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Qualitative study of a second order difference equation

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Abstract: In this paper, we study a second order rational difference equation. We analyze the stability of the unique positive equilibrium of the equation and prove the existence of a Neimark-Sacker bifurcation, validating our theoretical analysis via a numerical exploration of the system.

Key words: Difference equations, Neimark-Sacker bifurcation, stability

1. Introduction

Nonlinear systems of difference equations have applications in the study of systems in which the $(k+1)$ state depends on the previous k states. These type of equations appears in the modeling of multiple phenomena in biology, ecology, physics, economics, etc. [3, 6, 9, 12–14]. This is the reason why, recently, many scientists have devoted their work to the study of the theory of difference equations [1, 19], the boundedness, the periodicity, the global asymptotic stability of their solutions and the existence of bifurcations. A bifurcation occurs when a small smooth variation made to the parameters of a system causes a sudden qualitative change in its behavior. In a Neimark–Sacker bifurcation, a closed invariant curve emerges from a fixed point in discrete dynamical systems when the fixed point changes its stability through a pair of complex eigenvalues with unit modulus [8–11, 16]. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable closed invariant curve, respectively. Recently, many authors have focused their efforts on the study of the existence of this bifurcation in many difference equations. Thus, DeVault et al. [5] consider the difference equation

$$x_{n+1} = p + \frac{x_{n-k}}{x_n},$$

where $n \in \mathbb{N}_0$, $k \in \mathbb{N}_2$, $p > 0$, and the initial conditions $x_{-k}, \ldots, x_0 > 0$, and perform a detailed analysis of the boundedness and the stability of its solutions. Saleh and coworkers [17] analyze the same equation, showing that each positive solution of the equation is globally asymptotically stable. They also analyze some properties concerning the semicycles of that equation. Tasdemir [18] studies a similar equation, given by

$$x_{n+1} = p + q\frac{x_n}{x_{n-m}^2},$$

where $p, q > 0$, $n \in \mathbb{N}_0$, and $m \in \mathbb{N}_2$. In his work, the author studies the boundedness of the solutions to the equation, their periodicity and their global stability. Beso et al. [3] show that the equation introduced by

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Tasdemir presents a Neimark–Sacker bifurcation, giving an asymptotic approximation of the invariant curve in the vicinity of the equilibrium point. He and Qiu [7] also investigate the existence of a Neimark–Sacker bifurcation in a difference equation of the type

$$x_{n+1} = \frac{\beta x_n + \alpha x_{n-2}}{1 + x_{n-1}}, \tag{1.1}$$

where $\alpha, \beta > 0$ and the initial conditions $x_{-2}, x_{-1}, x_0 \in \mathbb{R}_0^+$.

In a recent paper, Berkal and Navarro [2] carry out a qualitative analysis of the system defined by

$$x_{n+1} = \frac{(1+ha)x_n}{1 + hbx_n + hy_n}, \quad y_{n+1} = \frac{y_n(1 + hcx_n)}{1+h}, \tag{1.2}$$

being $h, a, b, c > 0$. The authors prove the existence of a Neimark–Sacker bifurcation.

Here, we have used the following notations: $\mathbb{N}$ for the set of natural numbers, $\mathbb{N}_\nu$ for the set $\{n \in \mathbb{Z} : n \geq \nu\}$, $\mathbb{R}_0^+$ for $\{x \in \mathbb{R} : x \geq 0\}$, and $\mathbb{R}^+$ for $\{x \in \mathbb{R} : x > 0\}$.

The studies cited above have led us to analyze in this paper the qualitative behavior of the rational difference equation of second order given by

$$X_{n+1} = A + B\frac{X_n^m}{X_{n-1}^{m+1}}, \quad m \geq 1, \tag{1.3}$$

where A, B and the initial conditions x_{-1}, x_0 are positive real numbers. In order to carry out this qualitative study, we introduce the change of variable

$$x_n = \frac{X_n}{A}, \tag{1.4}$$

which transforms Eq. (1.3) into

$$x_{n+1} = 1 + p\frac{x_n^m}{x_{n-1}^{m+1}}, \quad m \geq 1. \tag{1.5}$$

Note that this equation has a unique positive equilibrium, given by

$$\bar{x} = \frac{1 + \sqrt{1+4p}}{2}.$$

In Section 2, we calculate the positive equilibrium of Eq. (1.5) and determine its stability. Section 3 is devoted to the study of the existence of a Neimark–Sacker bifurcation. Finally, in Section 4, we perform a numerical exploration of a particular case of Eq. (1.5) in order to illustrate the theoretical results.

