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A note on ss-supplement submodules

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Abstract: In this paper, we describe ss-supplement submodules in terms of a special class of endomorphisms. Let R be a ring with semisimple radical and P be a projective $R-$module. We show that there is a bijection between ss-supplement submodules of P and ss-supplement submodules of $End_R(P)$. Moreover, we define radical-s-projective modules as a generalization of projective modules. We prove that every ss-supplement submodule of a projective $R-$module is radical-s-projective over the ring R with semisimple radical. We show that over SSI-ring R, every radical-s-projective $R-$module is projective. We provide that over a ring R with semisimple radical, every ss-supplement submodule of a projective $R-$module is a direct summand if and only if every radical-s-projective $R-$module is projective.

Key words: ss-Supplement, radical-s-projective modules, endomorphism rings

1. Introduction

Throughout this study, R is associative ring with identity, all modules are unital left $R-$modules and homomorphisms operate on the right. Let M be an $R-$module. The notation $U \leq M$ means that U is a submodule of M. A submodule U of M is called *small* in M, denoted as $U \ll M$, if $M \neq U + K$ for every proper submodule K of M. An $R-$module homomorphism $f : M \to N$ is called a *small epimorphism* if it is an epimorphism with $Ker(f) \ll M$. A projective module P together with a small epimorphism $f : P \to M$ is called *projective cover* of M. $End_R(M)$ indicates the endomorphism ring of an $R-$module M. By $Rad(M)$, we denote the intersection of all maximal submodules of M, equivalently the sum of all small submodules of M. Moreover, $Soc(M)$ stands for the socle of a module M, i.e. the sum of all simple submodules of M. Note that $Soc(M)$ is the largest semisimple submodule of a module M (see [13]).

Let M be a module and U, V be submodules of M. V is called a *supplement* of U in M if V is minimal element in the set of submodules $L \leq M$ with $M = U + L$, equivalently $M = U + V$ and $U \cap V \ll V$ (see [13]).

Following [14], the sum of all simple submodules of M which are small in M is named with $Soc_s(M)$ for a module M, that is, $Soc_s(M) = \sum\{U \ll M \mid U \text{ is simple}\}$. Denote that $Soc_s(M) \leq Soc(M)$ and $Soc_s(M) \leq Rad(M)$. In [7], a submodule V is called an *ss-supplement* of U in a module M if $M = U + V$ and $U \cap V \leq Soc_s(V)$. In the same paper, it is proved that V is an ss-supplement of U in M if and only if $M = U + V$, $U \cap V \ll V$ and $U \cap V$ is semisimple if and only if $M = U + V$, $U \cap V \leq Rad(V)$ and $U \cap V$ is semisimple. Now it is clear that the following implication on submodules of a module holds:

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direct summands $\Longrightarrow$ *ss*-supplement submodules $\Longrightarrow$ supplement submodules

In [4], an element x of a ring R is called *left (CE) element* if there exists $a \in R$ such that $ax^2 = x$ and $x - x^2 \in Rad(R)$. An element x of a ring R is called *weak left (CE) element* if there exists $b \in R$ such that $bx^2 = x$ and for every $b' \in R$ with $b'x^2 = x$, there exists $c \in R$ with $cb'x = x$. Note that every left (CE) element is a weak left (CE) element. The author showed that there is a relationship between left (CE) elements of endomorphism ring of a projective module M and supplement submodules of M. In addition, he described radical-projective modules as a generalization of projective modules. A module M is called *radical-projective* if for every $R-$module epimorphism $g : A \to B$ and a homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$ such that $(M)(f - hg) \ll B$. The obtained results in [4] allow to extend some of the characterizations of the rings for which every supplement submodule of a finitely generated projective module is a direct summand to those in which every supplement submodule of a (nonnecessarily finitely generated) projective module is a direct summand.

In this paper, we start by searching what kind of connection there is between *ss*-supplements of a projective module M over the ring with semisimple radical and left (CE) elements of the ring $End_R(M)$. We show that *ss*-supplement submodules in a projective module M over the ring with semisimple radical are of the form $Im(f)$ for a left (CE) endomorphism f of M.

In Section 3, we define radical-s-projective modules as a generalization of projective modules. We prove that every *ss*-supplement submodule of a projective module is radical-projective. We also provide that *ss*-supplements in projective modules are radical-s-projective over the rings with semisimple radical. Over a ring with semisimple radical, we show that for a radical-s-projective module M, $Rad(M) = Rad(R)M$, $Rad(M)$ is semisimple and *ss*-supplements in M are radical-s-projective. We give some examples of projective modules whose *ss*-supplements are direct summands. Moreover, we give an example of a ring whose every radical-s-projective module is projective.

2. *ss*-Supplement submodules in projective modules

First of all, we state the rings whose every submodule is an ss-supplement in the following proposition.

Proposition 2.1 *The following statements are equivalent for a ring R:*

1. *R is a semisimple ring.*
2. *Every submodule of ${}_RR$ is an ss-supplement in ${}_RR$.*
3. *Every submodule of ${}_RR$ is a supplement in ${}_RR$.*

Proof (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) are clear.

