\in\mathcal{H}(M)$ and $sV,\ s^*V\in\mathcal{V}(M)$. From $(F^*)^*=F$, we find $(f^*)^*=f$, $(h^*)^*=h$, $(t^*)^*=t$ and $(s^*)^*=s$. Because of $F^2=I$ and $(F^*)^2=I$, we get

$$f^2 = I - th, \qquad hf + sh = 0, \qquad ft + ts = 0, \qquad s^2 = I - ht \tag{3.3}$$

and

$$(f^*)^2 = I - t^*h^*, \quad h^*f^* + s^*h^* = 0, \quad f^*t^* + t^*s^* = 0, \quad (s^*)^2 = I - h^*t^*. \tag{3.4}$$

Because of $g(FE, G) = g(E, F^*G)$ for any vector fields E and G on M, we find

$$g(fX, Y) = g(X, f^*Y), \quad g(fX, f^*Y) = g(X, Y) - g(hX, h^*Y), \tag{3.5}$$

$$g(hX, V) = g(X, t^*V), \tag{3.6}$$

$$g(tU, Y) = g(U, h^*Y), \tag{3.7}$$

$$g(sU, V) = g(U, s^*V), \qquad g(sU, s^*V) = g(U, V) - g(tU, t^*V). \tag{3.8}$$

Thus t (resp. s) vanishes identically if and only if so is h^* (resp. s^*), and f (resp. h) vanishes identically if and only if so is f^* (resp. t^*). Hence we have the following lemma.

Lemma 3.1 *We find for each* $p \in M$

(1) $F(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ *if and only if* $F^*(\mathcal{H}_p(M)) \subset \mathcal{H}_p(M)$.

(2) $F(\mathcal{H}_p(M)) \subset \mathcal{H}_p(M)$ *if and only if* $F^*(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$.

(3) $F(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$ *if and only if* $F^*(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$.

(4) $F(\mathcal{H}_p(M)) \subset \mathcal{V}_p(M)$ *if and only if* $F^*(\mathcal{H}_p(M)) \subset \mathcal{V}_p(M)$.

If $F(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (resp. $F^*(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$) for each $p \in M$, then $\overline{M}$ is said to be an F-invariant (resp. F^*-invariant) submanifold of M. Then t and h^* (resp. h and t^*) vanish identically. Because of $f^2 = I$ and $g(fX, f^*Y) = g(X, Y)$, the triple (M, g, f) is an almost product-like semi-Riemannian manifold and so is (M, g, f^*). If $F(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$ for each $p \in M$, then $\overline{M}$ is said to be an F-antiinvariant submanifold of M. Since $s = 0$ is equivalent to $s^* = 0$, $\overline{M}$ is an F-antiinvariant if and only if $\overline{M}$ is an F^*-antiinvariant. Thus, in this paper, it is simply referred to as antiinvariant.

Also, we have from (3.5)∼(3.8).

Lemma 3.2 *Let* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion. We find*

$$\begin{aligned}
&g((\mathcal{H}\nabla_X f)Y, Z) = g(Y, (\mathcal{H}\nabla^*_X f^*)Z), && g((\mathcal{H}\nabla_U f)X, Y) = g(X, (\mathcal{H}\nabla^*_U f^*)Y),\\
&g(\mathcal{V}\nabla_X h)Y, V) = g(Y, (\mathcal{H}\nabla^*_X t^*)V), && g((\overline{\nabla}_U h)Y, V) = g(Y, (\mathcal{H}\nabla^*_U t^*)V),\\
&g((\mathcal{H}\nabla_X t)V, Y) = g(V, (\mathcal{V}\nabla^*_X h^*)Y), && g((\mathcal{H}\nabla_U t)V, Y) = g(V, (\overline{\nabla}^*_U h^*)Y),\\
&g((\mathcal{V}\nabla_X s)U, V) = g(U, (\mathcal{V}\nabla^*_X s^*)V), && g((\overline{\nabla}_U s)V, W) = g(V, (\overline{\nabla}^*_U s^*)W),
\end{aligned}$$

where we put

$$\begin{aligned}
&(\mathcal{H}\nabla_X f)Y = \mathcal{H}\nabla_X(fY) - f(\mathcal{H}\nabla_X Y), && (\mathcal{H}\nabla_U f)X = \mathcal{H}\nabla_U(fX) - f(\mathcal{H}\nabla_U X),\\
&(\mathcal{V}\nabla_X h)Y = \mathcal{V}\nabla_X(hY) - h(\mathcal{H}\nabla_X Y), && (\overline{\nabla}_U h)X = \overline{\nabla}_U(hX) - h(\mathcal{H}\nabla_U X),\\
&(\mathcal{H}\nabla_X t)V = \mathcal{H}\nabla_X(tV) - t(\mathcal{V}\nabla_X V), && (\mathcal{H}\nabla_U t)V = \mathcal{H}\nabla_U(tV) - t(\overline{\nabla}_U V),\\
&(\mathcal{V}\nabla_X s)U = \mathcal{V}\nabla_X(sU) - s(\mathcal{V}\nabla_X U), && (\overline{\nabla}_U s)V = \overline{\nabla}_U(sV) - s(\overline{\nabla}_U V),
\end{aligned}$$

also, we set $(\mathcal{H}\nabla^*_X f^*)Y = \mathcal{H}\nabla^*_X(f^*Y) - f^*(\mathcal{H}\nabla^*_X Y)$*, etc.*

Hence we have the following corollary.

Corollary 3.3 *If* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion, then we get*

(1) $\mathcal{H}\nabla f = 0$ *is equivalent to* $\mathcal{H}\nabla^* f^* = 0$.

(2) $\mathcal{V}\nabla_X h = 0$ *(resp.* $\overline{\nabla}_U h = 0$*) is equivalent to* $\mathcal{H}\nabla^*_X t^* = 0$ *(resp.* $\mathcal{H}\nabla^*_U t^* = 0$*).*

(3) $\mathcal{H}\nabla_X t = 0$ *(resp.* $\mathcal{H}\nabla_U t = 0$*) is equivalent to* $\mathcal{V}\nabla^*_X h^* = 0$ *(resp.* $\overline{\nabla}^*_U h^* = 0$*).*

(4) $\mathcal{V}\nabla_X s = 0$ *(resp.* $\overline{\nabla}_U s = 0$*) is equivalent to* $\mathcal{V}\nabla^*_X s^* = 0$ *(resp.* $\overline{\nabla}^*_U s^* = 0$*).*

Because of $\nabla F = 0$ and Lemma 2.8, we get the following proposition.

