Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2023) 47: 1051 – 1072
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doi:10.55730/1300-0098.3412

Nilary group rings and algebras

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Received: 04.08.2022 • **Accepted/Published Online:** 10.01.2023 • **Final Version:** 16.05.2023

Abstract: A ring A is (principally) nilary, denoted (pr-)nilary, if whenever $XY = 0$, then there exists a positive integer n such that either $X^n = 0$ or $Y^n = 0$ for all (principal) ideals X, Y of A. We determine necessary and/or sufficient conditions for the group ring $A[G]$ to be (principally) nilary in terms of conditions on the ring A or the group G. For example, we show that: (1) If $A[G]$ is (pr-)nilary, then A is (pr-)nilary and either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A. (2) Assume that G is finite. Then G is nilpotent and $A[G]$ is (pr-)nilary if and only if G is a p-group, $char(A) = p^\alpha$ (p is a prime), and A is (pr-)nilary. (3) Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and F is a field with $char(F) = q$. Then $F[G]$ is nilary if and only if G is a q-group. Examples are provided to illustrate and delimit our results.

Key words: Nilary ring, indecomposable ring, group ring, group algebra, quasi-Frobenius ring

1. Introduction

Throughout, all rings are associative with a nonzero unity; R denotes such a ring; and G denotes a group. Also, we consider all groups to be nontrivial (i.e. of order greater than one) unless indicated otherwise. Recall that a ring R is called Quasi-Frobenius (denoted, QF) if R is right Artinian and right self-injective.

This paper is the second in a series of papers on the classification of indecomposable QF-rings. The following facts are well-known: (1) Every QF-ring is a finite direct sum of indecomposable QF-rings. (2) Let R be a commutative ring. Then R is an indecomposable QF-ring if and only if R is a local Artinian ring with simple socle. (3) R is a prime QF-ring if and only if R is isomorphic to a $n-$by$-n$ ($n \in \mathbb{N}$) matrix ring over a division ring. Note that a nonprime local QF-ring has a nonzero nilpotent Jacobson radical which is a prime ideal. So from the above facts and other properties of QF-rings, it seems reasonable to investigate the classification of indecomposable QF-rings.
As a first step in our investigation, we recognized that the classes of nilary rings and pr-nilary rings properly contain the class of all rings with a nilpotent prime ideal (e.g., all prime rings and all local rings with nilpotent Jacobson radical).

Recall from [2], a ring R is (principally) nilary, if whenever $XY = 0$ then there exists $n \in \mathbb{N}$ such that either $X^n = 0$ or $Y^n = 0$ for all (principal) ideals X, Y of R. We use pr-nilary to denote principally nilary.

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2010 *AMS Mathematics Subject Classification:* Primary 16L60, 16S34, 16D50; Secondary:16D70.

These classes of rings have the following useful properties (see [2]): (1) Every nilary ring is pr-nilary, but the converse is not true even for commutative rings [2, Example 1.10(ii)]. However, if R has ACC on ideals, then R is pr-nilary if and only if R is nilary [2, Proposition 1.4]. (2) Every pr-nilary ring is indecomposable [2, Proposition 1.4]. (3) The nilary and pr-nilary properties are Morita invariants [2, Theorem 3.5]. (4) If I is a nilpotent ideal of R and $\frac{R}{I}$ is a nilary ring, then R is a nilary ring [2, Proposition 1.6]. Thus, we can classify the indecomposable QF-rings into the class of nilary QF-rings and the class of nonnilary indecomposable QF-rings.

In [1, Theorem 2.2], we characterized the nilary QF-rings. Among the characterizations, the following is useful in this paper: Let R be a QF-ring. Then R is nilary if and only if there exists a set of orthogonal primitive idempotents $\{e_1, \cdots, e_m\}$, such that $e_1R, \cdots, e_mR$ represent a complete set of isomorphism classes of principal indecomposable modules with each Re_iR right essential in R.

The class of group algebras of the form, $R = F[G]$, where F is a field and G is a finite group are Frobenius algebras (hence QF-rings) which have important applications in group representation theory, algebraic coding theory, and physics through the symmetry group of a physical system. In this paper, after some basic results on (pr-)nilary (i.e. pr-nilary or nilary) rings and an example of an indecomposable QF-ring which is not nilary, we investigate the pr-nilary and nilary conditions on group rings of the form, $R = A[G]$, where A is a ring and G is a group, in Section 2. In particular, we find necessary and/or sufficient conditions on the ring A and the group G for R to be nilary or pr-nilary. For example, we show (let A be a ring, G be a group, and p be a prime): (1) If $A[G]$ is (pr-)nilary, then either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A (Theorem 2.10(i)). (2) Assume that G is finite. Then G is nilpotent and $A[G]$ is (pr-)nilary if and only if G is a p-group, $char(A) = p^\alpha$ (p is a prime and α is a positive integer), and A is (pr-)nilary (Corollary 2.30). (3) Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and $char(A) = q^\alpha$. Then $A[G]$ is (pr-)nilary if and only if G is a q-group (Theorem 2.31).

In Section 3, we apply the results of Sections 1 and 2 and [1] to investigate (pr-) nilary group algebras of the form $R = F[G]$ where F is a field and G is a group. In particular, we begin the search for a characterization of the nilary QF-algebra of the form $R = F[G]$ where F is a field and G is a finite group, in terms of the properties of F and G. Also, we characterize the nilary group algebras which have a nilpotent prime ideal (Theorem 3.5).

We use $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_n$ $(n > 1)$ to denote the set of positive integers, the ring of integers, and the ring of integers modulo n, respectively; $I \trianglelefteq A$ means that I is an ideal (a two-sided) of the ring A, $K \leq R_R$ and $K \leq^{ess} R_R$ denote that K is a right ideal of R and K is right essential in R (i.e. $K \cap Y \neq 0$ for each nonzero right ideal Y of R), respectively; and we use $< X >$ for the ideal generated by the nonempty subset X of A. $P(A)$, $J(A)$, $gcd(a,b)$, $U(A)$, $char(A)$, and $cent(A)$ denote the prime radical, the Jacobson radical of a ring A, the greatest common divisor of a and b, the units of A, the characteristic of A, and the center of A, respectively. The left (right) annihilator of the subset X of the ring A is denoted by $\ell_A(X) = \{a \in A \mid aX = 0\}$ ($r_A(X) = \{a \in A \mid Xa = 0\}$). For other terminology see [9, 10, 11].

2. nilary rings

Definition 2.1 *[2, Definition 1.1 (i)-(iii) and Definition 2.1]*

(i) An ideal I of a ring R is said to be a (principally) right primary ideal if whenever X and Y are (principal) ideals of R with $XY \subseteq I$, then either $X \subseteq I$ or $Y^n \subseteq I$ for some positive integer n depending

on X *and* Y.

(ii) *An ideal* I *is called a (principally) nilary ideal if whenever* X *and* Y *are (principal) ideals of* R *with* $XY \subseteq I$, *then either* $X^n \subseteq I$ *or* $Y^n \subseteq I$ *for some positive integer* n *depending on* X *and* Y.

(iii) *A ring* R *is said to be a (principally) right primary ring or (principally) nilary ring if the zero ideal is a (principal) right primary or a (principal) nilary ideal of* R, *respectively.*

(iv) *An ideal* I *of* R *is called a strongly pr-nilary ideal if* $\sqrt{I}$ *is a prime ideal of* R. R *is called a strongly pr-nilary ring if the zero ideal of* R *is a strongly pr-nilary. Every strongly pr-nilary ideal (ring) is a pr-nilary ideal (ring) [2, Proposition 2.4(i)].*

In this paper, p-groups are used in conjunction with the principally nilary concept. In order to avoid confusion, we use "pr-" as an abbreviation for "principally". Thus pr-nilary ring denotes principally nilary ring. Note that in [2] a principally nilary ring (ideal) is denoted as a p-nilary ring (ideal). Also, we use "(pr-) nilary" to denote "principally nilary or nilary, respectively".

In parts (i) and (iii) above, the left-sided version is defined analogously. Define $\sqrt{I} = \sum\{V \trianglelefteq R \mid V^n \subseteq I \text{ for some } n \in \mathbb{N}\}$; $\sqrt{I}$ is called the pseudo radical of I. Let $\sqrt{0_R}$, and $\sqrt{0_{A[G]}}$ denote the pseudo radical (i.e. Wedderburn radical) of R, and $A[G]$, respectively.

Observe, from Proposition 2.8, that if R has a nilpotent prime ideal, then R is nilary and strongly pr-nilary. For example, all prime rings and all local rings with nilpotent Jacobson radical are nilary and strongly pr-nilary. See [2] for more examples.

Lemma 2.2 *Let* I *be an ideal of a ring* R. *The following conditions are equivalent:*

(i) I *is a nilary ideal of* R.

(ii) $AB \subseteq I$ *implies that* $A^m \subseteq I$ *or* $B^m \subseteq I$ *for some* $m \in \mathbb{N}$ *and for all left ideals* A, B *of* R.

(iii) *Let* B *be any left ideal of* R *and* $(I : B) = \{r \in R | rB \subseteq I\}$. *Then* $B^m \subseteq I$ *or* $(I : B)^m \subseteq I$ *for some* $m \in \mathbb{N}$.