(3) $\Longrightarrow$ (1) follows from [9, Lemma 2.13]. □

In general, every supplement submodule of a projective module is not an *ss*-supplement submodule. For example; consider the ring $R = \mathbb{Z}_8$. The projective module ${}_RR$ is a supplement of $Rad({}_RR)$ in ${}_RR$; but ${}_RR$ is not an *ss*-supplement.

Lemma 2.2 *Let R be a ring with semisimple $Rad(R)$, P be a projective left $R-$module and $K, L \leq P$. Then K is an ss-supplement of L in P if and only if K is a supplement of L in P.*

Proof The necessity is clear.

Conversely, suppose that K is a supplement of L in P. Then $P = K + L$ and $K \cap L \ll K$. Thus, we have that $K \cap L \leq Rad(P) = Rad(R)P \leq Soc({}_RR)P = Soc(P)$, since P is projective and by the assumption. Hence, K is an *ss*-supplement of L in P from [7, Lemma 3]. □

Left (CE) elements were used by Zöschinger in [15] and their relationship with supplements in projective modules was given by Izurdiaga in [4]. Now we shall give the relationship of left (CE) elements with *ss*-supplements in projective modules over the rings whose radical is semisimple.

Proposition 2.3 *Let P be a projective $R-$module where $Rad(R)$ is semisimple and $f \in End_R(P)$. Then the following statements are equivalent:*

1. *f is a left (CE) element in $End_R(P)$.*
2. *$Im(f)$ is an ss-supplement of $Im(1_P - f)$ in P.*

Proof (1) $\Longrightarrow$ (2) Since $f \in End_R(P)$ is left (CE), then by [4, Proposition 1.4] $Im(f)$ is a supplement of $Im(1_P - f)$ in P. Thus, $Im(f)$ is an *ss*-supplement of $Im(1_P - f)$ in P from Lemma 2.2.

(2) $\Longrightarrow$ (1) $Im(f)$ is a supplement of $Im(1_P - f)$ in P, by the hypothesis. Thus, f is a left (CE) element in $End_R(P)$ by [4, Proposition 1.4]. □

Proposition 2.4 *Let P be a projective $R-$module where $Rad(R)$ is semisimple and $f \in End_R(P)$. Then the following statements are equivalent:*

1. *f is weak left (CE) element in $End_R(P)$.*
2. *$Im(f)$ is an ss-supplement of $Ker(f)$ in P.*

Proof (1) $\Longrightarrow$ (2) Let f be weak left (CE) endomorphism of P. Then by [4, Proposition 1.3], $Im(f)$ is a supplement of $Ker(f)$ in P. Thus, $Im(f)$ is an *ss*-supplement of $Ker(f)$ in P from Lemma 2.2.

(2) $\Longrightarrow$ (1) Let $Im(f)$ be an *ss*-supplement of $Ker(f)$ in P. Then $Im(f)$ is a supplement of $Ker(f)$ in P. Hence, f is a weak left (CE) element of $End_R(P)$ by [4, Proposition 1.3]. □

It is well known that if M is a module and f is an endomorphism of M, then $M = Im(f) \oplus Im(1_M - f)$ if and only if f is idempotent. The above propositions generalize this one in the case of projective modules over the rings whose radical is semisimple, *ss*-supplements and left (CE) endomorphisms.

The following result shows that when the module is projective over a ring with semisimple radical, every *ss*-supplement is of the form $Im(f)$ for a left (CE) endomorphism f. This is a generalization of the well known result about direct summands (see, for example, [1, 5.7]).

Theorem 2.5 *Let R be a ring with semisimple $Rad(R)$, P be projective $R-$module and $K, L \leq P$. Then the following statements are equivalent:*

1. *K is an ss-supplement of L in P.*
2. *There exists $f \in End_R(P)$ left (CE) such that:*

a) $Im(f) = K$.

b) $Im(1_P - f) \leq L$

c) $(L)f \ll K$ *and* $(L)f$ *is semisimple.*

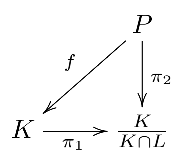
3. *There exists* $f \in End_R(P)$ *weak left* (CE) *such that:*

a) $Im(f) = K$.

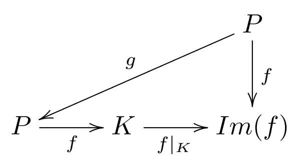
b) $Ker(f) \leq L$

c) $(L)f \ll K$.

Proof (1) $\Longrightarrow$ (2) We can easily write the decomposition $\frac{P}{K\cap L} = \frac{K}{K\cap L} \oplus \frac{L}{K\cap L}$, by the hypothesis. By using this decomposition and the projectivity of P, we can construct the following commutative diagram with canonical projections π_1 and π_2



and there exists a homomorphism $f : P \to K$ such that $f\pi_1 = \pi_2$. Since $(K)f\pi_1 = \frac{K}{K\cap L}$, $f\mid_K$ is an epimorphism by [1, 5.15], and so (a) holds. Since P is projective, then we get from the following diagram that there is a $g \in End_R(P)$ such that $gf^2 = f$.