Proposition 3.4 *If* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion, then we get*

$$\begin{aligned}
\overline{\nabla}_U(sV) + T_U(tV) &= h(T_U V) + s(\overline{\nabla}_U V),\\
\mathcal{H}\nabla_U(tV) + T_U(sV) &= f(T_U V) + t(\overline{\nabla}_U V),\\
\overline{\nabla}_U(hX) + T_U(fX) &= h(\mathcal{H}\nabla_U X) + s(T_U X),\\
\mathcal{H}\nabla_U(fX) + T_U(hX) &= f(\mathcal{H}\nabla_U X) + t(T_U X),\\
\mathcal{V}\nabla_X(sU) + A_X(tU) &= h(A_X U) + s(\mathcal{V}\nabla_X U),\\
\mathcal{H}\nabla_X(tU) + A_X(sU) &= f(A_X U) + t(\mathcal{V}\nabla_X U),\\
\mathcal{V}\nabla_X(hY) + A_X(fY) &= h(\mathcal{H}\nabla_X Y) + s(A_X Y),\\
\mathcal{H}\nabla_X(fY) + A_X(hY) &= f(\mathcal{H}\nabla_X Y) + t(A_X Y)
\end{aligned}$$

for any $U, V \in \mathcal{V}(M)$ *and* $X, Y \in \mathcal{H}(M)$.

From (3.3) and Proposition 3.4, we have the following lemma.

Lemma 3.5 *If* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion, then we get*

$$\begin{aligned}
T_U V &= f\{\mathcal{H}\nabla_U(tV) + T_U(sV)\} + t\{\overline{\nabla}_U(sV) + T_U(tV)\},\\
\overline{\nabla}_U V &= h\{\mathcal{H}\nabla_U(tV) + T_U(sV)\} + s\{\overline{\nabla}_U(sV) + T_U(tV)\},\\
\mathcal{H}\nabla_U X &= f\{\mathcal{H}\nabla_U(fX) + T_U(hX)\} + t\{\overline{\nabla}_U(hX) + T_U(fX)\},\\
T_U X &= h\{\mathcal{H}\nabla_U(fX) + T_U(hX)\} + s\{\overline{\nabla}_U(hX) + T_U(fX)\},\\
A_X U &= f\{\mathcal{H}\nabla_X(tU) + A_X(sU)\} + t\{\mathcal{V}\nabla_X(sU) + A_X(tU)\},\\
\mathcal{V}\nabla_X U &= h\{\mathcal{H}\nabla_X(tU) + A_X(sU)\} + s\{\mathcal{V}\nabla_X(sU) + A_X(tU)\},\\
\mathcal{H}\nabla_X Y &= f\{\mathcal{H}\nabla_X(fY) + A_X(hY)\} + t\{\mathcal{V}\nabla_X(hY) + A_X(fY)\},\\
A_X Y &= h\{\mathcal{H}\nabla_X(fY) + A_X(hY)\} + s\{\mathcal{V}\nabla_X(hY) + A_X(fY)\}
\end{aligned}$$

for any $U, V \in \mathcal{V}(M)$ *and* $X, Y \in \mathcal{H}(M)$.

Lemma 3.6 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is a locally product-like statistical submersion, then we obtain*

$$\begin{aligned}
h(T_U V) &= (\overline{\nabla}_U s)V + T_U(tV), \\
f(T_U V) &= (\mathcal{H}\nabla_U t)V + T_U(sV), \\
s(T_U X) &= (\overline{\nabla}_U h)X + T_U(fX), \\
t(T_U X) &= (\mathcal{H}\nabla_U f)X + T_U(hX), \\
h(A_X U) &= (\mathcal{V}\nabla_X s)U + A_X(tU), \\
f(A_X U) &= (\mathcal{H}\nabla_X t)U + A_X(sU), \\
s(A_X Y) &= (\mathcal{V}\nabla_X h)Y + A_X(fY), \\
t(A_X Y) &= (\mathcal{H}\nabla_X f)Y + A_X(hY).
\end{aligned}$$

Thus we have the following corollary.

Corollary 3.7 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion. We find*

(1) $\mathcal{H}\nabla f = 0$ *if and only if* $T_U(hX) = t(T_U X)$ *and* $A_X(hY) = t(A_X Y)$.

(2) $\mathcal{V}\nabla h = 0$ *if and only if* $T_U(fX) = s(T_U X)$ *and* $A_X(fY) = s(A_X Y)$.

(3) $\mathcal{H}\nabla t = 0$ *if and only if* $T_U(sV) = f(T_U V)$ *and* $A_X(sU) = f(A_X U)$.

(4) $\mathcal{V}\nabla s = 0$ *if and only if* $T_U(tV) = h(T_U V)$ *and* $A_X(tU) = h(A_X U)$.

Example 3.8 *We put* $B = \{ (x_2, x_3, x_4) \,|\, x_i \in \mathbb{R} \ (i = 2, 3, 4) \} = \mathbb{R}^3$ *and*

$$g_B = \begin{pmatrix} e^{-x_1} & 0 & 0 \\ 0 & e^{x_1} & 0 \\ 0 & 0 & e^{-x_1} \end{pmatrix}.$$

If we set

$$\begin{aligned}
&\widehat{\nabla}_{\partial_{2*}}\partial_{2*} = \widehat{\nabla}_{\partial_{4*}}\partial_{4*} = e^{x_1}\partial_{4*}, && \widehat{\nabla}_{\partial_{2*}}\partial_{3*} = \widehat{\nabla}_{\partial_{3*}}\partial_{2*} = -e^{x_1}\partial_{4*}, \\
&\widehat{\nabla}_{\partial_{2*}}\partial_{4*} = \widehat{\nabla}_{\partial_{4*}}\partial_{2*} = e^{x_1}\partial_{2*} - e^{-x_1}\partial_{3*}, && \widehat{\nabla}_{\partial_{3*}}\partial_{3*} = 0, \\
&\widehat{\nabla}_{\partial_{3*}}\partial_{4*} = \widehat{\nabla}_{\partial_{4*}}\partial_{3*} = -e^{x_1}\partial_{2*}
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\nabla}^*_{\partial_{2*}}\partial_{2*} = \widehat{\nabla}^*_{\partial_{4*}}\partial_{4*} = -e^{x_1}\partial_{4*}, && \widehat{\nabla}^*_{\partial_{2*}}\partial_{3*} = \widehat{\nabla}^*_{\partial_{3*}}\partial_{2*} = e^{x_1}\partial_{4*}, \\
&\widehat{\nabla}^*_{\partial_{2*}}\partial_{4*} = \widehat{\nabla}^*_{\partial_{4*}}\partial_{2*} = -e^{x_1}\partial_{2*} + e^{-x_1}\partial_{3*}, && \widehat{\nabla}^*_{\partial_{3*}}\partial_{3*} = 0, \\
&\widehat{\nabla}^*_{\partial_{3*}}\partial_{4*} = \widehat{\nabla}^*_{\partial_{4*}}\partial_{3*} = e^{x_1}\partial_{2*},
\end{aligned}$$

then $(B, \widehat{\nabla}, g_B)$ *is a statistical manifold and so is* $(B, \widehat{\nabla}^*, g_B)$, *where* $\partial_{i*} = \partial/\partial x_i \ (i = 2, 3, 4)$. *Let* (M_+, ∇, g, F) *and* (M_-, ∇, g, F) *be locally product-like statistical manifolds of Example 2.4. We define a locally product-like statistical submersion* $\pi : (M_\pm, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *by*

$$\pi(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4).$$

For $\partial_2, \partial_3, \partial_4 \in \mathcal{H}(M_\pm)$ *and* $\partial_1 \in \mathcal{V}(M_\pm)$, *it is clear that*

