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On generalized Darboux frame of a spacelike curve lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$

Dedicated to the memory of Professor Emeritus Krishan Lal Duggal

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Abstract: In this paper we introduce generalized Darboux frame of a spacelike curve α lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$. We prove that α has two such frames and obtain generalized Darboux frame's equations. We find the relations between the curvature functions k_g, k_n, τ_g of α with respect to its Darboux frame and the curvature functions $\tilde{k}_g$, $\tilde{k}_n$, $\tilde{\tau}_g$ with respect to generalized Darboux frames. We show that such frames exist along a spacelike straight line lying on a ruled surface which is not entirely lightlike, but contains some lightlike points. We define lightlike ruled surfaces on which the tangent and the binormal indicatrix of a null Cartan curve are the principal curvature lines having $\tilde{\tau}_g = 0$ and give some examples.

Key words: Generalized Darboux frame, spacelike curve, Darboux frame, lightlike surface, Minkowski space

1. Introduction

Lightlike (null, degenerate) submanifolds represent mathematical realizations of many basic concepts in physical theory, in particular in general relativity. Lightlike hypersurfaces provide models for studying different horizon types (event, Cauchy, Kruskal) and null geodesics are the paths of massless particles. The main property of null submanifolds which distinguish them from semi-Riemannians submanifolds, is that their normal bundle intersects their tangent bundle and that the induced metric on the submanifold is degenerate ([3]). Some characterizations of lightlike surfaces and hypersurfaces in Minkowski space can be found in [2, 4, 11]. Spacelike surfaces immersed in null hypersurfaces are of interest in singularity theory in general relativity and cosmology, where they appear as trapped surfaces ([15]).

Geometric properties of a spacelike curve lying on a lightlike surface in Minkowski space $\mathbb{E}_1^3$ can be expressed in the terms of its curvature functions—geodesic curvature, normal curvature and geodesic torsion, determined by its Darboux frame ([8, 14, 19]). Some special curves lying on surfaces, such as relatively normal-slant helices ([10]), isophote curves ([12]) and Lorentzian Darboux images ([5]), are defined in terms of Darboux frame's vector fields. In particular, the Darboux frame plays an important role in characterizations of k-type spacelike slant helices ([14]), studying normal Darboux images of spacelike curves lying on lightlike surfaces

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([18],[20]), investigations of generalized focal surfaces of spacelike curves lying in lightlike surfaces ([9]), obtaining the relations between pseudo-spherical normal Darboux images of spacelike curves and the lightlike surfaces along those curves ([6]), etc.

In this paper, we show that there is a more general way to frame a spacelike curve α lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$. Namely, we define two new frames along α in such way that α is geodesic, asymptotic, or principal curvature line if and only if the curvature function $\tilde{k}_g$, $\tilde{k}_n$ and $\tilde{\tau}_g$ of α, respectively, with respect to a new frame is equal to zero. Since such frames coincide with the Darboux frame in a special case, we called them generalized Darboux frames of the first and the second kind. We obtain generalized Darboux frame's equations and the relations between the curvature functions of α with respect to its Darboux frame and generalized Darboux frames. We show that such frames also exist along a spacelike straight line lying on a surface which is not entirely lightlike, but contains some lightlike points. We define lightlike ruled surfaces in $\mathbb{E}^3_1$ on which the tangent and the binormal indicatrix of a null Cartan curve are spacelike principal curvature lines having $\tilde{\tau}_g = 0$. Finally, we give some examples.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle\cdot,\cdot\rangle$ is an indefinite metric, a vector $x \neq 0$ in $\mathbb{E}^3_1$ can be spacelike, timelike, or null (lightlike), if $\langle x, x\rangle > 0, \langle x, x\rangle < 0$, or $\langle x, x\rangle = 0$, respectively ([13]). In particular, the vector $x = 0$ is said to be spacelike. The norm (length) of a vector x in $\mathbb{E}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$.

The vector product of vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{E}^3_1$ is defined by ([17])

$$u \times v = (u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can be spacelike, timelike, or null (lightlike), if all of its velocity vectors α' are spacelike, timelike, or null, respectively ([13]).

Frenet frame $\{T, N, B\}$ of spacelike curve α with a nonnull principal normal in $\mathbb{E}^3_1$ is an orthonormal frame consisting of a tangential vector field T, the principal normal vector field N, and the binormal vector field B, such that the next conditions hold ([17])

$$\langle T, T\rangle = 1, \quad \langle N, N\rangle = -\langle B, B\rangle = \epsilon = \pm 1, \quad \langle T, N\rangle = \langle T, B\rangle = \langle N, B\rangle = 0, \tag{2.1}$$

$$T \times N = -\epsilon B, \quad N \times B = T, \quad B \times T = \epsilon N. \tag{2.2}$$

The Frenet frame is positively oriented, if $[T, N, B] = det(T, N, B) = 1$. Frenet frame's equations read ([7])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon\kappa & 0 \\ -\kappa & 0 & -\epsilon\tau \\ 0 & -\epsilon\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where κ and τ are the curvature and the torsion of α, respectively.