(iv) $\frac{R}{I}$ *is a nilary ring.*

Proof

$(i) \Rightarrow (ii)$ Assume I is nilary and $AB \subseteq I$. Then $(A(RB))R \subseteq I$. So $(AR)(BR) \subseteq I$. Hence, $(AR)^m \subseteq I$ or $(BR)^m \subseteq I$, for some $m \in \mathbb{N}$. Therefore, $A^m \subseteq I$ or $B^m \subseteq I$.

$(ii) \Rightarrow (iii)$ Observe that $(B : I)$ is a two sided ideal of R and $(I : B)B \subseteq I$. So $(I : B)^m \subseteq I$ or $B^m \subseteq I$ for some positive integer m.

$(iii) \Rightarrow (i)$ Let A, B be ideals of R such that $AB \subseteq I$. Then $A \subseteq (I : B)$. Therefore, $A^m \subseteq I$ or $B^m \subseteq I$ for some positive integer m.

$(vi) \Leftrightarrow (i)$ The proof of this implication is routine.

□

Corollary 2.3 *The following conditions are equivalent:*

(i) R is a nilary ring.

(ii) $IJ = 0$ implies that $I^m = 0$ or $J^m = 0$ for some $m \in \mathbb{N}$ and for all left ideals I, J of R.

(iii) For any left ideal J of R either $J^m = 0$ or $\ell(J)^m = 0$ for some $m \in \mathbb{N}$.

(iv) For any right ideal I of R either $I^m = 0$ or $r(I)^m = 0$ for some $m \in \mathbb{N}$.

Results similar to Lemma 2.2 and Corollary 2.3 hold for pr-nilary ideals and rings, respectively. These results are used implicitly throughout the paper.

Lemma 2.4

(i) If R/I is a nilary ring and I is a nilpotent ideal, then R is a nilary ring.

(ii) If R/I is a pr-nilary ring and $\sqrt{I}$ is a sum of nilpotent ideals, then R is a pr-nilary ring.

(ii) $\sqrt{0_R}$ is nilpotent if and only if $P(R)$ is nilpotent, in either case, $\sqrt{0_R} = P(R)$.

Proof

(i) This is [2, Proposition 1.6(iii)].

(ii) This is [2, Proposition 1.6(iv)].

(iii) Clearly $\sqrt{0_R} \subseteq P(R)$; hence, $P(R)$ nilpotent implies $\sqrt{0_R} = P(R)$. So $\sqrt{0_R}$ is nilpotent. Now assume $\sqrt{0_R}$ is nilpotent. Then [7, P.184] yields $\sqrt{0_R} = P(R)$. So $P(R)$ is nilpotent.

□

Proposition 2.5 *(i) $P(R)$ is a (pr-)nilary ideal if and only if $P(R)$ is a prime ideal.*
(ii) $P(R) = \sqrt{0_R}$ and $P(R)$ is a pr-nilary ideal if and only if R is a strongly pr-nilary ring.
(iii) Assume $\sqrt{0_R}$ is a nilpotent pr-nilary ideal. Then $P(R) = \sqrt{0_R}$, $P(R)$ is a prime ideal, R is a nilary and strongly pr-nilary ring.
(iv) R has a nilpotent prime ideal if and only if $\sqrt{0_R}$ is a nilpotent pr-nilary ideal if and only if $P(R)$ is a nilpotent prime ideal.

Proof (i) Since $P(R)$ is a semiprime ideal, this part follows from [2, Proposition 1.3(i)].
(ii) Part(i) and [2, Proposition 2.3] yields this part.
(iii) Assume $\sqrt{0_R}$ is a nilpotent pr-nilary ideal. From Lemma 2.4(iii), $P(R) = \sqrt{0_R}$. By part (i), $P(R)$ is a prime ideal and R is strongly pr-nilary. From Lemma 2.4(i), R is a nilary ring.
(iv) Assume X is a nilpotent prime ideal. Then $X \subseteq \sqrt{0_R} \subseteq P(R) \subseteq X$. The remainder of the proof follows from Lemma 2.4(iii) and part(i).

□

Proposition 2.6 *Let R be a commutative ring. The following are equivalent.*
(i) R is pr-nilary.
(ii) $P(R)$ is a prime ideal.
(iii) R is a strongly pr-nilary.

Proof $(i) \Rightarrow (ii)$ Suppose R is pr-nilary. Let $x, y \in R$ such that $xy \in P(R)$. Since $P(R)$ is the set of nilpotent elements of R, there exists a positive integer m such that $(xy)^m = x^m y^m = 0$. Then $(x^m R)(y^m R) = 0$. Since R is pr-nilary, there exists a positive integer k such that $(x^m R)^k = 0$ or $(y^m R)^k = 0$. Then $x^{mk} = 0$ or $y^{mk} = 0$. Therefore, either $x \in P(R)$ or $y \in P(R)$, so $P(R)$ is a prime ideal of R.
$(ii) \Rightarrow (iii)$ Since R is commutative, $\sqrt{0_R} = P(R)$. So R is strongly pr-nilary.
$(iii) \Rightarrow (i)$ This implication follows from [2, Proposition 2.4(i)]

□

Note that Proposition 2.6 is not true for noncommutative rings. In Example 4.13 or [1, Example 2.13], it is shown that $R = \mathbb{Z}_3[S_3]$ is a nilary ring such that $P(R) = J(R)$ is nilpotent; but $\frac{R}{P(R)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Therefore, $P(R)$ is not a prime ideal of R.

Corollary 2.7 *Assume R is (pr-)nilary and S is a subring of R.*

(i) If $S \subseteq cent(R)$, then S is a (pr-)nilary ring and $\frac{S}{P(S)}$ is a domain.

(ii) If R is right duo (i.e. every right ideal is an ideal), then S is (pr-)nilary.

Proof

(i) Assume that R is nilary and $X, Y \trianglelefteq S$ such that $XY = 0$. Then $(XR)(YR) = 0$. So there exist $m \in \mathbb{N}$ such that $(XR)^m = 0$ or $(YR)^m = 0$. Hence, $X^m = 0$ or $Y^m = 0$. Therefore, S is nilary. The pr-nialry part is similar. By Proposition 2.6, $\frac{S}{P(S)}$ is a domain.

(ii) We show the nilary case. The pr-nilary case is similar. Let $X, Y \trianglelefteq S$ such that $XY = 0$. Then $XR, YR \leq R_R$. Hence, $(XR)(YR) = X(RYR) = XYR = 0$. So $(XR)^m = 0$ or $(YR)^m = 0$ for some $m \in \mathbb{N}$. Therefore, $X^m = 0$ or $Y^m = 0$.

□

Proposition 2.8 *(i) If R has a nilpotent prime ideal, then R is nilary and strongly pr-nilary. In particular, prime rings and local rings with nilpotent Jacobson radical are nilary and strongly pr-nilary.*
(ii) Let M be an (R,R)-bimodule and $\mathscr{T}(R,M)$ the trivial (also called the split-null) extension of M by R. If R is (pr-)nilary, then $\mathscr{T}(R,M)$ is (pr-)nilary.

Proof (i) This part follows from Proposition 2.5 (iii) and (iv).
(ii) Let $(0, M) = \{(0,m) | m \in M\}$. Then $(0, M)$ is a nilpotent ideal of $\mathscr{T}(R,M)$ and $\frac{\mathscr{T}(R,M)}{(0,M)} \cong R$. The remainder of the proof follows from Lemmas 2.2 and 2.4. □

Theorem 2.9 *Let R be a (pr-)nilary ring. Then either $char(R) = 0$ or $char(R) = p^\beta$ for some positive integer β, where p is a prime number. If R is semiprime and $char(R) \neq 0$, then $char(R) = p$.*

Proof Assume that $char(R) > 0$ and $char(R) = p^{\alpha}m$ for some $\alpha, m \in \mathbb{N}$, where p is a prime number such that $gcd(p, m) = 1$. Since $\langle p^{\alpha}\rangle\langle m\rangle = 0$ and R is a (pr-)nilary ring, either p or m is nilpotent in R. In case p is nilpotent in R, then $p \in J(R)$; this implies that $m \in U(R)$ because $gcd(p, m) = 1$. Now, we have

$$0 = (char(R) \cdot 1)m^{-1} = p^{\alpha}mm^{-1} = p^{\alpha};$$

thus, $p^{\alpha} = 0$ in R; but $char(R) = p^{\alpha}m$, so $m = 1$. In case m is nilpotent in R, then $m \in J(R)$; this implies that $p \in U(R)$ and $p^{\alpha} \in U(R)$ because $gcd(p^{\alpha}, m) = 1$. Now, we have

$$0 = p^{-\alpha}(1 \cdot char(R)) = p^{-\alpha}p^{\alpha}m = m;$$

hence, $m = 0$ in R; but $char(R) = p^{\alpha}m$, a contradiction. So either $char(R) = 0$ or $char(R) = p^{\alpha}$ for some $\alpha \in \mathbb{N}$. In the last part of the result, since R is semiprime and $p \in cent(R)$, $char(R) = p$. □

Example 2.10 *This example is an indecomposable Frobenius basic ring R with $u.dim(R_R) = 3$ which is not a nilary ring (see [11, Example 16.19(4)]). In fact, this example can be extended to an indecomposable Frobenius basic ring R with $u.dim(R_R) = n$ for $n \geq 3$ which is not a nilary ring. Let K be a division ring, and*

$$R = \left\{ \begin{bmatrix} a_1 & x_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & x_2 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & x_3 \\ 0 & 0 & 0 & 0 & 0 & a_1 \end{bmatrix} | a_i, x_i \in K \, for \; i = 1, 2, 3 \right\}.$$

Let $E = \{e_1, e_2, e_3\}$ where $e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, *and* $e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. *Then E is a complete set of orthogonal primitive idempotents for R. To see that R is indecomposable, assume c is a nontrivial central idempotent of R. Then either c or $1 - c$ is in E. Since no element of E is central, R is indecomposable. To see that R is not nilary, let $I = Re_1R$. Then*

$$I = \left\{ \begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} | \, a, x_i \in K \, for \; i = 1, 2 \right\} \text{ and}$$

$$r(I) = Re_2R\left\{\begin{bmatrix} b & y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 & 0 & b \end{bmatrix} \mid b, y_i \in K \; for \; i = 1, 2\right\}.$$

Since neither I nor $r(I)$ are nilpotent, R cannot be nilary.

