Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2023) 47: 439 – 475
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doi:10.55730/1300-0098.3372

# Twisted Dirac operators and the Kastler-Kalau-Walze type theorem for five dimensional manifolds with boundary

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**Received:** 20.04.2021 • **Accepted/Published Online:** 23.12.2022 • **Final Version:** 09.03.2023

**Abstract:** In this paper, we prove the Kastler-Kalau-Walze type theorems for twisted Dirac operators on 5-dimensional manifolds with boundary.

**Key words:** Twisted Dirac operators, noncommutative residue Kastler-Kalau-Walze type theorems

## 1. Introduction

The noncommutative residue found in [4, 14] plays a prominent role in noncommutative geometry. For arbitrary closed compact $n-$dimensional manifolds, the noncommutative residue was introduced by Wodzicki in [14] using the theory of zeta functions of elliptic pseudodifferential operators. In [1], Connes used the noncommutative residue to derive a conformal $4-$dimensional Polyakov action analogue. Furthermore, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action in [2]. In [6], Kastler gave a brute-force proof of this theorem. In [5], Kaulau and Walze proved this theorem in the normal coordinates system simultaneously, which is called the Kastler-Kalau-Walze theorem now.

An important application of Riemannian geometry is to allow us to define the volume element of a Riemannian manifold $(M_n, g)$. The noncommutative residue of Wodzicki [14] and Guillemin [4] is a trace on the algebra of (integer order) $\Psi DOs$ on $M$. An important feature is that it allows us to extend to all $\Psi DOs$ by the Dixmier trace, which plays the role of the integral in the framework of noncommutative geometry. Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace in [3]. In [7], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For a spin manifold $M$ with boundary $\partial M$, by the composition formula in Boutet de Monvel's algebra and the definition of $\widetilde{\mathrm{Wres}}$ [11], $\widetilde{\mathrm{Wres}}[(\pi^+D^{-1})^2]$ should be the sum of two terms from interior and boundary of $M$, where $\pi^+D^{-1}$ is an element in Boutet de Mouvel's algebra [12].

In [8], Wang and Wang computed the lower volume $Vol_5^{(1,1)}$ and proved a Kastler-Kalau-Walze type theorem for the Dirac operator on 5-dimensional manifolds with boundary. In [9], Wang and Wang gave two lichnerowicz type formulas and the proof of the Kastler-Kalau-Walze type theorem for twisted Dirac operators

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2010 *AMS Mathematics Subject Classification:* 53C40; 53C42

and twisted signature operators on four dimensional manifolds with boundary. In the present paper, we shall restrict our attention to the cases $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})\circ\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})]$ and $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})\circ\pi^+((\widetilde{\mathrm{D}}_{\mathrm{H}}^{*})^{-1})]$ for 5-dimensional manifolds with boundary, where twisted Dirac operators $\widetilde{D}_H, \widetilde{D}_H^*$ are defined in (2.9), (2.10).

The paper is organized in the following way. In Section 2, we recall the definition of twisted Dirac operators and define lower dimensional volumes of compact Riemannian manifolds with boundary. In Section 3, we compute $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})\circ\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})]$, $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{\mathrm{D}}_{\mathrm{H}}^{-1})\circ\pi^+((\widetilde{\mathrm{D}}_{\mathrm{H}}^{*})^{-1}))]$ and get the Kastler-Kalau-Walze type theorem for twisted Dirac operators on 5-dimensional spin manifolds with boundary.

## 2. Twisted Dirac operators and lower-dimensional volumes of spin manifolds with boundary

Firstly, we give the expression of the Dirac operator. Let $\nabla^L$ denote the Levi-civita connection about $g^M$. In the local coordinates $x_i$; $1\leq i\leq n$ and the fixed orthonormal frame $\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1},\cdots,\widetilde{e_n})^t=(\omega_{s,t})(\widetilde{e_1},\cdots,\widetilde{e_n})^t. \tag{2.1}$$

The classical Dirac operator is defined on $S(TM)$ by

$$D=\sum_{j=1}^n c(\widetilde{e_j})[\widetilde{e_j}+\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_j})c(\widetilde{e_s})c(\widetilde{e_t})]. \tag{2.2}$$

Next, we consider an $n$-dimensional oriented Riemannian manifold $(M,g^M)$ equipped with a fixed spin structure. We recall twisted Dirac operators. Let $S(TM)$ be the spinors bundle and $H$ be an additional smooth vector bundle equipped with a nonunitary connection $\widetilde{\nabla}^H$. Let $\widetilde{\nabla}^{H,*}$ be the dual connection on $H$, and define

$$\nabla^H=\frac{\widetilde{\nabla}^H+\widetilde{\nabla}^{H,*}}{2},\quad \Phi=\frac{\widetilde{\nabla}^H-\widetilde{\nabla}^{H,*}}{2}; \tag{2.3}$$

then $\nabla^H$ is a metric connection and $\Phi$ is an endomorphism of $H$ with a 1-form coefficient. We consider the tensor product vector bundle $S(TM)\bigotimes H$, which becomes a Clifford module via the definition;

$$c(a)=c(a)\bigotimes id_H, a\in TM, \tag{2.4}$$

and which we equip with the compund connection:

$$\widetilde{\nabla}^{S(TM)\bigotimes H}=\nabla^{S(TM)}\bigotimes id_H+id_{S(TM)}\bigotimes\widetilde{\nabla}^H, \tag{2.5}$$

$$\nabla^{S(TM)\bigotimes H}=\nabla^{S(TM)}\bigotimes id_H+id_{S(TM)}\bigotimes\nabla^H, \tag{2.6}$$

then the spinor connection $\widetilde{\nabla}$ induced by $\widetilde{\nabla}^{S(TM)\bigotimes H}$ is locally given by

$$\widetilde{\nabla}^{S(TM)\bigotimes H}=\nabla^{S(TM)}\bigotimes id_H+id_{S(TM)}\bigotimes\nabla^H+id_{S(TM)}\bigotimes\Phi. \tag{2.7}$$

Let

$$D_H=\sum_{i=1}^n c(\widetilde{e_i})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_i}}. \tag{2.8}$$

Then we can obtain twisted Dirac operators $\widetilde{D}_H$ and $\widetilde{D}^*_H$ associated to the connection $\widetilde{\widetilde{\nabla}}$ as follows:

$$\widetilde{D}_H = D_H + \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \Phi(\widetilde{e_j})$$
$$= \sum_{j=1}^{n} c(\widetilde{e_j})\widetilde{e_j} + \frac{1}{4}\sum_{j=1}^{n} \omega_{st}(\widetilde{e_j})c(\widetilde{e_j})c(\widetilde{e_s})c(\widetilde{e_t}) + \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \sigma^H(\widetilde{e_j}) + \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \Phi(\widetilde{e_j}). \tag{2.9}$$

$$\widetilde{D}^*_H = D_H - \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \Phi^*(\widetilde{e_j})$$
$$= \sum_{j=1}^{n} c(\widetilde{e_j})\widetilde{e_j} + \frac{1}{4}\sum_{j=1}^{n} \omega_{st}(\widetilde{e_j})c(\widetilde{e_j})c(\widetilde{e_s})c(\widetilde{e_t}) + \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \sigma^H(\widetilde{e_j}) - \sum_{j=1}^{n} c(\widetilde{e_j}) \bigotimes \Phi^*(\widetilde{e_j}). \tag{2.10}$$

where $\sigma^H$ is the connection matrix of $\nabla^H$ and $\Phi^*$ is the transport matrix of $\Phi$.

**Lemma 2.1** *Let $\{\widetilde{e_i}\}(1 \leq i, j \leq n)(\{\partial_i\})$ be the orthonormal frames (natural frames respectively) on TM,*

$$D_H = \sum_{i,j} g^{ij} c(\partial_i)\nabla^{S(TM)\bigotimes H}_{\partial_i} = \sum_{j=1}^{n} c(\widetilde{e_j})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_j}}, \tag{2.11}$$

*where* $\nabla^{S(TM)\bigotimes H}_{\partial_i} = \partial_j + \sigma^s_j + \sigma^H_j$ *and* $\sigma^s_j = \frac{1}{4}\Sigma_{j,k} < \nabla^L_{\partial_i}\widetilde{e_j}, \widetilde{e_k} > c(\widetilde{e_j})c(\widetilde{e_k})$, $\sigma^H_j$ *is the connection matrix of* $\nabla^H$.

**Theorem 2.2** [3]**(Fedosov-Golse-Leichtnam-Schrohe)** *Let $M$ and $\partial M$ be connected,* $\dim M = n \geq 3$, $A = \begin{pmatrix} \pi^+P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$ *, and denote by $p$, $b$, and $s$ the local symbols of $P, G$, and $S$ respectively. Define:*

$$\widetilde{\text{Wres}}(A) = \int_X \int_{\mathbf{S}} \text{tr}_E \left[p_{-n}(x,\xi)\right] \sigma(\xi) dx$$
$$+ 2\pi \int_{\partial X} \int_{\mathbf{S}'} \left\{\text{tr}_E \left[(\text{tr} b_{-n})(x',\xi')\right] + \text{tr}_F \left[s_{1-n}(x',\xi')\right]\right\} \sigma(\xi') dx', \tag{2.12}$$

*where $\widetilde{\text{Wres}}$ denotes the noncommutative residue of an operator in the Boutet de Monvel's algebra.*
*Then a) $\widetilde{\text{Wres}}([A,B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.*

**Definition 2.3** [12] *Lower dimensional volumes of spin manifolds with boundary are defined by*

$$\text{Vol}^{(p_1,p_2)}_n M := \widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]. \tag{2.13}$$

By [12], we get

$$\widetilde{\text{Wres}}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\wedge^* T^* M \bigotimes \mathbb{C}}[\sigma_{-n}(D^{-p_1-p_2})]\sigma(\xi) dx + \int_{\partial M} \Psi, \tag{2.14}$$

and

$$\Psi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\times \mathrm{trace}_{\wedge^*T^*M\bigotimes\mathbb{C}}[\partial_{x_n}^{j}\partial_{\xi'}^{\alpha}\partial_{\xi_n}^{k}\sigma_r^{+}(D^{-p_1})(x',0,\xi',\xi_n) \times\partial_{x'}^{\alpha}\partial_{\xi_n}^{j+1}\partial_{x_n}^{k}\sigma_l(D^{-p_2})(x',0,\xi',\xi_n)]d\xi_n\sigma(\xi')dx', \tag{2.15}$$

where the sum is taken over $r+l-k+|\alpha|-j-1=-n,\quad r\leq -p_1, l\leq -p_2$.

## 3. A Kastler-Kalau-Walze type theorem for 5-dimensional spin manifolds with boundary

In this section, we compute the lower dimensional volume for $\widetilde{D}_H$ and $\widetilde{D}_H^*$ on five dimensional compact spin manifolds with boundary and get a Kastler-Kalau-Walze type theorem in this case.