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad s = 0.$$

Thus each fiber is antiinvariant of $M_\pm$. *The structure* f *satisfies* $f^3 - f = 0$. *Moreover, we obtain*

$$\begin{aligned}
&T_{\partial_1}\partial_1 = 0, && \overline{\nabla}_{\partial_1}\partial_1 = e^{x_1}\partial_1, \\
&\mathcal{H}\nabla_{\partial_1}\partial_2 = -e^{x_1}\partial_2, && T_{\partial_1}\partial_2 = 0, \\
&\mathcal{H}\nabla_{\partial_1}\partial_3 = e^{x_1}\partial_3, && T_{\partial_1}\partial_3 = 0, \\
&\mathcal{H}\nabla_{\partial_1}\partial_4 = -e^{x_1}\partial_4, && T_{\partial_1}\partial_4 = 0, \\
&A_{\partial_2}\partial_1 = -e^{x_1}\partial_2, && \mathcal{V}\nabla_{\partial_2}\partial_1 = 0, \\
&A_{\partial_3}\partial_1 = e^{x_1}\partial_3, && \mathcal{V}\nabla_{\partial_3}\partial_1 = 0, \\
&A_{\partial_4}\partial_1 = -e^{x_1}\partial_4, && \mathcal{V}\nabla_{\partial_4}\partial_1 = 0, \\
&\mathcal{H}\nabla_{\partial_2}\partial_2 = \mathcal{H}\nabla_{\partial_4}\partial_4 = e^{x_1}\partial_4, && A_{\partial_2}\partial_2 = A_{\partial_4}\partial_4 = -e^{-x_1}\partial_1, \\
&\mathcal{H}\nabla_{\partial_2}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_2 = -e^{x_1}\partial_4, && A_{\partial_2}\partial_3 = A_{\partial_3}\partial_2 = 0, \\
&\mathcal{H}\nabla_{\partial_2}\partial_4 = \mathcal{H}\nabla_{\partial_4}\partial_2 = e^{x_1}\partial_2 - e^{-x_1}\partial_3, && A_{\partial_2}\partial_4 = A_{\partial_4}\partial_2 = 0, \\
&\mathcal{H}\nabla_{\partial_3}\partial_3 = 0, && A_{\partial_3}\partial_3 = e^{x_1}\partial_1, \\
&\mathcal{H}\nabla_{\partial_3}\partial_4 = \mathcal{H}\nabla_{\partial_4}\partial_3 = -e^{x_1}\partial_2, && A_{\partial_3}\partial_4 = A_{\partial_4}\partial_3 = 0.
\end{aligned}$$

Therefore π *has isometric fiber. Also, we find* $T_U(tV) = h(T_U V)$ *and* $A_X(tU) = h(A_X U)$.

Example 3.9 *We denote subsets of* $\mathbb{R}^2$ *by*

$$B_+ = \{\, (x_1, x_3) \,|\, x_1 > 0 \,\}, \qquad B_- = \{\, (x_1, x_3) \,|\, x_1 < 0 \,\}$$

and we set

$$g_B = \begin{pmatrix} -1 + e^{x_1} & 0 \\ 0 & e^{x_1} \end{pmatrix}.$$

If we put

$$\widehat{\nabla}_{\partial_{1*}}\partial_{1*} = \widehat{\nabla}_{\partial_{3*}}\partial_{3*} = e^{x_1}\partial_{1*}, \qquad \widehat{\nabla}_{\partial_{1*}}\partial_{3*} = \widehat{\nabla}_{\partial_{3*}}\partial_{1*} = e^{x_1}\partial_{3*}$$

and

$$\begin{aligned}
&\widehat{\nabla}^*_{\partial_{1*}}\partial_{1*} = -e^{x_1}(e^{x_1} - 2)(e^{x_1} - 1)^{-1}\,\partial_{1*}, \quad \widehat{\nabla}^*_{\partial_{1*}}\partial_{3*} = \widehat{\nabla}^*_{\partial_{3*}}\partial_{1*} = -(e^{x_1} - 1)\,\partial_{3*}, \\
&\widehat{\nabla}^*_{\partial_{3*}}\partial_{3*} = -e^{2x_1}(e^{x_1} - 1)^{-1}\partial_{1*},
\end{aligned}$$

then $(B_+, \widehat{\nabla}, g_B)$ *(resp.* $(B_-, \widehat{\nabla}, g_B)$*) is a statistical manifold and so is* $(B_+, \widehat{\nabla}^*, g_B)$ *(resp.* $(B_-, \widehat{\nabla}^*, g_B)$*), where* $\partial_{i*} = \partial/\partial x_i$ $(i = 1, 3)$*. Let* (M_+, ∇, g, F) *and* (M_-, ∇, g, F) *be locally product-like statistical manifolds of Example 2.4. We define a locally product-like statistical submersion* $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ *by*

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_3).$$

For $\partial_1, \partial_3 \in \mathcal{H}(M_\pm)$ *and* $\partial_2, \partial_4 \in \mathcal{V}(M_\pm)$*, it is easy to see that*

$$f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h = 0, \qquad t = 0, \qquad s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus each fiber is F*-invariant and* F^**-invariant of* $M_\pm$*. Moreover we get*

$$\begin{aligned}
&T_{\partial_2}\partial_2 = T_{\partial_4}\partial_4 = -e^{-x_1}\partial_1, && \overline{\nabla}_{\partial_2}\partial_2 = \overline{\nabla}_{\partial_4}\partial_4 = e^{x_1}\partial_4,\\
&T_{\partial_2}\partial_4 = T_{\partial_4}\partial_2 = -e^{-x_1}\partial_3, && \overline{\nabla}_{\partial_2}\partial_4 = \overline{\nabla}_{\partial_4}\partial_2 = e^{x_1}\partial_2,\\
&\mathcal{H}\nabla_{\partial_2}\partial_1 = \mathcal{H}\nabla_{\partial_4}\partial_3 = 0, && T_{\partial_2}\partial_1 = T_{\partial_4}\partial_3 = -e^{x_1}\partial_2,\\
&\mathcal{H}\nabla_{\partial_2}\partial_3 = \mathcal{H}\nabla_{\partial_4}\partial_1 = 0, && T_{\partial_2}\partial_3 = T_{\partial_4}\partial_1 = -e^{x_1}\partial_4,\\
&A_{\partial_1}\partial_2 = A_{\partial_3}\partial_4 = 0, && \mathcal{V}\nabla_{\partial_1}\partial_2 = \mathcal{V}\nabla_{\partial_3}\partial_4 = -e^{x_1}\partial_2,\\
&A_{\partial_1}\partial_4 = A_{\partial_3}\partial_2 = 0, && \mathcal{V}\nabla_{\partial_1}\partial_4 = \mathcal{V}\nabla_{\partial_3}\partial_2 = -e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_1}\partial_1 = \mathcal{H}\nabla_{\partial_3}\partial_3 = e^{x_1}\partial_1, && A_{\partial_1}\partial_1 = A_{\partial_3}\partial_3 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_1 = e^{x_1}\partial_3, && A_{\partial_1}\partial_3 = A_{\partial_3}\partial_1 = 0.
\end{aligned}$$