3. Group rings that are (pr-)nilary

If A is a ring and G is a group, $A[G]$ will denote the group ring of G over A. Consider the function $\varepsilon : A[G] \to A$ defined by $\varepsilon(\Sigma_{g\in G} a_g g) = \Sigma_{g\in G} a_g$. This function is called the augmentation map, and ε is a ring homomorphism that maps $A[G]$ onto A. $Ker(\varepsilon) = \{\alpha = \Sigma_{g\in G} a_g g \in A[G] \mid \varepsilon(\alpha) = \Sigma_{g\in G} a_g = 0\}$. $Ker(\varepsilon)$ is a nontrivial ideal called the augmentation (fundamental) ideal of $A[G]$ and is denoted by $\Delta(G)$. The ideal $\Delta(G)$ consists of the elements of the form $a_1(1-g_1) + \cdots + a_k(1-g_k)$ with each $a_i \in A$, each $g_i \in G$, and k a positive integer. From the above, it is clear that $A[G]/\Delta(G) \cong A$. Let H be a normal subgroup of G. Then the natural homomorphism $G \to G/H$ mapping g to gH induces a ring homomorphism $\varepsilon_H : A[G] \to A[G/H]$ defined by $\varepsilon_H(\Sigma_{g\in G} a_g g) = \Sigma_{g\in G} a_g gH \in A[G/H]$. Also, $Ker(\varepsilon_H) = \Delta(G, H)$ for the kernel of this homomorphism. The ideal $\Delta(G, H)$ consists of the elements of the form $a_1g_1(1-h_1) + \cdots + a_kg_k(1-h_k)$ with each $a_i \in A$, each $g_i \in G$, $h_i \in H$, and k a positive integer. $A[G]\Delta(H)$ is the kernel of ε_H (i.e. $\Delta(G, H) = A[G]\Delta(H)$). In particular, if $H = G$, then $\varepsilon = \varepsilon_G$, and we write $\Delta(G) = \Delta(G, G)$. For a nonempty subset I of A, we have I is a right ideal of A if and only if $I[G]$ ($IA[G] = I[G]$) is a right ideal of $A[G]$; if I is an ideal, then $I[G]$ is an ideal and $A[G]/I[G] \cong (A/I)[G]$. We use $\nu(G)$ to denote the set of orders of all finite normal subgroups; $Z(G)$ to denote the center of a group G; $\rho(G)$ is the set of $g \in G$ which have only a finite number of conjugates; $\sigma(G)$ denotes the set of $g \in \rho(G)$ of finite order; groups with $\sigma(G) = G$ are called locally normal, an equivalent definition being every finite subset is contained in a finite normal subgroup; S_n is the symmetric group and A_n its alternating subgroup; C_n is used for the cyclic group of order n ($n \geq 1$); the subgroup $\langle g \rangle$ is called the cyclic subgroup of G generated by g; a group G is called a p-group if the order of each element of G is a power of p; $|G|$ denotes the order of G; the order of an element g is denoted by $o(g)$; G is called a torsion group if, for every $g \in G$ there exists a nonzero $n \in \mathbb{N}$ with $o(g) = n$; G is prime if it satisfies either one of the following two equivalent conditions: (i) $\sigma(G) = 1$, (ii) $\nu(G) = \{1\}$, i.e. G contains no finite normal subgroup except 1; G is called a Dedekind group if every subgroup of G is normal. Also, for a prime p and a finite group G we denote by $O_p(G)$ the maximal normal subgroup of G such that its order is divisible by p; and $O_{p'}(G)$ the maximal normal subgroup of G such that its order is not divisible by p. These definitions and concepts may be found in [5, 11, 15].

This section is devoted to obtaining and investigating results related to nilary group rings. The results of Connell, in [5], which relate to prime rings are (partially) generalized to the class of nilary rings. We start with finding necessary conditions on A and G so that $A[G]$ is nilary.

Lemma 3.1 *Let A be a ring and G be a group. The following statements are equivalent:*

(i) $\sqrt{0_{A[G]}} = \Delta(G)$;

(ii) $P(A[G]) = \sqrt{0_{A[G]}} = \Delta(G)$;

(iii) G is a locally normal p-group, A is semiprime, and $p = 0$ in A.

Proof See [5, p. 682, Theorem 10]. □

Recall that if G is a group and H is a finite subgroup, then $\hat{H} = \sum_{h\in H} h$.

Lemma 3.2 *[11, Lemma 3.4.3] and [5, p. 651, Propsition 1]*
Let H be a subgroup of a group G and let A be a ring. Then $\ell_{A[G]}(\Delta(G,H)) \neq 0$ if and only if H is finite. In this case, we have

$$\ell_{A[G]}(\Delta(G,H)) = A[G]\hat{H}.$$

Furthermore, if H is normal in G, then the element $\hat{H}$ is central in $A[G]$ and we have

$$\ell_{A[G]}(\Delta(G,H)) = r_{A[G]}(\Delta(G,H)) = \hat{H}A[G].$$

Lemma 3.3 *[5, p. 656, Proposition 4(ii)] The left and right annihilator ideals of $\Delta(G)$ coincide and are given by*

$$(\Delta(G))^* = \begin{cases} 0 & \text{if } G \text{ is infinite,} \\ A\sum_{g\in G} g & \text{if } G = \{g_1, g_2, ..., g_n\}. \end{cases}$$

In the latter case

$$\Delta(G) \cap (\Delta(G))^* = \{a\sum_{g\in G} g \mid a \in A, na = 0\}.$$

From Lemma 3.3, observe that if $R = A[G]$ is nilary, then either $\ell(\Delta(G)) = r(\Delta(G)) = 0$, $\Delta(G)$ is nilpotent, or $\hat{G}$ is a central nilpotent element of R.

Proposition 3.4 *Let A be a ring, G be a group, and $H \trianglelefteq G$.*

(i) $\Delta(G,H)$ is a (pr-)nilary ideal if and only if $A[G/H]$ is a (pr-)nilary ring.

(ii) $\Delta(G)$ is a (pr-)nilary ideal if and only if A is a (pr-)nilary ring.

Proof (i) This part follows from Lemma 2.2 and the fact that $A[G/H] \cong A[G]/\Delta(G,H)$ for any normal subgroup H of G [11, Corollary 3.3.5].
(ii) Put $H = G$ in part(i). □

Lemma 3.5 *[5, p. 681, Theorem 9] Let A be a ring and G be a group. Then $\Delta(G)$ is nilpotent if and only if*

(i) G is a finite p-group, and

(ii) p is nilpotent in A.

Proposition 3.6 *Let A be a ring, G a group and $H \trianglelefteq G$.*
(i) H is a finite p-subgroup and p is nilpotent in A if and only if $\Delta(G,H)$ is nilpotent.
(ii) Assume H is a finite p-subgroup and p is nilpotent in A. If $A[\frac{G}{H}]$ is (pr-)nilary, then $A[G]$ is (pr-) nilary.
(iii) Let $G = H \rtimes K$ where H is a finite p-subgroup and p is nilpotent in A. If $A[K]$ is (pr-)nilary, then $A[G]$ is (pr-)nilary.

Proof (i) ($\Rightarrow$) Since

$$\Delta(G,H) = A[G]\Delta(H) = \Delta(H)A[G],$$

we have $\Delta(G,H))^n = (A[G]\Delta(H))^n = A[G](\Delta(H))^n$. The result follows from Lemma 3.5.

($\Leftarrow$) Since $\Delta(H) \subseteq \Delta(G,H)$. If $\Delta(G,H)$ is nilpotent, then $\Delta(H)$ is nilpotent. By Lemma 3.5, then H is a finite p-subgroup and p is nilpotent in A.
(ii) By part(i), $\Delta(G,H)$ is nilpotent. From [11, Corollary 3.3.5], $\frac{A[G]}{\Delta(G,H)} = A[\frac{G}{H}]$. Now, Lemma 2.4 yields the result.
(iii) This part follows from part(ii).