2. Stability analysis

In this section, we perform a local stability analysis of the positive equilibrium of Eq. (1.5). In order to discuss the stability of the equilibrium, we use the following Lemma [12]:

Lemma 2.1 *Let $\rho(\lambda) = \lambda^2 - C\lambda + D$, $\rho(1) > 0$, and λ_1, λ_2 be the roots of $\rho(\lambda) = 0$. Then:*

1. *$|\lambda_1| < 1$ and $|\lambda_2| < 1$ if and only if $\rho(-1) > 0$ and $\rho(0) < 1$.*

2. $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$, *or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *if and only if* $\rho(-1) < 0$.

3. $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $\rho(-1) > 0$ *and* $\rho(0) > 1$.

4. $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $\rho(-1) = 0$ *and* $\rho(0) \neq \pm 1$.

5. λ_1 *and* λ_2 *are complex conjugates, with* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $|C| < 2$ *and* $\rho(0) = 1$.

Let us start this analysis by converting Eq. (1.5) into a two dimensional system. To this end, we define $y_n = x_{n-1}$, so that Eq. (1.5) adopts the following form,

$$x_{n+1} = 1 + p\frac{x_n^m}{y_{n-1}^{m+1}},$$
$$y_{n+1} = x_n. \tag{2.1}$$

Eq. (2.1) has a positive fixed point, given by $E = (\bar{x}, \bar{x})$. In order to discuss the linear stability of E, we determine the Jacobian matrix of this system evaluated at this equilibrium:

$$J(x,y) = \begin{pmatrix} mp\dfrac{x^{m-1}}{y^{m+1}} & -(m+1)p\dfrac{x^m}{y^{m+2}} \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

The evaluation of this matrix at the equilibrium

$$E = \left(\frac{1+\sqrt{1+4p}}{2}, \frac{1+\sqrt{1+4p}}{2}\right)$$

results

$$J(\bar{x},\bar{x}) = \begin{pmatrix} \dfrac{mp}{\bar{x}^2} & \dfrac{-(m+1)p}{\bar{x}^2} \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

with characteristic polynomial

$$\rho(\lambda) = \lambda^2 - \frac{mp}{\bar{x}^2}\lambda + \frac{(m+1)p}{\bar{x}^2}. \tag{2.4}$$

Lemma 2.2 *System (2.1) has a unique positive equilibrium point,*

$$E = \left(\frac{1+\sqrt{1+4p}}{2}, \frac{1+\sqrt{1+4p}}{2}\right)$$

and

1. E *is locally asymptotically stable (stable sink) if*

$$p < \frac{m+1}{m^2}.$$

2. E *is unstable (source) if*

$$p > \frac{m+1}{m^2}.$$

3. The roots of equation $\rho(\lambda)=0$ *are complex numbers with modulus one if*

$$C<2$$

and

$$p=\frac{m+1}{m^2}\,.$$

Proof From Eq. (2.3), it follows that

$$\rho(0)=\frac{(m+1)p}{\bar{x}^2}>0\,,\quad \rho(1)=1+\frac{p}{\bar{x}^2}>0$$

and

$$\rho(-1)=1+\frac{(2m+1)p}{\bar{x}^2}>1\,.$$

Taking into account that

$$\bar{x}=\frac{1+\sqrt{1+4p}}{2}\,,$$

we can express $\rho(-1)$ and $\rho(0)$ as

$$\rho(-1)=4p(m+1)+\sqrt{1+4p}+1\,,\quad \rho(0)=p+1-\frac{m+1}{m^2}\,.$$

Moreover, $\mid C\mid<2$ because

$$\frac{m+1}{m+1+\left(1+\dfrac{2}{m}+m\sqrt{1+\dfrac{4(m+1)}{m^2}}\right)}<1\,,$$

where $p=\dfrac{m+1}{m^2}$ and, then, the application of Lemma 2.1 concludes the proof. □

3. Neimark-Sacker bifurcation

In this section, we study the existence of a Neimark-Sacker bifurcation in Eq. (2.1). A Neimark-Sacker bifurcation occurs when a closed invariant curve emerges from an equilibrium point in a discrete dynamical system and, then, the stability of the equilibrium changes via a pair of complex eigenvalues with unit modulus [9–11, 16].

