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Notes on totally geodesic foliations of a complete semi-Riemannian manifold

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Abstract: In this paper, we prove that the orthogonal complement $\mathcal{F}^{\perp}$ of a totally geodesic foliation $\mathcal{F}$ on a complete semi-Riemannian manifold (M,g) satisfying a certain inequality between mixed sectional curvatures and the integrability tensor of $\mathcal{F}^{\perp}$ is totally geodesic. We also obtain conditions for the existence of totally geodesic foliations on a complete semi-Riemannian manifold (M,g) with bundle-like metric g.

Key words: Complete semi-Riemannian manifold, totally geodesic foliation, mixed sectional curvature

1. Introduction

The theory of foliations of manifolds has begun with the study of C. Ehresmann and G. Reeb [6] on a joint of differential equations and the differential topology. In particular, totally geodesic foliations and the foliations with bundle-like metric g on an ambient manifold (M,g) are substantial research subjects for topologists and geometers. When all the leaves of a foliation $\mathcal{F}$ are totally geodesic submanifolds of M, we call $\mathcal{F}$ a *totally geodesic foliation* on (M,g). One important line of research on totally geodesic foliations focuses on the existence of these foliations and several results have been obtained by solving a Riccati type differential equation or finding a Riemannian metric g on M such that a given foliation becomes totally geodesic (see [3, 5, 8, 11, 13]). In the Lorentzian context, the authors of [4, 10] recently studied the geometric properties and existence of totally geodesic foliations of codimension one in a spacetime.

Another major research direction on totally geodesic foliations is the integrability of a transversal distribution and total geodesicity of the orthogonal complement of a totally geodesic foliation, which has been intensively discussed with mixed sectional curvatures of the ambient manifold. S. Tanno [14] showed that if all the mixed sectional curvatures of a Riemannian manifold M^n vanish identically on M and the transversal distribution of a totally geodesic foliation $\mathcal{F}$ is integrable, then the foliation $\mathcal{F}^{\perp}$ defined by the transversal distribution is also totally geodesic(see also [1]). G. Oshikiri [13] studied such subjects on compact or complete Riemannian manifolds with some curvature constraints. In [2, 3], it was proved that on a positively or negatively curved semi-Riemannian manifold, there exists no totally geodesic foliation with bundle-like metric on M such that the orthogonal complement distribution of the foliation is integrable.

In this paper, we firstly investigate total geodesicity of the orthogonal complement $\mathcal{F}^{\perp}$ of a totally geodesic foliation $\mathcal{F}$ on a complete semi-Riemannian manifold (M,g). Unlike Riemannian manifolds, because

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the metric g does not need to be positive definite, it is necessary to find appropriate conditions accordingly. In particular, we introduce special functions associated with integrable tensors A and T. Using ordinary differential equations and inequalities related to the mixed sectional curvatures and the integrable tensor A of $\mathcal{F}^\perp$, we show that the orthogonal complement $\mathcal{F}^\perp$ of a totally geodesic foliation $\mathcal{F}$ on a complete semi-Riemannian manifold (M, g) satisfying some suitable conditions is also totally geodesic. As an application, we prove generalized results of the aforementioned theorems in [2, 13, 14]. Subsequently, we discuss the existence of totally geodesic foliations on a complete semi-Riemannian manifold (M, g) with bundle-like metric g. We find a key function related to the tensor A and show that if the mixed sectional curvatures of M satisfy an appropriate inequality with the key function, then there exists a totally geodesic foliation on (M, g) with bundle-like metric g.

2. Preliminaries

Let (M, g) be an n-dimensional semi-Riemannian manifold of index r and TM the tangent bundle of M. We often use $\langle\ ,\ \rangle$ as an alternative notation for g. Denote by ∇ the Levi-Civita connection of (M, g), and by $\Gamma(TM)$ the $C^\infty(M)$-module of smooth sections of TM, where $C^\infty(M)$ is the set of all real-valued smooth functions on M. Mappings, vector fields, and manifolds are sufficiently differentiable.

