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Bi-periodic incomplete Horadam numbers

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Abstract: In this paper, we introduce bi-periodic incomplete Horadam numbers as a natural generalization of incomplete Horadam numbers. We study their basic properties and provide recurrence relations. In particular, we derive the generating function of these numbers.

Key words: Fibonacci sequence, Horadam sequence, bi-periodic Horadam sequence, bi-periodic incomplete Horadam sequence

1. Introduction

The Fibonacci sequence is one of the most famous and most studied sequences in mathematics. Its nth term F_n, also called as the nth Fibonacci number, is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$ are the initial values. This recurrence relation also defines the Lucas sequence for the initial values $L_0 = 2$ and $L_1 = 1$. It is well known that F_{n+1} counts the number of tilings of an n-board using either square tiles or two-square-wide dominoes [3]. It can be expressed as

$$F_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}.$$

This expression gives rise to a fascinating class of integers called the incomplete Fibonacci numbers. They were introduced by Flipponi [7] for integers n and k with $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ as

$$F_n(k) = \sum_{i=0}^{k} \binom{n-1-i}{i}.$$

Combinatorially, $F_{n+1}(k)$ counts the number of ways to tile an n-board with at most k dominoes [2]. Flipponi [7] also defined the incomplete Lucas numbers for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ as

$$L_n(k) = \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i}.$$

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Incomplete Fibonacci and Lucas numbers have many interesting properties. They generalize the Fibonacci and Lucas numbers. In other words, incomplete Fibonacci numbers reduce to Fibonacci numbers when $k = \lfloor \frac{n-1}{2} \rfloor$, and incomplete Lucas numbers reduce to Lucas numbers when $k = \lfloor \frac{n}{2} \rfloor$.

Horadam sequence $\{W_n\}$ with arbitrary integer initial values W_0 and W_1 is defined by the recurrence relation $W_n = pW_{n-1} + qW_{n-2}$ for $n \geq 2$. Its terms are called the Horadam numbers and they provide a generalization for Fibonacci numbers and Lucas numbers. Indeed, $\{W_n\}$ reduces to $\{F_n\}$ for $p = q = 1$ and $W_0 = 0, W_1 = 1$, and to $\{L_n\}$ for $p = q = 1$ and $W_0 = 2, W_1 = 1$. With this in mind, a question arises whether or not incomplete Fibonacci and Lucas numbers extend to Horadam-like numbers. Belbachir and Belkhir [1] responded this question by introducing incomplete Horadam numbers for $n \geq 2$ and $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ as

$$W_n(k) = \sum_{i=0}^{k} \frac{(n-2i)W_1 + piW_0}{n-i} \binom{n-i}{i} p^{n-2i-1} q^i, \tag{1.1}$$

where p and q are integers. They also introduced hyper-Horadam numbers and provided a connection between Horadam numbers, incomplete Horadam numbers, and hyper-Horadam numbers.

The bi-periodic Horadam sequence $\{w_n\}$ is a natural generalization of the Horadam sequence. For arbitrary initial values w_0 and w_1, its terms are defined recursively for $n \geq 2$ by

$$w_n = a^{\xi(n+1)} b^{\xi(n)} w_{n-1} + c w_{n-2}, \tag{1.2}$$

where a, b, and c are nonzero real numbers. Here, $\xi(n) = \frac{1-(-1)^n}{2}$. It can easily be seen that the bi-periodic Fibonacci sequence, the generalized bi-periodic Fibonacci sequence, the bi-periodic Lucas sequence, the generalized bi-periodic Lucas sequence, and the classical Horadam sequence are special cases of the bi-periodic Horadam sequence. For example, $\{w_n\}$ reduces to $\{W_n\}$ when $a = p$, $b = p$, and $c = q$. For details, we refer to [4–6, 12–14].

