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On the relation between oscillation of solutions of differential equations and corresponding equations on time scales

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Abstract: This paper studies oscillatory properties of solutions of a dynamic equation on the set of time scales $\mathbf{T}_\lambda$ provided that the graininess function μ_λ approaches zero as $\lambda \to 0$. We derived the conditions under which oscillation of solutions of differential equations implies that of solutions of the corresponding equations defined on time scales with the same initial data, and vice versa.

Key words: Time scale, graininess function, oscillation, $\Delta-$derivative, generalized zero

1. Introduction

The theory of differential (dynamic) equations on time scales was introduced by Hilger in [16] as a way to unify the fields of discrete and continuous dynamical systems. This theory was further developed in [5, 6]. Analogues of classical results were obtained in the theory of ordinary differential equations, the optimal control theory [9, 11, 12], stochastic systems on time scales [7], etc. Some interesting results on the topological structure of Cantor-type time scales were obtained in [21]. We are especially interested in the behavior of the solutions of dynamic equations that are defined on a family of time scales $\mathbb{T}_\lambda$ when the graininess function μ_λ goes to zero as $\lambda \to 0$. In such a case, the intervals of the time scale $[t_0, t_1]_\lambda = [t_0, t_1] \cap \mathbb{T}_\lambda$ approach $[t_0, t_1]$ (for instance, in the Hausdorff metric). The question naturally arises as to whether solutions of equations on time scales and the corresponding differential equations share the same properties.

In this paper, we study the relation between the oscillatory behavior of solutions of second-order linear differential equations and solutions of corresponding equations on time scales. We show that if the graininess function is small enough, oscillation of solutions of a differential equation is equivalent to that of the corresponding equation on a time scale.

Oscillatory properties of solutions of differential equations are essential in various areas since the time of Sturm: partial differential equations (Sturm-Liouville problems; see, for example, [14]), spectral theory [24], quantum mechanics [15], etc. The fundamentals of Sturm's theory on oscillation of solutions of second-order linear differential equations are provided in numerous classic textbooks; see, for instance, [22]. The basics of

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the analogous theory for equations on time scales are well presented in the monographs [1, 2] and papers [3, 4]. The issues considered in this paper were previously studied in [8, 23] for a special case of Eulerian time scales $\mathbb{T} = nh$, $n \in N$, $h > 0$, i.e. for difference equations. In the present work, we provide stronger results.

Firstly, in the aforementioned papers, the time scales are discrete: all points on scales are isolated, and the Δ-derivative turns into a difference ratio. Here, we consider general time scales that allow the presence of limit points as well, which greatly complicates the study.

Secondly, the main result of [8, 23] generally means the following: if a solution of a differential equation has at least three zeros on an interval, then the solution of the corresponding difference equation with a sufficiently small step oscillates on the interval (has at least two generalized zeros), and vice versa. In the present paper, we obtain a more accurate result: for a sufficiently small graininess function μ_λ, the numbers of zeros of the corresponding solutions of differential and dynamic equations on the interval coincide, and these zeros approach each other as $\mu_\lambda \to 0$.

We note that the issues of preserving various properties of solutions in the transition from differential to dynamical systems and vice versa were considered earlier in the following papers. In [18, 19], the connection between the existence of solutions of differential equations and dynamical equations bounded on the axis is studied. The paper [20] considers the optimal control problems for differential equations and corresponding dynamical equations on time scales.

The rest of this paper is organized as follows. Section 2.1 provides some basic concepts of the theory of time scales. In Section 2.2, we state the problem and obtain some auxiliary results. We state the main results and prove them in Section 2.3. Section 2.4 considers a weakly nonlinear case. A numerical illustration of the results obtained is given on the example of the Airy equation in Section 2.5.

2. Auxiliary and main results

2.1. Basic concepts of the theory of time scales

We present some basic notions of the theory of time scales [5].

- A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real axis.
- For every set $A \subset \mathbb{R}$, we define $A_{\mathbb{T}} := A \cap \mathbb{T}$.
- For a time scale $\mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}$, and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} | s < t\}$.

  Hereinafter we assume $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$.
- The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) := \sigma(t) - t$.
- A point $t \in \mathbb{T}$ is called *left-dense* (LD) (*left-scattered* (LS), *right-dense* (RD), or *right-scattered* (RS)), if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively).
- If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.
- A function $f : \mathbb{T} \to \mathbb{R}^d$ is called Δ-*differentiable* at $t \in \mathbb{T}^k$ if the finite limit

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(t)}{\sigma - t}$$

exists in $\mathbb{R}^d$, and the number $f^{\Delta}(t)$ is called the Δ-*derivative* at the point t.

We cite some known results [5]:

(a) If $t \in \mathbb{T}^k$ is a right-dense point of a time scale $\mathbb{T}$, then f is Δ-differentiable at t iff the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists in $\mathbb{R}^d$.

(b) If $t \in \mathbb{T}^k$ is a right-scattered point of a time scale $\mathbb{T}$ and f is continuous at t, then f is Δ-differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

2.2. Problem statement and auxiliary results

We consider the second-order linear equation on the interval $[0, a]$

$$\ddot{x} + p(t)x = 0, \tag{2.1}$$

with $p \in C^1([0, a])$, and the corresponding dynamic equation defined in the set of time scales $\mathbb{T}_\lambda$

$$x_\lambda^{\Delta\Delta} + p(t)x_\lambda = 0, \tag{2.2}$$

where $t \in \mathbb{T}_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}$, $\lambda = 0$ is a limit point of the set Λ, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, and $x_\lambda^{\Delta}(t)$ is the Δ-derivative of $x_\lambda(t)$ in $\mathbb{T}_\lambda$.

Let $\mu_\lambda := \sup_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda : \mathbb{T}_\lambda \to [0, \infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ approaches the continuous time scale $\mathbb{T}_0 = [0, a]$.

Since $p(t)$ is continuous on $[0,\ a]$, equation (2.1) with arbitrary initial data at $t_0 \in [0, a]$ has a unique solution on $[0,\ a]$ (see [22]).

Definition 2.1 *A solution $x_\lambda(t)$ of equation (2.2) has a generalized zero at $t \in \mathbb{T}_\lambda$, if one of the following conditions holds:*

(i) $x_\lambda(t) = 0$*;*

(ii) if t is right-scattered, then $x_\lambda(t) \cdot x_\lambda(\sigma(t)) < 0$.

Remark 2.2 *In monograph [6], a generalized zero is defined in the following way: a solution $x(t)$ of a dynamic equation has a generalized zero at a point t if $x(t) = 0$, or $x(\rho(t))x(t) < 0$ provided t is left-scattered. However, the definition via the forward jump operator is more suitable for our reasoning.*

Definition 2.3 *A solution $x_\lambda(t)$ is called oscillatory on an interval if it has at least two generalized zeros on this interval.*

Definition 2.4 *The solutions $x(t)$ and x_λ of equations (2.1) and (2.2), respectively, are called corresponding solutions if $x(t_0) = x_\lambda(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$.*

In what follows, we will need the following result on corresponding solutions.

Let us consider the system of differential equations

$$\frac{dx}{dt} = X(t,x), \; t \in [0,a], \tag{2.3}$$

where $x \in D, \; D \subset \mathbb{R}^d$ is a domain in the space $\mathbb{R}^d$, and the corresponding system of equations defined on $\mathbb{T}_\lambda$

$$x_\lambda^\Delta = X(t, x_\lambda). \tag{2.4}$$

.

Here system (2.3) corresponding to differential equation (2.1) is of the form

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = -p(t)x, \end{cases} \tag{2.5}$$

and system (2.4) corresponding to dynamic equation (2.2) is of the form

$$\begin{cases} \frac{\Delta x}{\Delta t} = y_\lambda, \\ \frac{\Delta y}{\Delta t} = -p(t)x_\lambda. \end{cases} \tag{2.6}$$

Assume that $X(t,x)$ is continuously differentiable and bounded with its partial derivatives, i.e. there exists $C > 0$ such that

$$|X(t,x)| + \left|\frac{\partial X(t,x)}{\partial t}\right| + \left\|\frac{\partial X(t,x)}{\partial x}\right\| \le C \tag{2.7}$$

for $t \in \mathbb{T}_\lambda$ and $x \in D$. Here $\frac{\partial X}{\partial x}$ is the corresponding Jacobian matrix, $|\cdot|$ is the Euclidian norm of a vector, and $||\cdot||$ is the norm of a matrix.

Let $t_0 \in \mathbb{T}_\lambda$, and let $x(t)$ and $x_\lambda(t)$ be solutions of (2.1) and (2.2), and the respective systems (2.3) on $[t_0, t_0 + T]$ and (2.4) on $[t_0, t_0 + T]_{\mathbb{T}_\lambda}$.

Lemma 2.5 [19] *If x_λ and $x(t)$ are the corresponding solutions of (2.4) and (2.3), then the inequality*

$$|x(t) - x_\lambda(t)| \le \mu(\lambda)K(T) \tag{2.8}$$

holds for $t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$. Here K is a constant depending on T and $\mu(\lambda) = \sup_{t\in[t_0,t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t)$.

Since the matrix X of system (2.4) is rd-continuous, bounded, regressive and Lipschitz continuous for sufficiently small μ_λ, it follows from Theorem 8.16 in [5] that the solution of the Cauchy problem $x(s) = x_0 \in D, \; s \in \mathbb{T}_\lambda$, can be continued both to the right and to the left of the point t_0. Hence, the matrix exponential e_X is nonsingular, and solutions of the system can be continued backward to the point $t_0 = 0$. Thus, all solutions are determined by the initial data at $t_0 = 0$.