A null curve β is called a null Cartan curve, if it is parameterized by pseudo-arc s such that holds $\langle \beta''(s), \beta''(s)\rangle = 1$ ([1]). Cartan frame $\{T, N, B\}$ of null Cartan curve β is a pseudo orthonormal frame consisting of the null tangential vector field T, the spacelike principal normal vector field N, and the null binormal vector field B, satisfying the conditions

$$\langle T, T\rangle = \langle B, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0, \quad \langle N, N\rangle = 1, \quad \langle T, B\rangle = \epsilon_0 = \pm 1, \tag{2.4}$$

$$T \times N = \epsilon_0 T, \quad N \times B = \epsilon_0 B, \quad B \times T = N. \tag{2.5}$$

Cartan frame's equations read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \epsilon_0\tau & 0 & -\epsilon_0\kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.6}$$

where κ and τ are the curvature and the torsion of β, respectively.

A surface M in $\mathbb{E}^3_1$ is called lightlike (null, degenerate), if each tangent plane at regular points of the surface is lightlike ([3],[6]). A point (u_0, t_0) is regular (resp. singular) point of the lightlike surface M in $\mathbb{E}^3_1$ with parametrization $x(u, t)$, if $x_u \times x_t|_{(u_0,t_0)} \neq 0$ (resp. $x_u \times x_t|_{(u_0,t_0)} = 0$).

Darboux frame $\{T, \zeta, \eta\}$ of a spacelike curve α with a nonnull principal normal lying on a lightlike surface with parametrization $x(u, t)$ in $\mathbb{E}^3_1$ is a pseudo orthonormal frame consisting of a spacelike tangential vector field $T = \alpha'$, the null normal vector field

$$\eta = x_u \times x_t|_\alpha$$

and the null vector field ζ satisfying the conditions ([14])

$$\langle T, T\rangle = 1, \quad \langle \eta, \eta\rangle = \langle \zeta, \zeta\rangle = \langle T, \zeta\rangle = \langle T, \eta\rangle = 0, \quad \langle \zeta, \eta\rangle = \epsilon_1 = \pm 1, \tag{2.7}$$

$$T \times \zeta = \epsilon_1 \zeta, \quad \zeta \times \eta = T, \quad \eta \times T = \epsilon_1 \eta. \tag{2.8}$$

Darboux frame's equations read

$$\begin{bmatrix} T' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_1 k_n & \epsilon_1 k_g \\ -k_g & \epsilon_1\tau_g & 0 \\ -k_n & 0 & -\epsilon_1\tau_g \end{bmatrix} \begin{bmatrix} T \\ \zeta \\ \eta \end{bmatrix}, \tag{2.9}$$

where k_n, k_g and τ_g are normal curvature, geodesic curvature and geodesic torsion of α, respectively.

The curve α is called geodesic curve, asymptotic curve and principal curvature line, if its curvature function $k_g(s) = 0$, $k_n(s) = 0$ and $\tau_g(s) = 0$, respectively, for each s. Throughout the next sections, $\mathbb{R}\setminus\{0\}$ will be denoted by $\mathbb{R}_0$.

3. Generalized Darboux frame of a spacelike curve lying on a lightlike surface

In this section, we show that there is a more general way to frame a spacelike curve lying on a lightlike surface in $\mathbb{E}^3_1$ by introducing two new frames along such curve. We call them generalized Darboux frames of the first and the second kind, due to the fact that they coincide with the Darboux frame in a special case. We obtain the generalized Darboux frame's equations and the relations between the curvature functions with respect to Darboux frame and generalized Darboux frames. We also define lightlike ruled surfaces on which the tangent and the binormal indicatrix of a null Cartan curve are spacelike principal curvature lines having the corresponding curvature function with respect to generalized Darboux frame equal to zero.

Throughout this section, denote by $\alpha(s)$ a spacelike curve parameterized by arclength s, with a nonnull principal normal lying on a lightlike surface in Minkowski space $\mathbb{E}^3_1$. The curve α can be geodesic, asumptotic, or principal curvature line. If α has the first curvature $\kappa(s) = 0$, relations (2.3) and (2.9) imply $T'(s) = \epsilon_1 k_n \zeta + \epsilon_1 k_g \eta = 0$. Hence $k_n(s) = k_g(s) = 0$. In particular, if α has $\kappa(s) \neq 0$, by using relations (2.3), and (2.9), we obtain

$$T' = \epsilon \kappa N = \epsilon_1 k_n \zeta + \epsilon_1 k_g \eta.$$

By using the last equation and the condition $\langle N, N\rangle = \epsilon = \pm 1$, we get $\langle T', T'\rangle = \epsilon\kappa^2 = 2\epsilon_1 k_n k_g \neq 0$. It follows $k_n \neq 0$ and $k_g \neq 0$, which proves the first theorem.

Theorem 3.1 *The curve α is geodesic and asymptotic curve if and only if it is a straight line.*

We will define two new frames along α in such way that α is geodesic, asymptotic, or principal curvature line if and only if the corresponding curvature function of α with respect to a new frame is equal to zero at each point of the curve.

In relation to that, let us consider positively oriented pseudo orthonormal frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ along α, satisfying the conditions

$$\tilde{\eta}(s) = \mu\eta, \tag{3.1}$$

$$\langle \tilde{T}, \tilde{T}\rangle = 1, \quad \langle \tilde{T}, \tilde{\zeta}\rangle = \langle \tilde{T}, \tilde{\eta}\rangle = \langle \tilde{\zeta}, \tilde{\zeta}\rangle = \langle \tilde{\eta}, \tilde{\eta}\rangle = 0, \quad \langle \tilde{\zeta}, \tilde{\eta}\rangle = \epsilon_2 = \pm 1, \tag{3.2}$$

$$\tilde{T} \times \tilde{\zeta} = \epsilon_2 \tilde{\zeta}, \quad \tilde{\zeta} \times \tilde{\eta} = \tilde{T}, \quad \tilde{\eta} \times \tilde{T} = \epsilon_2 \tilde{\eta}, \tag{3.3}$$

where $\mu \neq 0$ is some differentiable function. In the next theorem, we obtain the relation between the frames $\{T, \zeta, \eta\}$ and $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$.