Ramírez [10] defined the bi-periodic incomplete Fibonacci numbers for $n \geq 1$ and $0 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ as

$$q_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}. \tag{1.3}$$

In this spirit, Tan and Ekin [12] introduced the bi-periodic incomplete Lucas numbers for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ by

$$p_n(k) = a^{\xi(n)} \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}. \tag{1.4}$$

Motivated by the above studies, we introduce in this paper the bi-periodic incomplete Horadam numbers. In particular, we give some recurrence relations and provide a connection between bi-periodic incomplete Fibonacci numbers and bi-periodic incomplete Horadam numbers. We then derive the generating function of these numbers. This new generalization shall give us a unified approach for many celebrated incomplete Fibonacci-like sequences such as bi-periodic incomplete Fibonacci and Lucas sequences, incomplete Fibonacci and Lucas sequences, incomplete balancing and Lucas-balancing sequences.

2. Main results

In this section, we shall introduce bi-periodic incomplete Horadam numbers. To this purpose, we begin with the following lemma. It provides a combinatorial expression for the bi-periodic Horadam numbers.

Lemma 2.1 *For $n \geq 1$, the bi-periodic Horadam numbers satisfy*

$$w_n = a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i.$$

Proof We will use induction on n. Clearly, the equality holds for $n = 1$. Now suppose that the lemma is true for any integer m with $1 \leq m \leq n$. Then by the inductive hypothesis, we can write

$$\begin{aligned} w_{n+1} &= a^{\xi(n)} b^{\xi(n+1)} w_n + c w_{n-1} \\ &= a^{\xi(n)} b^{\xi(n+1)} a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i + \\ & \quad a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1-2i)w_1 + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i - 1} c^{i+1}. \end{aligned}$$

Since $\xi(n-1) = \xi(n+1)$, we have

$$\begin{aligned} a^{-\xi(n)} w_{n+1} &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i + \xi(n+1)} c^i + \\ & \quad \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1-2i)w_1 + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i - 1} c^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i + \\ & \quad \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(n-1-2i)w_1 + biw_0}{n-1-i} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i - 1} c^{i+1} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i + \\ & \quad \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(n-2i+1)w_1 + b(i-1)w_0}{n-i} \binom{n-i}{i-1} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i \end{aligned}$$

$$
\begin{aligned}
&= w_1(ab)^{\lfloor \frac{n}{2} \rfloor} + \xi(n)(ab)^{-\xi(n)} b w_0 c^{\lfloor \frac{n+1}{2} \rfloor} + \\
&\quad \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left[\frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} + \right. \\
&\qquad \left. \frac{(n-2i+1)w_1 + b(i-1)w_0}{n-i} \binom{n-i}{i-1} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i \\
&= \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(n-2i+1)\, w_1 + biw_0}{n-i+1} \binom{n-i+1}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i .
\end{aligned}
$$

Thus, the given formula is true for any positive integer n. □

In the light of Lemma 2.1, we can define bi-periodic incomplete Horadam numbers as follows.

Definition 2.2 *Let n and k be positive integers such that $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$. We define the bi-periodic incomplete Horadam numbers as*

$$
w_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \frac{(n-2i)w_1 + biw_0}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i .
$$

Note that the incomplete Horadam numbers in (1.1) are a special case of this definition. They are obtained for $a = p$, $b = p$, and $c = q$.

It can easily be seen that $w_n(0) = a^{\xi(n-1)} w_1 (ab)^{\lfloor \frac{n}{2} \rfloor}$ and $w_n\left(\lfloor \frac{n}{2} \rfloor\right) = w_n$ for $n \geq 1$. Similarly,

$$
\begin{aligned}
w_n(1) &= a^{\xi(n-1)} \left(w_1(ab)^{\lfloor \frac{n}{2} \rfloor} + \left[(n-2)w_1 + bw_0\right] (ab)^{\lfloor \frac{n-3}{2} \rfloor} c \right), \\
w_n\left(\left\lfloor \frac{n-2}{2} \right\rfloor \right) &= \begin{cases} w_n - w_0 c^{\frac{n}{2}}, & \text{if } n \text{ is even}, \\ w_n - \left[w_1 + \left(\frac{n-1}{2}\right) bw_0 \right] c^{\frac{n-1}{2}}, & \text{if } n \text{ is odd}, \end{cases}
\end{aligned}
$$

for $n \geq 2$.