In what follows, we consider a foliation $\mathcal{F}$ of dimension p (or codimension $q = n - p$) of M. The tangent distribution to $\mathcal{F}$ and its complementary orthogonal distribution in TM are denoted by $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. We assume that the induced metric tensor on $\mathcal{D}$ is nondegenerate and of the constant index. Then let $dim(\mathcal{F})$ be the dimension of $\mathcal{D}_m$ and $ind(\mathcal{F})$ be the index of $\mathcal{D}_m$ at any $m \in M$. In this case, we say that $\mathcal{F}$ is a *nondegenerate foliation* and $(M, g, \mathcal{F})$ is a *foliated semi-Riemannian manifold*.

Let $\pi^\perp : TM \to \mathcal{D}$ and $\pi : TM \to \mathcal{D}^\perp$ be the natural projections. Tensors A and T of type $(1, 2)$ are defined as follows:

$$T_V W = \pi(\nabla_{\pi^\perp(V)} \pi^\perp(W)) + \pi^\perp(\nabla_{\pi^\perp(V)} \pi(W))$$

and

$$A_V W = \pi(\nabla_{\pi(V)} \pi^\perp(W)) + \pi^\perp(\nabla_{\pi(V)} \pi(W))$$

for $V, W \in \Gamma(TM)$. $T_V W$ is the second fundamental form of the leaves of $\mathcal{F}$, and is symmetric. If $T_V W = 0$ for all $V, W \in \Gamma(\mathcal{D})$, then $T = 0$. The vanishing of T is thus equivalent to the property that all the leaves of $\mathcal{F}$ are totally geodesic submanifolds of (M, g). Such a foliation is said to be *totally geodesic*. The following lemma is well-known and can be found in [2] (see also [9]).

Lemma 2.1 *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold. Then*

(i) $A_X Y = A_Y X$ for all $X, Y \in \Gamma(\mathcal{D}^\perp)$ if and only if $\mathcal{D}^\perp$ is integrable.

(ii) $A = 0$ if and only if the foliation $\mathcal{F}^\perp$ defined by $\mathcal{D}^\perp$ is totally geodesic.

The Riemannian curvature tensor R of M is defined by

$$R(X, Y)Z = \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z$$

for $X, Y, Z \in \Gamma(TM)$. If X and Y span a nondegenerate plane Π, the sectional curvature $K(X,Y)$ of Π is defined by

$$K(X,Y) = \frac{\langle R(X,Y)X, Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

$K(V,X)$ is called the *mixed sectional curvature* determined by $V \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$ (see [2, 12]).

In the remaining of this section, we give a generalization of a formula in [13] to the semi-Riemannian manifolds by direct calculations. Unless otherwise stated, the indices i, j run through $\{1, \cdots, p = dim(\mathcal{F})\}$, and α, β through $\{p+1, \cdots, n = dim(M)\}$. Let $\{E_i\}$ and $\{X_\alpha\}$ be orthonormal frame fields for $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively, where $\epsilon_i = \langle E_i, E_i\rangle$ and $\epsilon_\alpha = \langle X_\alpha, X_\alpha\rangle$. For unit sections $E \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$, we have

$$\begin{aligned}
\langle R(E,X)X,E\rangle &= \langle \nabla_{[E,X]}X, E\rangle - \langle \nabla_E\nabla_X X, E\rangle + \langle \nabla_X\nabla_E X, E\rangle \\
&= \sum_i \epsilon_i \langle [E,X], E_i\rangle\langle \nabla_{E_i}X, E\rangle + \sum_\alpha \epsilon_\alpha\langle [E,X], X_\alpha\rangle\langle \nabla_{X_\alpha}X, E\rangle \\
&\quad - \langle \nabla_E\nabla_X X, E\rangle + X\langle \nabla_E X, E\rangle - \langle \nabla_E X, \nabla_X E\rangle \\
&= \sum_i \epsilon_i\langle \nabla_E X, E_i\rangle\langle \nabla_E X, E_i\rangle - \sum_i \epsilon_i \langle \nabla_X E, E_i\rangle\langle \nabla_E X, E_i\rangle \\
&\quad + \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle \nabla_{X_\alpha}X, E\rangle + \sum_\alpha \epsilon_\alpha\langle E, \nabla_X X_\alpha\rangle\langle \nabla_{X_\alpha}X, E\rangle \\
&\quad - \sum_i \epsilon_i\langle \nabla_E X, E_i\rangle\langle \nabla_X E, E_i\rangle - \sum_\alpha \epsilon_\alpha \langle \nabla_E X, X_\alpha\rangle\langle \nabla_X E, X_\alpha\rangle \\
&\quad - \langle \nabla_E\nabla_X X, E\rangle - X\langle X, \nabla_E E\rangle \\
&= \langle T_E X, T_E X\rangle - \langle \nabla_E\nabla_X X, E\rangle - X\langle X, T_E E\rangle \\
&\quad - 2\sum_i \epsilon_i\langle \nabla_X E, E_i\rangle\langle \nabla_E X, E_i\rangle + \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle \\
&\quad + \sum_\alpha \epsilon_\alpha\langle E, A_X X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle + \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_X X_\alpha, E\rangle.
\end{aligned} \tag{2.1}$$