First of all, we should remark that the roots of (2.4) are conjugate complex numbers if and only if

$$p=\frac{m+1}{m^2}\,. \tag{3.1}$$

Let us define then $\mathcal{N}_s$ as the set of parameters of Eq. (2.1) satisfying the condition (3.1),

$$\mathcal{N}_s=\left\{m\in\mathbb{R}^+\,:\,p=\frac{m+1}{m^2}\right\}\,.$$

The change of variable defined by

$$u_n = x_n - \bar{x}\,, \quad v_n = y_n - \bar{x}\,,$$

transforms the system (2.1) into

$$\begin{aligned} u_{n+1} &= 1 + p\frac{(u_n+\bar{x})^m}{(v_{n-1}+\bar{x})^{m+1}} - \bar{x}\,,\\ v_{n+1} &= u_n\,. \end{aligned} \tag{3.2}$$

For any $p \in \mathcal{N}_s$, we define the function

$$F(u,v) \;\; \to \begin{pmatrix} 1 + p\dfrac{(u+\bar{x})^m}{(v+\bar{x})^{m+1}} - \bar{x} \\ u \end{pmatrix}, \tag{3.3}$$

where $p = \dfrac{m+1}{m^2}$. The origin is the unique fixed point of $F(u,v)$, and the Jacobian matrix of $F(u,v)$ evaluated at $(0,0)$ is given by

$$J_F(0,0) = \begin{pmatrix} \dfrac{mp}{\bar{x}^2} & \dfrac{-(m+1)p}{\bar{x}^2} \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of $J_F(0,0)$ are $\lambda(p,m) = \alpha(p,m) + i\beta(p,m)$ and $\bar{\lambda}(p,m) = \alpha(p,m) - i\beta(p,m)$, being

$$\alpha(p,m) = \frac{mp}{(1+2p+\sqrt{1+4p})}$$

and

$$\beta(p,m) = \frac{\sqrt{m^2p^2 - 2p(m+1)(1+2p+\sqrt{1+4p})}}{(1+2p+\sqrt{1+4p})}\,.$$

If we assume that F has the following form near the origin,

$$F(p,u,v) \;\; = \begin{pmatrix} \dfrac{mp}{\bar{x}^2} & \dfrac{-(m+1)p}{\bar{x}^2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(p,u,v) \\ f_2(p,u,v) \end{pmatrix}, \tag{3.4}$$

then

$$\begin{pmatrix} 1 + p\dfrac{(u+\bar{x})^m}{(v+\bar{x})^{m+1}} - \bar{x} \\ u \end{pmatrix} \;\; = \;\; F(p,u,v) \quad , \tag{3.5}$$

from which

$$f_1(p,u,v) = 1 + p\frac{(u+\bar{x})^m}{(v+\bar{x})^{m+1}} - \bar{x} - \frac{mp}{\bar{x}^2}u + \frac{-(m+1)p}{\bar{x}^2}v$$

and

$$f_2(p,u,v) = 0\,.$$

Let p_0 be

$$p_0 = \frac{m+1}{m^2}.$$

For $p = p_0$, it is easy to see that $\bar{x} = \dfrac{m+1}{m}$ and

$$J_F(0,0) = J_0 = \begin{pmatrix} \frac{m}{m+1} & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of J_0 are

$$\lambda(m), \bar{\lambda}(m) = \frac{m \pm i\sqrt{3m^2+8m+4}}{2(m+1)}, \tag{3.6}$$

and their corresponding eigenvectors can be written as

$$\mu(m), \bar{\mu}(m) = \left(\frac{m \mp i\sqrt{3m^2+8m+4}}{2(m+1)}, 1 \right).$$

Taking into account Eq. (3.6), one can easily check that $\mid \lambda(m) \mid = 1$ and

$$\lambda^2(m) = \frac{-(m^2+4m+2) + i\sqrt{3m^2+8m+4}}{2(m+1)^2},$$

$$\lambda^3(m) = -\frac{(m^3+7m^2+10m+4) + i(m^2+3m+2)\sqrt{3m^2+8m+4}}{4(m+1)^3},$$

$$\lambda^4(m) = \frac{(m^4+5m^3+12m^2+12m+4) - i(m^3+5m^2+6m+2)\sqrt{3m^2+8m+4}}{4(m+1)^3},$$

from which follows that $\lambda^k(m) \neq 1$ for any $k = 1,2,3,4$.