□

From [1, Example 2.13 or Example 4.3], $\mathbb{Z}_3[S_3]$ is nilary. By Proposition 3.6(iii), $\mathbb{Z}_3[C_{3^n} \times S_3]$ is nilary for each positive integer n.

Lemma 3.7 *Let A be a ring and G be a group.Then, $A[G]$ is prime if and only if A is prime and G is prime.*

Proof See [5, p. 675, Theorem 8]. □

Lemma 3.8 *Let A be a ring and G be a group. If G is prime, then $A[G]$ is semiprime if and only if A is semiprime.*

Proof See [5, p. 676]. □

Theorem 3.9 *Let A be a ring and H a subgroup of G such that $H \subseteq Z(G)$.*

(i) If J is a nilary ideal of $A[G]$, then $J \cap A[H]$ is a nilary ideal of $A[H]$.

(ii) If J is a pr-nilary ideal of $A[G]$, then $J \cap A[H]$ is a pr-nilary ideal of $A[H]$.

(iii) If $A[G]$ is a (pr-)nilary ring, then $A[H]$ is a (pr-)nilary ring.

(iv) If $A[G]$ is a (pr-)nilary ring, then A is a (pr-)nilary ring.

(v) If $A[G]$ is a (pr-)nilary, then $A[Z(G)]$ is a (pr-)nilary ring.

Proof

(i) Let $R = A[G]$ and let $R' = A[H]$. Assume that $I, K \trianglelefteq R'$ with $IK \subseteq J \cap R'$. Now, we have that $(IK)R \subseteq (J \cap R')R \subseteq JR = J$. Since $H \subseteq Z(G)$, we get, $RK = KR$ and so $(IK)R = (IR)(KR)$. Then $(IR)(KR) \subseteq J$. Hence, $(IR)^n \subseteq J$ or $(KR)^n \subseteq J$ for some $n \in \mathbb{N}$ because J is a nilary ideal. Assume that $(IR)^n \subseteq J$. Since R is a unitary ring, $I^n \subseteq (IR)^n \subseteq J$. Therefore, $I^n \subseteq J \cap R'$. If $(KR)^n \subseteq J$, then $K^n \subseteq J \cap R'$. Hence, $J \cap R'$ is a nilary ideal of R'.

(ii) Notice that if I is a (principal) finitely generated ideal of $A[H]$, then $IA[G]$ is a (principal) finitely generated ideal of $A[G]$. By using part(i) and [2, Proposition 1.3(iii)], we have the result.

(iii) Put $J = \{0\}$ in Part(i) and (ii).

(iv) Put $H = \{1\}$ in Part(iii).

(v) Put $H = Z(G)$ in Part(iii).

□

Theorem 3.10 *Let A be a ring, p be a prime, and G be a group.*

(i) If $A[G]$ is (pr-)nilary, then either G is prime or the order of each finite nontrivial normal subgroup of G is nilpotent in A.

(ii) If $char(A) = 0$ and $A[G]$ is (pr-)nilary, then G is prime (i.e. $\nu(G) = 1$).

(iii) If $char(A) = p^\alpha$, and $A[G]$ is (pr-)nilary, then p divides $|H|$ for each nontrivial finite normal subgroup H of G.

(iv) If G is finite and $char(A) = p^\alpha$, and $A[G]$ is (pr-)nilary, then $O_{p'}(G) = 1$.

Proof

(i) Assume that G is not prime. Then there exists a finite nontrivial normal subgroup H of G. Now, we have that $\Delta(G, H)$ is a finitely generated ideal of $A[G]$, and by Lemma 3.2 $0 \neq r_{A[G]}(\Delta(G, H))$ is a principal ideal of $A[G]$ generated by $\hat{H} = \sum_{i=1}^{n} h_i$, with $|H| = n$. Now, we have

$$(\Delta(G, H))\ r_{A[G]}(\Delta(G, H)) = 0.$$

Since $A[G]$ is (pr-)nilary, either $\Delta(G, H)$ or $r_{A[G]}(\Delta(G, H))$ is nilpotent (see [2, Proposition1.3(iii)]), so either $(\Delta(G, H))^m = 0$ or $(r_{A[G]}(\Delta(G, H)))^m = 0$ for some $m \in \mathbb{N}$, by Lemma 2.3(ii).

Suppose $(\Delta(G, H))^m = 0$. By Proposition 3.6, H is a p-group and p is a nilpotent in A. Hence, n is nilpotent in A. Since n was arbitrary, this implies it is nilpotent in A for each $1 \neq n \in \nu(G)$.

Now suppose $(r_{A[G]}(\Delta(G, H)))^m = 0$. Since $x = \hat{H} \in r_{A[G]}(\Delta(G, H))$, $x^m = 0$. Also, $x^2 = \hat{H}x = \sum_{i=1}^{n} h_i x = nx$; hence, $x^3 = (x^2)x = (nx)x = n(x^2) = n(nx) = (n \cdot n)x = (n^2)x = n^2 x$. Therefore, $x^m = n^{m-1}x$, since $x^m = 0$. Then $n^{m-1}x = 0$. So $n^{m-1}\hat{H} = n^{m-1}(h_1 + h_2 + h_3 + \cdots + h_n) = 0$. Since H is linearly independent over A, $n^{m-1} = 0$ in A. Since n is arbitrary, this implies that the order of each finite nontrivial normal subgroup of G is nilpotent in A.

(ii) Assume that G is not prime. There is a nontrivial finite normal subgroup H of G. Since $A[G]$ is (pr-)nilary, either $\Delta(G, H)$ or $r_{A[G]}(\Delta(G, H))$ is nilpotent. By using part(i), we find $|H|$ is nilpotent in A. However, $|H| \neq 0$, and $char(A) = 0$, a contradiction. Hence, $\sigma(G) = 1$. Therefore, G is prime.

(iii) Assume that G has a nontrivial finite normal subgroup H. By part(i) we have $|H|$ is nilpotent in A. However, $|H| \neq 0$, and $char(A) = p^{\alpha}$. Hence, $o(1) = p^{\alpha}$ in the additive group A. So $p^{\alpha}||H|$. Therefore, $p||H|$, since p is a prime.

(iv) Assume that $O_{p'}(G)$ is nontrivial. Hence, $O_{p'}(G)$ is a nontrivial normal subgroup of G. Then p divides $|O_{p'}(G)|$. By Cauchy's Theorem, we find that $O_{p'}(G)$ contains an element of order p, a contradiction.

□

The following examples illustrate and delimit Theorem 3.10.

Example 3.11 *Let A be a domain (e.g., $A = \mathbb{Q}$, the rational numbers) and $G = S_\infty$ (the infinite symmetric group). Notice that A is prime and G is prime, and hence $A[G]$ is a prime ring by Lemma 3.7. Therefore, $A[G]$ is (pr-)nilary. However, G has infinitely many nontrivial finite subgroups which are not normal and A contains no nonzero nilpotent elements.*

Example 3.12 *From [1, Proposition 1.4(iii)], a (pr-) nilary ring is indecomposable. Then it is free of nontrivial central idempotents. Hence, for any (pr-)nilary group algebra $F[G]$, the principal block is the unique block of G over the field F. So if the group algebra $F[G]$ has more than one block, then it is not (pr-)nilary. By using [6, Theorem 4], we find that $\mathbb{Z}_2[A_5]$ is not nilary because it has two bockls. Therefore, the converse of Theorem 3.10 part(i) is false, since A_5 has only one nontrivial normal subgroup and its order is nilpotent in $\mathbb{Z}_2$.*

Corollary 3.13 *Let A be a ring and G be a finite group.*
If $A[G]$ is (pr-)nilary, then A is (pr-)nilary and $|G|$ is nilpotent in A.

Proof The proof follows from Theorems 3.9 part(iv) and 3.10(i). □

Corollary 3.14 *Let A be a semiprime ring and G be a group. If either G is prime or $char(A) = 0$, then $A[G]$ is a (pr-)nilary ring if and only if $A[G]$ is a prime ring.*

Proof Suppose $A[G]$ is (pr-)nilary. First, assume that $char(A) = 0$. From Theorem 3.10(ii), G is prime. Since $A[G]$ is a (pr-)nilary ring, A is a (pr-)nilary ring, by Theorem 3.9(iv) . From [2, Proposition 1.3(i)] A is prime. Hence, $A[G]$ is prime because A is prime and G is prime, (see, Lemma 3.7). The converse is clear. Next, assume that G is prime. This proof is similar to that used for $char(A) = 0$. The converse is routine.

□

Remark 3.15 *Let $A = \mathbb{Z}_{p^n}$ with $n \in \mathbb{N}$, and p be a prime number.*
(i) If $n = 1$ then A is prime and hence nilary.
(ii) If $n > 1$ then A is nilary, but it is not prime.

The next result provides examples of (pr-)nilary group rings $A[G]$ where G is a prime group, but $A[G]$ is not a prime ring.