Theorem 3.2 *The frames $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$ and $\{T, \zeta, \eta\}$ of α are related by:*

(i)

$$\begin{aligned} &\tilde{T}(s) = T(s) + \lambda(s)\eta(s), \\ &\tilde{\zeta}(s) = -\epsilon_2 \frac{\lambda(s)}{\mu(s)} T(s) + \frac{1}{\mu(s)}\zeta(s) - \epsilon_2 \frac{\lambda^2(s)}{2\mu(s)}\eta(s), \\ &\tilde{\eta}(s) = \mu(s)\eta(s), \end{aligned} \tag{3.4}$$

where $\mu \neq 0$, $\lambda \neq 0$ are some differentiable functions and $\epsilon_2 = \epsilon_1 = \pm 1$;

(ii)

$$\begin{aligned} &\tilde{T}(s) = T(s), \\ &\tilde{\zeta}(s) = \frac{1}{\mu(s)}\zeta(s), \\ &\tilde{\eta}(s) = \mu(s)\eta(s), \end{aligned} \tag{3.5}$$

where $\mu \neq 0$ is some differentiable function.

Proof Assume that α has Darboux frame whose vector fields satisfy relations (2.7) and (2.8) and pseudo-orthonormal frame $\{\tilde{T}, \tilde{\zeta}, \tilde{\eta}\}$, whose vector fields satisfy relations (3.1), (3.2), and (3.3). Since the vector field $\tilde{\eta}$ is a lightlike, its orthogonal complement $\tilde{\eta}^{\perp}$ is also lightlike and represents a lightlike tangent plane of the surface. The conditions (2.7), (3.1) and (3.2) imply $\langle \tilde{T}, \tilde{\eta} \rangle = \langle \eta, \tilde{\eta} \rangle = \langle T, \tilde{\eta} \rangle = 0$. Hence $\tilde{T}, \eta, T \in \tilde{\eta}^{\perp}$, so the unit spacelike vector field $\tilde{T}$ can be written as

$$\tilde{T}(s) = T(s) + \lambda(s)\eta(s), \tag{3.6}$$

where $\lambda(s)$ is some differentiable function. If $\lambda(s) \neq 0$ for each s, by using (3.1), (3.2), (3.3) and (3.6), we get

$$\tilde{\zeta}(s) = -\epsilon_2 \frac{\lambda(s)}{\mu(s)} T(s) + \frac{1}{\mu(s)} \zeta(s) - \epsilon_2 \frac{\lambda^2(s)}{2\mu(s)} \eta(s), \tag{3.7}$$

where $\langle \tilde{\zeta}, \tilde{\eta} \rangle = \epsilon_2 = \langle \zeta, \eta \rangle = \epsilon_1 = \pm 1$. Relations (3.1), (3.6), and (3.7) give relation (3.4). If $\lambda(s) = 0$ for each s, by using (3.1), (3.2) and (3.6), we find $\tilde{T}(s) = T(s)$ and $\tilde{\zeta}(s) = \frac{1}{\mu(s)} \zeta(s)$. Hence we get the frame given by relation (3.5), where $\langle \tilde{\zeta}, \tilde{\eta} \rangle = \epsilon_2 = \langle \zeta, \eta \rangle = \epsilon_1 = \pm 1$. □

Remark 3.3 *The frames given by relations (3.4) and (3.5) coincide with the Darboux frame $\{T, \zeta, \eta\}$, if $\mu(s) = 1$ and $\lambda(s) = 0$ for each s.*

Remark 3.4 *In the proof of Theorem 3.2, assuming that the unit spacelike vector field $\tilde{T}$ can be written as $\tilde{T} = T - \lambda\eta$ and using the conditions (3.1), (3.2), and (3.3), we obtain*

$$\tilde{\zeta}(s) = \epsilon_2 \frac{\lambda(s)}{\mu(s)} T(s) + \frac{1}{\mu(s)} \zeta(s) - \epsilon_2 \frac{\lambda^2(s)}{2\mu(s)} \eta(s).$$

Consequently, we get the frame which coincides with the frame given by relation (3.4), up to isometries of $\mathbb{E}^3_1$.

We define the curvature functions with respect to new frames given by (3.4) and (3.5) as follows.

Definition 3.5 *The curvature functions of α defined by*

$$\tilde{k}_n = \langle \tilde{T}', \tilde{\eta} \rangle, \quad \tilde{k}_g = \langle \tilde{T}', \tilde{\zeta} \rangle, \quad \tilde{\tau}_g = \langle \tilde{\zeta}', \tilde{\eta} \rangle. \tag{3.8}$$

are called generalized normal curvature, generalized geodesic curvature and generalized geodesic torsion respectively.

By using relation (3.8), we obtain the next theorem which can be easily proved; therefore, we omit its proof.