**Proposition 3.1** *[13] The following identity holds:*

$$1) When\ \ p_1+p_2=n,\ Vol_n^{(p_1,p_2)}M=c_0VolM; \tag{3.1}$$

$$2) When\ \ p_1+p_2\equiv\ n\ mod\ 1,\ Vol_n^{(p_1,p_2)}M=\int_{\partial M}\Psi. \tag{3.2}$$

Next, we compute $Vol_5^{(1,1)}$ for $\widetilde{D}_H$ on 5-dimensional spin manifolds with boundary, then we have

$$\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+(\widetilde{D}_H^{-1})]=\int_{\partial M}\Psi. \tag{3.3}$$

Write

$$D_x^{\alpha} = (-i)^{|\alpha|}\partial_x^{\alpha};\ \sigma(\widetilde{D}_H)=p_1+p_0;\ \sigma(\widetilde{D}_H^{-1})=\sum_{j=1}^{\infty}q_{-j}. \tag{3.4}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned} 1=\sigma(\widetilde{D}_H\circ\widetilde{D}_H^{-1}) &= \sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}[\sigma(\widetilde{D}_H)]D_x^{\alpha}[\sigma(\widetilde{D}_H^{-1})] \\ &= (p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots) \\ &\quad +\sum_j(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0)(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots) \\ &= p_1q_{-1}+(p_1q_{-2}+p_0q_{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}q_{-1})+\cdots, \end{aligned} \tag{3.5}$$

so

$$q_{-1}=p_1^{-1}; \tag{3.6}$$

$$q_{-2}=-p_1^{-1}[p_0p_1^{-1}+\sum_j\partial_{\xi_j}p_1D_{x_j}(p_1^{-1})]; \tag{3.7}$$

$$q_{-3}=-p_1^{-1}[p_0q_{-2}+\sum_{j=1}^{n-1}c(dx_j)\partial_{\xi_j}q_{-2}+c(dx_n)\partial_{x_n}q_{-2}]. \tag{3.8}$$

Since $\Psi$ is a global form on $\partial M$, for any fixed point $x_0\in\partial M$, we choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Psi(x_0)$ in the coordinates $\widetilde{U}=U\times[0,1)\subset M$ and the metric $\frac{1}{h(x_n)}g^{\partial M}+dx_n^2$. The dual metric of $g^M$ on $\widetilde{U}$ is $h(x_n)g^{\partial M}+dx_n^2$. Write $g^M_{ij}=g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j})$; $g^{ij}_M=g^M(dx_i,dx_j)$, then

$$[g^M_{ij}]=\begin{bmatrix}\frac{1}{h(x_n)}[g^{\partial M}_{ij}] & 0\\ 0 & 1\end{bmatrix};\quad [g^{ij}_M]=\begin{bmatrix}h(x_n)[g^{ij}_{\partial M}] & 0\\ 0 & 1\end{bmatrix}, \tag{3.9}$$

and

$$\partial_{x_s}g^{\partial M}_{ij}(x_0)=0, 1\le i,j\le n-1;\quad g^M_{ij}(x_0)=\delta_{ij}. \tag{3.10}$$

Let $\{e_1,\cdots,e_n\}$ be an orthonormal frame field in $U$ about $g^{\partial M}$ which is parallel along geodesics and $e_i=\frac{\partial}{\partial x_i}(x_0)$, then $\{\widetilde{e_1}=\sqrt{h(x_n)}e_1,\cdots,\widetilde{e_{n-1}}=\sqrt{h(x_n)}e_{n-1},\widetilde{e_n}=\frac{\partial}{\partial x_n}\}$ is the orthonormal frame field in $\widetilde{U}$ about $g^M$. Locally $S(TM)|\widetilde{U}\cong\widetilde{U}\times\wedge^*_C(\frac{n}{2})$. Let $\{f_1,\cdots,f_n\}$ be the orthonormal basis of $\wedge^*_C(\frac{n}{2})$. Take a spin frame field $\sigma:\widetilde{U}\to Spin(M)$ such that $\pi\sigma=\{\widetilde{e_1},\cdots,\widetilde{e_n}\}$ where $\pi:Spin(M)\to O(M)$ is a double covering, then $\{[\sigma,f_i],1\le i\le 4\}$ is an orthonormal frame of $S(TM)|_{\widetilde{U}}$. In the following, since the global form $\Psi$ is independent of the choice of the local frame, so we can compute $tr_{S(TM)}$ in the frame $\{[\sigma,f_i],1\le i\le 4\}$. Let $\{\widehat{e_1},\cdots,\widehat{e_n}\}$ be the canonical basis of $R^n$ and $c(\widehat{e_i})\in Hom(\wedge^*_C(\frac{n-1}{2}),\wedge^*_C(\frac{n-1}{2}))$ be the Clifford action. By [15], then

$$c(\widetilde{e_i})=[\sigma,c(\widehat{e_i})];\quad c(\widetilde{e_i})[\sigma,f_i]=[\sigma,c(\widehat{e_i})f_i];\quad \frac{\partial}{\partial x_i}=[\sigma,\frac{\partial}{\partial x_i}]. \tag{3.11}$$

then we have $\frac{\partial}{\partial x_i}c(\widetilde{e_i})=0$ in the above frame. By Lemma 2.2 in [12], we have:

**Lemma 3.2** [12] *With the metric $g^M$ on $M$ near the boundary*

$$\partial_{x_j}(|\xi|^2_{g^M})(x_0) = \begin{cases}0, & \text{if } j<n,\\ h'(0)|\xi'|^2_{g^{\partial M}}, & \text{if } j=n;\end{cases} \tag{3.12}$$

$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases}0, & \text{if } j<n,\\ \partial x_n(c(\xi'))(x_0), & \text{if } j=n,\end{cases} \tag{3.13}$$

*where $\xi=\xi'+\xi_n dx_n$.*

By Lemma 2.1 in [12], we have:

**Lemma 3.3** *The symbols of twisted Dirac operators:*

$$\sigma_{-1}(\widetilde{D}_H^{-1}) = \sigma_{-1}((\widetilde{D}_H^{*})^{-1}) = \frac{ic(\xi)}{|\xi|^2}; \tag{3.14}$$

$$\sigma_{-2}(\widetilde{D}_H^{-1}) = \frac{c(\xi)\sigma_0(\widetilde{D}_H)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)\Big]; \tag{3.15}$$

$$\sigma_{-2}((\widetilde{D}_H^{*})^{-1}) = \frac{c(\xi)\sigma_0(\widetilde{D}_H^{*})c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)\Big[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)\Big]; \tag{3.16}$$

$$\sigma_{-3}(\widetilde{D}_H^{-1}) = -\frac{1}{p_1}[p_0 q_{-2} + \sum_{j=1}^{n-1} c(dx_j)\partial_{x_j} q_{-2} + c(dx_n)\partial_{x_n} q_{-2}]; \tag{3.17}$$

$$\sigma_{-3}((\widetilde{D}_H^{*})^{-1}) = -\frac{1}{p_1}[p_0^* q_{-2}^* + \sum_{j=1}^{n-1} c(dx_j)\partial_{x_j} q_{-2}^* + c(dx_n)\partial_{x_n} q_{-2}^*]; \tag{3.18}$$

(3.19)

*where*

$$\sigma_0(\widetilde{D}_H)(x_0) = -h'(0)c(dx_n) + \sum_{j=1}^{n} c(\widetilde{e_j})(\sigma_j^H + \Phi(\widetilde{e_j}))(x_0); \tag{3.20}$$

$$\sigma_0(\widetilde{D}_H^{*})(x_0) = -h'(0)c(dx_n) + \sum_{j=1}^{n} c(\widetilde{e_j})(\sigma_j^H - \Phi^*(\widetilde{e_j}))(x_0). \tag{3.21}$$

Let us consider $\sigma_{-3}(\widetilde{D}_H^{-1})$ of twisted Dirac operators. From Lemma 3.7 in [8], we get:

**Lemma 3.4** *[8] For the Dirac operators, the following identity holds:*

$$\begin{aligned}
\sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} &= \frac{-ih'^2}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
&+ \frac{ih'}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)c(dx_n)\partial_{x_n}[c(\xi)](x_0) \\
&+ \frac{-ih'^2}{(1+\xi_n^2)^4}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
&+ \frac{1}{8}c(\xi)\sum_{\beta is\alpha} R^{\partial M}_{\beta is\alpha}(x_0)\frac{-i}{(1+\xi_n^2)^3}c(\xi)c(\widetilde{e_i})c(\xi)c(\widetilde{e_\beta})c(\widetilde{e_s})c(\widetilde{e_\alpha})c(\xi) \\
&+ \frac{1}{6}\sum_{l,t<n}\Big(R^{\partial M}_{tilj}(x_0)+R^{\partial M}_{tjli}(x_0)\Big)\frac{-i}{(1+\xi_n^2)^3}c(\xi)c(\widetilde{e_i})c(\xi)c(dx_j)c(\widetilde{e_t}) \\
&+ \frac{1}{3}\sum_{\alpha,\beta<n}\Big(R^{\partial M}_{i\alpha j\beta}(x_0)+R^{\partial M}_{i\beta j\alpha}(x_0)\Big)\xi_\alpha\xi_\beta\frac{i}{(1+\xi_n^2)^4}c(\xi)c(\widetilde{e_i})c(\xi)c(dx_j)c(\xi) \\
&+ i\bigg(\frac{h'}{(1+\xi_n^2)^3}+\frac{h'}{(1+\xi_n^2)^4}\bigg)c(\xi)c(dx_n)\partial_{x_n}[c(\xi)](x_0)c(dx_n)c(\xi) \\
&- i\bigg(\frac{h'^2-h''}{(1+\xi_n^2)^3}+\frac{2h'^2-h''}{(1+\xi_n^2)^4}+\frac{3h'^2}{(1+\xi_n^2)^5}\bigg)c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
&+ i\bigg(\frac{h'}{(1+\xi_n^2)^3}+\frac{3h'}{(1+\xi_n^2)^4}\bigg)c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi) \\
&+ \frac{-i}{(1+\xi_n^2)^3}c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]c(dx_n)\partial_{x_n}[c(\xi)](x_0) \\
&+ \bigg(\frac{3h'^2}{4}-\frac{h''}{2}\bigg)\frac{-i}{(1+\xi_n^2)^4}c(\xi)c(dx_n)c(\xi)c(dx_n)c(\xi'). \qquad (3.22)
\end{aligned}$$

**Lemma 3.5** *For twisted Dirac operators, the following identity holds:*

$$\begin{aligned}
\sigma_{-3}(\widetilde{D}_H^{-1})(x_0)|_{|\xi'|=1} &= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + \frac{ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^6} \\
&+ \frac{-ic(\xi)\gamma c(\xi)\gamma c(\xi)}{|\xi|^6} + \frac{-ih'c(\xi)\gamma c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^8} \\
&+ \frac{-ic(\xi)}{|\xi|^6}\sum_{j=1}^{n-1}c(\widetilde{e_j})c(\xi)\partial_{x_i}(\gamma)c(\xi) + \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\gamma c(\xi)}{|\xi|^6} \\
&+ \frac{2ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^6} \\
&+ \frac{-ic(\xi)c(dx_n)c(\xi)\gamma\partial_{x_n}[c(\xi)]}{|\xi|^6} \\
&:= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + Q_{-3}(x_0)|_{|\xi'|=1}, \qquad (3.23)
\end{aligned}$$

$$\sigma_{-3}((\widetilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1} = \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + \frac{ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^6}$$
$$+ \frac{-ic(\xi)\tau c(\xi)\tau c(\xi)}{|\xi|^6} + \frac{-ih'c(\xi)\tau c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6} + \frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^8}$$
$$+ \frac{-ic(\xi)}{|\xi|^6}\sum_{j=1}^{n-1} c(\widetilde{e_j})c(\xi)\partial_{x_i}(\tau)c(\xi) + \frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\tau c(\xi)}{|\xi|^6}$$
$$+ \frac{2ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^8} + \frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^6}$$
$$+ \frac{-ic(\xi)c(dx_n)c(\xi)\tau\partial_{x_n}[c(\xi)]}{|\xi|^6}$$
$$:= \sigma_{-3}(D^{-1})(x_0)|_{|\xi'|=1} + R_{-3}(x_0)|_{|\xi'|=1}, \tag{3.24}$$

*where* $\gamma = \sum_{j=1}^n c(\widetilde{e_j})(\sigma_j^H + \Phi(\widetilde{e_j}))$, *and* $\tau = \sum_{j=1}^n c(\widetilde{e_j})(\sigma_j^H - \Phi^*(\widetilde{e_j}))$.

When $n=5$, then $\mathrm{tr}_{S(TM)\bigotimes H}[\mathtt{id}] = 4dimH$, where tr as shorthand of trace, the sum is taken over $-r-l+k+j+|\alpha|=4,\ \ r\leq -1, l\leq -1$, then we have the following fifteen cases:
**case (1)** $r=-1,\ l=-1,\ k=0,\ j=|\alpha|=1$.