For $p = p_0$ and $\bar{x} = \dfrac{m+1}{m}$, (3.5) adopts the form

$$F(u,v) = \begin{pmatrix} \frac{m}{m+1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix},$$

where

$$g_1(u,v) = f_1(p,u,v) = \frac{(m+1)(u+\frac{m+1}{m})^m}{m^2(v+\frac{m+1}{m})^{m+1}} - \frac{1}{m} - \frac{m}{(m+1)}u + v$$

and

$$g_2(u,v) = f_2(p,u,v) = 0.$$

Hence, (for $p = p_0$), (3.2) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{m}{m+1} & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}.$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = T \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix},$$

where

$$T = \begin{pmatrix} -1 & 0 \\ \dfrac{-m}{2(m+1)} & \dfrac{-\sqrt{3m^2+8m+4}}{2(m+1)} \end{pmatrix}$$

and

$$T^{-1} = \begin{pmatrix} -1 & 0 \\ \dfrac{-m}{\sqrt{3m^2+8m+4}} & \dfrac{-2(m+1)}{\sqrt{3m^2+8m+4}} \end{pmatrix}.$$

Using this transformation, the normal form of (3.2) is computed as

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{m}{2(m+1)} & \frac{-\sqrt{3m^2+8m+4}}{2(m+1)} \\ \dfrac{5m^2+8m+4}{2(m+1)\sqrt{3m^2+8m+4}} & \dfrac{-2m}{2(m+1)} \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + T^{-1} G \left(T \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} \right),$$

where

$$G \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}.$$

Let

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h_1(u,v) \\ h_2(u,v) \end{pmatrix} = T^{-1} G \left(T \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

After some calculations, we obtain that

$$h_1(u,v) = \frac{m+1}{m^2}\Pi(u,v) + \frac{1}{m} - \frac{m}{2(m+1)}u - \frac{\sqrt{3m^2+8m+4}}{2(m+1)}v,$$

being

$$\Pi(u,v) = -\left(\frac{m+1}{m} - u\right)^m \left(\frac{m+1}{m} - \frac{m}{2(m+1)}u - \frac{\sqrt{3m^2+8m+4}}{2(m+1)}v\right)^{-(m+1)},$$

and

$$h_2(u,v) = \frac{-m}{\sqrt{3m^2+8m+4}} h_1(u,v).$$

Next, in order to determine the direction of the appearance of the invariant curve in a system exhibiting a Neimark–Sacker bifurcation, we consider the first Lyapunov coefficients at the point $(u,v,p_0) = \left(0, 0, \dfrac{m+1}{m}\right)$, given by

$$L(m) = \left(Re(\bar{\lambda}(m)\xi_{21}) - Re\left(\frac{(1-2\lambda(m))\bar{\lambda}(m)^2}{1-\lambda(m)}\xi_{20}\xi_{11}\right) - \frac{1}{2} \mid \xi_{11} \mid^2 - \mid \xi_{02} \mid^2 \right),$$

where

$$\lambda(m)=\frac{m+i\sqrt{3m^2+8m+4}}{2(m+1)}\,,\quad \bar{\lambda}(m)=\frac{m-i\sqrt{3m^2+8m+4}}{2(m+1)}\,,$$

and

$$\begin{aligned}
\xi_{20}&=\frac{1}{8}\left[h_{1uu}-h_{1vv}+2h_{2uv}+i\left(h_{2uu}-h_{2vv}-2h_{1uv}\right)\right]\,,\\
\xi_{11}&=\frac{1}{4}\left[h_{1uu}+h_{1vv}+i\left(h_{2uu}+h_{2vv}\right)\right]\,,\\
\xi_{02}&=\frac{1}{8}\left[h_{1uu}-h_{1vv}-2h_{2uv}+i\left(h_{2uu}-h_{2vv}+2h_{1uv}\right)\right]\,,\\
\xi_{21}&=\frac{1}{16}\left[h_{1uuu}+h_{1uvv}+h_{2uuv}+h_{2vvv}+i\left(h_{2uuu}+h_{2uvv}-h_{1uuv}-h_{1vvv}\right)\right]\,,
\end{aligned}$$

being

$$h_{1uu}=\frac{\partial h_1^2}{\partial u^2}\,,\quad h_{2uv}=\frac{\partial h_2^2}{\partial u\partial v}\,,\quad h_{1uuv}=\frac{\partial h_1^3}{\partial u^2\partial v}\,,\quad \ldots$$