Thus,

$$\begin{aligned}
K(E,X) = -\frac{1}{\langle E,E\rangle\langle X,X\rangle}\Big\{&\langle T_E X, T_E X\rangle - \langle \nabla_E\nabla_X X, E\rangle - X\langle X, T_E E\rangle \\
&- 2\sum_i \epsilon_i\langle \nabla_X E, E_i\rangle\langle \nabla_E X, E_i\rangle + \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle \\
&+ \sum_\alpha \epsilon_\alpha\langle E, A_X X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle + \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_X X_\alpha, E\rangle\Big\}
\end{aligned}$$

If $\mathcal{F}$ is totally geodesic, this equality becomes

$$\begin{aligned}
K(E,X) = \frac{1}{\langle E,E\rangle\langle X,X\rangle}\Big\{&\langle \nabla_E\nabla_X X, E\rangle - \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle \\
&- \sum_\alpha \epsilon_\alpha\langle E, A_X X_\alpha\rangle\langle A_{X_\alpha}X, E\rangle - \sum_\alpha \epsilon_\alpha\langle \nabla_E X, X_\alpha\rangle\langle A_X X_\alpha, E\rangle\Big\}
\end{aligned}$$

for unit sections $E \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$. In particular, we get

$$\sum_\alpha K(E, X_\alpha) = -\sum_{\alpha,\beta} \frac{\epsilon_\beta}{\langle E, E\rangle\epsilon_\alpha}\Big\{\langle\nabla_E X_\alpha, X_\beta\rangle\langle A_{X_\beta}X_\alpha, E\rangle + \langle E, A_{X_\alpha}X_\beta\rangle\langle A_{X_\beta}X_\alpha, E\rangle + \langle\nabla_E X_\alpha, X_\beta\rangle\langle A_{X_\alpha}X_\beta, E\rangle\Big\} + \sum_\alpha \frac{1}{\langle E, E\rangle\epsilon_\alpha}\langle\nabla_E\nabla_{X_\alpha}X_\alpha, E\rangle,$$

for any local orthonormal frame field $\{X_\alpha\}$ of $\mathcal{D}^\perp$. Since $\langle\nabla_E X_\alpha, X_\beta\rangle$ is skew-symmetric in α and β, we have

$$\sum_\alpha K(E, X_\alpha) = -\sum_{\alpha,\beta} \frac{\epsilon_\beta}{\langle E, E\rangle\epsilon_\alpha}\langle A_{X_\alpha}X_\beta, E\rangle\langle A_{X_\beta}X_\alpha, E\rangle + \sum_\alpha \frac{1}{\langle E, E\rangle\epsilon_\alpha}\langle\nabla_E\nabla_{X_\alpha}X_\alpha, E\rangle. \tag{2.2}$$

3. Main theorems

In this section, we state the conditions on M which guarantee that $\mathcal{F}^\perp$ associated with a totally geodesic foliation $\mathcal{F}$ is also totally geodesic. Indeed, we will show that some conditions on the mixed sectional curvature K of M play a central role in determining geometric properties of $\mathcal{F}^\perp$.