Proposition 3.16 *Assume that G is prime. Let A be a commutative ring with a nonzero nilpotent prime ideal I. Then $A[G]$ is (pr-)nilary which is not prime.*

Proof Put $B = A/I$, since I is prime then B is a prime ring. Hence, $B[G] = (A/I)[G] \cong A[G]/I[G]$, by [5, p. 654, (9)]. Since B is prime and G is prime, $B[G]$ is prime, by Lemma 3.7. Therefore, $B[G] \cong A[G]/I[G]$ is prime. Hence, $A[G]/I[G]$ is (pr-)nilary. Since $I[G]$ is nilpotent, $A[G]$ is (pr-)nilary, by Lemma 2.4. Since A is not prime, $A[G]$ is not prime, by Lemma 3.7.

□

For a particular example, take $A = \mathbb{Z}_{p^m}$ and $G = S_\infty$, for some $m \in \mathbb{N}$ and $m > 1$.

Proposition 3.17 *Let A be a ring, G be a group, $H \lhd G$, and and $R = A[G]$.*
(i) If $|G| = \infty$, then $\Delta(G) \leq^{ess} R_R$ and $\Delta(G) \leq^{ess} {}_RR$.
(ii) If $|G| = m < \infty$, R is pr-nilary, and $A[\frac{G}{H}]$ is semiprime, then A is a prime ring, $char(A) = p$, $\Delta(G,H) \leq^{ess} R_R$ and $\Delta(G,H) \leq^{ess} {}_RR$.

Proof Let $X \leq R_R$ such that $X \cap \Delta(G) = 0$. Then $X \subseteq \ell(\Delta(G,H)) = \hat{H}R$ by Lemma 3.2.
(i) Since $|G| = \infty$, $X = 0$ by Lemma 3.3. Therefore, $\Delta(G,H) \leq^{ess} R_R$. Similarly, $\Delta(G,H) \leq^{ess}_R R$.
(ii) Since $\frac{R}{\Delta(G,H)} \cong A[\frac{G}{H}]$, A is semiprime and $\Delta(G,H)$ is a semiprime ideal of R. Hence, $P(R) \subseteq \Delta(G,H)$. By Theorem 3.9(iv), A is pr-nilary ring. From‘[2, definition 1.1(iii) and Proposition 1.3(i)], A is a prime ring. Theorem 2.9 yields that $char(A) = 0$ or $char(A) = p$ for some prime p. If $char(A) = 0$, then G is prime by Theorem 3.10(ii). This is a contradiction to $|G| = m < \infty$. So $char(A) = p$. By Theorem 3.10(iii), $p||H|$. Then $X^2 \subseteq (\hat{H}R)^2 = |H|\hat{H}R = 0$. So $X \subseteq X \cap P(R) \subseteq X \cap \Delta(G,H) = 0$. Then $\Delta(G,H) \leq^{ess} R_R$. A similar argument yields $\Delta(G,H) \leq^{ess} {}_RR$. □

Proposition 3.18 *Let A be a ring and G be a Dedekind group. Assume $A[G]$ is (pr-)nilary, then $T(G)$ is trivial or $T(G)$ is a p-group where p is a prime number.*

Proof From Theorem 2.9, if A is nilary then either $char(A) = 0$ or $char(A) = p^\alpha$ for some prime p and positive integer α. If $char(A) = 0$, then by Theorem 3.10 G is prime. Hence, $T(G)$ is trivial. Assume $char(A) = p^\alpha$ and $T(G)$ is nontrivial. Let $g \in G$ of finite order and $g \neq 1$. Let $H =< g >$. Since G is Dedekind, then H is a finite normal subgroup of G. By Theorem 3.10, p divides $|H|$. Assume that there is a prime number $q \neq p$ such that q divides $|H|$. Notice that H is cyclic, so abelian; thus, there is a subgroup K of H such that the order of K is q. Also, K is a subgroup G; therefore , K is normal because G is Dedekind. Again, by Theorem 3.10, p divides $|K|$, a contradiction. Therefore, $T(G)$ is a p-group.

□

A group G is hypercentral if there exists a smallest ordinal α such that $Z_\alpha = G$, where $Z_0 = 1, Z_1 = Z(G)$ if λ is a limit ordinal $Z_\lambda = \bigcup_{\beta<\lambda} Z_\beta$ and $\frac{Z_{\beta+1}}{Z_\beta} = Z(\frac{G}{Z_\beta})$. Furthermore, α is called the class of G. If α is finite, G is also called nilpotent.

If π is a set of primes, a group is said to be π-free if it contains no nontrivial elements whose order is a π-number (i.e. a product of primes in π).

Lemma 3.19 *[13, Theorem 2.2.12] Let G be a group with π-free center. Then each upper central factor, and therefore the hypercenter of G, $h(G)$, is π-free.*

Theorem 3.20 *Let A be a ring and G be any group.*

(i) If $A[G]$ is (pr-)nilary, then $T(h(G))$ is either trivial or a p-group for some prime number p.

(ii) If $char(A) = 0$ and $A[G]$ is (pr-)nilary, then $h(G)$ is either trivial or torsion-free.

Proof

(i) Assume that $A[G]$ is (pr-)nilary. Then, by Theorem 3.9(v), $A[Z(G)]$ is (pr-)nilary. By using Proposition 3.18, if $T(Z(G))$ is nontrivial, then it is a p-group for some prime number p. Assume that $T(h(G))$ is nontrivial. Now, if $T(Z(G))$ is trivial, then it is q-free for any prime q, by Lemma 3.19. $T(h(G))$ is also q-free for any prime q. Therefore, $T(h(G))$ is trivial which is contrary to our assumption. Thus, $T(Z(G)$ is nontrivial. So we have that $T(Z(G))$ is a p-group for some prime number p. From Lemma 3.19, we have that $T(h(G))$ is a p-group.

(ii) Assume that $Z(G) \neq \{1\}$. If $Z(G)$ is not torsion-free, then there is $g \in Z(G)$ with $o(g) < \infty$. Put $H = \langle g \rangle$. Then $H \trianglelefteq Z(G)$. By using Theorem 3.10, we find that $|H|$ is nilpotent in A, contrary to $char(A) = 0$. So $Z(G)$ is torsion-free. Hence, it is q-free for each prime number q. From Lemma 3.19, we have that $T(h(G))$ is torsion-free.

□

Theorem 3.21 *Let A be a ring and G be a nontrivial nilpotent group.*

(i) If G is torsion and $A[G]$ is (pr-)nilary, then $char(A) = p^\alpha$ and G is a p-group for some prime number p.

(ii) If $char(A) = p^\alpha$ and $A[G]$ is a (pr-)nilary ring, then $T(G)$ is either trivial or a p-group.

(iii) If $char(A) = 0$ and $A[G]$ is a (pr-)nilary ring, then $T(G)$ is trivial and G is torsion-free.

Proof

(i) Let G be a torsion nilpotent group; hence, G is not prime. By Theorem 3.10(iii) and Theorem 3.9, we find that $char(A) = p^\alpha$ for some prime number p. Since G is torsion and nilpotent, $h(G) = T(h(G)) = G$. From Theorem 3.20(ii), we find that G is a p-group.

(ii) Since G is a nilpotent group, $T(G) = T(h(G))$. By using Theorem 3.20 (ii), we find that $T(G)$ is either trivial or a p-group.

(iii) This part follows directly from Theorem 3.20 part(iii).

□

Remark 3.22 *Theorem 2.21 (i) and (ii) are false, if we replace the condition "G is nontrivial nilpotent" with "G is nontrivial solvable." To see this, observe that for $A = \mathbb{Z}_3$ and $G = S_3$, then $A[G]$ is nilary (see Example 3.13), G is solvable, $T(G) = G$, $char(A) = 3$; but G is not a p-group.*

Now, we give some sufficient conditions on A and G so that $A[G]$ is (pr-)nilary ring.

Lemma 3.23 *Let G be a nontrivial locally normal p-group, and A a pr-nilary ring such that p is nilpotent in A. Then*
(i) $\Delta(G) \subseteq \sqrt{0_{A[G]}}$.
(ii) $\sqrt{0_{A[G]}} \subseteq \Delta(G) \Leftrightarrow$ *A is semiprime $\Leftrightarrow A$ is prime.*

Proof (i) Let $g \in G$ and H be the normal closure $< g >$ (i.e. the smallest normal subgroup of G containing g). Then, $\Delta_A(H)$ is nilpotent, say $(\Delta_A(H))^m = 0$, by Lemma 3.5. Now an element of $\Delta(G,H)$ is a sum of terms of the form $(1-g_1)r_1(1-g_2)r_2\cdots(1-g_m)r_m$, $g_i \in H$,$r_i \in A[G]$. From the normality of H, $x \in \Delta(G,H)$ is a sum of terms of the form

$$y = (1-g'_l)(1-g'_2)\cdots(1-g'_n)r, \;\; g'_i \in H, r \in A[G];$$

hence, $y = 0$, since $(1-g'_1)\ \cdots(1-g'_n) \in (\Delta_A(H))^n$. Thus, $\Delta_A(G,H)$ is nilpotent and $1-g \in \Delta(G,H)$. Therefore, $\Delta(G) \subseteq \sqrt{0_{A[G]}}$.
(ii) Assume $\sqrt{0_{A[G]}} \subseteq \Delta(G)$. From part(i), $\Delta(G) = \sqrt{0_{A[G]}}$. By Lemma 3.1, A is semiprime. Now assume A is semiprime, by [2, Proposition 1.3(i)], A is a prime ring.