Now for every $x \in P$, $x = k + l$ where $k \in K$ and $l \in L$. Then we have that $(x)f\pi_1 = (k+l)f\pi_1 = (k+l)\pi_2$. Thus, the equality $(x)f + K\cap L = k + K\cap L$ implies that $k - (x)f \in K\cap L$. Since $k + l - (x)f - l \in K\cap L \leq L$, we deduce that $x - (x)f \in L$. Thus, (b) holds; and so we get that $(P)(1_P - f)f\pi_1 = (P)(f - f^2)\pi_1 = 0$. It follows immediately that $(P)(f - f^2) \leq Ker(\pi_1) = K\cap L \ll K \leq P$; and thus $f - f^2 \in Rad(End_R(P))$ by [1, 17.11]. On the other hand, since $(L)f\pi_1 = (L)f + K\cap L = (L)\pi_2 = 0$, then we get that $(L)f \leq K\cap L \ll K$ and $(L)f$ is semisimple as a submodule of $K\cap L$ from [6, 8.1.5]; hence, (c) holds.

(2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (1) $Im(f)$ is an *ss*-supplement of $Ker(f)$ in P from Proposition 2.4. Thus, we have that $P = K + L$. Now let X be a submodule of K such that $P = X + L$. Since $(L)f \ll K$, then from the equality $(X)f + (L)f = (P)f = K$, we get that $K = (X)f$. Therefore, $P = X + Ker(f)$. Since $K = Im(f)$ is a supplement of $Ker(f)$ in P, we obtain that $X = K$. Consequently, K is an *ss*-supplement of L in P by Lemma 2.2. □

Corollary 2.6 *Let* R *be a ring with semisimple* $Rad(R)$ *and* K, L *be left ideals of* R. *Then the following statements are equivalent:*

1. *K is an ss-supplement of L in ${}_RR$.*
2. *There exists left (CE) element x of R such that:*
   a) *$Rx = K$.*
   b) *$R(1-x) \subseteq L$*
   c) *$Lx \ll K$ and Lx is semisimple.*
3. *There exists a weak left (CE) element x of R such that:*
   a) *$Rx = K$.*
   b) *$\{r \in R \mid rx = 0\} \subseteq L$*
   c) *$Lx \ll K$.*

Proof It follows from Theorem 2.5. □

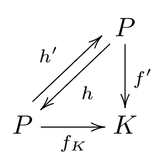
We should note that the left (CE) endomorphisms which are obtained in Theorem 2.5 is not uniquely determined by K and L. Now we give an example of this case.

Example 2.7 *Let consider the ring $R = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$. Let $K = \begin{bmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{bmatrix}$, $L = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & 0 \end{bmatrix}$ and $a, b \in \mathbb{Z}_3$ distinct and nonzero. Note that $L = Soc({}_RR)$. K is an ss-supplement of L in ${}_RR$. Also, since $f = \begin{bmatrix} 0 & a \\ 0 & 1 \end{bmatrix}$ and $g = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$ are idempotent elements of R, they are distinct left (CE) endomorphisms of R that satisfy the conditions of Theorem 2.5.*

Now we shall give the relationship between *ss*-supplement submodules of a projective module M over a ring with semisimple radical and *ss*-supplement submodules of endomorphism ring of M. For a module M, we denote by $S_s(M)$ the set of *ss*-supplement submodules of M.

Proposition 2.8 *Let R be a ring with semisimple $Rad(R)$. For every projective $R-$module P with endomorphism ring $E = End_R(P)$, there is a bijective function between the sets $S_s(P)$ and $S_s({}_EE)$.*

Proof Let take any *ss*-supplement submodule of P, say K, we have left (CE) endomorphism f_K of P such that $Im(f_K) = K$ by Theorem 2.5. Now consider the left ideal $E.f_K$ of E. By Corollary 2.6, the left ideal $E.f_K$ is an *ss*-supplement submodule of ${}_EE$. We claim that the left ideal $E.f_K$ does not depend on the election of the endomorphism f_K. For every left (CE) endomorphism f' with $Im(f') = K$, we can construct the following commutative diagram



that means $h'f' = f_K$ and $hf_K = f'$ as P is projective. Thus, for every $g \in E$, we obtain that $gf_K = g(h'f') = (gh')f' \in E.f'$. Conversely, for every $g' \in E$, we get that $g'f' = g'(hf_K) = (g'h)f_K \in E.f_K$. Hence, we deduce that $E.f_K = E.f'$.

Analogously, if we take an *ss*-supplement submodule of ${}_E E$, say L, then there exists a left (CE) element x_L in E such that $L = E.x_L$ by Corollary 2.6. In this case, $Im(x_L)$ is an *ss*-supplement submodule in P by Proposition 2.3 and this submodule does not depend on the election of x_L, as the reason can be checked as above with the help of the projectivity of P.

Now we define the maps $\Phi : S_s(P) \to S_s({}_E E)$ via $\Phi(K) = E.f_K$ where $Im(f_K) = K$ for all $K \in S_s(P)$ and $\Psi : S_s({}_E E) \to S_s(P)$ via $\Psi(L) = Im(x_L)$ where $L = E.x_L$ for all $L \in S_s({}_E E)$. It can be easily checked that Φ and Ψ are mutually inverse. $\square$

3. Radical-s-projective modules

We shall define radical-s-projective modules as a generalization of radical-projective modules.