From now on, we suppose that an n-dimensional semi-Riemannian manifold M of index r is complete, that is, every geodesic of M can be extended on the entire real line, and a nondegenerate foliation $\mathcal{F}$ of dimension p (or codimension $q = n - p$) is totally geodesic.

Given $m \in M$, let $\{x_\alpha\}$ be an orthonormal basis of $\mathcal{D}_m^\perp$. Let γ be a unit-speed geodesic along $\mathcal{F}$ with $\gamma(0) = m$ and $\{X_\alpha\}$ the orthonormal frame field along γ obtained by the parallel translation of $\{x_\alpha\}$ of $\mathcal{D}_m^\perp$. As $\mathcal{F}$ is totally geodesic, $\{X_\alpha\}$ is an orthonormal frame field for $\mathcal{D}^\perp$ along γ. From (2.2) or direct calculation using $\nabla_{\gamma'}\gamma' = 0$ and $\nabla_{\gamma'}X_\alpha = 0$, we obtain

$$\sum_\alpha K(\gamma', X_\alpha) = -\sum_{\alpha,\beta} \frac{\epsilon_\beta}{\langle\gamma', \gamma'\rangle\epsilon_\alpha}\langle A_{X_\alpha}X_\beta, \gamma'\rangle\langle A_{X_\beta}X_\alpha, \gamma'\rangle + \sum_\alpha \frac{1}{\langle\gamma', \gamma'\rangle\epsilon_\alpha}\gamma'\langle\nabla_{X_\alpha}X_\alpha, \gamma'\rangle. \tag{3.1}$$

Set $G = \pi^\perp(\sum_\alpha \epsilon_\alpha\nabla_{X_\alpha}X_\alpha)$. Then (3.1) can be expressed as

$$\sum_\alpha \langle\gamma', \gamma'\rangle K(\gamma', X_\alpha) = -\sum_{\alpha,\beta}\epsilon_\beta\epsilon_\alpha\langle A_{X_\alpha}X_\beta, \gamma'\rangle\langle A_{X_\beta}X_\alpha, \gamma'\rangle + \gamma'\langle G, \gamma'\rangle.$$

Let $f(t) = \langle G, \gamma'\rangle$. We have

$$(f(t))^2 = \Big(\sum_\alpha \epsilon_\alpha\langle A_{X_\alpha}X_\alpha, \gamma'\rangle\Big)^2 \leq q\sum_\alpha\langle A_{X_\alpha}X_\alpha, \gamma'\rangle^2 \quad (q = dim(\mathcal{F}^\perp)).$$

Hence

$$\sum_\alpha \langle\gamma', \gamma'\rangle K(\gamma', X_\alpha) \leq f'(t) - \frac{1}{q}f(t)^2 - \sum_{\alpha\neq\beta}\epsilon_\beta\epsilon_\alpha\langle A_{X_\alpha}X_\beta, \gamma'\rangle\langle A_{X_\beta}X_\alpha, \gamma'\rangle. \tag{3.2}$$

For the proofs of main theorems, we need the following lemma.

Lemma 3.1 *Let $h \in C^1(\mathbb{R})$. If there is a constant $k > 0$ such that*

$$h'(t) - kh(t)^2 \geq 0$$

for every $t \in \mathbb{R}$, then h is identically zero.

Proof Suppose there is $t_0 \in \mathbb{R}$ with $h(t_0) = c \neq 0$. We may assume $t_0 = 0$ and $c > 0$. As $h(t)$ is nondecreasing, $h(t) \geq c > 0$ for $t \geq 0$. This gives $\frac{h'(t)}{h(t)^2} \geq k$, and integrating both sides, we get

$$-\frac{1}{h(t)} + \frac{1}{c} \geq kt.$$

This does not hold for $t \geq \frac{1}{kc}$ since $h(t) > 0$, a contradiction. Hence, $h(t) = 0$ for all $t \in \mathbb{R}$. □

In this setting, we can prove our main theorems.