Theorem 3.6 *If α has the frame given by relation (3.4) or (3.5), then the frame's equations read*

$$\begin{bmatrix} \tilde{T}' \\ \tilde{\zeta}' \\ \tilde{\eta}' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 \tilde{k}_n & \epsilon_2 \tilde{k}_g \\ -\tilde{k}_g & \epsilon_2 \tilde{\tau}_g & 0 \\ -\tilde{k}_n & 0 & -\epsilon_2 \tilde{\tau}_g \end{bmatrix} \begin{bmatrix} \tilde{T} \\ \tilde{\zeta} \\ \tilde{\eta} \end{bmatrix}. \tag{3.9}$$

Now it is natural to ask what is the relation between the curvature functions k_g, k_n, τ_g of α with respect to Darboux frame and the curvature functions $\tilde{k}_g$, $\tilde{k}_n$ and $\tilde{\tau}_g$ with respect to introduced frames.

Theorem 3.7 *The curvature functions of α with respect to the frame given by (3.4) or (3.5) and the curvature functions with respect to Darboux frame, are related by*

$$\begin{aligned}
\tilde{k}_n &= \mu k_n, \\
\tilde{\tau}_g &= -\epsilon_2 \lambda k_n - \epsilon_2 \frac{\mu'}{\mu} + \tau_g, \\
\tilde{k}_g &= \frac{1}{\mu} k_g + \epsilon_2 \frac{1}{\mu} \lambda' - \frac{1}{\mu} \lambda \tau_g + \epsilon_2 \frac{1}{2\mu} \lambda^2 k_n,
\end{aligned} \tag{3.10}$$

where $\mu \neq 0$, $\lambda = 0$ if α has the frame given by (3.5), or $\mu \neq 0$, $\lambda \neq 0$ if α has the frame given by (3.4).

Proof Differentiating the relation $\tilde{\eta} = \mu\eta$ with respect to s and using (2.9), we obtain

$$\tilde{\eta}' = (-\mu k_n)T + (\mu' - \epsilon_1 \mu \tau_g)\eta. \tag{3.11}$$

According to relations (3.4) and (3.9), we have

$$\tilde{\eta}' = -\tilde{k}_n \tilde{T} - \epsilon_2 \tilde{\tau}_g \tilde{\eta} = (-\tilde{k}_n)T + (-\lambda \tilde{k}_n - \epsilon_2 \mu \tilde{\tau}_g)\eta. \tag{3.12}$$

Relations (3.11) and (3.12) give $\tilde{k}_n = \mu k_n$ and $\tilde{\tau}_g = -\epsilon_2 \lambda k_n - \epsilon_2 \frac{\mu'}{\mu} + \tau_g$. From (3.4) and (3.8) we obtain

$$\tilde{k}_g = \langle \tilde{T}', \tilde{\zeta} \rangle = \langle T' + \lambda'\eta + \lambda\eta', -\epsilon_2 \frac{\lambda}{\mu} T + \frac{1}{\mu}\zeta - \epsilon_2 \frac{\lambda^2}{2\mu}\eta \rangle.$$

By using (2.7), (2.9) and the last equation, we get

$$\tilde{k}_g = \frac{1}{\mu} k_g + \epsilon_2 \frac{1}{\mu} \lambda' - \frac{1}{\mu} \lambda \tau_g + \epsilon_2 \frac{1}{2\mu} \lambda^2 k_n,$$

which completes the proof. □

We define generalized Darboux frame of the first kind of α as follows.

Definition 3.8 *Generalized Darboux frame of the first kind of α is the frame given by (3.4), where the function $\lambda \neq 0$ satisfies Riccati differential equation*

$$2\epsilon_2 \lambda'(s) - 2\lambda(s)\tau_g(s) + \epsilon_2 \lambda^2(s) k_n(s) = 0, \tag{3.13}$$

and the function $\mu \neq 0$ satisfies differential equation

$$\mu'(s) + \mu(s)\lambda(s)k_n(s) = 0. \tag{3.14}$$

By using relations (3.12), (3.13) and (3.14), we get the next statement.

Theorem 3.9 *The curvature functions of* α *with respect to the Darboux frame and generalized Darboux frame of the first kind, are related by*

$$\begin{aligned} \tilde{k}_n(s) &= \mu(s)k_n(s), \\ \tilde{\tau}_g(s) &= \tau_g(s), \\ \tilde{k}_g(s) &= \frac{1}{\mu(s)}k_g(s), \end{aligned} \tag{3.15}$$

where $\mu \neq 0$ *satisfies differential equation (3.14).*

We analogously define the generalized Darboux frame of the second kind of α.

Definition 3.10 *Generalized Darboux frame of the second kind of* α *is the frame given by (3.5), where* $\mu(s) \in \mathbb{R}_0$ *for each* s.

In particular, substituting $\mu(s) = \mu_0$ and $\lambda = 0$ in (3.10), we get the following statement.

Theorem 3.11 *The curvature functions of* α *with respect to the Darboux frame and generalized Darboux frame of the second kind, are related by*

$$\begin{aligned} \tilde{k}_n(s) &= \mu_0 k_n(s), \\ \tilde{\tau}_g(s) &= \tau_g(s), \\ \tilde{k}_g(s) &= \frac{1}{\mu_0}k_g(s), \end{aligned} \tag{3.16}$$

where $\mu_0 \in \mathbb{R}_0$.

It can be easily seen that Theorems 3.9 and 3.11 imply the next property.

Theorem 3.12 *The curve* α *is geodesic, asymptotic, or principal curvature line if and only if it has the curvature function* $\tilde{k}_g(s) = 0$, $\tilde{k}_n(s) = 0$ *and* $\tilde{\tau}_g(s) = 0$, *respectively, for each* s.