Example 2.3 *For $a = 3, b = 2, c = 1, w_0 = 4, w_1 = 2$ and $1 \leq n \leq 10$, all the values of $w_n(k)$ are displayed in the table on the next page.*

Proposition 2.4 *Consider the bi-periodic incomplete Horadam numbers $w_n(k)$. For $0 \leq k \leq \frac{n-3}{2}$, they satisfy the nonlinear recurrence relation*

$$
w_n(k) = a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + c w_{n-2}(k-1).
$$

Proof Suppose n is even. Since $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$, we have

$$
a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + c w_{n-2}(k-1)
$$

Table. Examples of a few bi-periodic incomplete Horadam numbers.

n/k	0	1	2	3	4	5
1	2					
2	6	10				
3	12	22				
4	36	72	76			
5	72	156	174			
6	216	504	594	598		
7	432	1080	1344	1370		
8	1296	3456	4536	4704	4708	
9	2592	7344	10152	10752	10786	
10	7776	23328	33912	36792	37062	37066

$$
\begin{aligned}
&= aa^{\xi(n)}\sum_{i=0}^{k}\frac{(n-2i-1)w_1+biw_0}{n-i-1}\binom{n-i-1}{i}(ab)^{\lfloor\frac{n-2}{2}\rfloor-i}c^i+\\
&\quad a^{\xi(n-1)}\sum_{i=0}^{k-1}\frac{(n-2i-2)w_1+biw_0}{n-i-2}\binom{n-i-2}{i}(ab)^{\lfloor\frac{n-1}{2}\rfloor-i-1}c^{i+1}\\
&= a^{\xi(n-1)}\sum_{i=0}^{k}\frac{(n-2i-1)w_1+biw_0}{n-i-1}\binom{n-i-1}{i}(ab)^{\lfloor\frac{n}{2}\rfloor-1-i}c^i+\\
&\quad a^{\xi(n-1)}\sum_{i=1}^{k}\frac{(n-2i)w_1+b(i-1)w_0}{n-i-1}\binom{n-i-1}{i-1}(ab)^{\lfloor\frac{n-1}{2}\rfloor-i}c^i\\
&= a^{\xi(n-1)}\sum_{i=0}^{k}\left[\frac{(n-2i-1)w_1+biw_0}{n-i-1}\binom{n-i-1}{i}+\right.\\
&\quad\left.\frac{(n-2i)w_1+b(i-1)w_0}{n-i-1}\binom{n-i-1}{i-1}\right](ab)^{\lfloor\frac{n-1}{2}\rfloor-i}c^i\\
&= a^{\xi(n-1)}\sum_{i=0}^{k}\left[w_1\binom{n-i-1}{i}+bw_0\binom{n-i-1}{i-1}\right](ab)^{\lfloor\frac{n-1}{2}\rfloor-i}c^i\\
&= a^{\xi(n-1)}\sum_{i=0}^{k}\frac{(n-2i)w_1+biw_0}{n-i}\binom{n-i}{i}(ab)^{\lfloor\frac{n-1}{2}\rfloor-i}c^i\\
&= w_n(k).
\end{aligned}
$$

The proof is similar when n is odd. This completes the proof. □

Proposition 2.4 can be transformed into a nonhomogeneous recurrence relation as follows:

$$\begin{aligned} w_n(k) &= a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + c w_{n-2}(k-1) \\ &= a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + c w_{n-2}(k) + c\big(w_{n-2}(k-1) - w_{n-2}(k)\big) \\ &= a^{\xi(n+1)} b^{\xi(n)} w_{n-1}(k) + c w_{n-2}(k) - \\ &\quad a^{\xi(n+1)} \frac{(n-2k-2)w_1 + bkw_0}{n-k-2} \binom{n-k-2}{k} (ab)^{\lfloor \frac{n-3}{2} \rfloor - k} c^{k+1}. \end{aligned} \tag{2.1}$$

Proposition 2.5 *For* $0 \leq k \leq \frac{n-s-1}{2}$, *we have*

$$\sum_{i=0}^{s} \binom{s}{i} w_{n+i}(k+i) a^{\lfloor \frac{i+\xi(n+1)}{2} \rfloor} b^{\lfloor \frac{i+\xi(n)}{2} \rfloor} c^{s-i} = w_{n+2s}(k+s). \tag{2.2}$$