Therefore each fiber is flat and $\mathcal{H}(M_\pm)$ *is integrable. We find* $T_U(fX) = s(T_UX)$*,* $T_U(sV) = f(T_UV)$*,* $\mathcal{H}\nabla f = 0$ *and* $\mathcal{V}\nabla s = 0$*.*

Example 3.10 *We denote subsets of* $\mathbb{R}^3$ *by*

$$B_+ = \{\, (x_1, x_2, x_3) \,|\, x_1 > 0 \,\}, \qquad B_- = \{\, (x_1, x_2, x_3) \,|\, x_1 < 0 \,\}$$

and we set

$$g_B = \begin{pmatrix} -1 + e^{x_1} & 0 & 0 \\ 0 & e^{-x_1} & 0 \\ 0 & 0 & e^{x_1} \end{pmatrix}.$$

If we put

$$\begin{aligned}
&\widehat{\nabla}_{\partial_{1*}}\partial_{1*} = \widehat{\nabla}_{\partial_{3*}}\partial_{3*} = e^{x_1}\partial_{1*}, && \widehat{\nabla}_{\partial_{1*}}\partial_{2*} = \widehat{\nabla}_{\partial_{2*}}\partial_{1*} = -e^{x_1}\partial_{2*},\\
&\widehat{\nabla}_{\partial_{1*}}\partial_{3*} = \widehat{\nabla}_{\partial_{3*}}\partial_{1*} = e^{x_1}\partial_{3*}, && \widehat{\nabla}_{\partial_{2*}}\partial_{2*} = -e^{-x_1}\partial_{1*},\\
&\widehat{\nabla}_{\partial_{2*}}\partial_{3*} = \widehat{\nabla}_{\partial_{3*}}\partial_{2*} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{\nabla}^*_{\partial_{1*}}\partial_{1*} = -e^{x_1}(e^{x_1} - 2)(e^{x_1} - 1)^{-1}\,\partial_{1*}, && \widehat{\nabla}^*_{\partial_{1*}}\partial_{2*} = \widehat{\nabla}^*_{\partial_{2*}}\partial_{1*} = (e^{x_1} - 1)\,\partial_{2*},\\
&\widehat{\nabla}^*_{\partial_{1*}}\partial_{3*} = \widehat{\nabla}^*_{\partial_{3*}}\partial_{1*} = -(e^{x_1} - 1)\,\partial_{3*}, && \widehat{\nabla}^*_{\partial_{2*}}\partial_{2*} = (e^{x_1} - 1)^{-1}\partial_{1*},\\
&\widehat{\nabla}^*_{\partial_{2*}}\partial_{3*} = \widehat{\nabla}^*_{\partial_{3*}}\partial_{2*} = 0, && \widehat{\nabla}^*_{\partial_{3*}}\partial_{3*} = -e^{2x_1}(e^{x_1} - 1)^{-1}\partial_{1*},
\end{aligned}$$

then $(B_+, \widehat{\nabla}, g_B)$ *(resp.* $(B_-, \widehat{\nabla}, g_B)$*) is a statistical manifold and so is* $(B_+, \widehat{\nabla}^*, g_B)$ *(resp.* $(B_-, \widehat{\nabla}^*, g_B)$*), where* $\partial_{i*} = \partial/\partial x_i$ $(i = 1, 2, 3)$*. Let* (M_+, ∇, g, F) *and* (M_-, ∇, g, F) *be locally product-like statistical manifolds of Example 2.4. We define a locally product-like statistical submersion* $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ *by*

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3).$$

For ∂_1*,* ∂_2*,* $\partial_3 \in \mathcal{H}(M_\pm)$ *and* $\partial_4 \in \mathcal{V}(M_\pm)$*, it is clear that*

$$f = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad s = 0.$$

Thus each fiber is antiinvariant of $M_\pm$*. The structure* f *satisfies* $f^3 - f = 0$*. Moreover, we obtain*

$$\begin{aligned}
&T_{\partial_4}\partial_4 = -e^{-x_1}\partial_1, && \overline{\nabla}_{\partial_4}\partial_4 = e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_4}\partial_1 = 0, && T_{\partial_4}\partial_1 = -e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_4}\partial_2 = e^{x_1}\partial_2 - e^{-x_1}\partial_3, && T_{\partial_4}\partial_2 = 0,\\
&\mathcal{H}\nabla_{\partial_4}\partial_3 = -e^{x_1}\partial_2, && T_{\partial_4}\partial_3 = 0,\\
&A_{\partial_1}\partial_4 = 0, && \mathcal{V}\nabla_{\partial_1}\partial_4 = -e^{x_1}\partial_4,\\
&A_{\partial_2}\partial_4 = e^{x_1}\partial_2 - e^{-x_1}\partial_3, && \mathcal{V}\nabla_{\partial_2}\partial_4 = 0,\\
&A_{\partial_3}\partial_4 = -e^{x_1}\partial_2, && \mathcal{V}\nabla_{\partial_3}\partial_4 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_1 = \mathcal{H}\nabla_{\partial_3}\partial_3 = e^{x_1}\partial_1, && A_{\partial_1}\partial_1 = A_{\partial_3}\partial_3 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_2 = \mathcal{H}\nabla_{\partial_2}\partial_1 = -e^{x_1}\partial_2, && A_{\partial_1}\partial_2 = A_{\partial_2}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_1 = e^{x_1}\partial_3, && A_{\partial_1}\partial_3 = A_{\partial_3}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_2}\partial_2 = -e^{-x_1}\partial_1, && A_{\partial_2}\partial_2 = e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_2}\partial_3 = \mathcal{H}\nabla_{\partial_3}\partial_2 = 0, && A_{\partial_2}\partial_3 = A_{\partial_3}\partial_2 = -e^{x_1}\partial_4.
\end{aligned}$$