Below we present some other necessary statements. In the following lemmas, we consider solutions of equation (2.1) with initial data $x(0) = x_0$, $\dot{x}(0) = x_1$, where

$$x_0^2 + x_1^2 = 1. \tag{2.9}$$

Let us denote this sphere by S. Let $x(t)$ be such a solution, and let t_k be its zeros on $(0, a)$.

Lemma 2.6 *There exists a number $\nu > 0$ such that for any zero t_k of any solution $x(t)$ of equation (2.1) with initial condition (2.9) the following inequality holds:*

$$|\dot{x}(t_k)| \geq \nu. \tag{2.10}$$

Proof We argue by contradiction. Let us assume that the statement of Lemma is not true. Then there exists a sequence $\nu_n \to 0$ such that for every n there exists a solution $x_n(t)$ with initial data $x_{n0} = x_n(0)$, $x_{n1} = \dot{x}_n(0)$, that has at least one zero t_n such that $x_n(t_n) = 0$, $|\dot{x}_n(t_n)| \leq \nu_n$. Since S is compact, there exist converging sequences $x_{n0} \to x_0$, $x_{n1} \to x_1$, where $x_0, x_1 \in S$.

Let us consider a solution $x^*(t)$ of equation (2.1) with initial data x_0, x_1 and zeros $t_n \to t^* \in [0, a]$. Due to continuous dependence of solutions on initial data, we have

$$sup_{t\in[0,a]} \left(|x_n(t) - x^*(t)| + |\dot{x}_n(t) - \dot{x}^*(t)|\right) \to 0, \quad n \to \infty. \tag{2.11}$$

Then

$$|x_n(t_n) - x^*(t^*)| \leq |x_n(t_n) - x^*(t_n)| + |x^*(t_n) - x^*(t^*)| \to 0, \quad n \to \infty, \tag{2.12}$$

which implies $x^*(t^*) = 0$.

In a similar way, we obtain $|\dot{x}_n(t_n) - \dot{x}^*(t^*)| \to 0$, but since $\dot{x}_n(t_n) \to 0, \quad n \to \infty$, we have $\dot{x}^*(t^*) = 0$. Hence, $x^*(t)$ is a trivial solution of equation (2.1), which contradicts the assumption.

We thus obtain

$$|\dot{x}(t_k)| \geq \nu.$$

□

Lemma 2.7 *There exists an ε-neighborhood of zeros t_k of all solutions $x(t)$ of equation (2.1) with initial data (2.9) such that for all $t \in [t_k - \varepsilon, t_k + \varepsilon]$ the following inequality holds:*

$$|\dot{x}(t)| > 0. \tag{2.13}$$

Proof Let us assume the contrary. Then there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and a sequence of solutions $x_n(t)$ of (2.1) with zero at t_n, in the ε_n-neighborhood of which there exists τ_n such that $\dot{x}_n(\tau_n) = 0$.

From the previous lemma it is known that zeros t_n of a solution $x(t)$ of (2.1) approach (in a subsequence) to t_0, which is the zero of the limit solution $x^*(t)$, but $\tau_n \to t_0$ as $n \to \infty$; hence, $x^*(t)$ is a trivial solution, which contradicts our assumption. □

Lemma 2.8 *For any $\delta \leq \varepsilon$, where ε is from Lemma 2.7, there exists $\gamma > 0$ such that for any zero t_n of the solution $x(t)$ of equation (2.1) with initial data (2.9) the following inequalities hold:*

$$|x(t_n - \delta)| \geq \gamma \quad \textit{and} \quad |x(t_n + \delta)| \geq \gamma. \tag{2.14}$$

Proof Assume the contrary, that is, there exist $\delta \le \varepsilon$, a sequence $\{\gamma_n\}$ ($\gamma_n \to 0$ as $n \to \infty$), and a sequence of solutions $x_n(t)$ with zeros at t_n, such that, e.g.,

$$|x(t_n - \delta)| < \gamma_n. \tag{2.15}$$

The sequences $x_n(t)$ and $\dot{x}_n(t)$ converge uniformly to $x_0(t)$ and $\dot{x}_0(t)$, respectively, for $t \in [0, a]$. The sequence of zeros t_n tends to t_0, and $x_0(t_0) = 0$. Then, by (2.15), we have $x_n(t_n - \delta) \to 0$, $n \to \infty$, but it follows from the uniform convergence of $x_n(t)$ that $x_0(t_0 - \delta) = 0$. Therefore, there exists a solution such that $x_0(t_0 - \delta) = x_0(t_0) = 0$. The δ-neighborhood of the point t_0 lies in the ε-neighborhood of t_0, where, by Lemma 2.7, the derivative $\dot{x}_0(t)$ does not vanish. However, since $x_0(t_0 - \delta) = x_0(t_0) = 0$, by Rolle's theorem, there exists at least one point $\tilde{t} \in (t_0 - \delta, t_0)$ such that $\dot{x}_0(\tilde{t}) = 0$. This contradiction proves the lemma. □

Let us now consider equation (2.2) for $t \in [0, a]_{\mathbb{T}_\lambda}$, $a > 0$, and $p \in C([0, a])$. We will study solutions of equation (2.2) with initial data similar to (2.9): $t_0 = 0$, $x_\lambda(0) = x_0$, and $x_\lambda^\Delta(0) = x_1$, where

$$x_0^2 + x_1^2 = 1. \tag{2.16}$$

Lemma 2.9 *There exists a number $\nu(\mu_\lambda) > 0$ such that for any generalized zero t_k of a solution $x_\lambda(t)$ of equation (2.2) with initial data (2.16) the following inequality holds:*

$$\left|x_\lambda^\Delta(t_k)\right| \ge \nu(\mu_\lambda). \tag{2.17}$$

Proof Let us assume to the contrary that (2.17) does not hold. Then there exists a sequence $\{\nu_n\}$, $\nu_n \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{N}$ there exists a solution $x_\lambda^{(n)}(t)$ with initial data $t_{n0} = 0$, $x_{n0} = x_\lambda^{(n)}(0)$, $x_{n1} = \left(x_\lambda^{(n)}\right)^\Delta(0)$, satisfying (2.16) and having at least one generalized zero t_n, i.e. one of the following conditions holds:

$$1) \quad x_\lambda^{(n)}(t_n) = 0; \tag{2.18}$$

$$2) \quad t_n \text{ are right-scattered and } \quad x_\lambda^{(n)}(t_n) \cdot x_\lambda^{(n)}(\sigma(t_n)) < 0; \tag{2.19}$$

and $\left|\left(x_\lambda^{(n)}\right)^\Delta(t_n)\right| \le \nu_n$.

Since the set of initial data (2.16) is compact, the sequence (x_{n0}, x_{n1}) has subsequences that converge to (x_0, x_1). Without loss of generality, we can assume that the sequence (x_{n0}, x_{n1}) is convergent:

$$(x_{n0}, x_{n1}) \to (x_0, x_1), \quad n \to \infty, \tag{2.20}$$

where ${x_0}^2 + {x_1}^2 = 1$.

Let us consider the nontrivial solution $x_\lambda^*(t)$ of equation (2.2) with initial data $x_\lambda^*(0) = x_0$, $(x_\lambda^*)^\Delta(0) = x_1$, $x_0^2 + x_1^2 = 1$. Without loss of generality, we suppose that the sequence $\{t_n\}$ is convergent, $t_n \to t^* \in [0, a]_\lambda$, $n \to \infty$.

Due to continuous dependence of solutions on initial data of the Cauchy problem on a finite interval [17, 3.2], we have

$$\sup_{t \in [0,a]_\lambda} \left(\left|x_\lambda^{(n)}(t) - x_\lambda^*(t)\right| + \left|\left(x_\lambda^{(n)}\right)^\Delta(t) - (x_\lambda^*)^\Delta(t)\right|\right) \to 0, \quad n \to \infty. \tag{2.21}$$

Then from

$$\left|x_\lambda^{(n)}(t_n) - x_\lambda^*(t^*)\right| \le \left|x_\lambda^{(n)}(t_n) - x_\lambda^*(t_n)\right| + |x_\lambda^*(t_n) - x_\lambda^*(t^*)| \to 0, \quad n \to \infty, \tag{2.22}$$

we obtain

$$x_\lambda^{(n)}(t_n) \to x_\lambda^*(t^*), \quad n \to \infty. \tag{2.23}$$

In a similar way, we get

$$\left(x_\lambda^{(n)}\right)^\Delta (t_n) \to (x_\lambda^*)^\Delta (t^*), \quad n \to \infty. \tag{2.24}$$

Let us consider such generalized zeros t_n that $x_\lambda^{(n)}(t_n) = 0$ and $\left|\left(x_\lambda^{(n)}\right)^\Delta (t_n)\right| \le \nu_n$. Then $x_\lambda^*(t^*) = 0$.

Since $\left(x_\lambda^{(n)}\right)^\Delta (t_n) \to 0, \quad n \to \infty$, we have $(x_\lambda^*)^\Delta (t^*) = 0$. Hence, $x_\lambda^*(t)$ is a trivial solution of equation (2.1), which contradicts our assumption.

We thus obtain

$$\left|x_\lambda^\Delta(t_k)\right| \ge \nu(\mu_\lambda).$$

Let us now consider generalized zeros t_n that are right-scattered and satisfy

$$x_\lambda^{(n)}(t_n) \cdot x_\lambda^{(n)}(\sigma(t_n)) < 0 \quad \text{and} \quad \left|\left(x_\lambda^{(n)}\right)^\Delta (t_n)\right| \le \nu_n.$$

Note that if a solution starting at zero is nontrivial, then, by [5, Th.8.16], there is no point on the time scale from which the solution degenerates into the trivial one.