Definition 3.1 *We call a module M is radical-s-projective if for every $R-$module epimorphism $g : A \to B$ and every homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$ such that $(M)(f - hg) \ll B$ and $(M)(f - hg)$ is semisimple.*

Now we have the following implications on modules under given definitions:

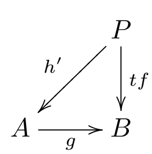
projective modules $\Longrightarrow$ radical-s-projective modules $\Longrightarrow$ radical-projective modules

Proposition 3.2 *Let P be a projective module and K be an ss-supplement submodule in P. Then K is radical-projective.*

Proof If K is an *ss*-supplement in P, then K is a supplement submodule in P. Hence, K is radical-projective from [4, Lemma 2.2]. $\square$

Proposition 3.3 *Let R be a ring with semisimple $Rad(R)$. Then every ss-supplement submodule of a projective $R-$module is radical-s-projective.*

Proof Let P be a projective $R-$module and K be an *ss*-supplement submodule of P. Then there exists $t \in End_R(P)$ left (CE) such that $Im(t) = K$ by Proposition 2.3. Let $g : A \to B$ be an $R-$module epimorphism and $f : K \to B$ be a homomorphism. Since P is projective, there exists a commutative diagram



such that $h'g = tf$. Note that, since t is left (CE) endomorphism, then $(P)(t - t^2) \ll P$ by [1, 17.11]. Now put $h = h' \mid_K$, then we deduce that

$$(K)(f - hg) = (K)t(f - hg) = (K)(1_B - t)hg = (K)(t - t^2)hg \ll B$$

by [13, 19.3(4)], since $(K)(t - t^2) \ll K$. Moreover, $(K)(t - t^2) \leq Rad(K) = Rad(R)K$ from [15, Lemma 2.1]. Therefore, we get that $(K)(f - hg) = (K)(t - t^2)hg \leq (Rad(R)K)hg = Rad(R)(K)hg \leq Soc({}_R R)(K)hg \leq Soc((K)hg)$ by the assumption. It completes the proof. $\square$

Remark 3.4 *Note that the homomorphism h that is obtained in Proposition 3.3 satisfies $(K)(f-hg) \leq Rad(R)B$ since $Rad(K)=Rad(R)K$ and $(K)(t-t^2) \leq Rad(R)K$ by [15, Lemma 2.1].*

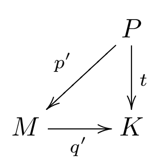
The following lemma is a slight generalization of [8, Lemma 2.3] for *ss*-supplement submodules.

Lemma 3.5 *Let R be a ring with semisimple $Rad(R)$, P be a projective $R-$module and $K \leq P$. Then the following statements are equivalent:*

1. *K is an ss-supplement in P.*
2. *There exist a small $R-$module epimorphism $q: M \to K$ and a homomorphism $p: P \to M$ such that qp is small epimorphism and $Ker(qp)$ is semisimple.*

Moreover, when these conditions are verified, for every small epimorphism $q': M \to K$ there exists $p': P \to M$ such that $q'p'$ is a small epimorphism with semisimple $Ker(q'p')$. In addition, the homomorphism p' can be obtained so that $p'q'$ is left (CE) endomorphism of P with image K.

Proof (1) $\Longrightarrow$ (2) If K is an *ss*-supplement in P, then there exists a left (CE) endomorphism t of P such that $Im(t)=K$, by Theorem 2.5. Thus, $Im(t)$ is an *ss*-supplement of $Ker(t)$ in P from Proposition 2.4. Thus, we can see $q = t\mid_K : K \to P$ is a small epimorphism because $Im(q)=K$ and $Ker(q)=K \cap Ker(t) \ll K$. Hence, the first part of (2) is verified. Note that $K \cap Ker(t)$ is semisimple, by the hypothesis. Now let $q': M \to K$ be a small epimorphism. Since P is projective, we can construct the following commutative diagram



from which we infer that there exists a homomorphism $p': P \to M$ such that $p'q' = t$. Since $(K)p'q' = (K)t = K$ and q' is small epimorphism, then $p'\mid_K$ is an epimorphism by [1, 5.15]. Since q is small epimorphism, i.e $Ker(q) = Ker(t\mid_K) \ll K$, it is obtained that $Ker(p'\mid_K) \ll K$. This means that $p'\mid_K$ is small epimorphism. Therefore, $q'p' = q'(p'\mid_K)$ is a small epimorphism as a composition of two such epimorphism from [13, 19.3(1)]. Moreover, since P is projective, $Ker(q'p') \leq Rad(P) = Rad(R)P \leq Soc({}_RR)P = Soc(P)$ by the assumption, and so $Ker(q'p')$ is semisimple.

(2) $\Longrightarrow$ (1) Let $q: M \to K$ be small epimorphism and $p: P \to M$ be a homomorphism such that qp is small epimorphism with semisimple $Ker(qp)$. We may assume that M is a submodule of K, and of P, too. In this case, since qp is an epimorphism by the hypothesis, then we get that $P = K + Ker(p)$. Since $K \cap Ker(p) \leq Ker(qp)$ is small in K, we obtain that $K \cap Ker(p) \ll K$. Hence, K is an *ss*-supplement in P by Lemma 2.2. □

Now we shall determine *ss*-supplement submodules of free modules in terms of radical-s-projective modules using left (CE) morphisms.