Proof We proceed by induction on s. The proof is clear for $s=0$ and $s=1$ from Proposition 2.4. So assume the relation in (2.2) holds for all positive $j < s+1$. We will only verify it for $j = s+1$ when n is even since the proof is similar when n is odd. Now,

$$\begin{aligned} &\sum_{i=0}^{s+1} \binom{s+1}{i} w_{n+i}(k+i) a^{\lfloor \frac{i+1}{2} \rfloor} b^{\lfloor \frac{i}{2} \rfloor} c^{s+1-i} \\ &= \sum_{i=0}^{s+1} \left[\binom{s}{i} + \binom{s}{i-1} \right] w_{n+i}(k+i) a^{\lfloor \frac{i+1}{2} \rfloor} b^{\lfloor \frac{i}{2} \rfloor} c^{s+1-i} \\ &= \sum_{i=0}^{s+1} \binom{s}{i} w_{n+i}(k+i) a^{\lfloor \frac{i+1}{2} \rfloor} b^{\lfloor \frac{i}{2} \rfloor} c^{s+1-i} + \\ &\quad \sum_{i=0}^{s+1} \binom{s}{i-1} w_{n+i}(k+i) a^{\lfloor \frac{i+1}{2} \rfloor} b^{\lfloor \frac{i}{2} \rfloor} c^{s+1-i} \\ &= \binom{s}{s+1} w_{n+s+1}(k+s+1) a^{\lfloor \frac{s+2}{2} \rfloor} b^{\lfloor \frac{s+1}{2} \rfloor} + \\ &\quad c \sum_{i=0}^{s} \binom{s}{i} w_{n+i}(k+i) a^{\lfloor \frac{i+1}{2} \rfloor} b^{\lfloor \frac{i}{2} \rfloor} c^{s-i} + \\ &\quad \sum_{i=-1}^{s} \binom{s}{i} w_{n+i+1}(k+i+1) a^{\lfloor \frac{i+2}{2} \rfloor} b^{\lfloor \frac{i+1}{2} \rfloor} c^{s-i} \\ &= \binom{s}{s+1} w_{n+s+1}(k+s+1) a^{\lfloor \frac{s+2}{2} \rfloor} b^{\lfloor \frac{s+1}{2} \rfloor} + c w_{n+2s}(k+s) + \\ &\quad \binom{s}{-1} w_n(k) c^{s+1} + a \sum_{i=0}^{s} \binom{s}{i} w_{n+i+1}(k+i+1) a^{\lfloor \frac{i}{2} \rfloor} b^{\lfloor \frac{i+1}{2} \rfloor} c^{s-i} \\ &= c w_{n+2s}(k+s) + a w_{n+2s+1}(k+s+1) = w_{n+2s+2}(k+s+1). \end{aligned}$$

Hence the theorem holds for all j. This completes the proof. □

Proposition 2.6 *For* $s \geq 2k+2$, *we have*

$$\begin{aligned}&\sum_{i=0}^{s-1} a^{\left\lfloor \frac{s-\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{s-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n+1)}{2} \right\rfloor} c w_{n+i}(k) \\ &\quad = w_{n+s+1}(k+1) - a^{\left\lfloor \frac{s+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{s+\xi(n)}{2} \right\rfloor} w_{n+1}(k+1).\end{aligned} \tag{2.3}$$

Proof We will use induction on s. We will only consider the case when n is odd since the proof is similar when n is even.

Suppose n is odd. Then $\xi(n) = 1$ and $\xi(n+1) = 0$. For $s = 2$, the right hand side of Equation 2.3 is $w_{n+3}(k+1) - abw_{n+1}(k+1)$, and it simplifies to $acw_n(k) + cw_{n+1}(k)$ by Proposition 2.4. This clearly equals the left hand side. Hence, the proposition is true for $s = 2$.