We consider the sequence $\{\sigma(t_n)\}$ for zeros t_n. Since $\left(x_\lambda^{(n)}\right)^\Delta (t_n) \to 0$ as $n \to 0$, from the definition of the $\Delta-$derivative at a right-scattered point we have

$$\left(x_\lambda^{(n)}\right)^\Delta (t_n) = \frac{x_\lambda^{(n)}(\sigma(t_n)) - x_\lambda^{(n)}(t_n)}{\mu_n} \to 0, \quad n \to 0.$$

Since $\mu_n \le C$, for some constant independent of n, we have $|x_\lambda^{(n)}(\sigma(t_n)) - x_\lambda^{(n)}(t_n)| \to 0$, $n \to \infty$. Thus, $x_\lambda^{(n)}(\sigma(t_n)) \to x_\lambda^*(t^*)$.

Passing to the limit in inequality (2.19), we have

$$x_\lambda^*(t^*) \cdot x_\lambda^*(t^*) \le 0, \tag{2.25}$$

that is $x_\lambda^*(t^*) = 0$.

It follows from $\left(x_\lambda^{(n)}\right)^\Delta (t_n) \to 0$, $n \to \infty$ that $(x_\lambda^*)^\Delta (t^*) = 0$. Hence, $x_\lambda^*(t)$ is the trivial solution of equation (2.1), which contradicts our assumption.

Thus, we obtain

$$\left|x_\lambda^\Delta(t_k)\right| \ge \nu(\mu_\lambda).$$

□

The following lemma applies to linear systems (2.3) and (2.4), namely, to the systems of the form

$$\frac{dx}{dt} = A(t)x \tag{2.26}$$

and

$$x_\lambda^\Delta = A(t)x_\lambda. \tag{2.27}$$

If the matrix $A(t)$ is continuous for $t \geq 0$, then all solutions of systems (2.26) and (2.27) are unboundedly continued to the right. We will consider solutions with initial data $x(t_0) = x_\lambda(t_0) = x_0$, $t_0 \geq 0$, and

$$|x_0| = 1. \tag{2.28}$$

Let us denote $M(T, t_0) = \max_{[t_0, t_0+T]} \|A(t)\|$, where $T > 0$ is fixed.

Lemma 2.10 *All solutions of the Cauchy problems for systems (2.26) and (2.27) with initial data (2.28) are uniformly bounded, i.e. there exists $R > 0$, depending only on T and $M(T, t_0)$, such that for all $t \in [t_0, t_0 + T]$ and $t \in [t_0, t_0 + T]_{\mathbb{T}_\lambda}$ the following inequalities hold:*

$$|x(t)| \leq R \quad \textit{and} \quad |x_\lambda(t)| \leq R. \tag{2.29}$$

Proof Any solution of system (2.26) has the integral representation

$$x(t) = x_0 + \int_{t_0}^{t} A(t_1)x(t_1)\, dt_1. \tag{2.30}$$

Using the method of successive approximations [13, p.73], we obtain the formal representation of the solution

$$x(t) = \Omega_{t_0}^t x_0, \tag{2.31}$$

where

$$\Omega_{t_0}^t = x_0 + \int_{t_0}^{t} A(t_1)x(t_0)\, dt_1 + \int_{t_0}^{t} A(t_1)\, ds \int_{t_0}^{t_1} A(t_2)x(t_2)\, dt_2 + \cdots .$$

Then, for $t \in [t_0, t_0 + T]$, in view of $|t - t_0| \leq T$, we obtain the estimate

$$|x(t)| \leq \|\Omega_{t_0}^t\||x_0| \leq 1 + MT + \frac{M^2T^2}{2!} + \cdots = e^{MT} = R.$$

Let us now consider solutions of system (2.27) that are represented as

$$x_\lambda(t) = x_0 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} A(t_1)x_\lambda(t_1)\, \Delta t_1.$$

Using the method of successive approximations and replacing $x_\lambda(t_1)$ with the sum

$$x_\lambda(t_1) = x_\lambda(t_0) + \int_{[t_0,t_1]_{\mathbb{T}_\lambda}} A(t_2)x_\lambda(t_2)\, \Delta t_2,$$

we obtain

$$x_\lambda(t) = x_0 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} A(t_1)x_\lambda(t_0)\,\Delta t_1 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} A(t_1)\Delta t_1 \int_{[t_0,t_1]_{\mathbb{T}_\lambda}} A(t_2)x_\lambda(t_2)\,\Delta t_2.$$

Repeating this process, an infinite number of times, we obtain a formal representation of the solution

$$x_\lambda(t) = x_0 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} A(t_1)x_\lambda(t_0)\,\Delta t_1 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} A(t_1)\Delta t_1 \int_{[t_0,t_1]_{\mathbb{T}_\lambda}} A(t_2)x_\lambda(t_2)\,\Delta t_2 + \cdots .$$

We then get the estimate

$$|x_\lambda(t)| \le |x_0| + \int_{[t_0,t]_{\mathbb{T}_\lambda}} \|A(t_1)\|\,|x_\lambda(t_0)|\;\Delta t_1 + \int_{[t_0,t]_{\mathbb{T}_\lambda}} \|A(t_1)\|\Delta t_1 \int_{[t_0,t_1]_{\mathbb{T}_\lambda}} \|A(t_2)\|\,|x_\lambda(t_2)|\;\Delta t_2 + \cdots .$$

In what follows, we use the following inequality (see Lemma 5 [10]):

$$\int\limits_{[t_0,t)_{\mathbb{T}}} (\tau - t_0)^k \Delta\tau \le \frac{(t-t_0)^{k+1}}{k+1}, \quad \text{for every } k \in \mathbb{N} \text{ and every } t \in \mathbb{T}.$$

Proceeding in a similar way, in view of

$$\int_{[t_0,t]_{\mathbb{T}_\lambda}} \|A(t_1)\|\,\Delta t_1 \le M\,|t-t_0|\,,$$

$$\int_{[t_0,t]_{\mathbb{T}_\lambda}} \|A(t_1)\|\,\Delta t_1 \int_{[t_0,t_1]_{\mathbb{T}_\lambda}} \|A(t_2)\|\,|x_\lambda(t_2)|\;\Delta t_2 \le \frac{M^2\,|t-t_0|^2}{2!},$$

and $|t-t_0| \le T$, we obtain

$$|x(t)| \le 1 + MT + \frac{M^2\,|t-t_0|^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{(MT)^n}{n!} = e^{MT} = R,$$

which proves the lemma. □

Lemma 2.11 *There exist μ_0 and B_0 such that for all $0 < \mu_\lambda \le \mu_0$ the following inequality holds:*

$$\nu(\mu_\lambda) \ge B_0, \tag{2.32}$$

where $\nu(\mu_\lambda)$ is from Lemma 2.9.

Proof Suppose, to the contrary, that the statement is not true. Then there exists a sequence of graininess functions $\{\mu_{\lambda n}(t)\}$. Let us assume that $\mu_n = \sup_{t\in[0,a]_{\lambda n}}(\mu_{\lambda n}(t))$ is such that $\mu_n > 0$, $\mu_n \to 0$, $n \to \infty$, and

$$\nu(\mu_n) \to 0, \quad n \to \infty. \tag{2.33}$$

Hence, it follows from (2.33) that for every μ_n there exists a solution x_{λ_n} of equation (2.2) (for simplicity, we denote it by x_n) with given μ_n and initial data $x_{0n} = x_{\lambda n}(0)$, $x_{1n} = x^{\Delta}_{\lambda n}(0)$ satisfying (2.16), such that for any $n \in \mathbb{N}$ we can choose a generalized zero t_n of the solution $x_n(t)$ such that

$$x_n^{\Delta}(t_n) \to 0, \quad n \to \infty. \tag{2.34}$$

We have either $x_n(t_n) = 0$ or $x_n(t_n) \cdot x_n(\sigma(t_n)) < 0$, if t_n is right-scattered. Taking into account $\sigma(t_n) = t_n + \mu_n(t_n)$, we rewrite the latter inequality as

$$x_n(t_n) \cdot x_n(t_n + \mu_n(t_n)) < 0. \tag{2.35}$$

It follows from $\mu_n \to 0$, $n \to \infty$, that $\mu_n(t_n) \to 0$. Using an analogue of Lagrange's formula (Theorem 1.22 [6]), we have

$$|x_n(t_n + \mu_n(t_n)) - x_n(t_n)| \le R \cdot \mu_n(t_n) \to 0, \quad n \to \infty. \tag{2.36}$$

Note that due to Lemma 2.10, the constant R does not depend on the time scale $\mathbb{T}_{\lambda_n}$. Thus, by (2.35) and (2.36), $x_n(t_n) \to 0$ as $n \to \infty$.

The sequence (x_{0n}, x_{1n}) can be considered convergent; hence,

$$(x_{0n}, x_{1n}) \to (x_0, x_1), \quad n \to \infty. \tag{2.37}$$

Let $t_n \in [0, a]_{\lambda n}$ on the time scale with given μ_n denote the argument at which the derivative possesses property (2.34). We thus have

$$x_{\lambda n}(t_n) \to 0, \quad n \to \infty \tag{2.38}$$

and

$$x^{\Delta}_{\lambda n}(t_n) \to 0, \quad n \to \infty. \tag{2.39}$$

The sequence $\{t_n\}$ also has a convergent subsequence. Without loss of generality, we assume that $\{t_n\}$ itself is convergent. Hence, $t_n \to t^*$, $n \to \infty$, $t^* \in [0, a]$.