Therefore we find $T_U(tV) = h(T_U V)$ *and* $A_X(tU) = h(A_X U)$*.*

Example 3.11 *We denote subsets of* $\mathbb{R}^2$ *by*

$$B_+ = \{\,(x_1, x_2) \,|\, x_1 > 0\,\}, \qquad B_- = \{\,(x_1, x_2) \,|\, x_1 < 0\,\}$$

and we set

$$g_B = \begin{pmatrix} -1 + e^{x_1} & 0 \\ 0 & e^{-x_1} \end{pmatrix}.$$

If we put

$$\widehat{\nabla}_{\partial_{1*}}\partial_{1*} = e^{x_1}\partial_{1*}, \qquad \widehat{\nabla}_{\partial_{1*}}\partial_{2*} = \widehat{\nabla}_{\partial_{2*}}\partial_{1*} = -e^{x_1}\partial_{2*}, \qquad \widehat{\nabla}_{\partial_{2*}}\partial_{2*} = -e^{-x_1}\partial_{1*}$$

and

$$\widehat{\nabla}^*_{\partial_{1*}}\partial_{1*} = -e^{x_1}(e^{x_1}-2)(e^{x_1}-1)^{-1}\,\partial_{1*}, \qquad \widehat{\nabla}^*_{\partial_{1*}}\partial_{2*} = \widehat{\nabla}^*_{\partial_{2*}}\partial_{1*} = (e^{x_1}-1)\,\partial_{2*},$$
$$\widehat{\nabla}^*_{\partial_{2*}}\partial_{2*} = (e^{x_1}-1)^{-1}\partial_{1*},$$

then $(B_+, \widehat{\nabla}, g_B)$ *(resp.* $(B_-, \widehat{\nabla}, g_B)$*) is a statistical manifold and so is* $(B_+, \widehat{\nabla}^*, g_B)$ *(resp.* $(B_-, \widehat{\nabla}^*, g_B)$*), where* $\partial_{i*} = \partial/\partial x_i$ $(i = 1, 2)$*. Let* (M_+, ∇, g, F) *and* (M_-, ∇, g, F) *be locally product-like statistical manifolds of Example 2.4. We define a locally product-like statistical submersion* $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ *by*

$$\pi(x_1, x_2, x_3, x_4) = (x_1, x_2).$$

For $\partial_1, \partial_2 \in \mathcal{H}(M_\pm)$ *and* $\partial_3, \partial_4 \in \mathcal{V}(M_\pm)$*, it is easy to see that*

$$f = 0, \qquad h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad s = 0.$$

Thus each fiber is antiinvariant of $M_\pm$*. Moreover we get*

$$\begin{aligned}
&T_{\partial_3}\partial_3 = e^{x_1}\partial_1, && \overline{\nabla}_{\partial_3}\partial_3 = 0,\\
&T_{\partial_3}\partial_4 = T_{\partial_4}\partial_3 = -e^{x_1}\partial_2, && \overline{\nabla}_{\partial_3}\partial_4 = \overline{\nabla}_{\partial_4}\partial_3 = 0,\\
&T_{\partial_4}\partial_4 = -e^{-x_1}\partial_1, && \overline{\nabla}_{\partial_4}\partial_4 = e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_3}\partial_1 = 0, && T_{\partial_3}\partial_1 = e^{x_1}\partial_3,\\
&\mathcal{H}\nabla_{\partial_3}\partial_2 = \mathcal{H}\nabla_{\partial_4}\partial_1 = 0, && T_{\partial_3}\partial_2 = T_{\partial_4}\partial_1 = -e^{x_1}\partial_4,\\
&\mathcal{H}\nabla_{\partial_4}\partial_2 = e^{x_1}\partial_2, && T_{\partial_4}\partial_2 = -e^{-x_1}\partial_3,\\
&A_{\partial_1}\partial_3 = 0, && \mathcal{V}\nabla_{\partial_1}\partial_3 = e^{x_1}\partial_3,\\
&A_{\partial_1}\partial_4 = A_{\partial_2}\partial_3 = 0, && \mathcal{V}\nabla_{\partial_1}\partial_4 = \mathcal{V}\nabla_{\partial_2}\partial_3 = -e^{x_1}\partial_4,\\
&A_{\partial_2}\partial_4 = e^{x_1}\partial_2, && \mathcal{V}\nabla_{\partial_2}\partial_4 = -e^{-x_1}\partial_3,\\
&\mathcal{H}\nabla_{\partial_1}\partial_1 = e^{x_1}\partial_1, && A_{\partial_1}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_1}\partial_2 = \mathcal{H}\nabla_{\partial_2}\partial_1 = -e^{x_1}\partial_2, && A_{\partial_1}\partial_2 = A_{\partial_2}\partial_1 = 0,\\
&\mathcal{H}\nabla_{\partial_2}\partial_2 = -e^{-x_1}\partial_1, && A_{\partial_2}\partial_2 = e^{x_1}\partial_4.
\end{aligned}$$

Therefore each fiber is flat. We find $T_U(hX) = t(T_U X)$*,* $A_X(hY) = t(A_X Y)$*,* $T_U(tV) = h(T_U V)$*,* $A_X(tU) = h(A_X U)$*,* $\mathcal{V}\nabla h = 0$ *and* $\mathcal{H}\nabla t = 0$*.*

Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion. If the curvature with respect to the affine connection ∇ of the total space M satisfies the condition (2.9), then we find from Theorem

2.9

$$
\begin{aligned}
&g(\overline{R}(U,V)W,W') + g(T_U W, T_V^* W') - g(T_V W, T_U^* W') \\
&= c\,[\,g(V,W)g(U,W') - g(U,W)g(V,W') + g(V,sW)g(sU,W') - g(U,sW)g(sV,W') \\
&\quad + \{\,g(sU,V) - g(U,sV)\}g(sW,W')\,], \qquad (3.9)\\
&g((\nabla_U T)_V W, X) - g((\nabla_V T)_U W, X) \\
&= c\,[\,g(V,sW)g(tU,X) - g(U,sW)g(tV,X) + \{\,g(sU,V) - g(U,sV)\}g(tW,X)\,], \qquad (3.10)\\
&g((\nabla_U T)_V X, W) - g((\nabla_V T)_U X, W) \\
&= c\,[\,g(V,hX)g(sU,W) - g(U,hX)g(sV,W) + \{\,g(sU,V) - g(U,sV)\}g(hX,W)\,], \qquad (3.11)\\
&g((\nabla_U A)_X V, Y) - g((\nabla_V A)_X U, Y) + g(T_U X, T_V^* Y) - g(T_V X, T_U^* Y) - g(A_X U, A_Y^* V) + g(A_X V, A_Y^* U) \\
&= c\,[\,g(V,hX)g(tU,Y) - g(U,hX)g(tV,Y) + \{\,g(sU,V) - g(U,sV)\}g(fX,Y)\,], \qquad (3.12)\\
&g([\mathcal{V}\nabla_X, \overline{\nabla}_U]V, W) - g(\nabla_{[X,U]}V, W) - g(T_U V, A_X^* W) + g(T_U^* W, A_X V) \\
&= c\,[\,g(U,sV)g(hX,W) - g(X,tV)g(sU,W) + \{\,g(hX,U) - g(X,tU)\}g(sV,W)\,], \qquad (3.13)\\
&g((\nabla_X T)_U V, Y) - g((\nabla_U A)_X V, Y) + g(A_X U, A_Y^* V) - g(T_U X, T_V^* Y) \\
&= c\,[\,g(U,V)g(X,Y) + g(U,sV)g(fX,Y) - g(X,tV)g(tU,Y) + \{g(hX,U) - g(X,tU)\}g(tV,Y)], \qquad (3.14)\\
&g((\nabla_X T)_U Y, V) - g((\nabla_U A)_X Y, V) + g(T_U X, T_V Y) - g(A_X U, A_Y V) \\
&= c\,[\,g(X,Y)g(U,V) + g(U,hY)g(hX,V) - g(X,fY)g(sU,V) + \{\,g(hX,U) - g(X,tU)\}g(hY,V)\,], \qquad (3.15)\\
&g((\nabla_X A)_Y U, Z) - g(T_U X, A_Y^* Z) - g(T_U Y, A_X^* Z) + g(A_X Y, T_U^* Z) \\
&= c\,[\,g(U,hY)g(fX,Z) - g(X,fY)g(tU,Z) + \{\,g(hX,U) - g(X,tU)\}g(fY,Z)\,], \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
&g([\mathcal{V}\nabla_X, \mathcal{V}\nabla_Y]U, V) - g(\nabla_{[X,Y]}U, V) + g(A_X U, A_Y^* V) - g(A_Y U, A_X^* V) \\
&= c\,[\,g(Y,tU)g(hX,V) - g(X,tU)g(hY,V) + \{\,g(fX,Y) - g(X,fY)\}g(sU,V)\,], \qquad (3.17)\\
&g((\nabla_X A)_Y U, Z) - g((\nabla_Y A)_X U, Z) + g(T_U^* Z, \theta_X Y) \\
&= c\,[\,g(Y,tU)g(fX,Z) - g(X,tU)g(fY,Z) + \{\,g(fX,Y) - g(X,fY)\}g(tU,Z)\,], \qquad (3.18)\\
&g((\nabla_X A)_Y Z, U) - g((\nabla_Y A)_X Z, U) - g(T_U Z, \theta_X Y) \qquad (3.19)\\
&= c\,[\,g(Y,fZ)g(hX,U) - g(X,fZ)g(hY,U) + \{\,g(fX,Y) - g(X,fY)\}g(hZ,U)\,], \qquad (3.20)\\
&g(\widehat{R}(X,Y)Z,Z') - g(A_Y Z, A_X^* Z') + g(A_X Z, A_Y^* Z') + g(\theta_X Y, A_Z^* Z') \\
&= c\,[\,g(Y,Z)g(X,Z') - g(X,Z)g(Y,Z') + g(Y,fZ)g(fX,Z') - g(X,fZ)g(fY,Z') \\
&\quad + \{\,g(fX,Y) - g(X,fY)\}g(fZ,Z')\,] \qquad (3.21)
\end{aligned}
$$