In the next two theorems, we prove that Cartan frame's vector fields of a null Cartan curve β generate a lightlike ruled surfaces on which the tangent and the binormal indicatrices of β are the spacelike principal curvature lines having the curvature $\tilde{\tau}_g(s) = 0$.

Theorem 3.13 *Let* $\beta(s)$ *be a null Cartan curve in* $\mathbb{E}^3_1$ *parameterized by pseudo arc* s *with torsion* $\tau_\beta(s) \neq$ *constant and Cartan frame* $\{T_\beta, N_\beta, B_\beta\}$. *Then the surface* M *with parametrization*

$$x(s,t) = T_\beta(s) + tB_\beta(s), \tag{3.17}$$

where $t \neq \frac{1}{\tau_\beta(s)}$ *is a lightlike ruled surface, on which tangent indicatrix* T_β *of* β *is a spacelike principal curvature line having generalized geodesic torsion* $\tilde{\tau}_g(s) = 0$.

Proof Assume that $\beta(s)$ is a null Cartan curve parameterized by pseudo arc s with Cartan frame $\{T_\beta, N_\beta, B_\beta\}$ and torsion $\tau_\beta(s) \neq$ *constant*. Let us consider a ruled surface M with parametrization (3.17). By taking partial derivatives of relation (3.17) with respect to s and t and using (2.5) and (2.6), we find that the normal vector field of M is given by

$$N(s,t) = x_s \times x_t = \epsilon_0(1 - t\tau_\beta(s))B_\beta(s).$$

Assume that $t \neq \frac{1}{\tau_\beta(s)}$. Then $N(s,t)$ is lightlike vector field, so M is a lightlike ruled surface. Denote by $\alpha(s) = T_\beta(s)$ tangent indicatrix of β and by $\{T, \zeta, \eta\}$ the Darboux frame along α. Relations (2.4) and (2.6) imply

$$T(s) = N_\beta(s), \quad \zeta(s) = \epsilon_1 T_\beta(s), \quad \eta(s) = -\epsilon_1 B_\beta(s). \tag{3.18}$$

Since $\langle T_\beta, B_\beta \rangle = \epsilon_0 = -\langle \zeta, \eta \rangle = -\epsilon_1$, relations (2.6) and (3.18) yield $T'(s) = \epsilon_0(\tau_\beta(s)T_\beta(s) - B_\beta(s))$. Thus $\langle T', T' \rangle = -2\epsilon_0 \tau_\beta \neq 0$, which implies that α has a nonnull principal normal. From (2.4),(2.6) and (3.18), we get $\tau_g(s) = \langle \zeta'(s), \eta(s) \rangle = -\epsilon_0 \langle N_\beta, B_\beta \rangle = 0$. Hence α is the principal curvature line. Theorems 3.9 and 3.11 imply that α has generalized geodesic torsion $\tilde{\tau}_g(s) = 0$. □

Theorem 3.14 *Let $\beta(s)$ be a null Cartan helix in $\mathbb{E}^3_1$ with torsion $\tau_\beta(s) = c_0 \in \mathbb{R}_0$ and Cartan frame $\{T_\beta, N_\beta, B_\beta\}$. Then the surface M with parametrization*

$$x(s,t) = B_\beta(s) + tT_\beta(s), \tag{3.19}$$

where $t \neq c_0$ is a lightlike ruled surface, on which binormal indicatrix B_β of $\beta(s)$ is a spacelike principal curvature line having generalized geodesic torsion $\tilde{\tau}_g(s) = 0$.

Proof Assume that $\beta(s)$ is a null Cartan helix parameterized by pseudo arc s with Cartan frame $\{T_\beta, N_\beta, B_\beta\}$ and torsion $\tau_\beta(s) = c_0 \in \mathbb{R}_0$ for each s. Let us consider a ruled surface M with parametrization (3.19). The normal vector field of M is given by

$$N(s,t) = x_s \times x_t = \epsilon_0(c_0 - t)T_\beta(s).$$

Assume that $t \neq c_0$. Then $N(s,t)$ is lightlike vector field, which implies that M is a lightlike ruled surface. Denote by $\alpha(s) = B_\beta(s)$ binormal indicatrix of β and by $\{T, \zeta, \eta\}$ the Darboux frame along α. By using (2.6), we obtain

$$T = -sgn(c_0)N_\beta, \quad \zeta = -\epsilon_0 \frac{sgn(c_0)}{c_0} B_\beta, \quad \eta = \epsilon_0 c_0 T_\beta, \tag{3.20}$$

where $\epsilon_1 = \epsilon_0$. From (2.6) and (3.20) we get $T' = sgn(c_0)\epsilon_0(B_\beta - c_0 T_\beta)$ and thus $\langle T', T' \rangle = -2\epsilon_0 c_0 \neq 0$. Consequently, α has a nonnull principal normal. By using (2.4), (2.6) and (3.20), we find $\tau_g(s) = \langle \zeta'(s), \eta(s) \rangle = 0$. Hence α is the principal curvature line. According to Theorems 3.9 and 3.11, it follows that α has generalized geodesic torsion $\tilde{\tau}_g(s) = 0$. □

4. Some examples

In this section, we give some examples of generalized Darboux frames of spacelike curves with a nonnull principal normal lying on the lightlike surfaces.