We can see that

$$\mid\lambda(p)\mid=\sqrt{\alpha(p)^2+\beta(p)^2}=\frac{2(m+1)p}{1+2p+\sqrt{1+4p}}$$

and, thus,

$$\left(\frac{d\mid\lambda(p)\mid}{dp}\right)_{p=p_0}=\left(\frac{2(m+1)}{\sqrt{p(4p+1)}(1+\sqrt{1+4p})}\right)_{p=p_0}\,.$$

Since

$$\left(\frac{d\mid\lambda(p)\mid}{dp}\right)_{p=p_0}=\frac{m^2}{m+1}>0\,,$$

the above analysis leads to the following result [10, 15, 16, 21].

Theorem 3.1 *Suppose that $L\neq 0$ and the parameter p changes its value in a small vicinity of $\mathcal{N}_s$. Then, Eq. (2.1) presents a Neimark-Sacker bifurcation at the positive equilibrium $E=\left(\frac{1+\sqrt{1+4p}}{2},\frac{1+\sqrt{1+4p}}{2}\right)$. Moreover, if a $L>0$ (respectively $L<0$), there exists a unique repelling (respectively attracting) invariant closed curve Υ_s which bifurcates from E.*

4. Illustrative example

In this section, we illustrate the results obtained in the previous sections by analyzing the solution to the difference equation given by

$$x_{n+1}=1+p\frac{x_n^2}{x_{n-1}^3}\,, \tag{4.1}$$

with initial conditions x_0 and x_{-1}.

In Eq. (4.1), $m = 2$ and $p_0 = 0.75$,

$$h_1(u,v) = -\frac{3}{4}\left(\frac{3}{2} - u\right)^2 \left(\frac{3}{2} - \frac{1}{3}u - \frac{2\sqrt{2}}{3}v\right)^{-3} + \frac{1}{2} - \frac{1}{3}u - \frac{2\sqrt{2}}{3}v$$

and

$$h_2(u,v) = \frac{-\sqrt{2}}{4} h_1(u,v)\,.$$

Furthermore, the calculation of the partial derivatives of h_1 and h_2 gives

$$h_{1uu} = \frac{4}{27}\,, \quad h_{1uv} = \frac{8\sqrt{2}}{27}\,, \quad h_{1vv} = \frac{-64}{27}\,, \quad h_{1uuu} = \frac{-8}{243}\,,$$

$$h_{1uuv} = \frac{80\sqrt{2}}{243}\,, \quad h_{1uvv} = \frac{128}{243}\,, \quad h_{1vvv} = \frac{-1280\sqrt{2}}{243}\,, \quad h_{2uu} = \frac{-\sqrt{2}}{27}\,,$$

$$h_{2uv} = \frac{-4}{27}\,, \quad h_{2vv} = \frac{16}{27}\,, \quad h_{2uuu} = \frac{2\sqrt{2}}{243}\,, \quad h_{2uuv} = \frac{-40}{243}\,,$$

$$h_{2uvv} = \frac{-32\sqrt{2}}{243}\,, \quad h_{2vvv} = \frac{640}{243}\,.$$

Now, we can obtain

$$\xi_{20} = \frac{5}{18} - i\frac{16 + 17\sqrt{2}}{216} = 0.27 - 0.132i\,,$$

$$\xi_{11} = \frac{5}{9} + i\frac{16 - \sqrt{2}}{216} = 0.555 + 0.151i\,,$$

$$\xi_{02} = \frac{19}{54} + i\frac{16 - 15\sqrt{2}}{216} = 0.018 + 0.024i\,,$$

$$\xi_{21} = \frac{720}{243} + i\frac{1160\sqrt{2}}{243} = 2.962 + 6.809i\,.$$

Also,

$$\lambda(2) = \frac{1 + i2\sqrt{2}}{3} = 0.33 + 0.440i$$

and

$$\bar{\lambda}(2) = \frac{1 - i2\sqrt{2}}{3} = 0.33 - 0.440i\,.$$

Finally,

$$L(m=2) = \left(Re(\bar{\lambda}(2)\xi_{21}) - Re\left(\frac{(1 - 2\lambda(2))\bar{\lambda}(2)^2}{1 - \lambda(2)}\xi_{20}\xi_{11}\right) - \frac{1}{2} \mid \xi_{11} \mid^2 - \mid \xi_{02} \mid^2\right) = 7.49966 > 0\,.$$