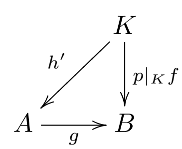
Theorem 3.6 *Let R be a ring with semisimple $Rad(R)$ and M be an $R-$module. Then the following statements are equivalent:*

1. *M is radical-s-projective.*

2. *There exist a free $R-$module F, an ss-supplement submodule K of F, and a small epimorphism $\varphi : M \to K$ with semisimple $Ker(\varphi)$.*

Proof (1) $\Longrightarrow$ (2) Let F be a convenient free module such that there exists an epimorphism $\psi : F \to M$. Since M is radical-s-projective, there exists a homomorphism $h : M \to F$ such that $(M)(1_M - h\psi) \ll M$ and $(M)(1_M - h\psi)$ is semisimple. Since the equality $M = (M)h\psi + (M)(1_M - h\psi)$ implies that $M = (M)h\psi$, $h\psi$ is an epimorphism. Moreover, since $Ker(h\psi) \leq Im(1_M - h\psi) \ll M$ and $(M)(1_M - h\psi)$ is semisimple, $h\psi$ is a small epimorphism with semisimple $Ker(h\psi)$. By Lemma 3.5, $Im(h)$ is an *ss*-supplement submodule of F. However, since $Ker(h) \leq Ker(h\psi)$, $h : M \to Im(h)$ is a small epimorphism with semisimple $Ker(h)$. Hence, the result follows, if we take $K = Im(h)$ and $\varphi = h$.

(2) $\Longrightarrow$ (1) By Lemma 3.5, there exists a homomorphism $p : F \to M$ such that φp is small epimorphism with semisimple $Ker(\varphi p)$, and in addition $p\varphi$ is a left (CE) element in $End_R(F)$ with $Im(p\varphi) = K$. Let $g : A \to B$ be an $R-$module epimorphism and $f : M \to B$ be a homomorphism. By Proposition 3.3, K is radical-s-projective, and so we can construct the following diagram



and $(K)(pf - h'g) \ll B$ and $(K)(pf - h'g)$ is semisimple. Put $h = \varphi h'$. Since $p\varphi$ is left (CE) with image K, $p\mid_K \varphi$ is an epimorphism, and so $p\mid_K$ is too, by [1, 5.15]. Since φp and φ are small epimorphisms, we obtain that

$$(M)(f - hg) = (M)\varphi p(f - hg) = (M)\varphi p(f - \varphi h'g) = (K)p(f - \varphi h'g) = (K)(pf - h'g + h'g - p\varphi h'g) \leq (K)(pf - h'g) + (K)(1_F - p\varphi)h'g. \quad (1)$$

There exists $t \in End_R(F)$ with $t(p\varphi)^2 = p\varphi$, since $p\varphi$ is left (CE). Note that $t(p\varphi - (p\varphi)^2) \in Rad(End_R(F))$ since $Rad(End_R(F))$ is a left ideal of $End_R(F)$. So that, by using $(K)p\varphi = K$ we obtain from [1, 17.11] that

$$(K)(1_F - p\varphi)h'g = (K)(p\varphi - (p\varphi)^2)h'g = (K)t(p\varphi - (p\varphi)^2)p\varphi h'g \ll B. \quad (2)$$

Therefore, by [13, 19.3(3)], we deduce from (1) and (2) that $(M)(f - hg) \leq (K)(pf - h'g) + (K)(1_F - p\varphi)h'g \ll B$. Moreover, since $(K)t(p\varphi - (p\varphi)^2)p\varphi \leq Rad(K) = Rad(R)K$ by [15, Lemma 2.1], then we get that $(K)t(p\varphi - (p\varphi)^2)p\varphi h'g \leq (Rad(R)K)h'g = Rad(R)(K)h'g \leq Soc(_RR)(K)h'g \leq Soc((K)h'g)$. Consequently, $(M)(f - hg)$ is semisimple by [6, 8.1.5]. □

Corollary 3.7 *Let R be a ring where $Rad(R)$ is semisimple and M be radical-s-projective module. Then;*

1. *$Rad(M) = Rad(R)M$ and $Rad(M)$ is semisimple.*

2. *ss-Supplement submodules of M are radical-s-projective.*

Proof (1) By Theorem 3.6, we have K that is an ss-supplement submodule of a free module and $\varphi : M \to K$ that is a small epimorphism with semisimple $Ker(\varphi)$, since M is radical-s-projective. By Lemma 3.5, there exists a homomorphism $p : K \to M$ such that φp is small epimorphism with semisimple $Ker(\varphi p)$. Since φ is small epimorphism, p is too, from [8, Lemma 2.1]. Thus, we obtain that $Rad(M) = (Rad(K))p = (Rad(R)K)p = Rad(R)(K)p = Rad(R)M$ by [1, 9.15] and [15, Lemma 2.1], as desired. On the other hand, since $Rad(R)$ is semisimple, $Rad(M) \le Soc(_R R)M \le Soc(M)$.