By (2.15), we get

$$\Psi_1 = \frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{x_n}\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.25}$$

By Lemma 3.3, for $i<n$, then

$$\partial_{x_i}\left(\sigma_{-1}(\widetilde{D}_H^{-1})(x_0)\right)(x_0) = \frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2} - \frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0, \tag{3.26}$$

so $\Psi_1 = 0$.

**case (2)** $r=-1,\ l=-1,\ k=|\alpha|=0,\ j=2$.

By (2.15), we get

$$\Psi_2 = \frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j=2}\mathrm{tr}[\partial_{x_n}^2\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^3\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.27}$$

By Lemma 3.2, we have

$$\partial_{\xi_n}^3\sigma_{-1}((\widetilde{D}_H)^{-1})(x_0)\mid_{|\xi'|=1} = ic(\xi')\frac{24\xi_n - 24\xi_n^3}{(1+\xi_n^2)^4} + ic(dx_n)\frac{-64\xi_n^4 + 36\xi_n^2 - 6}{(1+\xi_n^2)^4}; \tag{3.28}$$

From the Lemma 3.2-3.4, we get

$$\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}^{-1})|_{|\xi'|=1} = \left(\frac{3}{4}(h'(0))^2 - \frac{1}{2}h''(0)\right)\frac{c(\xi')}{2(\xi_n - i)} - h'(0)\frac{\xi_n - 2i}{2(\xi_n - i)}\partial_{x_n}[c(\xi')]$$
$$- h''(0)\left[\frac{\xi_n - 2i}{4(\xi_n - i)}c(\xi') + \frac{1}{4(\xi_n - i)^2}c(d_{x_n})\right]$$
$$+ 2i(h'(0))^2\left[\frac{-3i\xi_n^2 - 9\xi_n + 8i}{16(\xi_n - i)^3}c(\xi') + \frac{-i\xi_n - 3}{16(\xi_n - i)^3}c(d_{x_n})\right]. \tag{3.29}$$

By the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}BA$, we have the equalities:

$$\mathrm{tr}[c(\xi')c(dx_n)] = 0; \quad \mathrm{tr}[c(dx_n)^2] = -4dimH; \quad \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1} = -4dimH;$$
$$\mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)] = 0; \quad \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1} = -2h'(0)dimH. \tag{3.30}$$

Similarly, we have

$$\mathrm{tr}\left[\partial_{x_n}^2 \pi_{\xi_n}^+ \sigma_{-1}(\widetilde{D}_H^{-1})\partial_{\xi_n}^3 \sigma_{-1}(\widetilde{D}_H^{-1})\right](x_0)|_{|\xi'|=1}$$
$$= h'(0)^2\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1+\xi_n^2)^4}$$
$$+ h''(0)\frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1+\xi_n^2)^4}. \tag{3.31}$$

Then,

$$\Psi_2 = -\frac{1}{6}(h'(0))^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'$$
$$- \frac{1}{6}h''(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'$$
$$= -\frac{1}{6}(h'(0))^2\Omega_3\int_{\Gamma^+}\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{2(\xi_n - i)^3(1+\xi_n^2)^4}d\xi_n dx'$$
$$- \frac{1}{6}h''(0)\Omega_3\int_{\Gamma^+}\frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1+\xi_n^2)^4}d\xi_n dx'$$
$$= -\frac{1}{6}(h'(0))^2\frac{2\pi i}{6!}\left[\frac{3(33\xi_n^5 - 75i\xi_n^4 - 94\xi_n^3 + 90i\xi_n^2 + 57\xi_n - 3i)}{(1+\xi_n^2)^4}\right]^{(6)}\Omega_3 dx'$$
$$- \frac{1}{6}h''(0)\frac{2\pi i}{5!}\frac{6(-9\xi_n^4 + 12i\xi_n^3 + 14\xi_n^2 - 12i\xi_n - 1)}{2(\xi_n - i)^2(1+\xi_n^2)^4}\Omega_3 dx'$$
$$= \left(\frac{29}{64}(h'(0))^2 - \frac{3}{8}h'(0)\right)\pi\Omega_3 dx'. \tag{3.32}$$

where $\Omega_3$ is the canonical volume of $S^3$.

**case (3)** $r=-1,\ l=-1,\ j=0,\ |\alpha|=2,\ k=0$.

$$\begin{aligned}\Psi_3&=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=2}\mathrm{tr}[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{1}{4}s\partial_M\pi^3dx',\end{aligned}\tag{3.33}$$

where $\Sigma_{t,l<n}R^{\partial_M}_{tltl}(x_0)$ is the scalar curvature $s_{\partial M}$ (see [8]).

**case (4)** $r=-1,\ l=-1,\ k=1,\ j=1,\ |\alpha|=0$

$$\begin{aligned}\Psi_4&=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^{2}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^{3}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{5}{16}(h'(0))^2\pi\Omega_3dx'.\end{aligned}\tag{3.34}$$

**case (5)** $r=-1,\ l=-1,\ k=1,\ j=0,\ |\alpha|=1$.

$$\begin{aligned}\Psi_5&=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi'}^{\alpha}\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=0.\end{aligned}\tag{3.35}$$

**case (6)** $r=-1,\ l=-1,\ k=2,\ j=|\alpha|=0$.

$$\begin{aligned}\Psi_6&=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{k=2}\mathrm{tr}[\partial_{\xi_n}^{2}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial_{x_n}^{2}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\left(\frac{29}{64}(h'(0))^2-\frac{3}{8}h''(0)\right)\pi\Omega_3dx'.\end{aligned}\tag{3.36}$$

**case (7)** $r=-1,\ l=-2,\ k=0,\ j=1,\ |\alpha|=0$.

$$\begin{aligned}\Psi_7&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}^{2}\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}^{2}\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-2}((D)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&\quad-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}^{2}\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}(\widetilde{D}_H^{-1})\times\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.37}$$

By Lemma 3.3, we have

$$\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})(x_0)|_{|\xi'|=1} = \frac{1}{(\xi_n-i)^3}\partial_{x_n}[c(\xi')](x_0) + h'(0)\left[\frac{4i-\xi_n}{2(\xi_n-i)^4}c(\xi') - \frac{3}{2(\xi_n-i)^4}c(dx_n)\right]. \quad (3.38)$$

From Lemma 3.2 and Lemma 3.3. we have

$$\begin{aligned}\sigma_{-2}(\widetilde{D}_H^{-2}) &= \frac{c(\xi)\sigma_0(D_H)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)] \\ &= \sigma_{-2}(D^{-1}) + \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}. \end{aligned} \quad (3.39)$$

By the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}BA$, then we have the equalities:

$$\begin{aligned}&\mathrm{tr}[\partial_{x_n}[c(\xi')]c(\xi)\gamma c(\xi)] = 4\xi_n h'(0)\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})];\ \mathrm{tr}[c(\xi')c(\xi)\gamma c(\xi)] = 8\xi_n\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]; \\ &\mathrm{tr}[c(dx_n)c(\xi)\gamma c(\xi)] = -4(1-\xi_n^2)\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})];\ \mathrm{tr}[c(\xi')\gamma] = \mathrm{tr}[-\sum_{j=1}^{n-1}\xi_j(\sigma_j^H + \Phi(\widetilde{e_j}))]; \\ &\mathrm{tr}[c(dx_n)\gamma] = \mathrm{tr}[-id\bigotimes(\sigma_n^H + \Phi(\widetilde{e_n}))] = -4\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]. \end{aligned} \quad (3.40)$$

Considering for $i < n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi') = 0$, we get:

$$\begin{aligned}&\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1}) \times \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0) \\ &= \frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]c(\xi)\gamma c(\xi)](x_0) + \frac{h'(0)(4i-\xi_n)}{2(\xi_n-i)^4|\xi|^4}\mathrm{tr}[c(\xi')c(\xi)\gamma c(\xi)](x_0) \\ &- \frac{3h'(0)}{2(\xi_n-i)^4|\xi|^4}\mathrm{tr}[c(dx_n)c(\xi)\gamma c(\xi)](x_0) \\ &= \left\{\frac{4\xi_n h'(0)}{(\xi_n-i)^3|\xi|^4} + \frac{(16\xi_n i - 10\xi_n^2 + 6)h'(0)}{(\xi_n-i)^4|\xi|^4}\right\}\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]. \end{aligned} \quad (3.41)$$

So we have:

$$\begin{aligned}\Psi_7 &= \frac{39}{32}(h'(0))^2\pi\Omega_3 dx' \\ &- \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1}) \times \frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx' \\ &= \frac{39}{32}(h'(0))^2\pi\Omega_3 dx' + \frac{(5i-2)h'}{8}\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]\pi\Omega_3 dx'. \end{aligned} \quad (3.42)$$

**case (8)** $r=-1,\ l=-2,\ k=j=0,\ |\alpha|=1$.

From (2.15) in [12] and the Leibniz rule, we get:

$$\begin{aligned}\Psi_8 =& -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^{\alpha}_{x'}\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ =& \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^{\alpha}_{x'}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ =& \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^{\alpha}_{x'}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ &+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^{\alpha}_{x'}\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx'. \qquad (3.43)\end{aligned}$$

By Lemmas 3.2 and 3.4, computations show:

$$\partial_{\xi_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})(x_0)|_{|\xi'|=1}=\frac{-1}{2(\xi_n-i)^2}c(dx_i)-\xi_i\frac{3i-\xi_n}{2(\xi_n-i)^3}c(\xi')+\xi_i\frac{1}{(\xi_n-i)^3}c(dx_n). \qquad (3.44)$$

From Lemma 3.2, we have:

$$\partial^{\alpha}_{x'}\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}|_{|\xi'|=1}=\frac{c(\xi)}{|\xi|^4}\partial_{x_i}(\gamma)c(\xi). \qquad (3.45)$$

By the relation of the Clifford action and $trAB=trBA$, then we have the equalities at a fixed point $x_0$:

$$\begin{aligned}&\mathrm{tr}[c(dx_i)c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0)=-4(1-\xi_n^2+2\xi_i\xi_j)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\};\\ &\xi_i\mathrm{tr}[c(\xi')c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0)=4(1-\xi_n^2)\xi_i\xi_j\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\};\\ &\xi_i\mathrm{tr}[c(dx_n)c(\xi)\partial_{x_i}(\gamma)c(\xi)](x_0)=8\xi_n\xi_i\xi_j\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}; \qquad (3.46)\end{aligned}$$

where $\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}(x_0)=\sum_{j=1}^{n-1}\mathrm{tr}[\nabla^H_{\partial_{x_j}}(\sigma^H_j+\Phi(\widetilde{e_j}))](x_0)$.