Example 4.1 *Let us consider a translation surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 1)*

$$x(s,t) = \alpha(s) + \beta(t),$$

where $\alpha(s)$ *is a spacelike straight line given by* $\alpha(s) = (s, s\sqrt{2}, 7)$ *and the curve* $\beta(t)$ *has parameter equation*

$$\beta(t) = \frac{1}{4}(\sqrt{2}\sinh t - t^2, (1-\sqrt{2})t^2, t^2 - 2\sinh t), \qquad s, t \in \mathbb{R}.$$

The normal vector field of M *has the form*

$$x_s \times x_t = (\cosh t - t)(\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}).$$

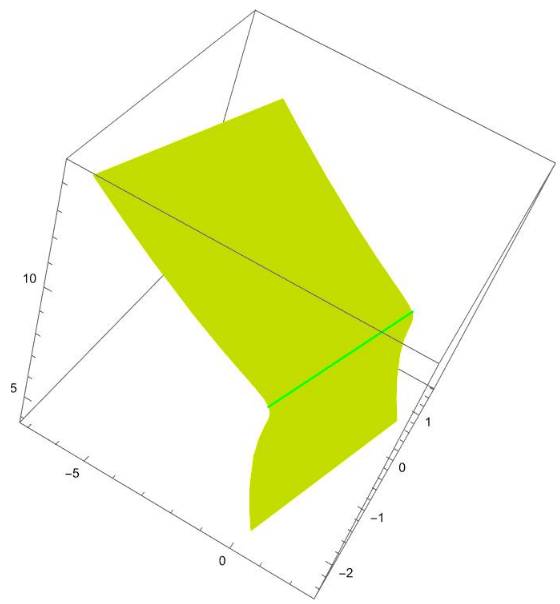

Figure 1. Translation surface M with a spacelike straight line α.

Since $x_s \times x_t \neq 0$ *and* $\langle x_s \times x_t, x_s \times x_t \rangle = 0$, *it follows that* M *is a lightlike translation surface. Darboux frame of* $\alpha(s) = x(s, 0)$ *has the form*

$$T(s) = (1, \sqrt{2}, 0), \quad \zeta(s) = (\sqrt{2}, 1, 1), \quad \eta(s) = x_s \times x_t|_\alpha = (\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}).$$

Hence the curvature functions of α *read*

$$k_g(s) = k_n(s) = \tau_g(s) = 0. \tag{4.1}$$

Substituting (4.1) in (3.13) and (3.14), we obtain $\lambda(s) = \lambda_0 \in \mathbb{R}_0$ *and* $\mu(s) = \mu_0 \in \mathbb{R}_0$. *According to Definitions 3.8 and 3.10, we find that generalized Darboux frame of the first kind of* α *reads*

$$\tilde{T} = T + \lambda_0\eta, \quad \tilde{\zeta} = -\frac{\lambda_0}{\mu_0}T + \frac{1}{\mu_0}\zeta - \frac{\lambda_0^2}{2\mu_0}\eta, \quad \tilde{\eta} = \mu_0\eta,$$

while generalized Darboux frame of the second kind has the form

$$\tilde{T} = T, \quad \tilde{\zeta} = \frac{1}{\mu_0}\zeta, \quad \tilde{\eta} = \mu_0\eta.$$

Since the vector fields $\tilde{T}$, $\tilde{\zeta}$ and $\tilde{\eta}$ are constant, we easily get

$$\tilde{k}_g(s) = \tilde{k}_n(s) = \tilde{\tau}_g(s) = 0. \tag{4.2}$$

Relations (4.1) and (4.2) imply that statement of Theorem 3.12 holds.

In the next example, we show that generalized Darboux frames exist along a spacelike straight line lying on the ruled surface which is not entirely lightlike, but contains some lightlike points, i.e. a regular points of the surface at which the tangent plane is a lightlike ([16]).

Example 4.2 *Let us consider ruled surface M in $\mathbb{E}^3_1$ with parametrization (see Figure 2)*

$$x(s,t) = (s\sinh s - \cosh s, s\cosh s - \sinh s, \frac{s^2}{2}) + t(\sinh s, \cosh s, s),$$

where $s \in \mathbb{R}$, $t \in \mathbb{R}_0$. A straightforward calculation shows that the normal vector field of M is given by

$$N(s,t) = x_s \times x_t = (s+t)(-s\sinh s + \cosh s, -s\cosh s + \sinh s, 1).$$

The last relation implies $\langle N(s,t), N(s,t)\rangle = s^2(s+t)^2$. Thus $N(s,t)$ is a lightlike vector field, if $s=0$ and $t \neq 0$. Hence M is not entirely the lightlike surface, but contains a set of regular points $x(0,t)$, at which the tangent plane is a lightlike.

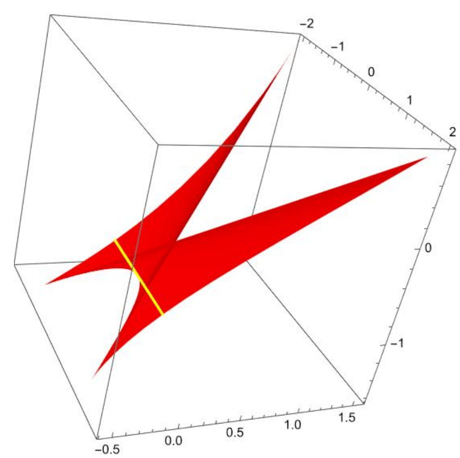

Figure 2. The ruled surface M with a spacelike straight line β.