Let us now consider the solution of the differential equation (2.1) with initial data (2.9). Obviously, it is nontrivial. We denote it by $x(t, x_0, x_1)$. We also denote by $x(t, x_{0n}, x_{1n})$ the solution of equation (2.1) with initial data (x_{0n}, x_{1n}).

Obviously,

$$x(t_n, x_0, x_1) \to x(t^*, x_0, x_1), \quad n \to \infty. \tag{2.40}$$

Since solutions of the Cauchy problem on a finite interval are continuously dependent on initial data, we have

$$\sup_{t \in [0,a]} |x(t, x_{0n}, x_{1n}) - x(t, x_0, x_1)| \to 0 \quad \text{if} \quad (x_{1n}, x_{0n}) \to (x_0, x_1).$$

Then

$$|x(t_n, x_{0n}, x_{1n}) - x(t_n, x_0, x_1)| \to 0, \quad n \to \infty. \tag{2.41}$$

By Lemma 2.10, all solutions of the Cauchy problems for equation (2.1) and dynamic equation (2.2) with initial data (2.9) are uniformly bounded on $[0, a]$ and $[0, a]_\lambda$, respectively. Hence, by Lemma 2.5, the uniform estimate (2.8) holds for the corresponding solutions of these equations. Uniformity here means that the constants $K(T)$ and R in Lemmas 2.5 and 2.10, respectively, can be chosen with respect to a and the maximum of the function $|p(t)|$ on $[0, a]$.

Consequently,

$$|x(t_n, x_{0n}, x_{1n}) - x_{\lambda n}(t_n)| \to 0, \quad n \to \infty. \tag{2.42}$$

Thus, from (2.38), (2.40-2.42), we obtain

$$x(t^*, x_0, x_1) = 0. \tag{2.43}$$

In a similar way, it can be shown that $\dot{x}(t^*, x_0, x_1) = 0$. This contradicts the fact that $x(t, x_0, x_1)$ is a nontrivial solution. □

Lemma 2.12 *There exist $\varepsilon > 0$ and $\mu_0 > 0$ such that for $0 < \mu_\lambda \leq \mu_0$, the Δ-derivative $x_\lambda^\Delta(t)$ of every solution $x_\lambda(t)$ of equation (2.2) with initial data (2.16) preserves its sign in ε-neighborhood of all generalized zeros t_k. That is, either*

$$\forall t \in [t_k - \varepsilon; t_k + \varepsilon]: \quad x_\lambda^\Delta(t) > 0, \tag{2.44}$$

or

$$\forall t \in [t_k - \varepsilon; t_k + \varepsilon]: \quad x_\lambda^\Delta(t) < 0. \tag{2.45}$$

Proof Let us suppose that condition (2.44) does not hold. Then, for every $\varepsilon > 0$ there exists a sequence $\{\mu_n(\varepsilon)\}$ such that $\mu_n(\varepsilon) \to 0$ as $n \to \infty$, where $\mu_n = \max_{t \in [0,a]_\lambda} \mu_\lambda(t)$ and $\left|x_\lambda^\Delta(t)\right| = 0$.

Hence, there is a sequence $\varepsilon_n \to 0$, $n \to 0$, for each term of which there exists a corresponding sequence $\{\mu_k(\varepsilon_n)\}$ such that $\mu_k(\varepsilon_n) \to 0$, $k \to \infty$, and $\left|x_\lambda^\Delta(t)\right| = 0$.

Using the diagonal method, we obtain that there exist a sequence $\varepsilon_n \to 0$, a corresponding sequence $\mu_n(\varepsilon_n)$, a solution $x_{\lambda_n}(t)$ with initial condition (x_{0n}, x_{1n}), and a generalized zero t_n such that there exists $\tau_n \in (t_n - \varepsilon_n; t_n + \varepsilon_n) \cap [0, a]_{\lambda_n}$ for which either $x_{\lambda_n}^\Delta(\tau_n) = 0$ or $x_{\lambda_n}^\Delta(\tau_n) \cdot x_{\lambda_n}^\Delta(t_n) < 0$.

Suppose $x_{\lambda_n}^\Delta(t_n) > 0$. By Lemma 2.11, there exists $B_0 > 0$ such that the inequality

$$x_{\lambda_n}^\Delta(t_n) \geq B_0 \tag{2.46}$$

holds uniformly on all time scales. Then either $x_{\lambda_n}^\Delta(\tau_n) = 0$ or $x_{\lambda_n}^\Delta(\tau_n) < 0$.

Let us now consider the solution $x(t, x_0, x_1)$ of the differential equation (2.1) with initial data (2.9), which is obviously nontrivial. We denote by $x(t, x_{0n}, x_{1n})$ the solution of equation (2.1) with initial data (x_{0n}, x_{1n}) such that (x_{0n}, x_{1n}) converges to (x_0, x_1) uniformly on all time scales.

Let $t_n \to t^*$ for $\varepsilon_n \to 0$, $n \to \infty$. Then $\tau_n \to t^*$, $n \to \infty$.

By Lemma 2.5, we have

$$|x_{\lambda_n}(t) - x(t, x_{0n}, x_{1n})| \leq K\mu_n(\varepsilon_n) \tag{2.47}$$

and

$$\left|x_{\lambda_n}^\Delta(t) - \dot{x}(t, x_{0n}, x_{1n})\right| \leq K\mu_n(\varepsilon_n). \tag{2.48}$$

Clearly, $|\dot{x}(t, x_{0n}, x_{1n}) - \dot{x}(t, x_0, x_1)| \to 0$, $n \to \infty$, whence

$$x_{\lambda_n}^\Delta(\tau_n) \to \dot{x}(t^*, x_0, x_1) \quad n \to \infty. \tag{2.49}$$

If $x_{\lambda_n}^\Delta(\tau_n) = 0$, then from (2.49), we have $\dot{x}(t^*, x_0, x_1) = 0$. However, $x_{\lambda_n}^\Delta(t_n) \to \dot{x}(t^*, x_0, x_1) = 0$ as $n \to \infty$, which contradicts (2.46).

If $x_{\lambda_n}^\Delta(\tau_n) < 0$, then $x_{\lambda_n}^\Delta(\tau_n) \to \dot{x}(t^*, x_0, x_1) \leq 0$ as $n \to \infty$. Since $x_{\lambda_n}^\Delta(t_n) > 0$, we have $x_{\lambda_n}^\Delta(t_n) \to \dot{x}(t^*, x_0, x_1) \geq 0$ as $n \to \infty$. We then obtain $\dot{x}(t^*, x_0, x_1) = 0$, which contradicts (2.46).

Thus, our assumption is not true, and the proof is complete. □

Lemma 2.13 *For any $\delta \le \varepsilon$, where ε is from Lemma 2.12, there exist μ_0 and $\gamma > 0$ such that for any $\mu_\lambda \le \mu_0$ and any generalized zero t_0 of a solution $x_\lambda(t)$ of equation (2.2) the following inequalities hold:*

$$|x_\lambda(t_l)| \ge \gamma \tag{2.50}$$

and

$$|x_\lambda(t_r)| \ge \gamma, \tag{2.51}$$

where $t_l = \inf\{t \in \mathbb{T}_\lambda | t > t_0 - \delta\}$, $t_r = \sup\{t \in \mathbb{T}_\lambda | t < t_0 + \delta\}$.

Remark 2.14 *If the points $t_0 \pm \delta$ belong to the time scale, then $t_l = t_0 - \delta$ and $t_r = t_0 + \delta$, respectively. Otherwise, for every fixed δ, by choosing a small graininess function $\mu_\lambda(t) \le \mu_\lambda \le \mu_0$, we can always guarantee the existence of scale points, different from t_0, in the δ-neighborhood of the point t_0.*

Proof Let us suppose that the statement of Lemma does not hold, i.e. there exists $\delta \le \varepsilon$, where ε is from Lemma 2.12, such that for any μ_0 and γ there exists the time scale $\mu_\lambda < \mu_0$ for which the inequalities (2.50) and (2.51) are not true. Then for any sequence $\gamma_k \to 0,\ k \to \infty$, there exists a sequence of time scales $\{\mu_k(n)\}$ such that the statement is not true.

Again we use the diagonal method. Then there exist sequences $\gamma_n \to 0,\ n \to \infty$, and corresponding sequences $\{\mu_n(n)\}$ such that on any such time scale there is a solution $x_{\lambda_n}^{(n)}(t)$ with initial data (x_{0n}, x_{1n}) and the generalized zero $t_n^{(n)}$ such that

$$\text{either} \quad \left| x_{\lambda_n}^{(n)}\left(t_{n,l}^{(n)}\right)\right| < \gamma_n, \quad \text{or} \quad \left| x_{\lambda_n}^{(n)}\left(t_{n,r}^{(n)}\right)\right| < \gamma_n, \tag{2.52}$$

where $t_{n,l}^{(n)} = \inf\{t \in \mathbb{T}_\lambda | t > t_n^{(n)} - \delta\}$ and $t_{n,r}^{(n)} = \sup\{t \in \mathbb{T}_\lambda | t < t_n^{(n)} + \delta\}$.

We again consider the solution $x(t, x_0, x_1)$ of the differential equation (2.1) with initial data (2.9). Obviously, it is not trivial. We denote by $x(t, x_{0n}, x_{1n})$ the solution of equation (2.1) with initial data (x_{0n}, x_{1n}) that converge to (x_0, x_1) uniformly on all time scales.