Combining (3.44)-(3.46) and direct computations, we obtain:

$$\begin{aligned}&\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^{\alpha}_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^{\alpha}_{x'}\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0)\\ &=\frac{-1}{2(\xi_n-i)^2|\xi|^4}\mathrm{tr}[c(dx_i)c(\xi)\partial_{x_i}(\gamma)c(\xi)]-\xi_i\frac{3i-\xi_n}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi')c(\xi)\partial_{x_i}(\gamma)c(\xi)]\\ &+\xi_i\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(dx_n)c(\xi)\partial_{x_i}(\gamma)c(\xi)]\\ &=\left(\frac{2}{(\xi_n-i)^3(\xi_n+i)}-\xi_i\xi_j\frac{4}{(\xi_n-i)^4(\xi_n+i)^2}+2\xi_i\xi_j\frac{3\xi_n^2i-\xi_n^3-3i+5\xi_n}{(\xi_n-i)^5(\xi_n+i)^2}\right)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}. \qquad (3.47)\end{aligned}$$

Considering for $i<n$, $\int_{|\xi'|=1}\int_{i_1}\int_{i_2}\cdots\int_{i_{2q+1}}\sigma(\xi')=0$, and $\int_{S^3}\xi_i\xi_j=\frac{\pi^2}{2}\delta^{ij}$ see ([6]). Therefore,

$$\begin{aligned}\Psi_8&=\left(\frac{3}{16}+\frac{3i}{32}\right)s_{\partial M}\pi\Omega_3dx'\\&+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left(\frac{2}{(\xi_n-i)^3(\xi_n+i)}-\xi_i\xi_j\frac{4}{(\xi_n-i)^4(\xi_n+i)^2}+2\xi_i\xi_j\frac{3\xi_n^2i-\xi_n^3-3i+5\xi_n}{(\xi_n-i)^5(\xi_n+i)^2}\right)\\&\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}(x_0)d\xi_n\sigma(\xi')dx'\\&=\left(\frac{3}{16}+\frac{3i}{32}\right)s_{\partial M}\pi\Omega_3dx'-(4+\frac{\pi^2}{2})\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'.\end{aligned}\tag{3.48}$$

**case (9)** $r=-1,\ l=-2,\ k=1,\ j=|\alpha|=0$.
From (2.15) and the Leibniz rule, we get:

$$\begin{aligned}\Psi_9&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}\left[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\right](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.49}$$

By (2.2.29) in [12], we have:

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\mid_{|\xi'|=1}=\frac{1}{(\xi_n-i)^3}c(\xi')+\frac{i}{2(\xi_n-i)^3}c(dx_n).\tag{3.50}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{\partial_{x_n}[c(\xi')]\gamma c(\xi)}{|\xi|^4}-\frac{2h'c(\xi)\gamma c(\xi)}{|\xi|^6}+\frac{c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^4}+\frac{c(\xi)\gamma\partial_{x_n}[c(\xi')]}{|\xi|^4}.\tag{3.51}$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then we have the equalities:

$$\begin{aligned}&\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(\xi')](x_0)=8\xi_n\mathrm{tr}[\partial_{x_n}(\sigma_n^H+\Phi(\widetilde{e_n}))];\\&\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(dx_n))](x_0)=-4(1-\xi_n^2)\mathrm{tr}[\partial_{x_n}(\sigma_n^H+\Phi(\widetilde{e_n}))].\end{aligned}\tag{3.52}$$

Combining (3.50)-(3.52) and direct computations, we obtain:

$$\begin{aligned}
&\sum_{|\alpha|=1}\mathrm{tr}\left[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\right]\\
&=\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(dx_n)]\\
&-\frac{2h'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(\xi')]-\frac{ih'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(dx_n)]\\
&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)dx_n)]\\
&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(\xi)]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(dx_n)]\\
&=\left(\frac{(4\xi_n-2i)h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{h'(0)(4i-16\xi_n-4i\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^3}\right)\mathrm{tr}[\sigma^H_n+\Phi(\widetilde{e_n})]\\
&+\frac{8\xi_n+2\xi_n^2i-2i}{(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma^H_n+\Phi(\widetilde{e_n}))]. \qquad (3.53)
\end{aligned}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. Therefore,

$$\begin{aligned}
\Psi_9&=\left(\frac{-367}{128}(h'(0))^2+\frac{103}{64}h''(0)\right)\pi\Omega_3dx'\\
&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left(\frac{(4\xi_n-2i)h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{h'(4i-16\xi_n-4i\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^3}\right)\mathrm{tr}[\sigma^H_n+\Phi(\widetilde{e_n})](x_0)d\xi_n\sigma(\xi')dx'\\
&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{8\xi_n+2\xi_n^2i-2i}{(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma^H_n+\Phi(\widetilde{e_n}))](x_0)d\xi_n\sigma(\xi')dx'\\
&=\left(\frac{-367}{128}(h'(0))^2+\frac{103}{64}h''(0)\right)\pi\Omega_3dx'-\frac{153h'}{128}\mathrm{tr}[\sigma^H_n+\Phi(\widetilde{e_n})]\pi\Omega_3dx'\\
&-\frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma^H_n+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'. \qquad (3.54)
\end{aligned}$$

**case (10)** $r=-2,\ l=-1,\ k=0,\ j=1,\ |\alpha|=0$.
From (2.15), we have:

$$\begin{aligned}
\Psi_{10}&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \qquad (3.55)
\end{aligned}$$

By the Leibniz rule, trace property and " + +" and " − −" vanishing after the integration over $\xi_n$ in [3], then

$$\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\right]d\xi_n$$
$$=\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\right]d\xi_n-\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\right]d\xi_n. \quad (3.56)$$

By Lemma 3.2, a simple computation shows:

$$\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})(x_0)|_{|\xi'|=1}=\frac{6\xi_n^2-2}{(1+\xi_n^2)^3}ic(\xi')+\frac{2\xi_n^3-6\xi_n}{(1+\xi_n^2)^3}ic(dx_n). \quad (3.57)$$

Similarly,

$$\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{\partial_{x_n}[c(\xi')]\gamma c(\xi)}{|\xi|^4}-\frac{2h'c(\xi)\gamma c(\xi)}{|\xi|^6}+\frac{c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^4}+\frac{c(\xi)\gamma\partial_{x_n}[c(\xi')]}{|\xi|^4}. \quad (3.58)$$

Combining (3.57), (3.58) and direct computations, we obtain

$$\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})\right]$$
$$=\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(dx_n)]$$
$$-\frac{2h'}{(\xi_n-i)^3|\xi|^6}tr[c(\xi)\gamma c(\xi)c(\xi')]-\frac{ih'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(dx_n)]$$
$$+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(dx_n)]$$
$$+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(\xi)]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(dx_n)]$$
$$=\frac{16h'i(\xi_n^3-2\xi_n^2-\xi_n^5)}{(\xi_n-i)^5(\xi_n+i)^5}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))]+\frac{8i(2\xi_n^3+\xi_n^5+\xi_n)}{(\xi_n-i)^5(\xi_n+i)^5}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]. \quad (3.59)$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. Therefore,

$$\Psi_{10}=\left(-\frac{367}{128}h'^2+\frac{103}{64}h''\right)\pi\Omega_3dx'$$
$$-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\left(-\frac{367}{128}h'^2+\frac{103}{64}h''\right)\pi\Omega_3dx'-\frac{129h'}{128}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'$$
$$-\frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'. \quad (3.60)$$

**case (11)** $r=-2,\ l=-1,\ k=j=|\alpha|=0$.
From (2.15), we have:

$$\Psi_{11}=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi'}^{\alpha}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.61}$$

By Lemma 3.2 and Lemma 3.3, for $i<n$, we have:

$$\partial_{x_i}\sigma_{-1}((\widetilde{D}_H)^{-1})(x_0)=\partial_{x_i}\left(\frac{ic(\xi)}{|\xi|^2}\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|)^2(x_0)}{|\xi|^4}=0. \tag{3.62}$$

So $\Psi_{11}=0$.
**case (12)** $r=-2,\ l=-1,\ k=1,\ j=|\alpha|=0$
From (2.15), we have:

$$\begin{aligned}\Psi_{12}&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi_{\xi_n}^{+}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\Psi_7+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\sigma_{-2}(D^{-1})\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned} \tag{3.63}$$

From Lemma 3.2-Lemma 3.3 and direct computations, we obtain:

$$\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})(x_0)|_{|\xi'|=1}=\frac{6i\xi_n^2-2i}{(1+\xi_n^2)^3}\partial_{x_n}[c(\xi')](x_0)+ih'(0)\left[\frac{4(1-5\xi_n^2)}{(1+\xi_n^2)^4}c(\xi')-\frac{12\xi_n(\xi_n^-1)}{(1+\xi_n^2)^4}c(dx_n)\right]. \tag{3.64}$$

Combining (3.64) and direct computations, we obtain:

$$\begin{aligned}&\mathrm{tr}\left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})\right]\\&=\frac{6i\xi_n^2-2i}{(1+\xi_n^2)^5}\mathrm{tr}[c(\xi)\gamma c(\xi)\partial_{x_n}[c(\xi')]]+ih'(0)\frac{4(1-5\xi_n^2)}{(1+\xi_n)^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(\xi')]-\frac{12\xi_n(\xi_n^2-1)}{(1+\xi_n^2)^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(dx_n)]\\&=\left(\frac{4\xi_n i(6\xi_n^2-2)h'(0)}{(1+\xi_n^2)^5}+\frac{-16i(\xi_n+4\xi_n^3+3\xi_n^5)h'(0)}{(1+\xi_n^2)^6}\right)\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})].\end{aligned} \tag{3.65}$$

Combining these assertions, we get:

$$\begin{aligned}\Psi_{12}&=\Psi_7+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{39}{32}(h'(0)^2)\pi\Omega_3dx'+\frac{(5i+6)h'}{8}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'.\end{aligned} \tag{3.66}$$

**case (13)** $r=-2,\ l=-2,\ k=j=|\alpha|=0$.
From (2.15), we have:

$$\begin{aligned}\Psi_{13}&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'\\&+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.67}$$

By [9], we obtain:

$$\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{(-i\xi_n-2)c(\xi')\gamma c(\xi')-i[c(dx_n)\gamma c(\xi')+c(\xi')\gamma c(dx_n)]-i\xi_nc(dx_n)\gamma c(dx_n)}{4(\xi_n-i)^2}.\tag{3.68}$$

$$\sigma_{-2}(D^{-1})=-h'\frac{\xi_n^2+2}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)+\frac{1}{(1+\xi_n^2)^3}c(\xi)c(dx_n)\partial_{x_n}[c(\xi')].\tag{3.69}$$

Then we have:

$$\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{(\xi_ni+3)c(\xi')\gamma c(\xi')}{4(\xi_n-i)^3}+\frac{(\xi_ni-1)c(dx_n)\gamma c(\xi')}{4(\xi_n-i)^3}+\frac{ic(dx_n)\gamma c(\xi')}{2(\xi_n-i)^3}\frac{ic(\xi')\gamma c(dx_n)}{2(\xi_n-i)^3}.\tag{3.70}$$

$$\begin{aligned}\partial_{\xi_n}\pi^+_{\xi_n}(\sigma_{-2}(D^{-1}))(x_0)|_{|\xi'|=1}&=h'(0)\frac{-i\xi_n-3}{4(\xi_n-i)^3}c(\xi')c(dx_n)c(\xi')+h'(0)\frac{i}{(\xi_n-i)^3}c(\xi')\\&+h'(0)\frac{i\xi_n-1}{4(\xi_n-i)^3}c(dx_n)+\frac{-i}{2(\xi_n-i)^3}\partial_{x_n}[c(\xi')](x_0)\\&+\frac{i\xi_n+3}{4(\xi_n-i)^3}c(\xi')c(dx_n)\partial_{x_n}[c(\xi')](x_0)\\&+h'(0)\frac{-2i\xi_n-8}{8(\xi_n-i)^4}c(\xi')+h'(0)\frac{i\xi_n^2+4\xi_n-9i}{8(\xi_n-i)^4}c(dx_n).\end{aligned}\tag{3.71}$$

So

$$\begin{aligned}&i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{13h'}{16}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'.\end{aligned}\tag{3.72}$$

Similarly, we obtain:

$$i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{73h'}{32}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'. \tag{3.73}$$

And

$$i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'$$
$$=-\frac{1}{16}\sum_{i=1}^{n}\mathrm{tr}[(\sigma_i^H+\Phi(\widetilde{e_i}))^2]\pi\Omega_3dx'. \tag{3.74}$$

Combining these assertions, we see:

$$\Psi_{13}=-\frac{821}{256}\pi\Omega_3dx'+\frac{h'}{4}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'-\frac{1}{16}\sum_{i=1}^{n}\mathrm{tr}[(\sigma_i^H+\Phi(\widetilde{e_i}))^2]\pi\Omega_3dx'. \tag{3.75}$$

**case (14)** $r=-1,\ l=-3,\ k=j=|\alpha|=0.$

From (2.15), we have:

$$\Psi_{14}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-3}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-3}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times Q_{-3}|_{|\xi'|=1}](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.76}$$