Theorem 3.2 *Let $\mathcal{F}$ be a totally geodesic nondegenerate foliation of codimension 1 on a complete semi-Riemannian manifold M. If all the mixed sectional curvatures of M vanish identically on M then the foliation $\mathcal{F}^\perp$ defined by $\mathcal{D}^\perp$ is totally geodesic.*

Proof It is enough to show that $A_X Y = 0$ for all $X, Y \in \mathcal{D}^\perp$ by Lemma 2.1. As the codimension of $\mathcal{F}$ is 1, the index α has no choice but $\alpha = n$. Given $m \in M$, let γ be a unit-speed geodesic along $\mathcal{F}$ with $\gamma(0) = m$. Then $\{x_\alpha\}$ and $\{X_\alpha\}$, defined in the beginning of this section, are just the singletons $\{x_n\}$ and $\{X_n\}$. Also, as $\mathcal{F}$ is totally geodesic, we use (3.2) and get

$$\langle \gamma', \gamma' \rangle K(\gamma', X_n) = f'(t) - f(t)^2.$$

Since all the mixed sectional curvatures of M vanish identically, we have $f'(t) = f(t)^2$. By Lemma 3.1, $f(t) = 0$ for every $t \in \mathbb{R}$ and every γ along $\mathcal{F}$, which therefore concludes that $\mathcal{F}^\perp$ is totally geodesic.

□

For higher codimension cases, we define a function associated with the tensor A whose value on any triple of $u, v, w \in T_m M$ for any $m \in M$ is

$$A(u, v, w) = \langle A_u v, w \rangle.$$

Note that the supremum of $|A_u v|$ over unit vectors u, v is related to the turbulence of $\mathcal{F}$ on a Riemannian manifold (M, g) with bundle-like metric g (see [11]). Since in semi-Riemannian manifolds, however, the Cauchy-Schwarz inequality is not available, that is, $|A_u v|$ can be less than $|A(u, v, w)|$ over some unit vectors, we thus must impose conditions and compute the result in terms of $A(u, v, w)$ itself, so that we establish Theorem 3.3. The function $T(u, v, w) = \langle T_u v, w \rangle$ over $u, v, w \in T_m M$ for any $m \in M$ is also defined in the same way. Here, we note that authors [4] investigated some conditions for a foliation $\mathcal{F}$ to be totally geodesic by introducing a special number $\mathcal{Q}_F$.

Theorem 3.3 *Let $\mathcal{F}$ be a totally geodesic nondegenerate foliation of codimension $q > 1$ on a complete semi-Riemannian manifold M. Suppose that for any $m \in M$ and for any unit vector $u \in \mathcal{D}_m$, the mixed sectional curvature satisfies the condition*

$$\sum_\alpha \langle u, u \rangle K(u, x_\alpha) \geq q^2 \max_{p+1 \leq \iota, \kappa \leq n} (A(x_\iota, x_\kappa, u))^2$$

where $\{x_\alpha\}$ is an orthonormal frame at m for $\mathcal{D}_m^\perp$. Then $\mathcal{D}^\perp$ is integrable and the foliation $\mathcal{F}^\perp$ defined by $\mathcal{D}^\perp$ is totally geodesic.

Proof Since $\mathcal{F}$ is totally geodesic, to apply (3.2), we consider the setting described in the beginning of this section; for a given $m \in M$, let γ be a unit-speed geodesic and $\{X_\alpha\}$ the orthonormal frame field along γ.

Using the assumption, from (3.2) we get

$$\begin{aligned} f'(t) - \frac{1}{q} f(t)^2 &\geq \sum_\alpha \langle \gamma', \gamma' \rangle K(\gamma', X_\alpha) + \sum_{\alpha \neq \beta} \epsilon_\alpha \epsilon_\beta \langle A_{X_\alpha} X_\beta, \gamma' \rangle \langle A_{X_\beta} X_\alpha, \gamma' \rangle \\ &\geq q^2 \max(\langle A_{X_\alpha} X_\beta, \gamma' \rangle)^2 + \sum_{\alpha \neq \beta} \epsilon_\alpha \epsilon_\beta \langle A_{X_\alpha} X_\beta, \gamma' \rangle \langle A_{X_\beta} X_\alpha, \gamma' \rangle \\ &\geq \max(\langle A_{X_\alpha} X_\beta, \gamma' \rangle)^2 + \sum_{\alpha < \beta} (\epsilon_\alpha \langle A_{X_\alpha} X_\beta, \gamma' \rangle + \epsilon_\beta \langle A_{X_\beta} X_\alpha, \gamma' \rangle)^2 \geq 0. \end{aligned}$$

By Lemma 3.1, $f = 0$, so, $\langle A_{X_\alpha} X_\beta, \gamma' \rangle = 0$ for any unit-speed geodesic γ along $\mathcal{F}$. Consequently, $\mathcal{D}^\perp$ is integrable and $\mathcal{F}^\perp$ is also totally geodesic.