(2) Let L be an ss-supplement submodule of M. Since M is radical-s-projective, there are free module F, an ss-supplement submodule K of F and a small epimorphism $\varphi : M \to K$ with semisimple $Ker(\varphi)$, by Theorem 3.6. Note that $\frac{M}{Ker(\varphi)} \cong \varphi(M) = K$. Consider the canonical projection $\pi : M \to \frac{M}{Ker(\varphi)}$. By the way, if L is an ss-supplement in M, then $\varphi(L)$ is an ss-supplement submodule of K, because $\frac{L+Ker(\varphi)}{Ker(\varphi)}$ is an ss-supplement in $\frac{M}{Ker(\varphi)}$. Thus, $\varphi(L)$ is a supplement submodule of K. Since K is a supplement submodule of F, by [3, 20.6(2)] $\varphi(L)$ is a supplement submodule of F. Note that $\varphi(L)$ is an ss-supplement submodule of F from Lemma 2.2. Since $Ker(\varphi) \ll M$, then we get that $L \cap Ker(\varphi) \ll L$ from [13, 41.1(5)]. Therefore, $\varphi |_L : L \to \varphi(L)$ is a small epimorphism with semisimple kernel by [6, 8.1.5]. Hence, L is radical-s-projective from Theorem 3.6. □

Recall from [13] that an ideal I of a ring R is *left t-nilpotent* if for every sequence $a_1, a_2, ...$ of elements in I, there is a $k \in \mathbb{Z}^+$ with $a_k a_{k-1} ... a_1 = 0$.

In [8] a module M is said to be $J(R)$*-projective* if for every $R-$module epimorphism $g : A \to B$ with $Rad(R)B = 0$ and every homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$ such that $f = hg$.

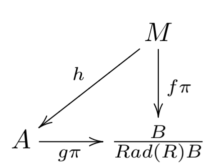
If a module M is radical-s-projective, then Remark 3.4 follows that the homomorphism h that is obtained verifies $(M)(f - hg) \le Rad(R)B$. Thus if $Rad(R)B = 0$, then $f = hg$. Hence, M is $J(R)$-projective, but the converse is not true in general, as the following example shows:

Example 3.8 *Let $R = \mathbb{Z}_2[[x]]$ the ring of formal power series in x with coefficients in $\mathbb{Z}_2$. Then R is left noetherian ring and $Rad(R)$ is not left t-nilpotent. Thus, there exists a left $R-$module M such that $Rad(R)M = M$ from [1, 28.3]. Therefore, M is $J(R)-$projective, but M is not radical-s-projective module from [4, 3.11].*

Now we give necessary conditions for $J(R)$-projective modules to be radical-s-projective.

Proposition 3.9 *Let R be a ring with semisimple $Rad(R)$ and M be a finitely generated $R-$module. Then M is radical-s-projective if and only if M is $J(R)$-projective.*

Proof Let M be $J(R)$-projective module. Let $g : A \to B$ be an $R-$module epimorphism and $f : M \to B$ be a homomorphism. Consider the canonical projection $\pi : B \to \frac{B}{Rad(R)B}$. Since M is $J(R)$-projective, then there exists a commutative diagram



such that there exists a homomorphism $h : M \to A$ with $hg\pi = f\pi$. Thus, we obtain that $0 = (M)(f\pi - hg\pi) = (M)(f - hg)\pi$, and so $(M)(f - hg) \leq Rad(R)B \leq Rad(B)$. Since M is finitely generated, $(M)(f - hg)$ is finitely generated. Since finitely generated submodules that are contained in radical are small, $(M)(f - hg) \ll B$. Moreover, we have that $(M)(f - hg) \leq Rad(R)B \leq Soc(_RR)B \leq Soc(B)$ by the assumption. Hence, M is radical-s-projective. □

Rings for which every supplement submodule of a finitely generated (in [4] not necessarily finitely generated) projective module is a direct summand have been widely studied (see, for example, [8], [15]). Now we shall give the analogous fact for *ss*-supplement submodules of projective modules.

Lemma 3.10 *Let R be a ring with semisimple $Rad(R)$, P be a projective $R-$module, and K be an ss-supplement submodule of P. Then the following statements are equivalent:*

1. *K is a direct summand.*
2. *K is projective.*
3. *For every $K' \leq P$ such that K is an ss-supplement of K', the factor module $\frac{P}{K'}$ has a projective cover whose kernel is semisimple.*
4. *K has a projective cover whose kernel is semisimple.*

Proof (1) $\Longrightarrow$ (2) follows from [13, 18.1]

(2) $\Longrightarrow$ (1) By Theorem 2.5 there exists a left (CE) element f in $End_R(P)$ with $Im(f) = K$. Since $K \cong \frac{P}{Ker(f)} \cong \frac{K}{K \cap Ker(f)}$, we deduce that $K \cap Ker(f)$ is a direct summand of K. On the other hand, $K \cap Ker(f)$ is small in K, by Proposition 2.4. Therefore, $K \cap Ker(f) = 0$. Again from Proposition 2.4, we get that $P = K \oplus Ker(f)$.

(2) $\Longrightarrow$ (3) Let K' be a submodule of P such that K is an *ss*-supplement of K'. Thus, we can write that $\frac{P}{K'} \cong \frac{K}{K \cap K'}$. Then the canonical projection $\pi : K \to \frac{K}{K \cap K'}$ is a projective cover with semisimple $Ker(\pi) = K \cap K'$, as required.