By (2.2.29) in [12], we have:

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})=-\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}. \tag{3.77}$$

In the orthonormal frame filed, we have:

$$Q_{-3}|_{|\xi'|=1}=\frac{ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^6}+\frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^6}+\frac{-ic(\xi)\gamma c(\xi)\gamma c(\xi)}{|\xi|^6}$$
$$+\frac{-ih'c(\xi)\gamma c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6}+\frac{ih'c(\xi)\gamma c(\xi)c(dx_n)c(\xi)}{|\xi|^8}+\frac{-ic(\xi)}{|\xi|^6}\sum_{j=1}^{n-1}c(\widetilde{e_j})c(\xi)\partial_{x_i}(\gamma)c(\xi)$$
$$+\frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\gamma c(\xi)}{|\xi|^6}+\frac{2ih'c(\xi)c(dx_n)c(\xi)\gamma c(\xi)}{|\xi|^8}+\frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^6}$$
$$+\frac{-ic(\xi)c(dx_n)c(\xi)\gamma\partial_{x_n}[c(\xi)]}{|\xi|^6}. \tag{3.78}$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then we have the equalities:

$$\mathrm{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\gamma c(\xi)]=(\xi_n^3-3\xi_n)tr[c(dx_n)\gamma]=-4(\xi_n^3-3\xi_n)\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})];$$
$$\mathrm{tr}[c(dx_n)c(\xi)\gamma c(\xi)\gamma c(\xi)]=4(3\xi_n-\xi_n^3)\{\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]-4(3\xi_n-\xi_n^3)\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2];$$
$$\mathrm{tr}[c(dx_n)c(\xi)\sum_{j=1}^{n-1}c(\widetilde{e_j})c(\xi)\partial_{x_i}(\gamma)c(\xi)]=4(\xi_n^3-3\xi_n)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\};$$
$$\mathrm{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\partial_{x_n}(\gamma)c(\xi)]=4(3\xi_n-\xi_n^3)\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]. \tag{3.79}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. From Lemma 3.4, combining (3.77)-(3.79) and direct computations, we obtain:

$$\begin{aligned}&i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times Q_{-3}|_{|\xi'|=1}](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{51h'}{16}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'+\frac{3}{4}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\&-\frac{3}{4}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]+\frac{3}{4}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2].\end{aligned} \tag{3.80}$$

Therefore,

$$\begin{aligned}\Psi_{14}&=\left(\frac{239}{64}(h'(0))^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3dx'\\&-\frac{51h'}{16}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'+\frac{3}{4}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\&-\frac{3}{4}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'\\&+\frac{3}{4}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'.\end{aligned} \tag{3.81}$$

**case (15)** $r=-3,\ l=-1,\ k=j=|\alpha|=0$.
From (2.15), we have:

$$\begin{aligned}\Psi_{15}&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-3}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-3}(D^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned} \tag{3.82}$$

By the Leibniz rule, trace property and "++" and "–" vanishing after the integration over $\xi_n$ in [3], then

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\\ &=\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n-\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n.\end{aligned}\tag{3.83}$$

By Lemma 3.2, a simple computation shows:

$$\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})=\frac{-2i\xi_nc(\xi')+i(1-\xi_n^2)c(dx_n)}{(\xi_n^2+1)^2}.\tag{3.84}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. From Lemma 3.4, combining (3.84) and direct computations, we obtain:

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ &=\frac{17h'}{48}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'-\frac{11}{24}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\ &+\frac{11}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'-\frac{11}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'\\ &+\frac{11}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'.\end{aligned}\tag{3.85}$$

Therefore, we obtain:

$$\begin{aligned}\Psi_{15}=&\left(\frac{239}{64}(h'(0))^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3dx'\\ &-\frac{17h'}{6}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'+\frac{7}{24}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\ &-\frac{7}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'+\frac{7}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'\\ &-\frac{7}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'.\end{aligned}\tag{3.86}$$

Now $\Psi$ is the sum of the $\Psi_{(1,2,\cdots,15)}$, then we obtain:

$$\begin{aligned}\sum_{i=1}^{15}\Psi_i=&\left(\frac{399}{256}(h'(0))^2-\frac{29}{32}h''(0)+(\frac{71}{96}+\frac{3i}{32})s_{\partial M}\right)\pi\Omega_3dx'\\ &+\frac{(240i-529)h'}{192}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'+\frac{23}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\ &-\frac{8\pi^2+89}{16}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'-\frac{1}{16}\sum_{j=1}^{n}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'\\ &-\frac{25}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'+\frac{25}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'.\end{aligned}\tag{3.87}$$

Hence,

**Theorem 3.6** *Let $M$ be a 5-dimensional compact spin manifold with the boundary $\partial M$, and twisted Dirac operator* $\widetilde{D}_H=\sum_{j=1}^n c(\widetilde{e_j})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_j}}+\sum_{j=1}^n c(\widetilde{e_j})\bigotimes\Phi(\widetilde{e_j})$, *then*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+(\widetilde{D}_H^{-1})]=\int_{\partial M}\Bigg[&\bigg(\frac{399}{256}(h'(0))^2-\frac{29}{32}h''(0)+(\frac{71}{96}+\frac{3i}{32})s_{\partial M}\bigg)\\&+\frac{(240i-529)h'}{192}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]+\frac{23}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\\&-\frac{8\pi^2+89}{16}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}-\frac{1}{16}\sum_{j=1}^n\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\\&-\frac{25}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]+\frac{25}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\Bigg]\pi\Omega_3 dvol_{\partial M}.\end{aligned}\tag{3.88}$$

Next we recall the Einstein-Hilbert action for manifolds with boundary (see[12] or [13]),

$$I_{Gr}=\frac{1}{16\pi}\int_M s dvol_M+2\int_{\partial M}K s dvol_{\partial M}:=I_{Gr,i}+I_{Gr,b},\tag{3.89}$$

where

$$K=\sum_{1\leq i,j\leq n-1}K_{ij}g^{ij}_{\partial M};\quad K_{ij}=-\Gamma^n_{ij},\tag{3.90}$$

and $K_{ij}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma $A.2$ in [12], $K_{ij}(x_0)=\sum_{ij}K_{ij}(x_0)g^{ij}_{\partial M}(x_0)=\sum_{i=1}^4 K_{ii}(x_0)=-2h'(0)$ Then

$$I_{Gr,b}=-4h'(0)vol_{\partial M}.\tag{3.91}$$

Then, we have:

**Theorem 3.7** *Let $M$ be a 5-dimensional compact spin manifold with the boundary $\partial M$, and twisted Dirac operator* $\widetilde{D}_H=\sum_{j=1}^n c(\widetilde{e_j})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_j}}+\sum_{j=1}^n c(\widetilde{e_j})\bigotimes\Phi(\widetilde{e_j})$. *The following identity holds:*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+(\widetilde{D}_H^{-1})]=\int_{\partial M}\Bigg[&\bigg(\frac{225}{64}K^2+\frac{29}{8}s_M|_{\partial M}+(\frac{197}{24}+\frac{3i}{2})s_{\partial M}\bigg)\\&+\frac{(529-240i)}{384}tr[\sigma_n^H+\Phi(\widetilde{e_n})]K+\frac{23}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\\&-\frac{8\pi^2+89}{16}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}-\frac{1}{16}\sum_{j=1}^n\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\\&-\frac{25}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]+\frac{25}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\Bigg]\pi\Omega_3 dvol_{\partial M}.\end{aligned}\tag{3.92}$$

*where $s_M, s_{\partial M}$ are respectively scalar curvatures on $M$ and $\partial M$.*

On the other hand, we compute $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+((\widetilde{D}_H^*)^{-1})]$ for 5-dimensional spin manifolds with boundary, then we have:

$$\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+((\widetilde{D}_H^*)^{-1})]=\int_{\partial M}\widetilde{\Psi}. \tag{3.93}$$

When $n=5$, then $\mathrm{tr}_{S(TM)\bigotimes H}[\mathtt{id}]=4dimH$, where tr as shorthand of trace, the sum is taken over $-r-l+k+j+|\alpha|=4,\ \ r\leq -1, l\leq -1,$ then we have the following fifteen cases:

**case (1)** $r=-1,\ l=-1,\ k=0,\ j=|\alpha|=1$.

By (2.15), we get:

$$\widetilde{\Psi}_1=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{x_n}\partial_{\xi'}^{\alpha}\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.94}$$

By Lemma 3.3, for $i<n$, then

$$\partial_{x_i}\left(\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0)\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4}=0, \tag{3.95}$$

so $\widetilde{\Psi}_1=0$.

**case (2)** $r=-1,\ l=-1,\ k=|\alpha|=0,\ j=2$.

By (2.15), we get:

$$\widetilde{\Psi}_2=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j=2}\mathrm{tr}[\partial_{x_n}^2\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^3\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.96}$$

By Lemma 3.2, we have:

$$\partial_{\xi_n}^3\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0)\mid_{|\xi'|=1}=ic(\xi')\frac{24\xi_n-24\xi_n^3}{(1+\xi_n^2)^4}+ic(dx_n)\frac{-64\xi_n^4+36\xi_n^2-6}{(1+\xi_n^2)^4}; \tag{3.97}$$

Similarly, we get:

$$\begin{aligned}\partial_{x_n}^2\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}^{-1})|_{|\xi'|=1}=&\left(\frac{3}{4}(h'(0))^2-\frac{1}{2}h''(0)\right)\frac{c(\xi')}{2(\xi_n-i)}-h'(0)\frac{\xi_n-2i}{2(\xi_n-i)}\partial_{x_n}[c(\xi')]\\&-h''(0)\left[\frac{\xi_n-2i}{4(\xi_n-i)}c(\xi')+\frac{1}{4(\xi_n-i)^2}c(d_{x_n})\right]\\&+2i(h'(0))^2\left[\frac{-3i\xi_n^2-9\xi_n+8i}{16(\xi_n-i)^3}c(\xi')+\frac{-i\xi_n-3}{16(\xi_n-i)^3}c(d_{x_n})\right].\end{aligned} \tag{3.98}$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, we have the equalities:

$$\begin{aligned}&\mathrm{tr}[c(\xi')c(dx_n)]=0;\quad \mathrm{tr}[c(dx_n)^2]=-4dimH;\quad \mathrm{tr}[c(\xi')^2](x_0)|_{|\xi'|=1}=-4dimH;\\&\mathrm{tr}[\partial_{x_n}c(\xi')c(dx_n)]=0;\quad \mathrm{tr}[\partial_{x_n}c(\xi')c(\xi')](x_0)|_{|\xi'|=1}=-2h'(0)dimH.\end{aligned} \tag{3.99}$$

Similarly, we have:

$$\begin{aligned}
&\mathrm{tr}\left[\partial_{x_n}^2\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\partial_{\xi_n}^3\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right](x_0)|_{|\xi'|=1}\\
&=h'(0)^2\frac{3(33\xi_n^5-75i\xi_n^4-94\xi_n^3+90i\xi_n^2+57\xi_n-3i)}{2(\xi_n-i)^3(1+\xi_n^2)^4}\\
&+h''(0)\frac{6(-9\xi_n^4+12i\xi_n^3+14\xi_n^2-12i\xi_n-1)}{2(\xi_n-i)^2(1+\xi_n^2)^4}. \qquad (3.100)
\end{aligned}$$

Then

$$\begin{aligned}
\widetilde{\Psi}_2&=-\frac{1}{6}(h'(0))^2\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3(33\xi_n^5-75i\xi_n^4-94\xi_n^3+90i\xi_n^2+57\xi_n-3i)}{2(\xi_n-i)^3(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'\\
&-\frac{1}{6}h''(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{6(-9\xi_n^4+12i\xi_n^3+14\xi_n^2-12i\xi_n-1)}{2(\xi_n-i)^2(1+\xi_n^2)^4}d\xi_n\sigma(\xi')dx'\\
&=\left(\frac{29}{64}(h'(0))^2-\frac{3}{8}h'(0)\right)\pi\Omega_3dx'. \qquad (3.101)
\end{aligned}$$

where $\Omega_3$ is the canonical volume of $S^3$.
**case (3)** $r=-1,\ l=-1,\ j=0,\ |\alpha|=2,\ k=0$.