□

The next result is the special case that $\mathcal{D}^\perp$ is integrable.

Corollary 3.4 *Let $\mathcal{F}$ be a totally geodesic nondegenerate foliation of codimension $q > 1$ on a complete semi-Riemannian manifold M with sectional curvature K. Suppose that $\mathcal{D}^\perp$ is integrable and $ind(M) = ind(\mathcal{F}^\perp)$. If $K \geq 0$ then the foliation $\mathcal{F}^\perp$ defined by $\mathcal{D}^\perp$ is totally geodesic.*

Proof When $ind(M) = ind(\mathcal{F}^\perp)$, all the leaves of $\mathcal{F}$ are totally geodesic spacelike submanifolds of M, so we can use the same setting in the proof of Theorem 3.3 for a spacelike geodesic. Since $\mathcal{D}^\perp$ is integrable, by Lemma 2.1, we obtain

$$f'(t) - \frac{f(t)^2}{q} \geq \sum_\alpha K(\gamma', X_\alpha) + \sum_{\alpha \neq \beta} \langle A_{X_\alpha} X_\beta, \gamma' \rangle^2,$$

for all unit-speed geodesic γ along $\mathcal{F}$.

When any sectional curvature of M is nonnegative, by Lemma 3.1, $f = 0$ and from (3.1)

$$\sum_\alpha K(\gamma', X_\alpha) = -\sum_{\alpha, \beta} \langle A_{X_\alpha} X_\beta, \gamma' \rangle^2 \geq 0.$$

Hence we have $\langle A_{X_\alpha} X_\beta, \gamma' \rangle = 0$ for all γ along $\mathcal{F}$, so $\mathcal{F}^\perp$ is totally geodesic. □

As inferred from the above proof, we can get the same conclusion by having the other hypotheses in Corollary 3.4 and replacing the nonnegativity of sectional curvatures of M with mixed sectional curvature $K(V, X) = 0$ for any $V \in \Gamma(\mathcal{D})$ and $X \in \Gamma(\mathcal{D}^\perp)$. The following corollary is directly deduced from Theorem 3.2 and Corollary 3.4 for Riemannian manifolds (see [2, 13, 14]) and Lorentzian manifolds.

Corollary 3.5 *Let $\mathcal{F}$ be a totally geodesic spacelike foliation on a complete Lorentzian (or Riemannian) manifold M. Suppose that $\mathcal{D}^\perp$ is integrable. If all the mixed sectional curvatures of M vanish identically on M then the foliation $\mathcal{F}^\perp$ defined by $\mathcal{D}^\perp$ is totally geodesic.*

4. Bundle-like metric

In this section, we discuss the existence of totally geodesic foliations with bundle–like metric g. The metric g is said to be bundle-like for $\mathcal{F}$ if each geodesic in (M,g) that is tangent to the normal distribution to $\mathcal{F}$ at one point remains tangent for its entire length (cf. [3]). We present the following properties of bundle-like metrics (see [2, 3, 9, 11]).

1. $\langle \nabla_X E, Y\rangle + \langle X, \nabla_Y E\rangle = 0$ for $E \in \Gamma(\mathcal{D})$ and $X, Y \in \Gamma(\mathcal{D}^\perp)$
2. $A_X Y = -A_Y X$ for all $X, Y \in \Gamma(\mathcal{D}^\perp)$ if and only if the metric g on M is bundle-like for $\mathcal{F}$.