Such points belong to a spacelike straight line $\beta(t) = x(0,t) = (-1,t,0)$. Since Darboux frame along β reads

$$T(t) = (0,1,0), \quad \zeta(t) = \left(-\frac{1}{2t}, 0, \frac{1}{2t}\right), \quad \eta(t) = x_s \times x_t|_\beta = (t,0,t),$$

the curvature functions of β have the form

$$k_g(t) = k_n(t) = 0, \quad \tau_g(t) = -\frac{1}{t}. \tag{4.3}$$

Substituting (4.3) in (3.13) and (3.14) and using relation $\epsilon_2 = \epsilon_1 = 1$, *we get*

$$\lambda(t) = \frac{1}{t}, \quad \mu(t) = \mu_0 \in \mathbb{R}_0. \tag{4.4}$$

By using relations (3.4), (3.5) and (4.4), we find that generalized Darboux frame of the first kind of β *reads*

$$\tilde{T}(t) = (1,1,1), \quad \tilde{\zeta}(t) = (-\frac{1}{\mu_0 t}, -\frac{1}{\mu_0 t}, 0), \quad \tilde{\eta}(t) = (\mu_0 t, 0, \mu_0 t),$$

and that the generalized Darboux frame of the second kind of β *is given by*

$$\tilde{T}(t) = (0,1,0), \quad \tilde{\zeta}(t) = \frac{1}{\mu_0}(-\frac{1}{2t}, 0, \frac{1}{2t}), \quad \tilde{\eta}(t) = \mu_0(t, 0, t).$$

It can be easily checked that curvature functions with respect to generalized Darboux frame of the first and the second kind have to form

$$\tilde{k}_g(t) = \tilde{k}_n(t) = 0, \quad \tilde{\tau}_g(t) = -\frac{1}{t}. \tag{4.5}$$

By using (4.3) and (4.5), it follows that statement of Theorem 3.12 holds.

Example 4.3 *Let us consider a lightlike cone* C *in* $\mathbb{E}^3_1$ *with parametrization (see Figure 3)*

$$x(s,t) = t\alpha(s),$$

where $(0,0,0) \notin C$ *and* $\alpha(s) = (\cosh s, \sinh s, 1)$ *is a spacelike circle with a timelike principal normal. The normal vector field of* C *has the form*

$$N(s,t) = x_s \times x_t = -t\alpha(s).$$

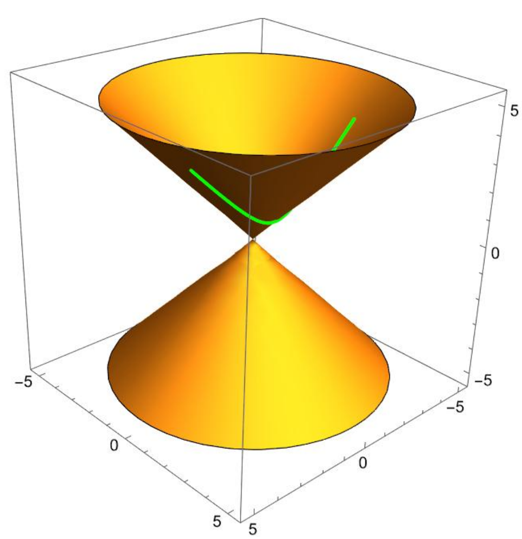

Figure 3. Lightlike cone C with a spacelike curve α.

Since $(0,0,0) \notin C$, it follows that $N(s,t) \neq (0,0,0)$. In particular, we have $\langle N(s,t), N(s,t)\rangle = 0$, which implies that $N(s,t)$ is a lightlike at each regular point of C. A straightforward calculation shows that Darboux frame along α reads

$$\begin{aligned}
&T(s) = (\sinh s, \cosh s, 0),\\
&\zeta(s) = \left(\frac{1}{2}\cosh s, \frac{1}{2}\sinh s, -\frac{1}{2}\right),\\
&\eta(s) = x_s \times x_t|_\alpha = (-\cosh s, -\sinh s, -1)\,.
\end{aligned}$$

The curvature functions of α with respect to Darboux frame are given by

$$k_g(s) = -\frac{1}{2}, \quad k_n(s) = 1, \quad \tau_g(s) = 0. \tag{4.6}$$

Since $\epsilon_2 = \epsilon_1 = 1$, from (3.13), (3.14) and (4.6), we find

$$\lambda(s) = \frac{2}{s}, \quad \mu(s) = \frac{1}{s^2}.$$

Substituting this in (3.4), it follows that generalized Darboux frame of the first kind of α reads

$$\tilde{T}(s) = T(s) + \frac{2}{s}\eta(s), \quad \tilde{\zeta}(s) = -2sT(s) + s^2\zeta(s) - 2\eta(s), \quad \tilde{\eta}(s) = \frac{1}{s^2}\eta(s). \tag{4.7}$$

By using (3.8) and (4.7), we find

$$\tilde{k}_g(s) = -\frac{s^2}{2}, \quad \tilde{k}_n(s) = \frac{1}{s^2}, \quad \tilde{\tau}_g(s) = 0. \tag{4.8}$$

According to Definition 3.10, generalized Darboux frame of the second kind of α has the form

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s), \tag{4.9}$$

where $\mu_0 \in \mathbb{R}_0$. By using (3.8) and (4.9), we get

$$\tilde{k}_g(s) = -\frac{1}{2\mu_0}, \quad \tilde{k}_n(s) = \mu_0, \quad \tilde{\tau}_g(s) = 0. \tag{4.10}$$

Consequently, relations (4.6), (4.8) and (4.10) imply that the statement of Theorem 3.12 holds.