Now suppose that the proposition is true for all $2 < s$. We prove it for s. Since $\left\lfloor \frac{s+1}{2} \right\rfloor = \left\lfloor \frac{s}{2} \right\rfloor + \xi(s)$, we have

$$\begin{aligned}
&\sum_{i=0}^{s} a^{\left\lfloor \frac{s+1-\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{s+1-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n+1)}{2} \right\rfloor} c w_{n+i}(k) \\
&= \sum_{i=0}^{s} a^{\left\lfloor \frac{s+1}{2} \right\rfloor - \left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor} c w_{n+i}(k) \\
&= \sum_{i=0}^{s-1} a^{\left\lfloor \frac{s+1}{2} \right\rfloor - \left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{i}{2} \right\rfloor} c w_{n+i}(k) + c w_{n+s}(k) \\
&= \sum_{i=0}^{s-1} a^{\left\lfloor \frac{s}{2} \right\rfloor + \xi(s) - \left\lfloor \frac{i+1}{2} \right\rfloor} b^{\left\lfloor \frac{s-1}{2} \right\rfloor + \xi(s+1) - \left\lfloor \frac{i}{2} \right\rfloor} c w_{n+i}(k) + c w_{n+s}(k) \\
&= a^{\xi(s)} b^{\xi(s+1)} \sum_{i=0}^{s-1} a^{\left\lfloor \frac{s+\xi(n+1)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n)}{2} \right\rfloor} b^{\left\lfloor \frac{s-\xi(n)}{2} \right\rfloor - \left\lfloor \frac{i+\xi(n+1)}{2} \right\rfloor} c w_{n+i}(k) + c w_{n+s}(k) \\
&= a^{\xi(s)} b^{\xi(s+1)} \left[w_{n+s+1}(k+1) - a^{\left\lfloor \frac{s+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{s+\xi(n)}{2} \right\rfloor} w_{n+1}(k+1) \right] + c w_{n+s}(k) \\
&= a^{\xi(s)} b^{\xi(s+1)} w_{n+s+1}(k+1) + c w_{n+s}(k) - a^{\xi(s) + \left\lfloor \frac{s}{2} \right\rfloor} b^{\xi(s+1) + \left\lfloor \frac{s+1}{2} \right\rfloor} w_{n+1}(k+1) \\
&= a^{\xi(s)} b^{\xi(s+1)} w_{n+s+1}(k+1) + c w_{n+s}(k) - a^{\left\lfloor \frac{s+1}{2} \right\rfloor} b^{\left\lfloor \frac{s+2}{2} \right\rfloor} w_{n+1}(k+1) \\
&= w_{n+s+2}(k+1) - a^{\left\lfloor \frac{s+1}{2} \right\rfloor} b^{\left\lfloor \frac{s+2}{2} \right\rfloor} w_{n+1}(k+1) \\
&= w_{n+s+2}(k+1) - a^{\left\lfloor \frac{s+1+\xi(n+1)}{2} \right\rfloor} b^{\left\lfloor \frac{s+1+\xi(n)}{2} \right\rfloor} w_{n+1}(k+1).
\end{aligned}$$

This completes the proof. □

We end this section by giving a connection between the generalized bi-periodic incomplete Fibonacci numbers $u_n(k)$ and the generalized bi-periodic incomplete Lucas numbers $v_n(k)$.

Proposition 2.7 *For* $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, *we have*

$$v_n(k) = u_{n+1}(k) + cu_{n-1}(k-1).$$

Proof Recall that

$$u_n(k) = a^{\xi(n-1)} \sum_{i=0}^{k} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} c^i, \quad 0 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$

and

$$v_n(k) = a^{\xi(n)} \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i, \quad 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor.$$

So we have,

$$\begin{aligned}
&u_{n+1}(k) + cu_{n-1}(k-1) \\
&= a^{\xi(n)} \sum_{i=0}^{k} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i + ca^{\xi(n)} \sum_{i=0}^{k-1} \binom{n-2-i}{i} (ab)^{\lfloor \frac{n-2}{2} \rfloor - i} c^i \\
&= a^{\xi(n)} \sum_{i=0}^{k} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i + a^{\xi(n)} \sum_{i=1}^{k} \binom{n-1-i}{i-1} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i \\
&= a^{\xi(n)} \sum_{i=0}^{k} \left[\binom{n-i}{i} + \binom{n-1-i}{i-1} \right] (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i \\
&= a^{\xi(n)} \sum_{i=0}^{k} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i} c^i \\
&= v_n(k).
\end{aligned}$$

□

3. The generating function

In this section, we shall derive the generating function for the bi-periodic incomplete Horadam numbers. To this purpose, we need the following lemma which can be obtained from [9] and [10]. We refer to Srivastava and Monacha [11] for a general treatment of generating functions of special functions.