Since $L(m) > 0$, by varying the value of p from $p < p_0$ to $p > p_0$, a supercritical Neimark–Sacker bifurcation arises at $p_0 = 0.75$ (see Figure 1). Namely, if $p = 0.748 < p_0$, the fixed point $\bar{x} = 1.5$ is asymptotically stable. In Figure 1a, we depict in red $900\,000$ iterations of the orbit with initial conditions $(x_0, x_{-1}) = (1.7, 1.7)$.

If $p > p_0$ (see Figure 1b–1d), we find an attracting closed invariant curve Υ_s encircling the fixed point. In Figure. 1b–1d, we depict some 900,000 iterations of the orbits with initial conditions $(x_0, x_{-1}) = (0.485, 0.485)$ (in blue and located in the interior of the invariant curve) and $(x_0, x_{-1}) = (1.7, 1.7)$ (in red and located at the outside of the invariant curve). The stable invariant curve Υ_s has been colored in green. In all these cases, (b), (c), and (d), the blue orbit leaves the unstable fixed point $\bar{x}$ and tends to the invariant curve Υ_s.

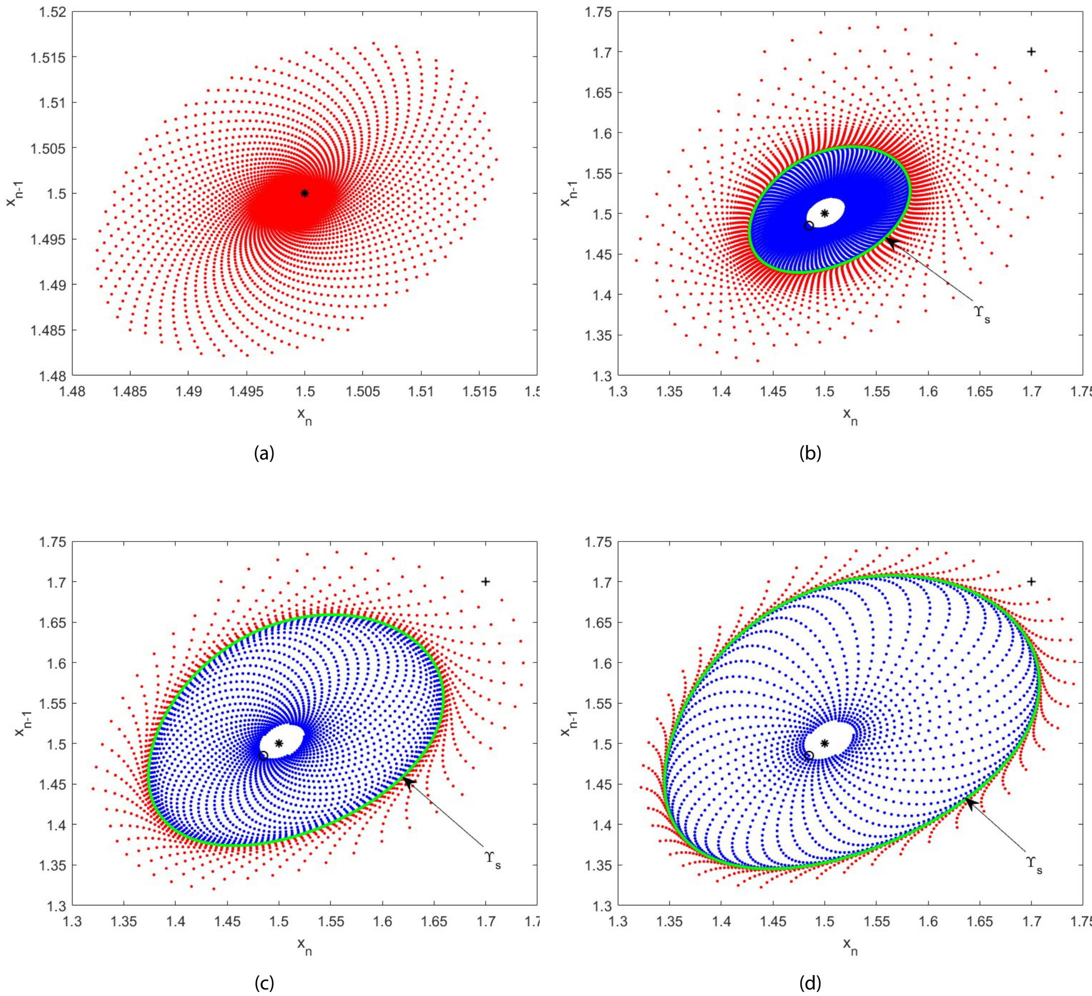

Figure 1. Trajectories (blue and red) for $m = 2$, $p_0 = 0.75$, $\bar{x} = 1.5$ and P: (a) $p = 0.748$, (b) $p = 0.7515$, (c) $p = 0.755$, (d) $p = 0.758$.

In Figure 2, we show the bifurcation diagram of Eq. (4.1), for values of the parameter p in the interval $(0, 1]$. Figure 3 shows an asymptotic approximation of the invariant curve. We can conclude that all the trajectories with initial conditions in the region enclosed by the invariant curve tends asymptotically to Υ_s, except the fixed point $\bar{x}$. The trajectories with initial conditions in the outside to the invariant curve tend to Υ_s. For values of the parameter p larger than p_0, the invariant curve Υ_s tends to expand as the value of p gets larger. Finally, for any $p \leq p_0$, the invariant curve reduces and tends to the fixed point.

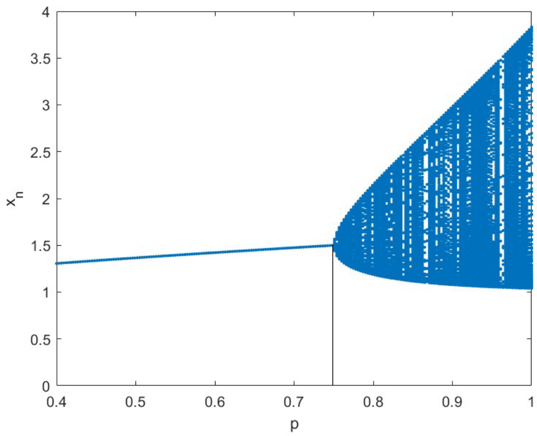

Figure 2. Bifurcation diagram in (p, x_n).

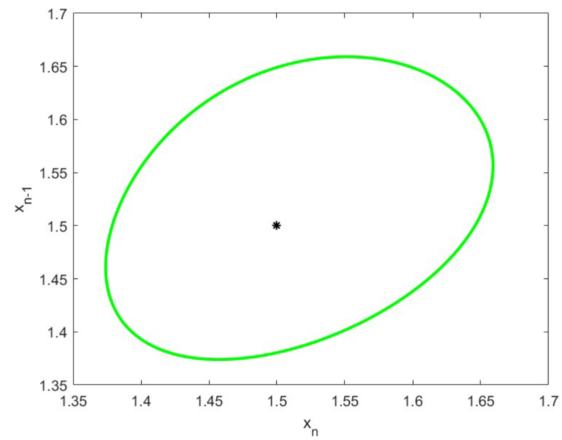

Figure 3. Asymptotic approximation of the invariant curve, for $p = 0.758$.

5. Conclusion

This paper focuses on a qualitative analysis of the solutions to the difference system given by Eq. (2.1). This system depends on two parameters, m and p, and has one positive equilibrium E. We have analyzed the stability of E, finding that the positive equilibrium is asymptotically stable if $p < (m+1)/m^2$ and unstable if $p > (m+1)/m^2$.

We have also proved the existence of a Neimark–Sacker bifurcation through the analysis of the normal form of the system, concluding that a Neimark–Sacker bifurcation occurs when the parameter p varies in a small vicinity of $p_0/(m+1)/m^2$. Finally, we have illustrated this theoretical result with the help of a numerical example.

Conflict of interest

The authors declare that they have no conflicts of interest.

Data availability statement

The datasets generated during the current study are available from the corresponding author on reasonable request.

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