Let $t_n^{(n)} \to t^*$ as $n \to \infty$, then, in view of $\mu_n(t_n^{(n)}) \to 0$, we obtain $\sigma\left(t_n^{(n)}\right) = t_n^{(n)} + \mu_n\left(t_n^{(n)}\right) \to t^*$, $n \to \infty$.

It follows from Lemma 2.5 that

$$\left| x_{\lambda_n}^{(n)}(t) - x(t, x_{0n}, x_{1n})\right| \le K\mu_n(t_n^{(n)}). \tag{2.53}$$

Clearly, $|\dot{x}(t, x_{0n}, x_{1n}) - \dot{x}(t, x_0, x_1)| \to 0,\ n \to \infty$, whence

$$\left| x_{\lambda_n}^{(n)}(t) - x(t, x_0, x_1)\right| \to 0, n \to \infty. \tag{2.54}$$

From (2.52),(2.54) and $\gamma_n \to 0,\ n \to \infty$, we obtain

$$x(t_l^*) = 0. \tag{2.55}$$

If $t_n^{(n)}$ is such that $x_{\lambda_n}^{(n)}(t_n^{(n)}) = 0$, then $x(t^*) = 0$.

If $t_n^{(n)}$ is right-scattered and such that $x_{\lambda_n}^{(n)}(t_n^{(n)}) \cdot x_{\lambda_n}^{(n)}(\sigma(t_n^{(n)})) < 0$, then, passing to the limit in the latter inequality, we obtain $x(t^*)^2 \le 0$, whence

$$x(t^*) = 0. \tag{2.56}$$

It follows from Lemmas 2.5 and 2.12 that

$$\dot{x}(t, x_0, x_1) \neq 0 \tag{2.57}$$

for any $t \in \left(t^* - \frac{\varepsilon}{2}, t^* + \frac{\varepsilon}{2}\right)$ (it may be assumed that $\delta < \varepsilon$).

However, due to (2.55) and (2.56), by Rolle's theorem, there exists at least one point on the interval (t_l^*, t^*) such that $\dot{x}(\,\tilde{t}\,) = 0$. This contradiction proves the lemma.

□

2.3. Main results

We now proceed with the main results on the relation between the oscillatory behavior of solutions of linear second-order differential equations (2.1) and corresponding dynamic equations (2.2).

Theorem 2.15 *For any ε, there exists $\mu_0 = \mu_0(\lambda)$ such that for all $\mu_\lambda \le \mu_0$ the following statement holds true:*

If $x(t)$ is a solution of equation (2.1) with initial data $x(0) = x_1$, $\dot{x} = x_1$, and $x_\lambda(t)$ is the corresponding solution of the dynamic equation (2.2) with initial data $x_\lambda(0) = x_0$, $x_\lambda^\Delta(0) = x_1$, then there is at least one zero $t_{0\lambda}$ of the corresponding solution of equation (2.2) in the ε-neighborhood of any zero t_0 of the solution $x(t)$.

Proof Let us choose an arbitrary $\varepsilon > 0$ and an arbitrary nontrivial solution $x(t)$ of equation (2.1). The function $y(t) = \frac{1}{\sqrt{x_0^2 + x_1^2}} x(t)$ is also a solution of equation (2.1), and zeros of $x(t)$ coincide with zeros of $y(t)$, but the initial data for $y(t)$ lie on S, a unit sphere in $\mathbb{R}^2$.

We choose μ_0 so that for $\mu_\lambda \le \mu_0$ the corresponding solutions of equations (2.1) and (2.2) satisfy the inequalities

$$|x_\lambda(t) - y(t)| < \frac{\gamma}{2}$$

and

$$\left|x_\lambda^\Delta(t) - \dot{y}(t)\right| < \frac{\gamma}{2}$$

for $t \in [0, a]_\lambda$. Then, by Lemma 2.8, in the ε-neighborhood of any zero t_0 of the solution $y(t)$ there exist numbers $t_1, t_2 \in \mathbb{T}_\lambda$ such that $x_\lambda(t_1) > 0$ and $x_\lambda(t_2) < 0$.

Thus, x_λ has a generalized zero in the ε-neighborhood of the zero of the solution $y(t)$, and hence of the zero of the solution $x(t)$. □

Remark 2.16 *Theorem 2.15 implies the following statement:*

If $\{t_n\}_1^N$ are zeros of an arbitrary nontrivial solution $x(t)$ of equation (2.1) on $[0, a]$, then the corresponding solution $x_\lambda(t)$ of equation (2.2) for sufficiently small μ_λ also has at least N zeros $\{t_{n_\lambda}\}$ on $[0, a]$, and

$$|t_{n_\lambda} - t_n| \to 0, \quad \lambda \to 0.$$

Remark 2.17 *If a solution $x(t)$ of equation (2.1) oscillates on $[0,a]$, then the corresponding solution $x_\lambda(t)$ of equation (2.2) oscillates on $[0,a]_\lambda$ as well.*

Theorem 2.18 *For any $\varepsilon > 0$, there exists $\mu_0 = \mu_0(\varepsilon)$ such that the following statement holds for all $\mu_\lambda \le \mu_0$:*

If $x_\lambda(t, x_0, x_1)$ is a solution of dynamic equation (2.2) with initial data $x_\lambda(0) = x_0$, $x_\lambda^\Delta(0) = x_1$, and $x(t)$ is the corresponding solution of differential equation (2.1) with the same initial data, then in the ε-neighborhood of a generalized zero $t_{0\lambda}$ of the solution $x_\lambda(t)$ there is at least one zero t_0 of the corresponding solution of equation (2.1).

Proof Let us choose an arbitrary $\varepsilon > 0$ and an arbitrary nontrivial solution $x_\lambda(t, x_0, x_1)$ of equation (2.2). The function $y_\lambda(t) = \frac{1}{\sqrt{x_0^2+x_1^2}} x_\lambda(t)$ also satisfies equation (2.2), and generalized zeros of the solution $x_\lambda(t)$ coincide with those of the solution $y_\lambda(t)$, but the initial data for $y_\lambda(t)$ lie on S, the unit sphere in $\mathbb{R}^2$.

We choose μ_0 so that for $\mu_\lambda \le \mu_0$ the corresponding solutions of (2.2) and (2.1) satisfy the inequalities

$$|x(t) - y_\lambda(t)| < \frac{\gamma}{2}$$

and

$$\left|\dot{x}(t) - y_\lambda^\Delta(t)\right| < \frac{\gamma}{2}$$

for $t \in [0,a]_\lambda$.

Then, by Lemmas 2.12 and 2.13, in the ε-neighborhood of an arbitrary zero $t_{0\lambda}$ of the solution $y_\lambda(t)$ there exist such numbers $t_1, t_2 \in \mathbb{R}$ that $x(t_1) > 0$ and $x(t_2) < 0$. Hence, due to continuity of the $x(t)$ on $\mathbb{R}$, there is a point $t_0 \in (t_1, t_2)$ such that $x(t_0) = 0$.

Thus, $x(t)$ has a zero in the ε-neighborhood of a generalized zero of the solution $y_\lambda(t)$, and hence in the ε-neighborhood of a zero of the solution $x_\lambda(t)$. □

Remark 2.19 *Theorem 2.18 implies the following statement: If $\{t_{n_\lambda}\}_1^N$ are zeros of an arbitrary nontrivial solution $x_\lambda(t)$ of equation (2.2), then the corresponding solution $x(t)$ of equation (2.1), for sufficiently small μ_λ, also has at least N zeros $\{t_n\}$ on $[0,a]$, and $|t_n - t_{n_\lambda}| \to 0$ as $\lambda \to 0$.*

Remark 2.20 *The above statement implies that if a solution $x_\lambda(t)$ of equation (2.2) oscillates on $[0,a]_\lambda$, then the corresponding solution $x(t)$ of equation (2.1) oscillates on $[0,a]$ as well.*

The following statement follows from Theorems 2.15 and 2.18.

Corollary 2.21 *For any $\varepsilon > 0$ there exists $\mu_0 = \mu_0(\varepsilon)$ such that the following statement holds true for all $\mu_\lambda \le \mu_0$:*

If $x(t)$ is a solution of equation (2.1) with initial data $x(0) = x_1$, $\dot{x} = x_1$, and $x_\lambda(t)$ is the corresponding solution of the dynamic equation (2.2) with initial data $x_\lambda(0) = x_0$, $x_\lambda^\Delta(0) = x_1$, then exactly one zero $t_{0\lambda}$ of $x_\lambda(t)$ lies in the ε-neighborhood of an arbitrary zero t_0 of $x(t)$, and vice versa.

2.4. Weakly nonlinear case

The following result relates to oscillatory properties of solutions of nonlinear equations.

We consider the nonlinear differential equation

$$\ddot{x} + p(t)x + \varepsilon f(t, x, \dot{x}) = 0, \tag{2.58}$$

where $\varepsilon > 0$ is a small parameter, $t \in [0, a]$.

The requirements on $f(t, x, y)$ and $p(t)$ for $x, y \in \mathbb{R}^1$, $t \in [0, a]$ are as follows:

1. $f(t, x, y)$ is continuous in all variables;
2. $f(t, x, y)$ is a linear growth function in x and y, i.e. there exists $N > 0$ such that the following inequality holds:
$$|f(t, x, y)| \leq N(1 + |x| + |y|);$$
3. $p(t) \geq 0$;
4. $p(t)$ satisfies the Lipschitz condition on $[0, a]$.

We consider equation (2.58) with initial data

$$x(0) = x_0, \quad y(0) = y_0,$$

that belong to a compact, $(x_0, y_0) \in K$, which does not contain the point $(0, 0)$.