for $X, Y, Z, Z' \in \mathcal{H}(M)$ and $U, V, W, W' \in \mathcal{V}(M)$. In the case of the locally product-like statistical submersion with isometric fiber, that is, $T = 0$, we get from (3.10)

$$c\,[\,g(V,sW)g(tU,X) - g(U,sW)g(tV,X) + \{\,g(sU,V) - g(U,sV)\}g(tW,X)\,] = 0,$$

which implies that $c = 0$ or

$$t\,[\,g(V,sW)U - g(U,sW)V + \{\,g(sU,V) - g(U,sV)\}W\,] = 0.$$

We assume that $g(V,sW)U - g(U,sW)V + \{g(sU,V) - g(U,sV)\}W = 0$, which yields that $s = 0$ $(\dim \overline{M} > 2)$. Thus we get $t = 0$ if $s \neq 0$. Hence we have the following theorem.

Theorem 3.12 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion with isometric fiber. If the total space satisfies the condition (2.9), then*

(1) the total space is flat, or

(2) each fiber is F-invariant of M, or

(3) each fiber is antiinvariant of M.

Corollary 3.13 *Let $\pi : (M, \nabla^*, g, F^*) \to (B, \widehat{\nabla}^*, g_B)$ be a locally product-like statistical submersion with isometric fiber. If the total space satisfies the condition (2.10), then*

(1) the total space is flat, or

(2) each fiber is F^*-invariant of M, or

(3) each fiber is antiinvariant of M.

Example 3.14 *Let $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion of Example 3.8. Then π is with isometric fiber.*

4. F-invariant locally product-like statistical submersions

The locally product-like statistical submersion $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is called an F-invariant locally product-like statistical submersion if $\overline{M}$ is an F-invariant submanifold of M, that is, $F(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (see (1) in Lemma 3.1). Thus we find $f^2 = I$, $hf + sh = 0$ and $s^2 = I$. From Lemmas 3.5, 3.6 and Corollary 3.7, we find the following lemma.

Lemma 4.1 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an F-invariant locally product-like statistical submersion, then we get*

$$\begin{aligned}
T_U V &= f(T_U(sV)), \\
\overline{\nabla}_U V &= h(T_U(sV)) + s(\overline{\nabla}_U(sV)), \\
\mathcal{H}\nabla_U X &= f\{\mathcal{H}\nabla_U(fX) + T_U(hX)\}, \\
T_U X &= h\{\mathcal{H}\nabla_U(fX) + T_U(hX)\} + s\{\overline{\nabla}_U(hX) + T_U(fX)\}, \\
A_X U &= f(A_X(sU)), \\
\mathcal{V}\nabla_X U &= h(A_X(sU)) + s(\mathcal{V}\nabla_X(sU)), \\
\mathcal{H}\nabla_X Y &= f\{\mathcal{H}\nabla_X(fY) + A_X(hY)\}, \\
A_X Y &= h\{\mathcal{H}\nabla_X(fY) + A_X(hY)\} + s\{\mathcal{V}\nabla_X(hY) + A_X(fY)\}.
\end{aligned}$$

Lemma 4.2 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an F-invariant locally product-like statistical submersion, then we obtain*

$$\begin{aligned}
(\overline{\nabla}_U s)V &= h(T_U V),\\
(\overline{\nabla}_U h)X + T_U(fX) &= s(T_U X),\\
(\mathcal{H}\nabla_U f)X + T_U(hX) &= 0,\\
(\mathcal{V}\nabla_X s)U &= h(A_X U),\\
(\mathcal{V}\nabla_X h)Y + A_X(fY) &= s(A_X Y),\\
(\mathcal{H}\nabla_X f)Y + A_X(hY) &= 0.
\end{aligned}$$

Corollary 4.3 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an F-invariant locally product-like statistical submersion. We find*

(1) $\mathcal{H}\nabla f = 0$ *if and only if* $T_U(hX) = 0$ *and* $A_X(hY) = 0$.

(2) $\mathcal{V}\nabla h = 0$ *if and only if* $T_U X = s(T_U(fX))$ *and* $A_X Y = s(A_X(fY))$.

(3) $\mathcal{V}\nabla s = 0$ *if and only if* $h(T_U V) = 0$ *and* $h(A_X U) = 0$.

We assume that $\mathcal{H}\nabla f = 0$. We put $V = hZ$ in (3.13). Then we obtain from (2.11), Lemma 2.7 and (1) in Corollary 4.3

$$-g(T_U((\mathcal{V}\nabla_X h)Z), Y) + g(A_X((\overline{\nabla}_U h)Z), Y) = c\{g(U, hZ)g(X, Y) + g(U, shZ)g(fX, Y)\},$$

which implies that $c = 0$ or $g(X, Y)hZ + g(fX, Y)shZ = 0$ if $\mathcal{V}\nabla h = 0$. From the second equation, we find $h = 0$ if $\mathrm{tr}\, f \neq \pm n$. Hence we have the following theorem.