Finally, assume A is a prime ring. Since p is a central nilpotent element of A, $p = 0$ in A. From Lemma 3.1, $\Delta(G) = \sqrt{0_{A[G]}}$.

□

Theorem 3.24 *Let G be a nontrivial locally normal p-group, A be a ring such that p is nilpotent in A, and either $\sqrt{0_{A[G]}} \subseteq \Delta(G)$ or A is semiprime. Consider the following conditions:*
(i) $A[G]$ is pr-nilary.
(ii) A is pr-nilary.
(iii) A is prime.
(iv) $P(A[G])$ is a prime ideal.
Then, $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(iii) \Rightarrow (iv)$. If A is semiprime, then $(iv) \Rightarrow (iii)$.

Proof $(i) \Rightarrow (ii)$ This implication follows from, Theorem 3.9(iv).
$(ii) \Leftrightarrow (iii)$ This equivalence is a consequence of Lemma 3.23.
$(ii) \Rightarrow (i)$ *and* (iv) Since A is pr-nilary, A is a prime ring by Lemma 3.23. Hence, $\Delta(G)$ is a prime ideal of $A[G]$. By Lemma 3.1 and Lemma 3.23, $\Delta(G) = \sqrt{0_{A[G]}} = P(A[G])$ is a prime ideal. Let X, Y be ideals of $A[G]$ such that $XY = 0$. Then $XY \subseteq \sqrt{0_{A[G]}}$. Hence, $X \subseteq \sqrt{0_{A[G]}}$ or $Y \subseteq \sqrt{0_{A[G]}}$. From [2, Proposition1.3(ii)], $A[G]$ is pr-nilary.
$(iii) \Rightarrow (iv)$ By Lemma 3.1, $P(A[G]) = \Delta(G)$. Hence, $\frac{A[G]}{P(A[G])} = \frac{A[G]}{\Delta(G)} \cong A$. Therefore, $P(A[G])$ is a prime ideal.
$(iv) \Rightarrow (iii)$ Assume A is semiprime and $P(A[G])$ is a prime ideal. From Lemma 3.1, $P(A[G]) = \Delta(G)$. Then $A \cong \frac{A[G]}{\Delta(G)} = \frac{A[G]}{P(A[G])}$. Therefore, A is a prime ring. □

Example 3.25 *Let F be a field such that $char(F) = p$, A be the ring of $n \times n$ matrices over F, and $G \cong C_p \bigoplus C_{p^2} + C_{p^3} \bigoplus \cdots \bigoplus C_{p^n} \bigoplus \cdots$. Then $A[G]$ is a pr-nilary ring by Theorem 3.24.*

Theorem 3.26 *Let G be a finite p-group and p be a nilpotent in A. Assume that $I \trianglelefteq A$. Then I is a (pr-)nilary ideal of A, if and only if $I[G]$ is a (pr-)nilary ideal of $A[G]$.*

Proof ($\Rightarrow$) Assume that I is a (pr-)nilary ideal of A, then $\bar{A} = \frac{A}{I}$ is a (pr-)nilary ring. Since p is nilpotent in A. Then $p^k = 0$ in A for some $k \in \mathbb{N}$. This implies that $p^k A = 0$. Hence, $p^k \bar{A} = 0$. Thus, $\Delta_{\bar{A}}(G)$ is nilpotent because G is a finite p-group and $p^k \bar{A} = 0$, by Lemma 3.5. Since $\bar{A} \cong \frac{\bar{A}[G]}{\Delta_{\bar{A}}(G)}$, $\bar{A}[G]$ is a (pr-)nilary ring, by Lemma 2.4. Since $\bar{A}[G] = (\frac{A}{I})[G] \cong \frac{A[G]}{I[G]}$. $\frac{A[G]}{I[G]}$ is a (pr-)nilary ring. Then $I[G]$ is a (pr-)nilary ideal of $A[G]$.
($\Leftarrow$) Assume that $I[G]$ is a (pr-)nilary ideal of $A[G]$. Then $\frac{A[G]}{I[G]} \cong (\frac{A}{I})[G]$ is (pr-)nilary. From Theorem 3.9(iv), $\frac{A}{I}$ is (pr-)nilary. Therefore, I is (pr-)nilary. □

Corollary 3.27 *Let A be a ring and G a finite p-group such that p is nilpotent in A. Then, $A[G]$ is a (pr-)nilary ring if and only if A is a (pr-)nilary ring.*

Proof Use Theorem 3.26 with $I = 0$. □

Theorem 3.28 *Let G be a locally normal p-group, A a ring such that p is nilpotent in A, $I \trianglelefteq A$, and either $\sqrt{0_{\bar{A}[G]}} \subseteq \Delta_{\bar{A}}(G)$ or $\bar{A}$ is semiprime, where $\bar{A} = \frac{A}{I}$. Then I is a pr-nilary ideal of A if and only if $I[G]$ is a pr-nilary ideal of $A[G]$.*

Proof ($\Rightarrow$) Assume I is a pr-nilary ideal of A. Then $\bar{A} = \frac{A}{I}$ is pr-nilary. Since p is nilpotent in A, $p^k = 0$ for some $k \in \mathbb{N}$. As in the proof of Theorem 3.26, $p^k \bar{A} = 0$. By Theorem 3.24, $\bar{A}[G] = \frac{A}{I}[G] = \frac{A[G]}{I[G]}$ is pr-nilary. Therefore, $I[G]$ is a pr-nilary ideal of $A[G]$.
($\Leftarrow$) Assume $I[G]$ is a pr-nilary ideal of $A[G]$. Then $\frac{A[G]}{I[G]} = \frac{A}{I}[G] = \bar{A}[G]$ is pr-nilary. By Theorem 3.9(iv), $\bar{A} = \frac{A}{I}$ is pr-nilary. Therefore, 3.9(iv), I is a pr-nilary ideal of A.

□

Theorem 3.29 *Let A be a ring with $char(A) = p^n$ and G be a nontrivial nilpotent group.*

(i) If G is finite, then $A[G]$ is a (pr-)nilary ring if and only if A is (pr-)nilary and G is a p-group.

(ii) If $T(G)$ is finite, and A is prime, then $A[G]$ is a (pr-)nilary ring if and only if $T(G)$ is a p-group.

(iii) Let G be a locally normal torsion group such that $\sqrt{0_{A[G]}} \subseteq \Delta(G)$ or A is semiprime. Then $A[G]$ is a pr-nialry ring if and only if A is (pr-)nilary and G is a p-group.

Proof

(i) Assume that G is a finite p-group and A is (pr-)nilary. Then by Corollary 3.27, it follows that $A[G]$ is a (pr-)nilary ring. Conversely, assume that $A[G]$ is a (pr-)nilary ring. Hence, A is (pr-)nilary, by Theorem 3.9(iv). Since G is nilpotent, by Theorem 3.21 it follows that G is a p-group.

(ii) Let A be a prime ring and G be a nilpotent group such that $T(G)$ is a finite p-group. If $\frac{G}{T(G)}$ is trivial, then G is finite p-group. From part(i), it follows that $A[G]$ is a (pr-)nilary ring. If $\frac{G}{T(G)}$ is nontrivial, then

$\frac{G}{T(G)}$ is torsion-free, hence a prime group. So $A[\frac{G}{T(G)}]$ is a prime ring, by Lemma 3.7. Hence, $A[\frac{G}{T(G)}]$ is (pr-)nilary. Now we have $A[\frac{G}{T(G)}] \cong \frac{A[G]}{\Delta(G,T(G))}$. Since $T(G)$ is a finite p-group, by using Proposition 3.6(i), it follows that the ideal $\Delta(G,T(G))$ is nilpotent. By Lemma 2.4, we conclude that $A[G]$ is a nilary ring. Conversely, if $A[G]$ is a nilary ring, then $T(G)$ is a finite p-group by Theorem 3.21.

(iii) Assume that $A[G]$ is pr-nilary. By Theorem 3.9(iv), A is pr-nilary. Then Theorem 3.21(i) yields that G is a p-group. Conversely, assume that A is pr-nilary and G is a p-group. By Lemma 3.23, A is prime. From Theorem 3.24, $A[G]$ is a pr-nilary ring.

□

Corollary 3.30 *Assume that G is a finite group and A is a ring. Then G is nilpotent and $A[G]$ is (pr-) nilary if and only if G is a p-group, $char(A) = p^{\alpha}$ (p is a prime), and A is (pr-) nilary.*

Proof This result follows from Theorems 3.9(iv), 3.21(i), 3.29(i). □

Theorem 3.31 *Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$; and A be a ring such that $char(A) = q^{\alpha}$ for some positive integer α. Then, $A[G]$ is (pr-)nilary if and only if A is (pr-)nilary and G is a p-group for some prime p.*

Proof ($\Rightarrow$) Assume $A[G]$ is (pr-)nilary. Then A is (pr-)nilary by Theorem 3.9(iv). Suppose G is not a p-group. Then G is not a q-group. From (p.16, Theorem 4.24, Subgroup series 2, https://kconrad.math.uconn.edu/blurbs/grouptheory/subgpseries2.pdf), the set N of all elements of order prime to q forms a normal subgroup of G. Since G is not a q-group, $|N| > 1$. By Theorem 3.10(i), $|N|$ is nilpotent in A. Hence, q divides $|N|$, a contradiction. Therefore, G is a p-group.
($\Leftarrow$) Assume A is (pr-)nilary and G is a p-group. Then G is a q-group. The result follows from Corollary 3.27. □

Observe that the condition, $char(A) = q^{\alpha}$, where q is the smallest prime divisor of $|G|$ is not superflous in Theorem 3.31. For example, S_3 is a finite supersolvable group. Let $R_1 = \mathbb{Z}_3[S_3]$. From Example 4.13 or [2, Example 2.13], R_1 is nilary; but S_3 is not a p-group. Also, from Theorem 3.31, we can conclude that $R_2 = \mathbb{Z}_{2^m}[S_3]$ is not nilary for any positive integer m.