Given a point $m \in M$, let γ be a unit-speed geodesic which is orthogonal to $\mathcal{D}$ with $\gamma(0) = m$. Choose local orthonormal frames $\{E_i\}$ and $\{X_\alpha\}$ along γ as usual. Based on the formula (2.1), we have an equation (which is a semi-Riemannian version of the Riccati type equations obtained in [11])

$$\begin{aligned}\sum_i \epsilon_i \langle R(E_i,\gamma')\gamma', E_i\rangle &= \sum_{i,j}\epsilon_i\epsilon_j\langle \gamma', \nabla_{E_i}E_j\rangle\langle \gamma', \nabla_{E_i}E_j\rangle - \sum_i \epsilon_i \gamma'\langle \gamma', \nabla_{E_i}E_i\rangle \\ &+ 2\sum_{i,j}\epsilon_i\epsilon_j\langle \nabla_{\gamma'}E_i, E_j\rangle\langle \gamma', \nabla_{E_i}E_j\rangle - \sum_{\alpha,i}\epsilon_\alpha\epsilon_i\langle E_i, A_{\gamma'}X_\alpha\rangle^2.\end{aligned}$$

Since $\langle \nabla_{\gamma'}E_i, E_i\rangle = 0$ and $\langle \gamma', [E_i,E_j]\rangle = 0$, we get

$$\begin{aligned}\sum_i \epsilon_i \langle R(E_i,\gamma')\gamma', E_i\rangle &= \sum_{i,j}\epsilon_i\epsilon_j\langle \gamma', \nabla_{E_i}E_j\rangle^2 - \sum_i \epsilon_i\gamma'\langle \gamma', \nabla_{E_i}E_i\rangle - \sum_{\alpha,i}\epsilon_\alpha\epsilon_i\langle E_i, A_{\gamma'}X_\alpha\rangle^2 \\ &= \sum_{i,j}\epsilon_i\epsilon_j\langle \gamma', \nabla_{E_i}E_j\rangle^2 - \sum_i \epsilon_i\gamma'\langle \gamma', \nabla_{E_i}E_i\rangle - \sum_{\alpha}\epsilon_\alpha\langle A_{\gamma'}X_\alpha, A_{\gamma'}X_\alpha\rangle.\end{aligned} \tag{4.1}$$

Let $G = \pi(\sum_i \epsilon_i \nabla_{E_i}E_i)$ and $f(t) = \langle G, \gamma'\rangle$. Then from (4.1)

$$f'(t) - \frac{1}{p}f^2(t) \geq \sum_i \langle \gamma', \gamma'\rangle K(E_i,\gamma') - \sum_\alpha \epsilon_\alpha\langle A_{\gamma'}X_\alpha, A_{\gamma'}X_\alpha\rangle + \sum_{i\neq j}\epsilon_i\epsilon_j\langle \gamma', T_{E_i}E_j\rangle^2 \tag{4.2}$$

To discuss the next result, consider the following function for the tensor A on a semi-Riemannian manifold (M,g) with bundle-like metric g

$$\Phi(u) = \sum_\alpha \epsilon_\alpha \langle A_{x_\alpha}u, A_{x_\alpha}u\rangle$$

over $u \in \mathcal{D}_m^\perp$ for any $m \in M$, where $\{x_\alpha\}$ is an orthonormal frame at m for $\mathcal{D}_m^\perp$ (see [7]). Note that the definition of $\Phi(u)$ is independent of the choice of the frame at m. This means if $\{y_\beta\}$ is related to the frame $\{x_\alpha\}$ by an orthogonal transformation of $\mathcal{D}_m^\perp$ then

$$\sum_\beta \epsilon_\beta\langle A_{y_\beta}u, A_{y_\beta}u\rangle = \sum_\alpha \epsilon_\alpha\langle A_{x_\alpha}u, A_{x_\alpha}u\rangle.$$

The next consequence is a semi-Riemannian version of a theorem by Kim–Tondeur [11].