Theorem 4.4 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an F-invariant locally product-like statistical submersion which the total space satisfies the condition (2.9). If $\mathcal{H}\nabla f = 0$ and $\mathcal{V}\nabla h = 0$, then we get*

(1) *the total space is flat, or*

(2) $h = 0$ *if* $\mathrm{tr}\, f \neq \pm n$.

Corollary 4.5 *Let $\pi : (M, \nabla^*, g, F^*) \to (B, \widehat{\nabla}^*, g_B)$ be an F^*-invariant locally product-like statistical submersion which the total space satisfies the condition (2.10). If $\mathcal{H}\nabla^* f^* = 0$ and $\mathcal{V}\nabla^* h^* = 0$, then the total space is flat or $h^* = 0$ if $\mathrm{tr}\, f \neq \pm n$.*

Example 4.6 *Let $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion of Example 3.9. Then π is F-invariant.*

5. F^*-invariant locally product-like statistical submersions

Next, the locally product-like statistical submersion $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is called an F^*-invariant locally product-like statistical submersion if $\overline{M}$ is an F^*-invariant submanifold of M, that is, $F^*(\mathcal{V}_p(M)) \subset \mathcal{V}_p(M)$ (see (2) in Lemma 3.1). Thus we find $f^2 = I$, $ft + ts = 0$ and $s^2 = I$. From Lemmas 3.5, 3.6 and Corollary 3.7, we find the following lemma.

Lemma 5.1 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an F^*-invariant locally product-like statistical submersion, then we get*

$$
\begin{aligned}
T_U V &= f\{\mathcal{H}\nabla_U(tV) + T_U(sV)\} + t\{\overline{\nabla}_U(sV) + T_U(tV)\},\\
\overline{\nabla}_U V &= s\{\overline{\nabla}_U(sV) + T_U(tV)\},\\
\mathcal{H}\nabla_U X &= f(\mathcal{H}\nabla_U(fX)) + t(T_U(fX)),\\
T_U X &= s(T_U(fX)),\\
A_X U &= f\{\mathcal{H}\nabla_X(tU) + A_X(sU)\} + t\{\mathcal{V}\nabla_X(sU) + A_X(tU)\},\\
\mathcal{V}\nabla_X U &= s\{\mathcal{V}\nabla_X(sU) + A_X(tU)\},\\
\mathcal{H}\nabla_X Y &= f(\mathcal{H}\nabla_X(fY)) + t(A_X(fY)),\\
A_X Y &= s(A_X(fY)).
\end{aligned}
$$

Lemma 5.2 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an F^*-invariant locally product-like statistical submersion, then we obtain*

$$
\begin{aligned}
(\overline{\nabla}_U s)V + T_U(tV) &= 0,\\
(\mathcal{H}\nabla_U t)V + T_U(sV) &= f(T_U V),\\
(\mathcal{H}\nabla_U f)X &= t(T_U X),\\
(\mathcal{V}\nabla_X s)U + A_X(tU) &= 0,\\
(\mathcal{H}\nabla_X t)U + A_X(sU) &= f(A_X U),\\
(\mathcal{H}\nabla_X f)Y &= t(A_X Y).
\end{aligned}
$$

Corollary 5.3 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an F^*-invariant locally product-like statistical submersion. We find*

(1) $\mathcal{H}\nabla f = 0$ *if and only if* $t(T_U X) = 0$ *and* $t(A_X Y) = 0$.

(2) $\mathcal{H}\nabla t = 0$ *if and only if* $T_U V = f(T_U(sV))$ *and* $A_X U = f(A_X(sU))$.

(3) $\mathcal{V}\nabla s = 0$ *if and only if* $T_U(tV) = 0$ *and* $A_X(tU) = 0$.

We assume that $\mathcal{V}\nabla s = 0$. We put $Y = tW$ in (3.15). Then we obtain from (2.11), Lemma 2.7 and (3) in Corollary 5.3

$$
-g(T_U((\mathcal{H}\nabla_X t)W), V) + g(A_X((\mathcal{H}\nabla_U t)W), V) = c\,\{\, g(X, tW)g(U, V) - g(X, ftW)g(sU, V)\},
$$

which implies that $c = 0$ or $g(X, tW)g(U, V) - g(X, ftW)g(sU, V) = 0$ if $\mathcal{H}\nabla t = 0$. From the second equation, we find $t = 0$. Hence we have the following theorem.

Theorem 5.4 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an F^*-invariant locally product-like statistical submersion which the total space satisfies the condition (2.9). If $\mathcal{H}\nabla t = 0$ and $\mathcal{V}\nabla s = 0$, then we get*

(1) the total space is flat, or

(2) $t = 0$.

Corollary 5.5 *Let $\pi : (M, \nabla^*, g, F^*) \to (B, \widehat{\nabla}^*, g_B)$ be an F-invariant locally product-like statistical submersion which the total space satisfies the condition (2.10). If $\mathcal{H}\nabla^* t^* = 0$ and $\mathcal{V}\nabla^* s^* = 0$, then the total space is flat or $t^* = 0$*

Example 5.6 *Let $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion of Example 3.9. Then π is F^*-invariant.*

6. Antiinvariant locally product-like statistical submersions

The locally product-like statistical submersion $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is called an antiinvariant locally product-like statistical submersion if $\overline{M}$ is an antiinvariant submanifold of M, that is, $F(\mathcal{V}_p(M)) \subset \mathcal{H}_p(M)$ (see (3) in Lemma 3.1). Thus we find $f^2 = I - th$, $hf = 0$, $ft = 0$ and $ht = I$. From Lemmas 3.5, 3.6 and Corollary 3.7, we find the following lemma.

Lemma 6.1 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an antiinvariant locally product-like statistical submersion, then we get*

$$\begin{aligned}
T_U V &= f(\mathcal{H}\nabla_U(tV)) + t(T_U(tV)),\\
\overline{\nabla}_U V &= h(\mathcal{H}\nabla_U(tV)),\\
\mathcal{H}\nabla_U X &= f\{\mathcal{H}\nabla_U(fX) + T_U(hX)\} + t\{\overline{\nabla}_U(hX) + T_U(fX)\},\\
T_U X &= h\{\mathcal{H}\nabla_U(fX) + T_U(hX)\},\\
A_X U &= f(\mathcal{H}\nabla_X(tU)) + t(A_X(tU)),\\
\mathcal{V}\nabla_X U &= h(\mathcal{H}\nabla_X(tU)),\\
\mathcal{H}\nabla_X Y &= f\{\mathcal{H}\nabla_X(fY) + A_X(hY)\} + t\{\mathcal{V}\nabla_X(hY) + A_X(fY)\},\\
A_X Y &= h\{\mathcal{H}\nabla_X(fY) + A_X(hY)\}.
\end{aligned}$$