Lemma 3.1 *Let $\{r_n\}_{n=0}^{\infty}$ be a given complex sequence, and let a, b, and c be complex numbers. Suppose that a complex sequence $\{s_n\}_{n=0}^{\infty}$ satisfies the nonhomogeneous and nonlinear recurrence relation*

$$s_n = \begin{cases} bs_{n-1} + cs_{n-2} + r_n, & \text{if } n \text{ is even,} \\ as_{n-1} + cs_{n-2} + ar_n, & \text{if } n \text{ is odd,} \end{cases}$$

for $n > 1$. Then the generating function $U(t)$ of $\{s_n\}_{n=0}^{\infty}$ is given by

$$U(t) = \frac{aG(t) + (s_0 - r_0) + [s_1 - a(s_0 + r_1)]\, t + (b-a)tf(t) + (1-a)G_1(t)}{1 - at - ct^2}$$

where $f(t)$, $G(t)$, and $G_1(t)$ are the generating functions of $\{s_{2n+1}\}_{n=0}^{\infty}$, $\{r_n\}_{n=0}^{\infty}$, and $\{r_{2n}\}_{n=0}^{\infty}$, respectively, and

$$f(t) = \frac{[s_1 - a(r_0 + r_1)]\, t + c\, [a(s_0 + r_1) - s_1]\, t^3 + atG_1(t) + a(1 - ct^2)G_2(t)}{1 - (ab + 2c)t^2 + c^2 t^4}$$

where $G_2(t)$ *denotes the generating function of* $\{r_{2n-1}\}_{n=1}^{\infty}$.

Proof Let $U(t)=\sum_{n=0}^{\infty} s_n t^n$ and $G(t)=\sum_{n=0}^{\infty} r_n t^n$. Then,

$$(1-at-ct^2)U(t)-aG(t)$$
$$=(s_0-ar_0)+[s_1-a(s_0+r_1)]\,t+\sum_{n=2}^{\infty}(s_n-as_{n-1}-cs_{n-2}-ar_n)t^n.$$

Let us simplify the summation above. Since $s_{2n+1}=as_{2n}+cs_{2n-1}+ar_{2n+1}$ and $s_{2m}=bs_{2m-1}+cs_{2m-2}+r_{2m}$, it follows that

$$\begin{aligned}\sum_{n=2}^{\infty}(s_n-as_{n-1}-cs_{n-2}-ar_n)t^n &= \sum_{m=1}^{\infty}(s_{2m}-as_{2m-1}-cs_{2m-2}-ar_{2m})t^{2m}\\ &= \sum_{m=1}^{\infty}\left[(b-a)s_{2m-1}+(1-a)r_{2m}\right]t^{2m}\\ &= (b-a)t\sum_{m=1}^{\infty}s_{2m-1}t^{2m-1}+(1-a)\sum_{m=1}^{\infty}r_{2m}t^{2m}\\ &= (b-a)tf(t)+(1-a)G_1(t)-(1-a)r_0.\end{aligned}$$

Hence,

$$(1-at-ct^2)U(t)-aG(t)=(s_0-r_0)+[s_1-a(s_0+r_1)]\,t+(b-a)tf(t)+(1-a)G_1(t).$$

Then the formula for the generating function follows by solving the above equation for $U(t)$.

Next, we calculate $f(t)$. For $m>2$, it is easy to see that

$$s_{2m-1}=(ab+2c)s_{2m-3}-c^2s_{2m-5}-a(cr_{2m-3}-r_{2m-2}-r_{2m-1}).$$

Moreover,

$$\begin{aligned}s_3-(ab+2c)s_1+a(cr_1-r_2-r_3) &= as_2-cs_1-abs_1+acr_1-ar_2\\ &= acs_0-cs_1+acr_1.\end{aligned}$$

Then we have,

$$\begin{aligned}&\left[1-(ab+2c)t^2+c^2t^4\right]f(t)-atG_1(t)+a(ct^2-1)G_2(t)\\ &\quad=\left[s_1-a(r_0+r_1)\right]t+c\left[a(s_0+r_1)-s_1)\right]t^3.\end{aligned}$$

The formula follows by solving the above equation for $f(t)$. □

Now, we are ready to state the generating function for the bi-periodic incomplete Horadam numbers.