Theorem 4.1 *Let $(M, g, \mathcal{F})$ be a foliated semi-Riemannian manifold, where $\mathcal{F}$ is a nondegenerate foliation of codimension q $(p = dim(\mathcal{F}))$ and g is a bundle-like metric for $\mathcal{F}$. Suppose that for any $m \in M$ and for any unit vector $u \in \mathcal{D}_m^{\perp}$, the mixed sectional curvature satisfies the condition*

$$\sum_i \langle u, u\rangle K(e_i, u) \geq \Phi(u) + 2p\ ind(\mathcal{F}) \max_{1\leq j,k\leq p} (T(e_j, e_k, u))^2,$$

where $\{e_i\}$ is an orthonormal frame at m for $\mathcal{D}_m$. Then $\mathcal{F}$ is totally geodesic.

Proof Given a point $m \in M$, let γ be a unit-speed geodesic which is orthogonal to $\mathcal{D}_m$ with $\gamma(0) = m$. Choose local orthonormal frames $\{E_i\}$ and $\{X_\alpha\}$ for $\mathcal{D}$ and $\mathcal{D}^{\perp}$, respectively, along γ with $X_n = \gamma'$, and so $A_{\gamma'} X_n = 0$. Since the number of cases where $\epsilon_i\epsilon_j = -1$ is not more than $2p\ ind(\mathcal{F})$, by (4.2),

$$\begin{aligned} f'(t) - \frac{1}{p} f^2(t) &\geq \sum_i \langle\gamma', \gamma'\rangle K(E_i, \gamma') - \sum_\alpha \epsilon_\alpha \langle A_{\gamma'} X_\alpha, A_{\gamma'} X_\alpha\rangle \\ &\quad + \sum_{i\neq j} \epsilon_i\epsilon_j \langle\gamma', T_{E_i} E_j\rangle^2 \\ &= \sum_i \langle\gamma', \gamma'\rangle K(E_i, \gamma') - \sum_\alpha \epsilon_\alpha \langle A_{\gamma'} X_\alpha, A_{\gamma'} X_\alpha\rangle \\ &\quad + \sum_{i\neq j,\ \epsilon_i\epsilon_j = 1} \epsilon_i\epsilon_j \langle\gamma', T_{E_i} E_j\rangle^2 + \sum_{\epsilon_i\epsilon_j = -1} \epsilon_i\epsilon_j \langle\gamma', T_{E_i} E_j\rangle^2 \\ &\geq \sum_i \langle\gamma', \gamma'\rangle K(E_i, \gamma') - \Phi(\gamma') - 2p\ ind(\mathcal{F}) \max_{1\leq j,k\leq p} (T(e_j, e_k, \gamma'))^2. \end{aligned} \tag{4.3}$$

By hypothesis, $f'(t) - \frac{1}{p} f^2(t) \geq 0$.

Thus, we conclude by Lemma 3.1 that f has only the trivial solution $f = 0$. Moreover, from (4.3),

$$\Big(2p\ ind(\mathcal{F}) \max_{i,j} (\langle T_{E_i} E_j, \gamma'\rangle)^2 - \sum_{\epsilon_i\epsilon_j = -1} \langle\gamma', \nabla_{E_i} E_j\rangle^2\Big) + \sum_{i\neq j,\ \epsilon_i\epsilon_j = 1} \langle\gamma', \nabla_{E_i} E_j\rangle^2 = 0.$$

Since all the terms are nonnegative, we have $\langle T_{E_i} E_j, \gamma'\rangle = 0$ for all i, j and any unit-speed geodesic γ, which means $T = 0$. Hence $\mathcal{F}$ is totally geodesic. □

If $\mathcal{F}$ is totally geodesic, that is, $T = 0$, then $\langle\gamma', \nabla_{E_i} E_j\rangle = 0$ for all i, j, so by using (4.1) we have

$$\sum_i \langle\gamma', \gamma'\rangle K(E_i, \gamma') = \sum_\alpha \epsilon_\alpha \langle A_{\gamma'} X_\alpha, A_{\gamma'} X_\alpha\rangle.$$

The integrability of $\mathcal{D}^{\perp}$ is equivalent to $A_X Y = 0$ for $X, Y \in \Gamma(\mathcal{D}^{\perp})$, and we thus deduce

Corollary 4.2 *[2] Let M be a semi-Riemannian manifold with positive(or negative) sectional curvature K. Then there exists no totally geodesic foliation with bundle-like metric on M such that $\mathcal{D}^{\perp}$ is integrable.*

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