4. Nilary group algebra

In this section, we determine necessary and/or sufficient conditions for a group algebra, $R = F[G]$, to be pr-nilary or nilary in terms of properties of the field F and the group G. For example, we show: (1) If $char(F) = 0$, then R is prime $\Leftrightarrow$ R is (pr-)nilary $\Leftrightarrow$ G is prime (Proposition 4.3). (2) If G is finite, then R is local $\Leftrightarrow$ $char(F) = p$ and G is a p-group $\Leftrightarrow$ $J(R) = \Delta(G)$ $\Leftrightarrow$ R is strongly pr-nilary $\Leftrightarrow$ G is nilpotent and $F[G]$ is nilary (Theorem 4.5). In the remainder of the section, we consider R where $char(F) = p$ and G is a finite solvable group. We apply our results to show that $\mathbb{Z}_3[S_3]$, $\mathbb{Z}_2[A_4]$, and $\mathbb{Z}_2[S_4]$ are nilary group rings which are not rings with a nilpotent prime ideal, hence neither prime nor local.

Proposition 4.1 *Let F be a field with $char(F) = 2$ and G be a finite simple nonabelian group. If $F[G]$ is nilary, then G is either M_{22} or M_{24}.*

Proof

Since any nilary ring is indecomposable as a ring, then it is free of central idempotents. Hence, for any group algebra $F[G]$, the principal block is the unique block of G over F. So if the group algebra $F[G]$ has more than one block, then it is not nilary. By using [6, Theorem 4], we find that G is either M_{22} or M_{24}.

□

Proposition 4.2 *Let F be a field, $char(F) = 0$; and let G be a group. Then the following conditions are equivalent:*
(i) G is a prime group;
(ii) $F[G]$ is a (pr-)nilary ring;
(iii) $F[G]$ is a prime ring.

Proof (iii)⇒ (ii) Clear. (ii)⇒ (i) Since $char(F) = 0$, $F[G]$ is semiprime. By [2, Proposition 1.3(i)], $F[G]$ is prime. From Lemma 3.7, G is prime. (i)⇒ (iii) Use Lemma 3.7. □

Proposition 4.3 *Let G be a finite group, F a field and $R = F[G]$.*
(i) R is a prime ring if and only if $|G| = 1$.
(ii) Assume $|G| > 1$. If R is nilary then $char(F) = p$, a prime integer, and p divides $|G|$. Also, $O_{p'}(G) = 1$.
(iii) If R is indecomposable and $|G| > 1$, then $Soc(R_R) \leq^{ess} J(R) \leq^{ess} \Delta(G) \leq^{ess} R_R$ and $Soc({}_RR) \leq^{ess} J(R) \leq^{ess} \Delta(G) \leq^{ess} {}_RR$.

Proof (i) This part follows from [5, p.675].
(ii) This part is a consequence of Theorem 3.10.
(iii) Since $R = \frac{R}{\Delta(G)} \cong F$, $\Delta(G)$ is a maximal left and right ideal. Hence, $J(R) \subseteq \Delta(G)$. From [10, Example 16.56], R is a Frobenius algebra (hence a QF-ring). Now the result follows from [1, Lemma 1.6].

□

Theorem 4.4 *Let F be a field.*
(i) Assume that $F[G]$ is (pr-)nilary and G is a nilpotent torsion group. Then $char(F) = p$ (a prime) and G is a p-group.
(ii) Assume that $char(F) = p$ (a prime), and G is a locally normal p-group. Then $F[G]$ is a pr-nilary local ring.

Proof (i) This part follows from Theorem 3.21.
(ii) By Theorem 3.24, $F[G]$ is a pr-nilary ring. From Lemma 3.1, $\sqrt{0_{F[G]}} = P(F[G]) = \Delta(G)$. Since $\frac{F[G]}{\Delta(G)} \cong F$, $\Delta(G)$ is a maximal left and right ideal of $F[G]$. Then $P(F[G]) \subseteq J(F[G]) \subseteq \Delta(G) \subseteq P(F[G])$, so $J(F[G]) = \Delta(G)$. By [9, Theorem 19.1], $F[G]$ is a local ring.

□

From [4], a ring is called an idempotent fine ring (denoted IF-ring) if each of its nonzero idempotents is a sum of a nilpotent element and a unit (e.g., any ring with trivial idempotents such as local rings and domains).

Theorem 4.5 *Let $R = F[G]$ where F is a field and G is a finite group such that $|G| > 1$. The following statements are equivalent.*

(i) R *is a local ring.*

(ii) R_R *is indecomposable.*

(iii) $\frac{R}{J(R)}$ *is a simple ring.*

(iv) $char(F) = p$ *and* G *is a p-group, where* p *is a prime.*

(v) $J(R) = \Delta(G)$.

(vi) R *is an IF-ring.*

(vii) R *is strongly pr-nilary.*

(viii) R *has a nilpotent prime ideal.*

(ix) G *is nilpotent and* R *is nilary.*

(x) G *is nilpotent and* R *is right primary.*

Proof $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ These equivalences are in [9, p. 294, Excercise 19.4].

$(i) \Rightarrow (v)$ From Proposition 4.3(iii), $J(R) \subseteq \Delta(G)$. Since R is local, $J(R) \subseteq \Delta(G)$.

$(v) \Rightarrow (i)$ Assume $J(R) = \Delta(G)$. Then $\frac{R}{J(R)} = \frac{R}{\Delta(G)} \cong F$. By [9, Theorem 19.1], R is local.

$(i) \Rightarrow (vi)$ Clear.

$(vi) \Rightarrow (vii)$ From [4, Proposition 7], every IF-ring is idempotent simple (i.e. $R = ReR$ for each $0 \neq e = e^2 \in R$). The concept of an idempotent-simple ring has been previously defined in [1, Definition 2.4] as a full ring. Since R is a QF-ring [10, Example 16.56], [1, Theorem 2.10] yields that R is strongly pr-nilary.

$(vii) \Rightarrow (viii)$ Assume that R is strongly pr-nilary. Since R is right Artinian, $\sqrt{0_R}$ is nilpotent. From Proposition 2.5(ii), R has a nilpotent prime ideal.

$(viii) \Rightarrow (vii)$ This implication follows from Proposition 2.8(i).

$(vii) \Rightarrow (iii)$ Since R is strongly pr-nilary, $\sqrt{0_R}$ is a prime ideal. Because R is right Artinian, $J(R)$ is nilpotent. So $\sqrt{0_R} \subseteq P(R) \subseteq J(R) \subseteq \sqrt{0_R}$; hence, $J(R)$ is a prime ideal. Therefore, $\frac{R}{J(R)}$ is simple.

$(iv) \Rightarrow (ix)$ This implication follows from Theorem 4.4(ii), since G is a finite p-group.

$(ix) \Rightarrow (iv)$ This implication follows from Theorem 3.21(ii).

$(vii) \Rightarrow (x)$ This implication follows from [2, Lemma 3.13].

$(x) \Rightarrow (ix)$ This implication is clear from the definitions of right primary and nilary rings.

□

Note that in Theorem 4.5 if R is local, then $\frac{R}{J(R)} \cong F$, since $J(R) = \Delta(G)$.

Corollary 4.6 *Let* F *be a field,* G *be a group and* H *a nontrivial locally finite subgroup of* $Z(G)$. *If* $F[G]$ *or* $F[Z(G)]$ *is (pr-)nilary, then* H *is a p-group,* $char(F) = p$ *(a prime) and* $F[H]$ *is a (pr-)nilary local ring.*

Proof

From Corollary 3.7, $F[H]$ is (pr-)nilary. Let $1 \neq x \in H$, and X be the subgroup generated by x. From Theorem 4.5, $char(F) = p$ and X is a p-group. Since x was arbitrary, H is a p-group. By Theorem 4.4, $F[H]$ is local. □

Proposition 4.7 *Let* G *be a group,* H *be a finite normal p-subgroup of* G, K *be a group such that* $\frac{G}{H} \cong K$ *and* F *be a field such that* $char(F) = p$, *where* p *is a prime. If* $F[K]$ *is (pr-)nilary, then* $F[G]$ *is (pr-)nilary.*

Proof This proof is a consequence of Proposition 3.6(ii). □

Corollary 4.8 *Let $G = H \rtimes K$ where H is a finite normal p-subgroup of G and F is a field such that $char(F) = p$, where p is a prime. If $F[K]$, is (pr-)nilary, then $F[G]$ is (pr-)nilary.*

Example 4.9 *Let $G = C_{3^m} \times S_3$ where $m \in \mathbb{N}$, and $F = \mathbb{Z}_3$. From [1, Example 2.13], $F[S_3]$ is nilary. Therefore, $F[G]$ is nilary by Corollary 4.8.*

Proposition 4.10 *Let G be a finite supersolvable group such that q is the smallest prime dividing $|G|$, and F is a field with $char(F) = q$. Then $F[G]$ is nilary if and only if G is a q-group.*

Proof This result is a corollary of Theorem 3.31. □

Theorem 4.11 *Let $R = F[G]$, where F is a field; and G is a group. Assume H is a finite subnormal subgroup of G such that $F[H]$ is nilary.*
(i) If $H \neq \{1\}$, then $J(F[H])R \subseteq J(R)$ and $J(F[H])R$ is left and right essential in R.
(ii) If I is a nonnilpotent ideal of $F[H]$, then IR is right essential in R. In particular, ReR is left and right essential in R for all $0 \neq e = e^2 \in F[H]$.
(iii) If $\{e_i | 1 \leq i \leq m\} \subseteq F[H]$ is an orthogonal set of idempotents such that $e_1R, \cdots, e_mR$ represents a complete set of isomorphism classes of the principal indecomposable modules of R, then R is nilary.