(3) $\Longrightarrow$ (4) Let K be an *ss*-supplement of K' in P. Note that $\frac{P}{K'} \cong \frac{K}{K \cap K'}$. Let $\varphi : Q \to \frac{K}{K \cap K'}$ be the projective cover of $\frac{K}{K \cap K'}$ whose kernel is semisimple. Since Q is projective, there exists a homomorphism $\psi : Q \to K$ such that $\psi\pi = \varphi$ where $\pi : K \to \frac{K}{K \cap K'}$ is the canonical projection. Then we have that $Ker(\psi) \leq Ker(\varphi) \ll Q$, and also $Ker(\psi)$ is semisimple. Hence ψ is the required projective cover.

(4) $\Longrightarrow$ (2) Let $\varphi : Q \to K$ be a projective cover of K whose kernel is semisimple. By Proposition 3.3, K is radical-s-projective. Thus, there exists a homomorphism $h : K \to Q$ such that $(K)(1_K - h\varphi) \ll K$ and $(K)(1_K - h\varphi)$ is semisimple. Since $K = (K)(1_K - h\varphi) + (K)h\varphi$, $h\varphi$ is an epimorphism. Since φ is a small epimorphism, then h is an epimorphism by [1, 5.15]. Moreover, since Q is projective, there exists a homomorphism $f : Q \to K$ such that $fh = 1_Q$, and so $K = Im(f) \oplus Ker(h)$. On the other hand, since $Ker(h) \leq (K)(1_K - h\varphi)$, $Ker(h)$ is small in K. Therefore, $Ker(h)$ has to be zero. Hence, h is an isomorphism and K is projective. □

For the modules M and N, N is said to be $M-$*cyclic* if there exists an epimorphism $\varphi : M \to N$.

Corollary 3.11 *Let R be a ring with semisimple $Rad(R)$, P be a projective $R-$module with endomorphism ring $E = End_R(P)$. Then the following statements are equivalent:*

1. *Every ss-supplement submodule of P is a direct summand.*
2. *Every ss-supplement submodule of ${}_EE$ is a direct summand.*
3. *Every $P-$cyclic radical-s-projective $R-$module is projective.*
4. *Every $P-$cyclic radical-s-projective $R-$module has a projective cover whose kernel is semisimple.*

Proof (1) $\Longrightarrow$ (2) Let L be a left ideal of E and suppose that L is an ss-supplement in E. Since there is a bijective function between ss-supplement submodules of P and ss-supplement submodules of ${}_EE$ from Proposition 2.8, by following the notation of the proof of Proposition 2.8, $\Psi(L)$ is a direct summand of P by assumption. Thus, there exists an idempotent element e in E with $\Psi(L) = Im(e)$ by [1, 5.8]. Again applying Proposition 2.8, we have that $L = \Phi\Psi(L) = E.e$. Hence, L is a direct summand of E.

(2) $\Longrightarrow$ (1) Let K be an ss-supplement submodule of P. By following the notation of the proof of Proposition 2.8, since there is a bijective function Φ from the set of ss-supplement submodules of P to the set of ss-supplement submodules of ${}_EE$ by Proposition 2.8, then $\Phi(K)$ is an ss-supplement submodule of ${}_EE$. By assumption, $\Phi(K)$ is a direct summand in ${}_EE$. Thus, we get that $\Phi(K) = E.f$ for some idempotent element f of E by [1, 5.8]. Now we deduce that $K = \Psi\Phi(K) = Im(f)$ is a direct summand in P.

(1) $\Longrightarrow$ (3) Let M be a $P-$cyclic radical-s-projective module. Then there exist an ss-supplement K of P and a small epimorphism $\varphi : M \to K$ with semisimple $Ker(\varphi)$ by Theorem 3.6. By assumption, K is a direct summand of P, and so K is projective from [13, 18.1]. Thus, there exists a homomorphism $f : K \to M$ such that $f\varphi = 1_K$. Therefore, $M = Ker(\varphi) \oplus Im(f)$, but since $Ker(\varphi)$ is small in M, then $Ker(\varphi)$ has to be zero. Hence, φ is an isomorphism and M is projective.

(3) $\Longrightarrow$ (4) Let M be a $P-$cyclic radical-s-projective module. By hypothesis, M is projective. Hence, 1_M is the desired projective cover of M.

(4) $\Longrightarrow$ (1) Let K be an ss-supplement submodule of P. From Theorem 2.5, there exists a left (CE) endomorphism f of P with $Im(f) = K$. Then K is $P-$cyclic. Also, from Proposition 3.3, K is radical-s-projective. Thus, K has a projective cover whose kernel is semisimple, by the assumption. Hence, K is a direct summand of P from Lemma 3.10. □

The following result gives a characterization of rings with semisimple radical for which every ss-supplement submodule of a projective module is a direct summand. For a set Γ, $\mathcal{RFM}_\Gamma(R)$ will indicate the ring of row finite $\Gamma-$matrices with entries in R.

Recall that a ring R is called *von-Neumann regular* if every element x can be written in the form xax, for some $a \in R$. A ring R is von-Neumann regular if and only if for every $x \in R$, Rx is a direct summand of ${}_RR$ (see [6, p. 38]).