$$\begin{aligned}
\widetilde{\Psi}_3&=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=2}\mathrm{tr}[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=-\frac{1}{4}s\partial_M\pi^3dx', \qquad (3.102)
\end{aligned}$$

where $\Sigma_{t,l<n}R_{tltl}^{\partial_M}(x_0)$ is the scalar curvature $s_{\partial M}$ (see [8]).
**case (4)** $r=-1,\ l=-1,\ k=1,\ j=1,\ |\alpha|=0$.

$$\begin{aligned}
\widetilde{\Psi}_4&=\frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=-\frac{5}{16}(h'(0))^2\pi\Omega_3dx'. \qquad (3.103)
\end{aligned}$$

**case (5)** $r=-1,\ l=-1,\ k=1,\ j=0,\ |\alpha|=1$.

$$\begin{aligned}
\widetilde{\Psi}_5&=\frac{i}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi'}^{\alpha}\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x'}^{\alpha}\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&=0. \qquad (3.104)
\end{aligned}$$

**case (6)** $r=-1,\ l=-1,\ k=2,\ j=|\alpha|=0$.

$$\widetilde{\Psi}_6 = \frac{i}{6}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{k=2}\mathrm{tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial^2_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$= \left(\frac{29}{64}(h'(0))^2 - \frac{3}{8}h''(0)\right)\pi\Omega_3 dx'. \tag{3.105}$$

**case (7)** $r=-1,\ l=-2,\ k=0,\ j=1,\ |\alpha|=0$.

$$\widetilde{\Psi}_7 = \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-2}((D)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$- \frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\frac{c(\xi)\tau c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.106}$$

By Lemma 3.3, we have:

$$\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})(x_0)|_{|\xi'|=1} = \frac{1}{(\xi_n-i)^3}\partial_{x_n}[c(\xi')](x_0) + h'(0)\left[\frac{4i-\xi_n}{2(\xi_n-i)^4}c(\xi') - \frac{3}{2(\xi_n-i)^4}c(dx_n)\right]. \tag{3.107}$$

From Lemma 3.2 and Lemma 3.3. we have:

$$\sigma_{-2}((\widetilde{D}_H^*)^{-2}) = \frac{c(\xi)\sigma_0(D_H)c(\xi)}{|\xi|^4} + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j)[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}(|\xi|^2)]$$
$$= \sigma_{-2}(D^{-1}) + \frac{c(\xi)\tau c(\xi)}{|\xi|^4}. \tag{3.108}$$

By the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}BA$, then we have the equalities:

$$\mathrm{tr}[\partial_{x_n}[c(\xi')]c(\xi)\tau c(\xi)] = 4\xi_n h'(0)\mathrm{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})];\ \mathrm{tr}[c(\xi')c(\xi)\tau c(\xi)] = 8\xi_n\mathrm{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})];$$
$$\mathrm{tr}[c(dx_n)c(\xi)\tau c(\xi)] = -4(1-\xi_n^2)tr[\sigma_n^H + \Phi(\widetilde{e_n})];\ \mathrm{tr}[c(\xi')\tau] = \mathrm{tr}[-\sum_{j=1}^{n-1}\xi_j(\sigma_j^H - \Phi^*(\widetilde{e_j}))];$$
$$\mathrm{tr}[c(dx_n)\tau] = \mathrm{tr}[-id\bigotimes(\sigma_n^H - \Phi^*(\widetilde{e_n}))] = -4\mathrm{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})]. \tag{3.109}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. Then, we get:

$$\begin{aligned}&\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\frac{c(\xi)\tau c(\xi)}{|\xi|^4}](x_0)\\&=\frac{1}{(\xi_n-i)^3|\xi|^4}tr[\partial_{x_n}[c(\xi')]c(\xi)\tau c(\xi)](x_0)+\frac{h'(0)(4i-\xi_n)}{2(\xi_n-i)^4|\xi|^4}\mathrm{tr}[c(\xi')c(\xi)\tau c(\xi)](x_0)\\&-\frac{3h'(0)}{2(\xi_n-i)^4|\xi|^4}\mathrm{tr}[c(dx_n)c(\xi)\tau c(\xi)](x_0)\\&=\left\{\frac{4\xi_nh'(0)}{(\xi_n-i)^3|\xi|^4}+\frac{(16\xi_ni-10\xi_n^2+6)h'(0)}{(\xi_n-i)^4|\xi|^4}\right\}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})].\end{aligned}\tag{3.110}$$

So we have:

$$\begin{aligned}\widetilde{\Psi}_7&=\frac{39}{32}(h'(0))^2\pi\Omega_3dx'\\&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^2_{\xi_n}\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{39}{32}(h'(0))^2\pi\Omega_3dx'+\frac{(5i-2)h'}{8}tr[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'.\end{aligned}\tag{3.111}$$

**case (8)** $r=-1,\ l=-2,\ k=0,\ j=0,\ |\alpha|=1$.

From 2.15 in [12] and the Leibniz rule, we get:

$$\begin{aligned}\widetilde{\Psi}_8&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^\alpha_{x'}\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^\alpha_{x'}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^\alpha_{x'}\frac{c(\xi)\tau c(\xi)}{|\xi|^4}](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.112}$$

By Lemmas 3.2 and 3.3, computations show:

$$\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})(x_0)|_{|\xi'|=1}=\frac{-1}{2(\xi_n-i)^2}c(dx_i)-\xi_i\frac{3i-\xi_n}{2(\xi_n-i)^3}c(\xi')+\xi_i\frac{1}{(\xi_n-i)^3}c(dx_n).\tag{3.113}$$

From Lemma 3.2, we have

$$\partial^\alpha_{x'}\frac{c(\xi)\tau c(\xi)}{|\xi|^4}|_{|\xi'|=1}=\frac{c(\xi)}{|\xi|^4}\partial_{x_i}(\tau)c(\xi).\tag{3.114}$$

By the relation of the Clifford action and $trAB = trBA$, then we have the equalities at a fixed point $x_0$:

$$\mathrm{tr}[c(dx_i)c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) = -4(1-\xi_n^2+2\xi_i\xi_j)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\};$$
$$\xi_i\mathrm{tr}[c(\xi')c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) = 4(1-\xi_n^2)\xi_i\xi_j\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\};$$
$$\xi_i\mathrm{tr}[c(dx_n)c(\xi)\partial_{x_i}(\tau)c(\xi)](x_0) = 8\xi_n\xi_i\xi_j\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}; \tag{3.115}$$

where $\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}(x_0) = \sum_{j=1}^{n-1}\mathrm{tr}[\nabla^H_{\partial_{x_j}}(\sigma^H_j-\Phi^*(\widetilde{e_j}))](x_0)$.

Combining (3.113)-(3.115) and direct computations, we obtain:

$$\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial^\alpha_{x'}\frac{c(\xi)\tau c(\xi)}{|\xi|^4}](x_0)$$
$$=\frac{-1}{2(\xi_n-i)^2|\xi|^4}\mathrm{tr}[c(dx_i)c(\xi)\partial_{x_i}(\tau)c(\xi)]-\xi_i\frac{3i-\xi_n}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi')c(\xi)\partial_{x_i}(\tau)c(\xi)]$$
$$+\xi_i\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(dx_n)c(\xi)\partial_{x_i}(\tau)c(\xi)]$$
$$=\left(\frac{2}{(\xi_n-i)^3(\xi_n+i)}-\xi_i\xi_j\frac{4}{(\xi_n-i)^4(\xi_n+i)^2}+2\xi_i\xi_j\frac{3\xi_n^2i-\xi_n^3-3i+5\xi_n}{(\xi_n-i)^5(\xi_n+i)^2}\right)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}. \tag{3.116}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$, and $\int_{S^3}\xi_i\xi_j=\frac{\pi^2}{2}\delta^{ij}$ (see [6]). Therefore,

$$\widetilde{\Psi}_8=\left(\frac{3}{16}+\frac{3i}{32}\right)s_{\partial M}\pi\Omega_3dx'$$
$$+\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left(\frac{2}{(\xi_n-i)^3(\xi_n+i)}-\xi_i\xi_j\frac{4}{(\xi_n-i)^4(\xi_n+i)^2}+2\xi_i\xi_j\frac{3\xi_n^2i-\xi_n^3-3i+5\xi_n}{(\xi_n-i)^5(\xi_n+i)^2}\right)$$
$$\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}(x_0)d\xi_n\sigma(\xi')dx'$$
$$=\left(\frac{3}{16}+\frac{3i}{32}\right)s_{\partial M}\pi\Omega_3dx'-(4+\frac{\pi^2}{2})\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}\pi\Omega_3dx'. \tag{3.117}$$

**case (9)** $r=-1,\ l=-2,\ k=1,\ j=|\alpha|=0$.

From (2.15) and the Leibniz rule, we get:

$$\widetilde{\Psi}_9=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\sigma_{-2}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}\left[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)\right](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.118}$$

By (2.2.29) in [12], we have:

$$\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\mid_{|\xi'|=1}=\frac{1}{(\xi_n-i)^3}c(\xi')+\frac{i}{2(\xi_n-i)^3}c(dx_n). \tag{3.119}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n}\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)=\frac{\partial_{x_n}[c(\xi')]\tau c(\xi)}{|\xi|^4}-\frac{2h'c(\xi)\tau c(\xi)}{|\xi|^6}+\frac{c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^4}+\frac{c(\xi)\tau\partial_{x_n}[c(\xi')]}{|\xi|^4}. \tag{3.120}$$

By the relation of the Clifford action and $trAB = trBA$, then we have the equalities:

$$\begin{aligned}&\mathrm{tr}[c(\xi)\partial_{x_n}(\tau)c(\xi)c(\xi')](x_0)=8\xi_n\mathrm{tr}[\partial_{x_n}(\sigma_n^H-\Phi^*(\widetilde{e_n}))];\\&\mathrm{tr}[c(\xi)\partial_{x_n}(\tau)c(\xi)c(dx_n))](x_0)=-4(1-\xi_n^2)\mathrm{tr}[\partial_{x_n}(\sigma_n^H-\Phi^*(\widetilde{e_n}))].\end{aligned} \tag{3.121}$$

Combining (3.119)-(3.121) and direct computations, we obtain:

$$\begin{aligned}&\sum_{|\alpha|=1}\mathrm{tr}\left[\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{x_n}\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)\right]\\&=\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\tau c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\tau c(\xi)c(dx_n)]\\&-\frac{2h'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\tau c(\xi)c(\xi')]-\frac{ih'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\tau c(\xi)c(dx_n)]\\&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\tau)c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\tau)c(\xi)dx_n)]\\&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\tau\partial_{x_n}[c(\xi')]c(\xi)]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\tau\partial_{x_n}[c(\xi')]c(dx_n)]\\&=\left(\frac{(4\xi_n-2i)h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{h'(0)(4i-16\xi_n-4i\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^3}\right)\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\\&+\frac{8\xi_n+2\xi_n^2i-2i}{(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))].\end{aligned} \tag{3.122}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. Therefore,

$$\begin{aligned}\widetilde{\Psi}_9&=\left(\frac{-367}{128}(h'(0))^2+\frac{103}{64}h''(0)\right)\pi\Omega_3dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\left(\frac{(4\xi_n-2i)h'(0)}{(\xi_n-i)^5(\xi_n+i)^2}+\frac{h'(4i-16\xi_n-4i\xi_n^2)}{(\xi_n-i)^6(\xi_n+i)^3}\right)\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})](x_0)d\xi_n\sigma(\xi')dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{8\xi_n+2\xi_n^2i-2i}{(\xi_n-i)^5(\xi_n+i)^2}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))](x_0)d\xi_n\sigma(\xi')dx'\\&=\left(\frac{-367}{128}(h'(0))^2+\frac{103}{64}h''(0)\right)\pi\Omega_3dx'-\frac{153h'}{128}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'\\&-\frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))]\pi\Omega_3dx'.\end{aligned} \tag{3.123}$$