When $\varepsilon = 0$, equation (2.58) turns into the equation

$$\ddot{x} + p(t)x = 0, \tag{2.59}$$

whose corresponding dynamic equation is of the form

$$x_\lambda^{\Delta\Delta} + p(t)x_\lambda = 0, \tag{2.60}$$

where $t \in \mathbb{T}_\lambda$, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, $x_\lambda^\Delta(t)$ is the Δ-derivative of $x_\lambda(t)$ on $\mathbb{T}_\lambda$. Initial conditions for equations (2.59) and (2.60) are $x(0) = x_0$, $\dot{x}(0) = y_0$ and $x_\lambda(0) = x_0$, $x_\lambda^\Delta(0) = y_0$, respectively.

Along with equation (2.60), we consider the nonlinear equation

$$x_\lambda^\Delta + p(t)x_\lambda + \varepsilon f(t, x_\lambda, x_\lambda^\Delta) = 0. \tag{2.61}$$

We are interested in conditions under which the oscillatory behavior of the solution of equation (2.60) implies that of the solution of equation (2.58), as well as in conditions under which the oscillatory behavior of the solution of equation (2.59) implies that of the solution of equation (2.61).

In what follows, we will need the following two auxiliary statements.

We consider solutions $x(t)$ of equation (2.59) with initial data $x(t_0) = x_0$, $x(t_0) = x_1$, where $t_0 \in [0, \bar{\mu}]$ and

$$x_0^2 + x_1^2 = 1. \tag{2.62}$$

Here $\bar{\mu}$ is fixed and satisfies $0 < \bar{\mu} < a$.

If such a solution oscillates on $(0,a)$, then it has at least two zeros. Let t_k and t_{k+1} denote two consecutive zeros on $(0,a)$ of this oscillating solution.

Let us introduce the following quantity:

$$M_k^x = \max_{t\in[t_k,t_{k+1}]} |x(t)|.$$

We will call this finite number sequence the sequence of amplitudes of oscillations of the solution $x(t)$ on the interval $(0,a)$.

Lemma 2.22 [8] *Let $p \in C([0,a])$ in equation (2.59). Then there exists $\Delta > 0$ such that an arbitrary oscillatory solution of equation (2.59) with initial data (2.62) satisfies the inequality*

$$M_k^x \geq \Delta. \tag{2.63}$$

Thus, for an arbitrary oscillatory on $(0,a)$ solution of equation (2.59) with initial data (2.62), the sequence of amplitudes of oscillations is bounded below by a number Δ that is independent of the solution .

We also consider solutions of equation (2.60) with initial data $x_\lambda(t_0) = x_0$, $\Delta x_\lambda(t_0) = x_1$, where $t_0 \in [0,a]_\lambda$ and

$$x_0^2 + x_1^2 = 1. \tag{2.64}$$

Let there be a unique solution $x_\lambda(t)$ on $[0,a]_\lambda$. If this solution oscillates on $[0,a]_\lambda$, then it has at least two generalized zeros. Let t_p and t_m denote two consecutive generalized zeros of $x_\lambda(t)$ $(t_p < t_m)$ on $[0,a]_\lambda$. We introduce the finite number sequence

$$M_p^x = \max_{t\in[t_p,t_m]} |x_\lambda(t)|\,,$$

which is called the sequence of amplitudes of oscillations of the solution $x_\lambda(t)$ on the interval $[0,a]_\lambda$.

Lemma 2.23 *Let $p(t) \in C([0,a]_\lambda)$ and $p(t) \geq 0$. Then there exists $\Delta(\mu_\lambda) > 0$ such that any oscillatory solution of equation (2.60) with initial data (2.64) satisfies the inequality*

$$M_p^x(\mu_\lambda) \geq \Delta(\mu_\lambda). \tag{2.65}$$

Proof To prove this lemma, it is necessary to prove an analogue of the Weierstrass theorem for continuous functions, namely: If $f(t) \in C([a,b]_\mathbb{T})$, then it is f bounded on $[a,b]_\mathbb{T}$ and reaches its largest and smallest values on this interval.

It is known that if a function $f : \mathbb{T} \to R$ is continuous then f is regulated [5, Th.1.60], and every regulated function on compact interval is bounded [5, Th.1.65]. Hence, $f(t) \in C([a,b]_\mathbb{T})$ is bounded on $[a,b]_\mathbb{T}$. Thus, f attains its least upper bound and its greatest lower bound on this interval.

Let us denote $M = \sup_{[a,b]_\mathbb{T}} f(t)$. Suppose, contrary to the claim of the theorem, that $f(t) < M$ for all $t \in [a,b]_\mathbb{T}$. Consider the function $\varphi(t) = \frac{1}{M-f(t)} > 0$.

It follows from $M - f(t) \neq 0$ that $\varphi(t) \in C([a,b]_\mathbb{T})$. Hence, $\varphi(t)$ is bounded on $[a,b]_\mathbb{T}$; that is, there is $M_0 > 0$ such that $\varphi(t) \leq M_0$ for all $t \in [a,b]_\mathbb{T}$.

Then, we have

$$M - f(t) \geq \frac{1}{M_0},$$

or

$$f(t) \geq M - \frac{1}{M_0},$$

which means that $M \neq \sup_{[a,b]_{\mathbb{T}}} f(t)$. Thus, the assumption is not true, and there is $t^* \in [a,b]_{\mathbb{T}}$ such that $f(t^*) = M$. Thus, the function $f(t)$ attains on $[a,b]_{\mathbb{T}}$ its least upper bound and hence its maximum value.

It can be shown in a similar way that $f(t)$ attains on $[a,b]_{\mathbb{T}}$ its greatest lower bound and hence its minimum value.

The statement is proven. Let us return to the proof of the lemma.

Suppose that (2.65) is not valid. Then there exists an infinite sequence of oscillatory solutions $x_\lambda^{(n)}(t)$ of equation (2.60) with initial data $t_{0n} \in [0,a]_\lambda$, x_{0n}, x_{1n}, satisfying (2.64), such that for every n an amplitude $M_{p(n)}^{x_\lambda^{(n)}}(\mu_\lambda)$ can be chosen from this sequence, so that

$$M_{p(n)}^{x_\lambda^{(n)}}(\mu_\lambda) \to 0, \quad n \to \infty. \tag{2.66}$$

Here $M_{p(n)}^{x_\lambda^{(n)}}(\mu_\lambda) = \max\limits_{t \in [t_{p(n)}, t_{m(n)}]} \left| x_\lambda^{(n)}(t) \right|$. Let $t_n^* \in [t_{p(n)}, t_{m(n)}]$ be a point, at which this maximum is attained. Then $\Delta x_\lambda^{(n)}(t_n^*) = 0$, $\left| x_\lambda^{(n)}(t_n^*) \right| = M_{p(n)}^{x_\lambda^{(n)}}(\mu_\lambda)$.

Since the set of initial data (2.64) is compact, then the sequence contains a convergent subsequence. Without loss of generality, let us assume that the sequence (t_{0n}, x_{0n}, x_{1n}) itself is convergent. Then,

$$(t_{0n}, x_{0n}, x_{1n}) \to (t_0, x_0, x_1), \quad n \to \infty \tag{2.67}$$

where $t_0 \in [0,a]_\lambda$, $x_0^2 + x_1^2 = 1$.

Let $x_\lambda(t)$ be the solution of equation (2.60) with initial conditions $x_\lambda(t_0) = x_0$, $\Delta x(t_0) = x_1$. Obviously, this solution is nontrivial.

We also select a convergent subsequence from the sequence t_n^* and denote it by t_n^*. Hence, $t_n^* \to t^* \in [0,a]_\lambda, n \to \infty$.

Due to continuous dependence of a solution on initial data for the Cauchy problem on a finite interval [17, Theorem 3.2], and in view of the inequality

$$\left| x_\lambda^{(n)}(t_n^*) - x_\lambda(t^*) \right| \leq \left| x_\lambda^{(n)}(t_n^*) - x_\lambda(t_n^*) \right| + |x_\lambda(t_n^*) - x_\lambda(t^*)|$$

it follows that

$$x_n(t_n^*) \to x(t^*), \qquad \Delta x_\lambda^{(n)}(t_n^*) \to \Delta x(t^*). \tag{2.68}$$

However, on the other hand, $x_\lambda^{(n)}(t_n^*) \to 0$ as $n \to \infty$, and $\Delta x_\lambda^{(n)}(t_n^*) = 0$ for any n. Consequently, $x_\lambda(t)$ is a trivial solution. This contradiction proves the lemma. □

Theorem 2.24 *If $f(t,x,y)$ and $p(t)$ satisfy conditions 1-4, then there exist $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and $0 < \mu_\lambda < \mu_0$ the following statement holds true:*

If a solution of equation (2.60) has at least three generalized zeros on the interval $[0,a]_{\mathbb{T}_\lambda}$, then the corresponding solution of equation (2.58) oscillates on $[0,a]$.

Proof Since the initial data (x_0, y_0) of equation (2.59) lie in the compact K, then, by Lemma 2.22, there is $\Delta > 0$ such that the amplitude of oscillations of an arbitrary solution $x(t)$ with initial data $(x_0, y_0) \in K$ is bounded below:

$$M_k^x \geq \Delta.$$

Let us show the proximity of solutions of equations (2.58) and (2.59).

We represent equation (2.58) as the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -p(t)x - \varepsilon f(t, x, y). \end{cases} \tag{2.69}$$

Let us denote $A(t) = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix}$, $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z(0) = z_0$. Then

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A(t)z - \varepsilon f(t, z). \tag{2.70}$$

We now rewrite equation (2.59) in the form

$$\dot{z_1} = A(t)z_1, \tag{2.71}$$

where $z_1(0) = z_0$.