Example 4.4 *Let β be a null Cartan curve in $\mathbb{E}^3_1$ with the parameter equation*

$$\beta(s) = \left(\frac{s^4}{4} + \frac{1}{16}\ln s, \frac{s^4}{4} - \frac{1}{16}\ln s, \frac{s^2}{4}\right),$$

where $s \in \mathbb{R}^+$. A straightforward calculation yields that Cartan frame of β has the form

$$\begin{aligned}
&T_\beta(s) = \left(s^3 + \frac{1}{16s}, s^3 - \frac{1}{16s}, \frac{s}{2}\right),\\
&N_\beta(s) = (3s^2 - \frac{1}{16s^2}, 3s^2 + \frac{1}{16s^2}, \frac{1}{2}),\\
&B_\beta(s) = \left(-\frac{9s}{2} - \frac{1}{32s^3}, -\frac{9s}{2} + \frac{1}{32s^3}, \frac{3}{4s}\right).
\end{aligned} \tag{4.11}$$

By using (2.4), (2.6) and (4.11), it follows that the torsion of β *reads*

$$\tau_\beta(s) = \langle N'_\beta(s), B_\beta(s)\rangle = \frac{3}{2s^2}.$$

Let M *be a ruled surface with parametrization (see Figure 4)*

$$x(s,t) = T_\beta(s) + tB_\beta(s),$$

where $s \in \mathbb{R}^+$ *and* $t \in \mathbb{R}$*. The normal vector field of* M *is given by*

$$N(s,t) = x_s \times x_t = \frac{2s^2 - 3t}{2s^2} B_\beta(s).$$

Assume that $t \neq \frac{1}{\tau_\beta(s)} = \frac{2s^2}{3}$*. Then* $N(s,t)$ *is a lightlike vector field, so* M *is a lightlike ruled surface. It can be easily verified that tangent indicatrix* $T_\beta(s)$ *of* β *is a spacelike curve with a nonnull principal normal.*

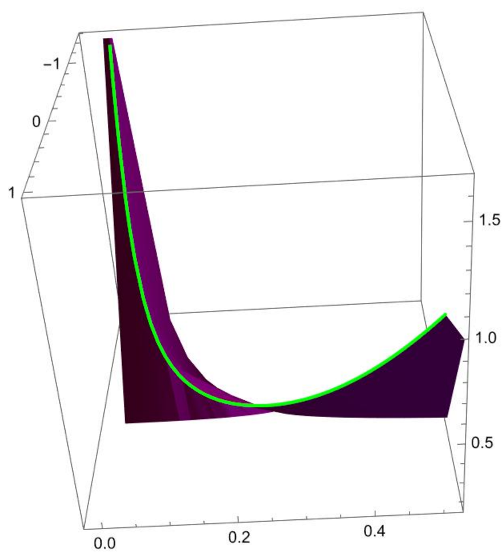

Figure 4. Lightlike ruled surface M with a spacelike base curve T_β.

Darboux frame $\{T, \zeta, \eta\}$ *along* T_β *reads*

$$T(s) = \left(3s^2 - \frac{1}{16s^2}, 3s^2 + \frac{1}{16s^2}, \frac{1}{2}\right),$$

$$\zeta(s) = -\left(s^3 + \frac{1}{16s}, s^3 - \frac{1}{16s}, \frac{s}{2}\right),$$

$$\eta(s) = \left(-\frac{9s}{2} - \frac{1}{32s^3}, -\frac{9s}{2} + \frac{1}{32s^3}, \frac{3}{4s}\right).$$

Consequently, the curvature functions of T_β *have the form*

$$k_g(s) = 1, \quad k_n(s) = \frac{3}{2s^2}, \quad \tau_g(s) = 0. \tag{4.12}$$

Since $\epsilon_2 = \epsilon_1 = -1$, by using (3.13), (3.14) and (4.12), we find

$$\lambda(s) = -\frac{4s}{3}, \quad \mu(s) = s^2.$$

According to Definition 3.8, generalized Darboux frame of the first kind of T_β reads

$$\begin{aligned}\tilde{T}(s) &= T(s) - \frac{4s}{3}\eta(s),\\ \tilde{\zeta}(s) &= -\frac{4}{3s}T(s) + \frac{1}{s^2}\zeta(s) + \frac{8}{9}\eta(s),\\ \tilde{\eta}(s) &= s^2\eta(s).\end{aligned} \tag{4.13}$$

Relations (3.8) and (4.13) yield

$$\tilde{k}_g(s) = \frac{1}{s^2}, \quad \tilde{k}_n(s) = \frac{3}{2}, \quad \tilde{\tau}_g(s) = 0. \tag{4.14}$$

In particular, generalized Darboux frame of the second kind of α has the form

$$\tilde{T}(s) = T(s), \quad \tilde{\zeta}(s) = \frac{1}{\mu_0}\zeta(s), \quad \tilde{\eta}(s) = \mu_0\eta(s), \tag{4.15}$$

where $\mu_0 \in \mathbb{R}_0$. By using relations (3.8), (4.12) and (4.15), we get

$$\tilde{k}_g(s) = \frac{1}{\mu_0}, \quad \tilde{k}_n(s) = \frac{3\mu_0}{2s^2}, \quad \tilde{\tau}_g(s) = 0. \tag{4.16}$$

Finally, relations (4.12), (4.14) and (4.16) imply that statement of Theorem 3.12 holds.

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