Lemma 6.2 *If $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ is an antiinvariant locally product-like statistical submersion,* *then we obtain*

$$\begin{aligned} T_U V &= t(T_U(tV)), \\ (\mathcal{H}\nabla_U t)V &= f(T_U V), \\ (\overline{\nabla}_U h)X + T_U(fX) &= 0, \\ (\mathcal{H}\nabla_U f)X + T_U(hX) &= t(T_U X), \\ A_X U &= t(A_X(tU)), \\ (\mathcal{H}\nabla_X t)U &= f(A_X U), \\ (\mathcal{V}\nabla_X h)Y + A_X(fY) &= 0, \\ (\mathcal{H}\nabla_X f)Y + A_X(hY) &= t(A_X Y). \end{aligned}$$

Corollary 6.3 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an antiinvariant locally product-like statistical submersion. We find*

(1) *$\mathcal{H}\nabla f = 0$ if and only if $T_U X = h(T_U(hX))$ and $A_X Y = h(A_X(hY))$.*

(2) *$\mathcal{V}\nabla h = 0$ if and only if $T_U(fX) = 0$ and $A_X(fY) = 0$.*

(3) *$\mathcal{H}\nabla t = 0$ if and only if $f(T_U V) = 0$ and $f(A_X U) = 0$.*

From Lemma 6.2 and Corollary 6.3, we have the following corollary.

Corollary 6.4 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an antiinvariant locally product-like statistical submersion. If $\mathcal{H}\nabla f = 0$, then we get*

$$f(T_U V) = 0, \qquad f(A_X U) = 0, \tag{6.1}$$

$$T_U(fX) = 0, \qquad A_X(fY) = 0. \tag{6.2}$$

We assume that $\mathcal{H}\nabla f = 0$. If we put $Z = fZ$ in (3.18), then we obtain from (6.2)

$$c\,\{\,g(Y, f^2 Z)g(hX, U) - g(X, f^2 Z)g(hY, U)\} = 0,$$

which implies that $c = 0$ or $h\,[\,g(Y, f^2 Z)X - g(X, f^2 Z)Y\,] = 0$. Because of $h \neq 0$, we find

$$g(Y, f^2 Z)X - g(X, f^2 Z)Y = 0.$$

Thus we obtain $f^2 = 0$ $(n \geq 2)$, which yields from (3.3) that $f = 0$. Hence we have the following theorem.

Theorem 6.5 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be an antiinvariant locally product-like statistical submersion. If the total space satisfies the condition (2.9) and $\mathcal{H}\nabla f = 0$, then we get*

(1) *the total space is flat, or*

(2) *$f = 0$ if $n \geq 2$.*

Corollary 6.6 *Let* $\pi : (M, \nabla^*, g, F^*) \to (B, \widehat{\nabla}^*, g_B)$ *be an antiinvariant locally product-like statistical submersion. If the total space satisfies the condition* (2.10) *and* $\mathcal{H}\nabla^* f^* = 0$, *then the total space is flat or* $f^* = 0$ *if* $n \geq 2$.

Example 6.7 *Let* $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ *be a locally product-like statistical submersion of Examples 3.8, 3.10 and 3.11. Then* π *is antiinvariant.*

7. Locally product-like statistical submersions satisfying $f = 0$

Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion satisfying $f = 0$, that is, $F(\mathcal{H}_p(M)) \subset \mathcal{V}_p(M)$ (see (4) in Lemma 3.1). Thus we find $th = I$, $sh = 0$, $ts = 0$ and $s^2 = I - ht$. From Lemmas 3.5, 3.6 and Corollary 3.7, we find the following lemma.

Lemma 7.1 *If* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion satisfying* $f = 0$, *then we get*

$$\begin{aligned}
T_U V &= t\{\overline{\nabla}_U(sV) + T_U(tV)\}, \\
\overline{\nabla}_U V &= h\{\mathcal{H}\nabla_U(tV) + T_U(sV)\} + s\{\overline{\nabla}_U(sV) + T_U(tV)\}, \\
\mathcal{H}\nabla_U X &= t(\overline{\nabla}_U(hX)), \\
T_U X &= h(T_U(hX)) + s(\overline{\nabla}_U(hX)), \\
A_X U &= t\{\mathcal{V}\nabla_X(sU) + A_X(tU)\}, \\
\mathcal{V}\nabla_X U &= h\{\mathcal{H}\nabla_X(tU) + A_X(sU)\} + s\{\mathcal{V}\nabla_X(sU) + A_X(tU)\}, \\
\mathcal{H}\nabla_X Y &= t(\mathcal{V}\nabla_X(hY)), \\
A_X Y &= h(A_X(hY)) + s(\mathcal{V}\nabla_X(hY)).
\end{aligned}$$

Lemma 7.2 *If* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *is a locally product-like statistical submersion satisfying* $f = 0$, *then we obtain*

$$\begin{aligned}
(\overline{\nabla}_U s)V + T_U(tV) &= h(T_U V), \\
(\mathcal{H}\nabla_U t)V + T_U(sV) &= 0, \\
(\overline{\nabla}_U h)X &= s(T_U X), \\
T_U(hX) &= t(T_U X), \\
(\mathcal{V}\nabla_X s)U + A_X(tU) &= h(A_X U), \\
(\mathcal{H}\nabla_X t)U + A_X(sU) &= 0, \\
(\mathcal{V}\nabla_X h)Y &= s(A_X Y), \\
A_X(hY) &= t(A_X Y).
\end{aligned}$$

Corollary 7.3 *Let* $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ *be a locally product-like statistical submersion satisfying* $f = 0$. *We find*

(1) *$\mathcal{V}\nabla h = 0$ if and only if $s(T_U X) = 0$ and $s(A_X Y) = 0$.*

(2) *$\mathcal{H}\nabla t = 0$ if and only if $T_U(sV) = 0$ and $A_X(sU) = 0$.*

(3) *$\mathcal{V}\nabla s = 0$ if and only if $T_U X = h(T_U(hX))$ and $A_X Y = h(A_X(hY))$.*

We assume that $\mathcal{V}\nabla s = 0$. It is easy to see from (3) in Corollary 7.3 that $T_U(sV) = 0$ and $A_X(sU) = 0$. If we put $V = sV$ in (3.13), then we obtain $c\,g(U, sV)g(X, Y) = 0$ from (2.11) and Lemma 2.7, which implies that $c = 0$ or $s = 0$. Hence we have the following theorem.

Theorem 7.4 *Let $\pi : (M, \nabla, g, F) \to (B, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion satisfying $f = 0$. If the total space satisfies the condition (2.9) and $\mathcal{V}\nabla s = 0$, then we get*

(1) *the total space is flat, or*

(2) *$s = 0$.*

Corollary 7.5 *Let $\pi : (M, \nabla^*, g, F^*) \to (B, \widehat{\nabla}^*, g_B)$ be a locally product-like statistical submersion satisfying $f^* = 0$. If the total space satisfies the condition (2.10) and $\mathcal{V}\nabla^* s^* = 0$, then the total space is flat or $s^* = 0$.*

Example 7.6 *Let $\pi : (M_\pm, \nabla, g, F) \to (B_\pm, \widehat{\nabla}, g_B)$ be a locally product-like statistical submersion of Example 3.11. Then π satisfies $f = 0$.*

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