**case (10)** $r=-2,\ l=-1,\ k=0,\ j=1,\ |\alpha|=0$.

From (2.15), we have:

$$\begin{aligned}\widetilde{\Psi}_{10}&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.124}$$

By the Leibniz rule, trace property and "$++$" and "$--$" vanishing after the integration over $\xi_n$ in [3], then,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right]d\xi_n\\&=\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right]d\xi_n-\int_{-\infty}^{+\infty}\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right]d\xi_n.\end{aligned}\tag{3.125}$$

By Lemma 3.2, a simple computation shows:

$$\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1}=\frac{6\xi_n^2-2}{(1+\xi_n^2)^3}ic(\xi')+\frac{2\xi_n^3-6\xi_n}{(1+\xi_n^2)^3}ic(dx_n).\tag{3.126}$$

From Lemma 3.2, a simple computation shows:

$$\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{\partial_{x_n}[c(\xi')]\gamma c(\xi)}{|\xi|^4}-\frac{2h'c(\xi)\gamma c(\xi)}{|\xi|^6}+\frac{c(\xi)\partial_{x_n}(\gamma)c(\xi)}{|\xi|^4}+\frac{c(\xi)\gamma\partial_{x_n}[c(\xi')]}{|\xi|^4}.\tag{3.127}$$

Combining (3.126), (3.127) and direct computations, we obtain:

$$\begin{aligned}&\mathrm{tr}\left[\partial_{x_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right]\\&=\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[\partial_{x_n}[c(\xi')]\gamma c(\xi)c(dx_n)]\\&-\frac{2h'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(\xi')]-\frac{ih'}{(\xi_n-i)^3|\xi|^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(dx_n)]\\&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(\xi')]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\partial_{x_n}(\gamma)c(\xi)c(dx_n)]\\&+\frac{1}{(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(\xi)]+\frac{i}{2(\xi_n-i)^3|\xi|^4}\mathrm{tr}[c(\xi)\gamma\partial_{x_n}[c(\xi')]c(dx_n)]\\&=\left(\frac{32i\partial_{x_n}(\sqrt{h})(x_0)(\xi_n^3+\xi_n)}{(\xi_n-i)^5(\xi_n+i)^5}+\frac{-16h'i(2\xi_n^2+\xi_n+\xi_n^5)}{(\xi_n-i)^5(\xi_n+i)^5}\right)\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))]\\&+\frac{8i(2\xi_n^3+\xi_n^5+\xi_n)}{(\xi_n-i)^5(\xi_n+i)^5}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))].\end{aligned}\tag{3.128}$$

Considering for $i<n$, $\int_{|\xi'|=1}\mathrm{tr}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. Therefore,

$$\begin{aligned}\widetilde{\Psi}_{10}=&\Big(-\frac{367}{128}h'^2+\frac{103}{64}h''\Big)\pi\Omega_3dx'\\&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\Big(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\Big)\times\partial^2_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{x_n}\Big(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\Big)\times\partial^2_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\=&\Big(-\frac{367}{128}h'^2+\frac{103}{64}h''\Big)\pi\Omega_3dx'-\frac{129h'}{128}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'\\&-\frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'.\end{aligned}\tag{3.129}$$

**case (11)** $r=-2,\ l=-1,\ k=j=|\alpha|=0$.
From (2.15), we have:

$$\widetilde{\Psi}_{11}=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial^\alpha_{\xi'}\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\tag{3.130}$$

By Lemmas 3.2 and 3.3, for $i<n$, we have:

$$\partial_{x_i}\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0)=\partial_{x_i}\Big(\frac{ic(\xi)}{|\xi|^2}\Big)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^2}-\frac{ic(\xi)\partial_{x_i}(|\xi|)^2(x_0)}{|\xi|^4}=0.\tag{3.131}$$

So $\widetilde{\Psi}_{11}=0$.
**case (12)** $r=-2,\ l=-1,\ k=1,\ j=|\alpha|=0$.
From (2.15), we have:

$$\begin{aligned}\widetilde{\Psi}_{12}=&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\=&\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial^2_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\=&\Psi_7+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\sigma_{-2}(D^{-1})\times\partial^2_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial^2_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'.\end{aligned}\tag{3.132}$$

From Lemmas 3.2 and 3.4 and direct computations, we obtain:

$$\partial^2_{\xi_n}\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})(x_0)|_{|\xi'|=1}=\frac{6i\xi_n^2-2i}{(1+\xi_n^2)^3}\partial_{x_n}[c(\xi')](x_0)+ih'(0)\Big[\frac{4(1-5\xi_n^2)}{(1+\xi_n^2)^4}c(\xi')-\frac{12\xi_n(\xi_n^2-1)}{(1+\xi_n^2)^4}c(dx_n)\Big].\tag{3.133}$$

Combining (3.133) and direct computations, we obtain:

$$\mathrm{tr}\left[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})\right]$$
$$=\frac{6i\xi_n^2-2i}{(1+\xi_n^2)^5}\mathrm{tr}[c(\xi)\gamma c(\xi)\partial_{x_n}[c(\xi')]]+ih'(0)\frac{4(1-5\xi_n^2)}{(1+\xi_n)^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(\xi')]-\frac{12\xi_n(\xi_n^2-1)}{(1+\xi_n^2)^6}\mathrm{tr}[c(\xi)\gamma c(\xi)c(dx_n)]$$
$$=\left(\frac{4\xi_n i(6\xi_n^2-2)h'(0)}{(1+\xi_n^2)^5}+\frac{-16i(\xi_n+4\xi_n^3+3\xi_n^5)h'(0)}{(1+\xi_n^2)^6}\right)\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]. \tag{3.134}$$

Combining these assertions, we get:

$$\widetilde{\Psi}_{12}=\Psi_7+\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\times\partial_{\xi_n}^2\partial_{x_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{39}{32}(h'(0)^2)\pi\Omega_3dx'+\frac{(5i+6)h'}{8}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'. \tag{3.135}$$

**case (13)** $r=-2,\ l=-2,\ k=j=|\alpha|=0$.

From (2.15), we have:

$$\widetilde{\Psi}_{13}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi_{\xi_n}^+\sigma_{-2}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-2}((\widetilde{D}_H)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-2}(D^{-1})\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-2}(D^{-1})\times\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'$$
$$+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi_{\xi_n}^+\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.136}$$

By [9], we obtain:

$$\pi_{\xi_n}^+\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{(-i\xi_n-2)c(\xi')\gamma c(\xi')-i[c(dx_n)\gamma c(\xi')+c(\xi')\gamma c(dx_n)]-i\xi_n c(dx_n)\gamma c(dx_n)}{4(\xi_n-i)^2}. \tag{3.137}$$

$$\sigma_{-2}(D^{-1})=-h'\frac{\xi_n^2+2}{(1+\xi_n^2)^3}c(\xi)c(dx_n)c(\xi)+\frac{1}{(1+\xi_n^2)^3}c(\xi)c(dx_n)\partial_{x_n}[c(\xi')]. \tag{3.138}$$

Then we have:

$$\partial_{\xi_n}\pi_{\xi_n}^+\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)=\frac{(\xi_n i+3)c(\xi')\gamma c(\xi')}{4(\xi_n-i)^3}+\frac{(\xi_n i-1)c(dx_n)\gamma c(\xi')}{4(\xi_n-i)^3}+\frac{ic(dx_n)\gamma c(\xi')}{2(\xi_n-i)^3}\frac{ic(\xi')\gamma c(dx_n)}{2(\xi_n-i)^3}. \tag{3.139}$$

$$\begin{aligned}\partial_{\xi_n}\pi^+_{\xi_n}(\sigma_{-2}(D^{-1}))(x_0)|_{|\xi'|=1} &= h'(0)\frac{-i\xi_n-3}{4(\xi_n-i)^3}c(\xi')c(dx_n)c(\xi')+h'(0)\frac{i}{(\xi_n-i)^3}c(\xi')\\ &+h'(0)\frac{i\xi_n-1}{4(\xi_n-i)^3}c(dx_n)+\frac{-i}{2(\xi_n-i)^3}\partial_{x_n}[c(\xi')](x_0)\\ &+\frac{i\xi_n+3}{4(\xi_n-i)^3}c(\xi')c(dx_n)\partial_{x_n}[c(\xi')](x_0)\\ &+h'(0)\frac{-2i\xi_n-8}{8(\xi_n-i)^4}c(\xi')+h'(0)\frac{i\xi_n^2+4\xi_n-9i}{8(\xi_n-i)^4}c(dx_n). \end{aligned}\tag{3.140}$$

So

$$\begin{aligned}& i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-2}(D^{-1})\times\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'\\ =&-\frac{13h'}{16}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'.\end{aligned}\tag{3.141}$$

Similarly, we obtain:

$$\begin{aligned}& i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\sigma_{-2}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\ =&\frac{73h'}{32}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'.\end{aligned}\tag{3.142}$$

And,

$$\begin{aligned}& i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\left(\frac{c(\xi)\gamma c(\xi)}{|\xi|^4}\right)\times\left(\frac{c(\xi)\tau c(\xi)}{|\xi|^4}\right)](x_0)d\xi_n\sigma(\xi')dx'\\ =&-\frac{1}{16}\sum_{i=1}^{n}\mathrm{tr}[(\sigma_i^H+\Phi(\widetilde{e_i}))(\sigma_i^H-\Phi^*(\widetilde{e_i}))]\pi\Omega_3dx'.\end{aligned}\tag{3.143}$$

Combining these assertions, we see:

$$\begin{aligned}\widetilde{\Psi}_{13}=&-\frac{821}{256}\pi\Omega_3dx'-\frac{13h'}{16}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'+\frac{73h'}{32}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'\\ &-\frac{1}{16}\sum_{i=1}^{n}\mathrm{tr}[(\sigma_i^H+\Phi(\widetilde{e_i}))(\sigma_i^H-\Phi^*(\widetilde{e_i}))]\pi\Omega_3dx'.\end{aligned}\tag{3.144}$$

**case (14)** $r=-1,\ l=-3,\ k=j=|\alpha|=0.$

From (2.15), we have:

$$
\begin{aligned}
\widetilde{\Psi}_{14} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-3}((\widetilde{D}_H^{*})^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-3}((\widetilde{D}_H^{*})^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times\sigma_{-3}(D^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\
&+ i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times R_{-3}|_{|\xi'|=1}](x_0)d\xi_n\sigma(\xi')dx'.
\end{aligned}\tag{3.145}
$$

By (2.2.29) in [12], we have:

$$
\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1}) = -\frac{c(\xi')+ic(dx_n)}{2(\xi_n-i)^2}.\tag{3.146}
$$