Let $X(t, s)$ be a fundamental system of solutions of (2.71). Then

$$z_1(t) = X(t, 0)z_0. \tag{2.72}$$

The integral representation of (2.70) is

$$z(t) = X(t, 0)z_0 + \varepsilon \int_0^t X(t, s) f(s, z(s)) ds. \tag{2.73}$$

Since $f(t, x, y)$ is a linear growth function, we have

$$|f(t, z)| \leq N \left(1 + |z|\right).$$

Then, for corresponding solutions of equations (2.70), (2.71),(2.72), and (2.73), we obtain

$$|z(t) - z_1(t)| \leq \varepsilon \int_0^t CN \left(1 + |z(s)|\right) ds, \tag{2.74}$$

where $||X(t, s)|| \leq C$, $t, s \in [0, a]$, C is a constant.

Let us now estimate $z(s)$. From (2.73), we obtain

$$|z(t)| \leq C|z_0| + \varepsilon NCa + \varepsilon NC \int_0^t |z(s)| \, ds. \tag{2.75}$$

Hence, by Gronwall's lemma, we obtain the following estimate:

$$|z(t)| \leq \left(C|z_0| + \varepsilon NCa\right) e^{\varepsilon NCa}, \quad t \in [0, a]. \tag{2.76}$$

Since $z_0 \in K$, there is $R > 0$ such that $|z_0| \le R$. Then

$$|z(t)| \le (CR + \varepsilon NCa)\, e^{\varepsilon NCa} \equiv B, \tag{2.77}$$

where B is a constant.

Hence, from (2.74), we obtain

$$|z(t) - z_1(t)| \le \varepsilon \int_0^t CN(1+B)ds \le \varepsilon NCa(1+B), \tag{2.78}$$

which estimates the proximity of the corresponding solutions of equations (2.58) and (2.59).

It follows from Theorem 2.18 that if a solution of equation (2.60) has N generalized zeros on $[0,a]_\lambda$, then the corresponding solution of equation (2.59) has at least N zeros on $[0,a]$.

Then, if the solution of equation (2.60) has three generalized zeros on $[0,a]_{\mathbb{T}_\lambda}$, then the solution of (2.59) has at least three zeros on $[0,a]$. Hence, taking into account estimate (2.78), Lemma 2.22, and the correspondence of solutions of equations (2.58) and (2.59), we obtain that there exist $\varepsilon_0 > 0$ and $\Delta > 0$ such that the solution of nonlinear equation (2.58) has at least two zeros on $[0,a]$, that is, the solution is oscillatory. $\square$

Theorem 2.25 *If $f(t,x,y)$ and $p(t)$ satisfy conditions 1-4, then there exist $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that for arbitrary $0 < \varepsilon < \varepsilon_0$ and $0 < \mu_\lambda < \mu_0$ the following statement holds true:*

If a solution of the linear equation (2.59) has at least three zeros on the interval $[0,a]$, then the corresponding solution of equation (2.61) oscillates on $[0,a]_{\mathbb{T}_\lambda}$.

Proof Let us estimate the norm of the difference between the corresponding solutions of equations (2.60) and (2.61). We represent equation (2.61) as the system

$$\begin{cases} x_\lambda^\Delta = y_\lambda, \\ y_\lambda^\Delta = -p(t)x_\lambda - \varepsilon f(t, x_\lambda, y_\lambda). \end{cases} \tag{2.79}$$

Let us denote $A(t) = \begin{pmatrix} 0 & 1 \\ -p(t) & 0 \end{pmatrix}$, $\varphi_\lambda = \begin{pmatrix} x_\lambda \\ y_\lambda \end{pmatrix}$, and $\varphi_\lambda(0) = \varphi_0$. Then

$$\varphi^\Delta = A(t)\varphi - \varepsilon f(t,\varphi). \tag{2.80}$$

Similarly, with the notation $\psi_\lambda = \begin{pmatrix} x_\lambda \\ y_\lambda \end{pmatrix}$, we represent equation (2.60) in the form

$$\psi_\lambda^\Delta(t) = A(t)\psi, \tag{2.81}$$

where $\psi_\lambda(0) = \psi_0$.

Let $X(t,s)$ be the fundamental system of solutions of equation (2.81) [5]. Hence,

$$\psi_\lambda(t) = X(t,0)\psi_0. \tag{2.82}$$

Rewriting (2.80) in an integral form, according to [6], we have

$$\varphi_\lambda(t) = X(t,0)\varphi_0 + \varepsilon \int_{[0,t]_\lambda} X(t,\sigma(s)) f(s,\varphi_\lambda(s))\Delta s. \tag{2.83}$$

It follows from (2.82) and (2.83) that

$$|\varphi_\lambda(t) - \psi_\lambda(t)| \le \varepsilon \int_{[0,t]_\lambda} |X(t,\sigma(s)) f(s,\varphi_\lambda(s))|\, \Delta s. \tag{2.84}$$

Since $f(t,x_\lambda,y_\lambda)$ is a linear growth function, we obtain

$$|f(t,\varphi_\lambda(t))| \le N(1+|\varphi_\lambda(t)|), \tag{2.85}$$

which yields

$$|\varphi_\lambda(t)| \le C_1|\varphi_0| + \varepsilon \int_{[0,t]_\lambda} C_1 N(1+\varphi_\lambda(s))\Delta s \equiv B_1, \tag{2.86}$$

where $||X(t,s)|| \le C_1$, $t,s \in [0,a]_{\mathbb{T}_\lambda}$.

Consequently,

$$|\varphi_\lambda(t) - \psi_\lambda(t)| \le \varepsilon N C_1 \int_{[0,t]_\lambda} (1+B_1)\Delta s. \tag{2.87}$$

We then have the estimate

$$|\varphi_\lambda(t) - \psi_\lambda(t)| \le \varepsilon N C_1 a(1+B_1). \tag{2.88}$$

Theorem 2.18 implies that if a solution of differential equation (2.59) has at least two zeros on the interval $[0,a]$, then there is $\mu_0 > 0$ such that the corresponding solution of equation (2.60) oscillates on this interval, i.e. it has at least two generalized zeros.

By Lemma 2.23, there is $\Delta(\mu_\lambda) > 0$ such that the amplitude of oscillations of any solution of equation (2.60) is bounded below:

$$M_p^x(\mu_\lambda) \ge \Delta(\mu_\lambda),$$

where $M_p^x = \max_{t\in[t_p,t_m]} |x_\lambda(t)|$, t_p and t_m are two consecutive generalized zeros of the solution x_λ.

Hence, from (2.88) it follows that there exist $\varepsilon > 0$ and $\mu_0 > 0$ such that if a solution of linear differential equation (2.59) has three zeros on $[0,a]$, then the corresponding solution of nonlinear dynamic equation (2.61) has at least two generalized zeros on the interval $[0,a]_{\mathbb{T}_\lambda}$, which means that this solution oscillates. □

2.5. Example

Let us illustrate the results obtained for the linear case by the following example.

We consider the Airy equation

$$\ddot{x} + t \cdot x = 0 \tag{2.89}$$

on the interval $[0, \frac{7\pi}{2}]$ with initial data

$$x(0) = 0; \quad \dot{x}(0) = 1, \tag{2.90}$$

satisfying condition (2.9), and the corresponding equation

$$x_\lambda^{\Delta\Delta} + t \cdot x_\lambda = 0 \tag{2.91}$$

on the set of scales $\mathbb{T}_\lambda = h\mathbb{Z}$ with the same initial data. Note that the graininess function of every such scale is constant and $\mu_\lambda = h$.

Equation (2.91) has the form of a difference equation:

$$\Delta_k^2 x + h^2 \cdot kh \cdot x(kh) = 0, \tag{2.92}$$

where $h > 0$ is the step size of the difference equation on $[0, \frac{7\pi}{2}]$, $kh = t_k \in \mathbb{T}_\lambda = h\mathbb{Z}$, $\Delta_k x = x(\sigma(t_k)) - x(t_k) = x(t_{k+1}) - x(t_k)$, $\Delta_k^2 x = \Delta_k(\Delta_k x)$, $k = 0, 1, 2, \ldots$.

Let us denote $x_k^h = x(t_k)$ and rewrite equations (2.92) in the form of systems

$$\begin{cases} x_{k+1}^h = x_k^h + h y_k^h, \\ y_{k+1}^h h = y_k^h - h \cdot t_k x_k^h, \end{cases} \tag{2.93}$$

with initial data

$$x_0^h = 0, \quad y_0^h = 1. \tag{2.94}$$

The solution of the Cauchy problem (2.89, 2.90) can be represented in the form [25]:

$$x(t) = -\frac{1}{2}\left(-\frac{1}{3}\right)^{\frac{2}{3}} \Gamma(\frac{1}{3}) \left(\sqrt{3}\mathrm{Bi}(-t) - 3\mathrm{Ai}(-t)\right), \tag{2.95}$$

where $\mathrm{Ai}(t)$ is the Airy function of the first kind, $\mathrm{Bi}(t)$ is the Airy function of the second kind, and $\Gamma(t)$ is the gamma function.

For $t \in \left[\frac{\pi}{2}, \frac{7\pi}{2}\right]$, by the comparison theorem [22], a solution of equation (2.89) has no less zeros as a solution of equation

$$\ddot{x} + \frac{\pi}{2} x = 0, \tag{2.96}$$

and no more zeros as a solution of equation

$$\ddot{x} + \frac{7\pi}{2} x = 0, \tag{2.97}$$

with the same initial data (2.90). The solution of equation (2.96) with initial data (2.90) is of the form

$$x(t) = \sqrt{\frac{2}{\pi}} \sin\left(\sqrt{\frac{\pi}{2}} t\right), \tag{2.98}$$

and has four zeros on $\left[\frac{\pi}{2}, \frac{7\pi}{2}\right]$. The solution of equation (2.97) with the same initial data is of the form

$$x(t) = \sqrt{\frac{2}{7\pi}} \sin\left(\sqrt{\frac{7\pi}{2}} t\right), \tag{2.99}$$

and has ten zeros on the given interval.