Theorem 3.2 *Consider the bi-periodic incomplete Horadam numbers* $w_n(k)$*. Let*

$$G_1(t) = -\frac{c^{k+1}\left(w_0 b - (w_0 b - w_1)t\right)}{2}\left[\frac{t^2}{\left(1-(ab)^{\frac{1}{2}}t\right)^{k+1}} + \frac{t^2}{\left(1+(ab)^{\frac{1}{2}}t\right)^{k+1}}\right]$$

$$G_2(t) = -\frac{c^{k+1}\left(w_0 b - (w_0 b - w_1)abt\right)}{2(ab)^{\frac{1}{2}}}\left[\frac{t^2}{\left(1-(ab)^{\frac{1}{2}}t\right)^{k+1}} - \frac{t^2}{\left(1+(ab)^{\frac{1}{2}}t\right)^{k+1}}\right].$$

Then, the generating function $W_k(t)$ *of* $w_n(k)$ *is given by*

$$W_k(t) = \sum_{n=0}^{\infty} w_n(k)t^n = \frac{aG(t) + w_{2k} + w_{2k-1}t + (b-a)tf(t) + (1-a)G_1(t)}{1 - at - ct^2}$$

where $G(t) = G_1(t) + G_2(t)$*, and*

$$f(t) = \frac{w_{2k+1}t - cw_{2k-1}t^3 + atG_1(t) + a(1-ct^2)G_2(t)}{1-(ab+2c)t^2 + c^2t^4}.$$

Proof Let k be a fixed positive integer. It is easy to see that $w_n(k) = 0$ for $0 \le n < 2k$, and $w_{2k}(k) = w_{2k}$ and $w_{2k+1}(k) = w_{2k+1}$. When n is even, it follows from the nonhomogeneous recurrence relation in Equation 2.1 that

$$\begin{aligned} w_n(k) &= aw_{n-1}(k) + cw_{n-2}(k) \\ &\quad - b^{-1}\frac{(n-2k-2)w_1 + bkw_0}{n-k-2}\binom{n-k-2}{k}(ab)^{\lfloor\frac{n-2}{2}\rfloor - k}c^{k+1}. \end{aligned}$$

Similarly, when n is odd

$$\begin{aligned} w_n(k) &= bw_{n-1}(k) + cw_{n-2}(k) \\ &\quad - \frac{(n-2k-2)w_1 + bkw_0}{n-k-2}\binom{n-k-2}{k}(ab)^{\lfloor\frac{n-2}{2}\rfloor - k}c^{k+1}. \end{aligned}$$

Now let us define

$$s_0 = w_{2k}(k) = w_{2k}, \quad s_1 = w_{2k+1}(k) = w_{2k+1}, \quad s_n = w_{2k+n}(k),$$

and

$$r_0 = r_1 = 0, \quad r_n = -\frac{(n-2)w_1 + bkw_0}{n+k-2}\binom{n+k-2}{k}(ab)^{\lfloor\frac{n}{2}\rfloor - 1}c^{k+1}.$$

Then,

$$\begin{aligned} G(t) &= G_1(t) + G_2(t) \\ &= -\frac{c^{k+1}t^2}{2}\left[\frac{\left[w_0 b - (w_0 b - w_1)t\right] + \left[w_0 b - (w_0 b - w_1)abt\right](ab)^{-\frac{1}{2}}}{\left(1-(ab)^{\frac{1}{2}}t\right)^{k+1}}\right. \\ &\qquad \left. + \frac{\left[w_0 b - (w_0 b - w_1)t\right] - \left[w_0 b - (w_0 b - w_1)abt\right](ab)^{-\frac{1}{2}}}{\left(1+(ab)^{\frac{1}{2}}t\right)^{k+1}}\right] \end{aligned}$$

is the generating function of the sequence $\{-r_n\}$. Thus the generating function of the sequence $\{w_n(k)\}_{n=0}^{\infty}$ follows from [9, Lemma 3.1]. This completes the proof. □

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