In the orthonormal frame filed, we have:

$$
\begin{aligned}
R_{-3}|_{|\xi'|=1} &= \frac{ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^6}+\frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^6}+\frac{-ic(\xi)\tau c(\xi)\tau c(\xi)}{|\xi|^6}\\
&+\frac{-ih'c(\xi)\tau c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]}{|\xi|^6}+\frac{ih'c(\xi)\tau c(\xi)c(dx_n)c(\xi)}{|\xi|^8}+\frac{-ic(\xi)}{|\xi|^6}\sum_{j=1}^{n-1}c(\widetilde{e_j})c(\xi)\partial_{x_i}(\tau)c(\xi)\\
&+\frac{i\xi c(\xi)c(dx_n)\partial_{x_n}[c(\xi)]\tau c(\xi)}{|\xi|^6}+\frac{2ih'c(\xi)c(dx_n)c(\xi)\tau c(\xi)}{|\xi|^8}+\frac{-ic(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)}{|\xi|^6}\\
&+\frac{-ic(\xi)c(dx_n)c(\xi)\tau\partial_{x_n}[c(\xi)]}{|\xi|^6}.
\end{aligned}\tag{3.147}
$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then we have the equalities:

$$
\begin{aligned}
&\mathrm{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\tau c(\xi)] = (\xi_n^3-3\xi_n)tr[c(dx_n)\tau] = -4(\xi_n^3-3\xi_n)\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})];\\
&\mathrm{tr}[c(dx_n)c(\xi)\tau c(\xi)\tau c(\xi)] = 4(3\xi_n-\xi_n^3)\{\mathrm{tr}[(\sigma_n^H-\Phi^*(\widetilde{e_n}))^2]-4(3\xi_n-\xi_n^3)\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H-\Phi^*(\widetilde{e_j}))^2];\\
&\mathrm{tr}[c(dx_n)c(\xi)\sum_{j=1}^{n-1}c(\widetilde{e_j})c(\xi)\partial_{x_i}(\tau)c(\xi)] = 4(\xi_n^3-3\xi_n)\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\};\\
&\mathrm{tr}[c(dx_n)c(\xi)c(dx_n)c(\xi)\partial_{x_n}(\tau)c(\xi)] = 4(3\xi_n-\xi_n^3)\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))].
\end{aligned}\tag{3.148}
$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. From Lemma 3.4, combining (3.146)-(3.148) and direct

computations, we obtain:

$$i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}(\widetilde{D}_H^{-1})\times R_{-3}|_{|\xi'|=1}](x_0)d\xi_n\sigma(\xi')dx'$$
$$=\frac{-51h'}{16}tr[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'+\frac{3}{4}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))]\pi\Omega_3dx'-\frac{3}{4}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}\pi\Omega_3dx'$$
$$-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H-\Phi^*(\widetilde{e_j}))^2]+\frac{3}{4}\mathrm{tr}[(\sigma_n^H-\Phi^*(\widetilde{e_n}))^2]. \tag{3.149}$$

Therefore,

$$\widetilde{\Psi}_{14}=\left(\frac{239}{64}(h'(0))^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3dx'$$
$$-\frac{-51h'}{16}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'+\frac{3}{4}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))]\pi\Omega_3dx'$$
$$-\frac{3}{4}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}\pi\Omega_3dx'-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H-\Phi^*(\widetilde{e_j}))^2]\pi\Omega_3dx'$$
$$+\frac{3}{4}\mathrm{tr}[(\sigma_n^H-\Phi^*(\widetilde{e_n}))^2]\pi\Omega_3dx'. \tag{3.150}$$

**case (15)** $r=-3,\ l=-1,\ k=j=|\alpha|=0.$
From (2.15), we have:

$$\widetilde{\Psi}_{15}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-3}(\widetilde{D}_H^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}\sigma_{-3}(D^{-1})\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'$$
$$-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{3.151}$$

By the Leibniz rule, trace property and "++" and "–" vanishing after the integration over $\xi_n$ in [3], then

$$\int_{-\infty}^{+\infty}\mathrm{tr}[\pi^+_{\xi_n}Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n$$
$$=\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n-\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n. \tag{3.152}$$

By Lemma 3.2, a simple computation shows

$$\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})=\frac{-2i\xi_nc(\xi')+i(1-\xi_n^2)c(dx_n)}{(\xi_n^2+1)^2}. \tag{3.153}$$

Considering for $i<n$, $\int_{|\xi'|=1}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2q+1}}\sigma(\xi')=0$. From Lemma 3.4, combining (3.153) and direct

computations, we obtain:

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{tr}[Q_{-3}|_{|\xi'|=1}\times\partial_{\xi_n}\sigma_{-1}((\widetilde{D}_H^*)^{-1})](x_0)d\xi_n\sigma(\xi')dx'\\&=\frac{17h'}{48}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'-\frac{11}{24}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\&+\frac{11}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'-\frac{11}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'+\frac{11}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'.\quad(3.154)\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\widetilde{\Psi}_{15}=&\left(\frac{239}{64}(h'(0))^2-\frac{27}{16}h''(0)-\frac{11}{192}s_{\partial M}\right)\pi\Omega_3dx'\\&-\frac{17h'}{6}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'+\frac{7}{24}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'\\&-\frac{7}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'+\frac{7}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'\\&-\frac{7}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'.\end{aligned}\tag{3.155}$$

Now $\widetilde{\Psi}$ is the sum of the $\widetilde{\Psi}_{(1,2,\cdots,15)}$, then we obtain:

$$\begin{aligned}\sum_{i=1}^{15}\widetilde{\Psi}_i=&\left(\frac{399}{256}(h'(0))^2-\frac{29}{32}h''(0)+(\frac{71}{96}+\frac{3i}{32})s_{\partial M}\right)\pi\Omega_3dx'\\&+\frac{(40i+17)h'}{64}\mathrm{tr}[\sigma_n^H-\Phi^*(\widetilde{e_n})]\pi\Omega_3dx'-(\frac{19}{4}+\frac{\pi^2}{2})\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\}\pi\Omega_3dx'\\&-\frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H-\Phi^*(\widetilde{e_n}))]\pi\Omega_3dx'+\frac{(240i-323)h'}{384}\mathrm{tr}[\sigma_n^H+\Phi(\widetilde{e_n})]\pi\Omega_3dx'\\&-\frac{13}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H+\Phi(\widetilde{e_n}))]\pi\Omega_3dx'-\frac{1}{16}\sum_{i=1}^{n}\mathrm{tr}[(\sigma_i^H+\Phi(\widetilde{e_i}))(\sigma_i^H-\Phi^*(\widetilde{e_i}))]\pi\Omega_3dx'\\&-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H-\Phi^*(\widetilde{e_j}))^2]\pi\Omega_3dx'+\frac{3}{4}\mathrm{tr}[(\sigma_n^H-\Phi^*(\widetilde{e_n}))^2]\pi\Omega_3dx'\\&-\frac{7}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H+\Phi(\widetilde{e_j}))^2]\pi\Omega_3dx'+\frac{7}{24}\mathrm{tr}[(\sigma_n^H+\Phi(\widetilde{e_n}))^2]\pi\Omega_3dx'\\&-\frac{7}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\pi\Omega_3dx'.\end{aligned}\tag{3.156}$$

Hence,

**Theorem 3.8** *Let $M$ be a 5-dimensional compact spin manifold with the boundary $\partial M$, and twisted Dirac* *operator* $\widetilde{D}_H^* = \sum_{j=1}^n c(\widetilde{e_j})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_j}} - \sum_{j=1}^n c(\widetilde{e_j})\bigotimes \Phi^*(\widetilde{e_j})$, *then*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+((\widetilde{D}_H^*)^{-1})] = \int_{\partial M}\Bigg[&\left(\frac{399}{256}(h'(0))^2 - \frac{29}{32}h''(0) + (\frac{71}{96}+\frac{3i}{32})s_{\partial M}\right)\\ &+\frac{(40i+17)h'}{64}\mathrm{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})] + \frac{(240i-323)h'}{384}\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]\\ &-(\frac{19}{4}+\frac{\pi^2}{2})\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\} - \frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H - \Phi^*(\widetilde{e_n}))]\\ &-\frac{13}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H + \Phi(\widetilde{e_n}))] - \frac{1}{16}\sum_{i=1}^n \mathrm{tr}[(\sigma_i^H + \Phi(\widetilde{e_i}))(\sigma_i^H - \Phi^*(\widetilde{e_i}))]\\ &-\frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H - \Phi^*(\widetilde{e_j}))^2] + \frac{3}{4}\mathrm{tr}[(\sigma_n^H - \Phi^*(\widetilde{e_n}))^2] - \frac{7}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H + \Phi(\widetilde{e_j}))^2]\\ &+\frac{7}{24}\mathrm{tr}[(\sigma_n^H + \Phi(\widetilde{e_n}))^2] - \frac{7}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\}\Bigg]\pi\Omega_3 dvol_{\partial M}. \qquad (3.157)\end{aligned}$$

Then, we have:

**Theorem 3.9** *Let* $M$ *be a 5-dimensional compact spin manifold with the boundary* $\partial M$, *and twisted Dirac operator* $\widetilde{D}_H^* = \sum_{j=1}^n c(\widetilde{e_j})\nabla^{S(TM)\bigotimes H}_{\widetilde{e_j}} - \sum_{j=1}^n c(\widetilde{e_j})\bigotimes \Phi^*(\widetilde{e_j})$. *The following identity holds:*

$$\begin{aligned}\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1})\circ\pi^+((\widetilde{D}_H^*)^{-1})] = \int_{\partial M}\Bigg[&\left(\frac{225}{64}K^2 + \frac{29}{8}s_M|_{\partial M} + (\frac{197}{24}+\frac{3i}{2})s_{\partial M}\right)\\ &+\left(\frac{(17+40i)}{128}\mathrm{tr}[\sigma_n^H - \Phi^*(\widetilde{e_n})] + \frac{(323-240i)}{768}\mathrm{tr}[\sigma_n^H + \Phi(\widetilde{e_n})]\right)K\\ &-(\frac{19}{4}+\frac{\pi^2}{2})\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H-\Phi^*)]\} - \frac{13}{48}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H + \Phi(\widetilde{e_n}))]\\ &-\frac{1}{16}\sum_{i=1}^n \mathrm{tr}[(\sigma_i^H + \Phi(\widetilde{e_i}))(\sigma_i^H - \Phi^*(\widetilde{e_i}))] - \frac{3}{4}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H - \Phi^*(\widetilde{e_j}))^2]\\ &+\frac{3}{4}\mathrm{tr}[(\sigma_n^H - \Phi^*(\widetilde{e_n}))^2] - \frac{7}{24}\sum_{j=1}^{n-1}\mathrm{tr}[(\sigma_j^H + \Phi(\widetilde{e_j}))^2] + \frac{7}{24}\mathrm{tr}[(\sigma_n^H + \Phi(\widetilde{e_n}))^2]\\ &-\frac{7}{24}\mathrm{tr}\{C_1^1[\nabla^{\partial M}(\sigma^H+\Phi)]\} - \frac{9}{16}\mathrm{tr}[\nabla^H_{\partial_{x_n}}(\sigma_n^H - \Phi^*(\widetilde{e_n}))]\Bigg]\pi\Omega_3 dvol_{\partial M}. \qquad (3.158)\end{aligned}$$

*where* $s_M, s_{\partial M}$ *are respectively scalar curvatures on* $M$ *and* $\partial M$.

## 4. Conclusion

In this paper, we mainly give the proof of the Kastler-Kalau-Walze type theorem for twisted Dirac operators on 5-dimensional compact manifolds with boundary. Firstly, we consider an $n$-dimensional oriented Riemannian manifold $(M, g^M)$ equipped with a fixed spin structure, which are twisted Dirac operators. Secondly, we compute the lower dimensional volume for $\widetilde{D}_H$ and $\widetilde{D}_H^*$ on five dimensional compact spin manifolds with boundary. In other words, we need to compute $Vol_5^{(1,1)}$ for $\widetilde{D}_H$ and $\widetilde{D}_H^*$ on 5-dimensional spin manifolds with boundary, where $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+(\widetilde{D}_H^{-1})] = \int_{\partial M} \Psi$, and $\widetilde{\mathrm{Wres}}[\pi^+(\widetilde{D}_H^{-1}) \circ \pi^+((\widetilde{D}_H^*)^{-1})] = \int_{\partial M} \widetilde{\Psi}$. Thirdly, we divide the boundary term $\int_{\partial M} \Psi$ and $\int_{\partial M} \widetilde{\Psi}$ into fifteen cases by using the boundary term formula of the manifolds with boundary, then the boundary terms of twisted Dirac operators are further given, the Kastler-Kalau-Walze type theorem for twisted Dirac operators on compact manifolds with boundary is proved.

**Acknowledgments**

The third author was supported in part by NSFC No.11771070. The authors are deeply grateful to the referees for their valuable comments and helpful suggestions.

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