Similarly, if we choose for the study an arbitrary $t_0 \in \left(0, \frac{\pi}{2}\right)$, we obtain that on the interval $\left(t_0, \frac{\pi}{2}\right]$ the solution of equation (2.89) with initial data (2.90) has no zeros, since the solution of equation (2.96) with the same initial data has no zeros as well. Thus, the solution of the Cauchy problem (2.89),(2.90) on the interval $(0, \frac{7\pi}{2})$ has no less than four zeros, but no more than ten zeros.

We have plotted an approximate graph of the solution (2.95) of equation (2.89) (see Figure 1) and obtained seven zeros on $\left(0, \frac{7\pi}{2}\right]$. Their approximate values t_n are listed in Table 1.

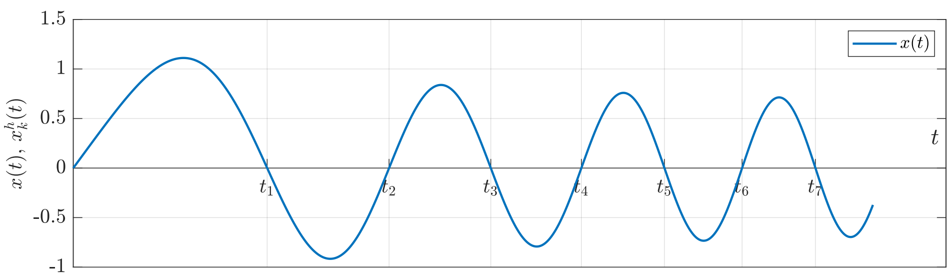

Figure 1. Approximate solution of Cauchy problem (2.89),(2.90)

Table 1. Approximate values of zeros of the solution of the Cauchy problem (2.89),(2.90)

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_n	2.666	4.342	5.741	6.986	8.128	9.196	10.204

Tables 2–5 provide the following values:

t_n, approximate values of zeros of the solution $x(t)$ of the Cauchy problem for equation (2.89) with initial data (2.90);

t_{n_λ}, generalized zeros in a neighborhood of t_n of the solution $x_k^h(t)$ of system (2.93) with initial data (2.94), for fixed h_λ;

$x_k^h(t_{\lambda n})$, values of the solution at generalized zeros for fixed h_λ;

$|t_n - t_{\lambda n}|$, differences between zeros of the differential equation and the dynamic equation for fixed h.

Table 2. Comparison of values when $h = \frac{7\pi}{94}$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_n	2.666	4.342	5.741	6.986	8.128	9.196	10.204
$t_{\lambda n}$	2.57343	4.44501	5.84870	7.25240	8.65609	9.82583	-
$x_k^h(t_{\lambda n})$	0.48026	–0.54678	2.16404	–4.31641	1.61439	–9.84481	-
$\|t_n - t_{\lambda n}\|$	0.09256	0.10301	0.10770	0.26640	0.52809	0.62983	-

Table 3. Comparison of values when $h = \frac{7\pi}{96}$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_n	2.666	4.342	5.741	6.986	8.128	9.196	10.204
$t_{\lambda n}$	2.74889	4.35241	5.95593	7.33038	8.47575	9.62112	10.7664
$x_k^h(t_{\lambda n})$	0.08729	–0.89673	1.00212	–1.89404	11.653	–37.4452	104.816
$\lvert t_n - t_{\lambda n}\rvert$	0.08289	0.01041	0.21493	0.34438	0.34775	0.42512	0.56249

Table 4. Comparison of values when $h = \frac{7\pi}{200}$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_n	2.666	4.342	5.741	6.986	8.128	9.196	10.204
$t_{\lambda n}$	2.63893	4.39822	5.82765	7.03716	8.24668	9.34623	10.33583
$x_k^h(t_{\lambda n})$	0.14208	–0.03916	0.031304	–0.51241	0.3775	–0.59169	2.79772
$\lvert t_n - t_{\lambda n}\rvert$	0.02706	0.05623	0.08665	0.05117	0.11868	0.15024	0.13184

Table 5. Comparison of values when $h = \frac{7\pi}{2000}$.

	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_n	2.666	4.342	5.741	6.986	8.128	9.196	10.204
$t_{\lambda n}$	2.66093	4.34325	5.73969	6.98219	8.12573	9.19230	10.2039
$x_k^h(t_{\lambda n})$	0.01472	–0.00646	0.012657	–0.02092	0.02143	–0.02634	0.02075
$\lvert t_n - t_{\lambda n}\rvert$	0.00507	0.00125	0.00131	0.00381	0.00227	0.00369	0.00017

As can be seen from Tables 2–5, there exists $h_0 = \frac{7\pi}{96}$ such that for all $h < h_0$ in neighborhoods of every zero t_n of the solution $x(t)$ of problem (2.89), (2.90) there is a generalized zero $t_{\lambda n}$ of the solution $x_k^h(t)$ of (2.93), (2.94).

Moreover, by calculating the generalized zeros of problem (2.93), (2.94) with $h = \frac{7\pi}{200}$ and $h = \frac{7\pi}{2000}$, it can be easily seen that the generalized zeros approach the zeros of the solution of problem (2.89), (2.90) as h approaches 0. This is illustrated by Figures 2–5.

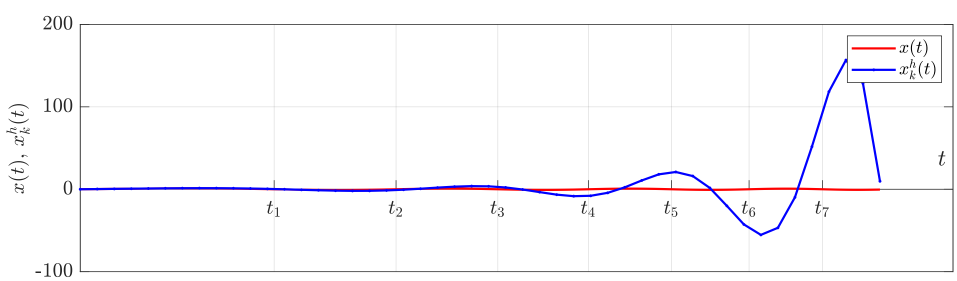

Figure 2. Comparing the solutions of (2.89) and (2.91) when $h = \frac{7\pi}{94}$.

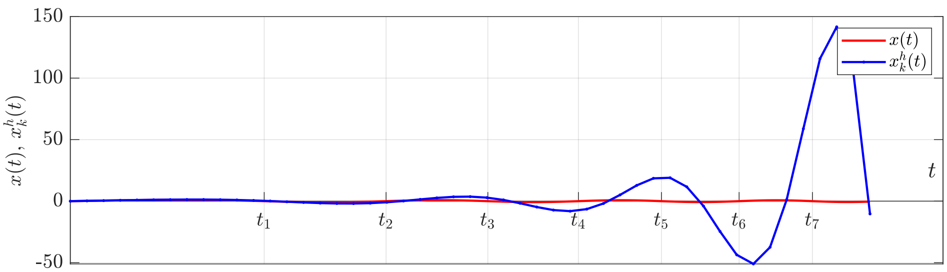

Figure 3. Comparing the solutions of (2.89) and (2.91) when $h = \frac{7\pi}{96}$.

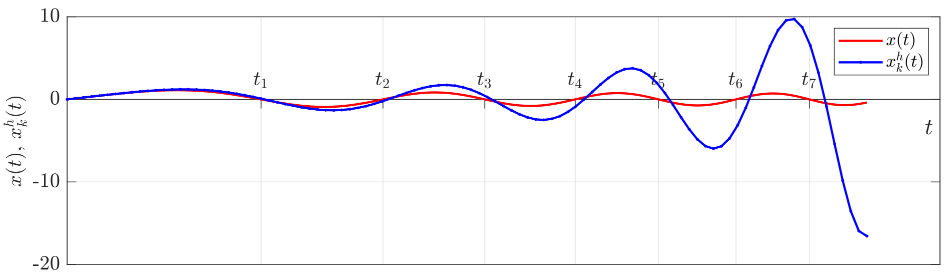

Figure 4. Comparing the solutions of (2.89) and (2.91) when $h = \frac{7\pi}{200}$.

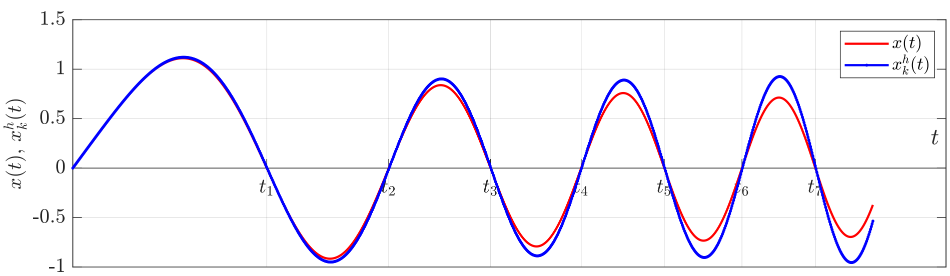

Figure 5. Comparing the solutions of (2.89) and (2.91) when $h = \frac{7\pi}{2000}$.

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