Corollary 3.12 *Let R be a ring with semisimple $Rad(R)$. Then the following statements are equivalent:*

1. *Every ss-supplement submodule of a projective $R-$module is a direct summand.*
2. *Every supplement submodule of a projective $R-$module is a direct summand.*
3. *For any set Γ, every weak left (CE) matrix $A \in \mathcal{RFM}_\Gamma(R)$ is (von-Neumann) regular.*

4. *For any set* Γ, *every left* (CE) *matrix* $A \in \mathcal{RFM}_{\Gamma}(R)$ *is (von-Neumann) regular.*

5. *For every set* Γ, *left* (CE) *matrix* $A \in \mathcal{RFM}_{\Gamma}(R)$ *and* $B \in \mathcal{RFM}_{\Gamma}(R)$ *with* $BA^2 = A$ *it is provided that* $ABA = A$.

6. *Every radical-s-projective* $R-$*module has a projective cover whose kernel is semisimple.*

7. *Every radical-s-projective* $R-$*module is projective.*

Proof It is clear from Lemma 2.2, Corollary 3.11, and [4, Corollary 3.4] . □

Recall that a module M is called *hereditary* (respectively, *semihereditary*) if every (respectively, finitely generated) submodule of M is projective. A ring R is called *left hereditary* (respectively, *left semihereditary*) if ${}_RR$ is hereditary (respectively, semihereditary) (see [13]).

Example 3.13 *Every (respectively, finitely generated) ss-supplement submodule of a hereditary (respectively, semihereditary) projective* $R-$*module is a direct summand where* $Rad(R)$ *is semisimple by Lemma 3.10. Moreover, if* R *is a hereditary (respectively, semihereditary) ring with semisimple* $Rad(R)$, *then every ss-supplement submodule of a (respectively, finitely generated) projective* $R-$*module is a direct summand by Lemma 3.10.*

In [5], a ring R is said to be a *left* V*-ring* if every simple left $R-$module is injective. It is well known that a ring R is a left V-ring if and only if $Rad(M) = 0$ for every left $R-$module M.

Example 3.14 *Consider the commutative ring* $\prod_{i\geq 1}^{\infty} F_i$ *where* $F_i = F$ *is any field. Let* R *be the subring of this ring of the formed by all sequences* $(r_n)_{n\in\mathbb{N}}$ *such that there exist* $r \in F$, $m \in \mathbb{N}$ *with* $r_n = r$ *for all* $n \geq m$. *Then* R *is a left* $V-$*ring such that* $Soc({}_RR)$ *is a maximal submodule of* ${}_RR$ *by [11, Example 2.5]. Clearly, every ss-supplement submodule of* ${}_RR$ *is a direct summand, since* $Rad(R) = 0$. *Thus, every radical-s-projective* $R-$*module is projective by Corollary 3.12.*

In [2], a ring R is called *SSI-ring* if every semisimple left $R-$module is injective.

Proposition 3.15 *Let* R *be SSI-ring and* M *be an* $R-$*module. If* M *is radical-s-projective, then* M *is projective.*

Proof Let $g : A \to B$ be an $R-$module epimorphism and $f : M \to B$ be any homomorphism. Then by the hypothesis, there exists a homomorphism $h : M \to A$ such that $(M)(f - hg) \ll B$ and $(M)(f - hg)$ is semisimple. Since R is *SSI*-ring, then $(M)(f - hg)$ is injective. Thus, $(M)(f - hg)$ is a direct summand of B by [10, Theorem 2.15], but since $(M)(f - hg)$ is small in B, it has to be zero. Hence, we obtain that $f = hg$, so M is a projective module. □

Recall from [12] that a module M is said to have *the exchange property* if for any module K and any two decompositions

$$K = M' \oplus L = \bigoplus_{i\in I} A_i$$

where $M' \cong M$, there are submodules $A_i' \leq A_i$ such that

$$K = M' \oplus (\bigoplus_{i\in I} A'_i).$$

Note that M has finite exchange property if this holds whenever the index set I is finite. A ring R is called an *exchange ring* if ${}_RR$ has the exchange property. For finitely generated modules, the exchange and finite exchange properties coincide.

Example 3.16 (1) *Let P be a projective module such that $E = End_R(P)$ is an exchange ring. Since every ss-supplement submodule of P is a supplement, then it is a direct summand of P from [4, Example 3.10(iii)] as idempotents lift modulo $Rad(E)$ by [15, Example 2].*

(2) *Let R be the ring of eventually constant sequences $r = (s_1, s_2, ..., s_n, t, t, t, ...)$ where $s_1, s_2, ..., s_n \in \mathbb{Q}$, t is in the set of integers localised at the prime ideal $2\mathbb{Z}$ and n depends on r. Then R is an exchange ring with $Rad(R) = 0$ from [3, 11.42(3)]. Since ${}_RR$ has finite exchange property, by [12, Theorem 2] $End_R({}_RR)$ is an exchange ring for all $0 \neq n \in \mathbb{N}$. So that every ss-supplement in a finitely generated projective $R-$module is a direct summand by [4, Example 3.10(iii)].*

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