Proof Note that $F[H]$ is a Frobenius algebra (hence a QF-ring) [10, pp. 442-443, Example 16.56].
(i) Using the subnormality of H, Proposition 4.3(iii) and an induction argument with [9, p.137, Excercise 8.5] and [12, p.467, Excercise 27], we obtain the result.
(ii) Since $F[H]$ is a QF-ring, [1, Theorem 2.2] yields that each nonnilpotent ideal I of $F[H]$ is left and right essential in $F[H]$. Now the result follows from the subnormality of H and [12, p 467, Excercise 27]
(iii) This part is a consequence of part (ii) and [1, Theorem 2.2].

□

For the ring $R = F[G]$ where F is a field and G is a finite solvable group, our next result provides a method for determining nilary subrings of R of the form $F[H]$ where H is a subnormal subgroup of G. The three examples following this result illustrate this method. Moreover, the examples are nilary group algebras which are not rings with a nilpotent prime ideal, hence neither prime nor local.

Theorem 4.12 *Let G be a finite solvable group which is not a p-group, p a prime, H_0 be a nontrivial subnormal p-subgroup of G, F be a field such that $char(F) = p$ and $R = F[G]$. Then there exists a composition series,*

$$\{1\} \trianglelefteq \cdots \trianglelefteq H_0 \trianglelefteq \cdots \trianglelefteq H_k \trianglelefteq \cdots \trianglelefteq H_m \trianglelefteq \cdots \trianglelefteq H_n = G$$

such that $R_i = F[H_i]$ and:
(i) $H_0, \cdots, H_{k-1}$ are p-groups;
(ii) R_i is a nilary local ring for all $i = 0, 1, \cdots, k-1$;
(iii) $|\frac{H_k}{H_{k-1}}| = q$, where q is a prime and $q \neq p$;
(iv) $\frac{F[H_k]}{\Delta(H_k, H_{k-1})} \cong F[\frac{H_k}{H_{k-1}}] \cong F[C_q] \cong \oplus_{j=1}^{t} F(j)$ where each $F(j)$ is a cyclotomic extension of F;
(v) there exists a complete set, $\{e_1, \cdots, e_t\}$, of orthogonal primitive idempotents of $F[H_k]$, so, $u.dim(R_k) = t$;
(vi) $J(R_k) = \Delta(H_k, H_{k-1})$, and R_k is a basic ring.

(vii) If $R_k e_j R_k$ is right essential in R_k for each e_j and each e_j is primitive in R_m for $j = 1, \cdots, t$ where $m \geq k$, then R_m is nilary.

Proof It is well known that the indicated composition series exists, see [14, p.80], and that it satisfies condition(i).

(ii) This part follows from Theorem 4.5.

(iii) This part is a property of the composition series [14, p.75].

(iv) The first isomorphism is due to [11, Corollary 3.3.5]. The second isomorphism follows from part (iii). For the third, see [11, p.144-145].

(v) From Proposition 3.6, $\Delta(H_k, H_{k-1})$ is nilpotent. Hence, the complete set of orthogonal primitive idempotents from the decomposition of $F[C_q]$ lift to $F[H_k]$, see [9, pp.319-321]. From [10, pp.442-443, Example 16.56], R_k is a symmetric algebra, hence a QF-ring. So each $e_f R_k$ is the injective hull of a minimal right ideal and $Soc(R_{k R_k})$ is right essential in R_k. Therefore, $u.dim(R_k) = t$.

(vi) Since $\Delta(H_k, H_{k-1})$ is nilpotent and $J(\frac{R_k}{\Delta(H_k, H_{k-1})}) = 0$ by part(iv), then $\Delta(H_k, H_{k-1}) = J(R_k)$, see [9, p.51,Proposition 4.6]. From [9, Proposition 25.10], R_k is a basic ring.

(vii) This part follows from Theorem 4.11(iii).

□

The following three examples of nilary nonlocal group algebras are applications of Theorem 4.12. Throughout these examples, we use the notation of Theorem 4.12, and σ denotes the permutation $(1\,2\,3)$. Recall from [1, Definition 2.4], R is antifull, if R has a nontrivial idempotent and for each $1 \neq e = e^2 \in R$, $ReR \neq R$.

Example 4.13 *Let $R = \mathbb{Z}_3[S_3]$. Then R is a nilary nonlocal basic antifull group algebra. To see this, observe that $H_0 =< \sigma >= \{1, \sigma, \sigma^2\}$, and $R_0 = \mathbb{Z}_3[C_3]$ is a nilary local ring by Theorem 4.5. Next, we have*

$$\frac{\mathbb{Z}_3[S_3]}{\Delta(S_3, H_0)} \cong \mathbb{Z}_3[C_2] \cong \frac{\mathbb{Z}_3[x]}{(x+1)} \oplus \frac{\mathbb{Z}_3[x]}{(x-1)} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$

Then $\{e, 1-e\}$ is a complete set of orthogonal primitive idempotents of R, where $e = \sigma + \sigma^2$; and $u.dim(R) = 2$. $J(R) = \Delta(S_3, H_0)$, and R is basic by [9, Proposition 25.10]. Since $ReR = eR \oplus (1-e)R \cap ReR$ and $(1-e)R \cap ReR \neq 0$, $ReR \leq^{ess} R_R$. Similarly, $R(1-e)R \leq^{ess} R_R$. So R is nilary. Alternatively, one can show that e is not central, so $1-e$ is not central; hence, R is indecomposable. By [1, Corollary 2.3], R is nilary. R is antifull from [1, Proposition 2.5].

Example 4.14 *Let $R = \mathbb{Z}_2[A_4]$. Then R is a nilary nonlocal basic antifull group algebra. To see this, observe that $H_0 = \{1, (12)(34), (13)(24), (14)(23)\} \cong K_4$, and $R_0 = \mathbb{Z}_2[K_4]$ is a nilary local ring by Theorem 4.5. Next,*

$$\frac{\mathbb{Z}_2[A_4]}{\Delta(A_4, H_0)} \cong \mathbb{Z}_2[C_3] \cong \frac{\mathbb{Z}_2[x]}{(x-1)} \oplus \frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$$

Then $\{e, 1-e\}$ is a complete set of orthogonal primitive idempotents of R where $e = \sigma + \sigma^2$; and $u.dim(R) = 2$. $J(R) = \Delta(A_4, H_0)$, and R is basic by [9, Proposition 25.10]. Again neither e nor $1-e$ is central in R. By [1, Corrolary 2.3], R is nilary, and R is antifull from [1, Proposition 2.5].

Example 4.15 *Let $R = \mathbb{Z}_2[S_4]$. Then, R is a nilary non-local group algebra. To see this, observe*

$$\{1\} \trianglelefteq \{1, (12)(34)\} \trianglelefteq H_0 \trianglelefteq H_1 \trianglelefteq H_2 = S_4$$

is a composition series where H_0 is as in Example 4.14, $H_1 = A_4$ and $H_2 = S_4$. Then,

$$\frac{\mathbb{Z}_2[S_4]}{\Delta(S_4, H_0)} \cong \mathbb{Z}_2[\frac{S_4}{H_0}] \cong \mathbb{Z}_2[S_3]$$

It can be shown by computer or hand calculations that $\{e, 1-e\}$ is a complete set of orthogonal primitive idempotents of $\mathbb{Z}_2[S_3]$, where $e = \sigma + \sigma^2$. Since $\Delta(S_4, H_0)$ is nilpotent by Proposition 3.6, $\{e, 1-e\}$ is a complete set of orthogonal primitive idempotents of $\mathbb{Z}_2[S_4]$. From Example 3.14, $R_1 = \mathbb{Z}_2[A_4]$ is nilary, so $R_1eR_1 \leq^{ess} R_{1R_1}$ and $R_1(1-e)R_1 \leq^{ess} R_{1R_1}$, from [1, Theorem 2.2]. By Theorem 4.12(vii), $R_2 = R$ is nilary and nonlocal.

Corollary 4.16 *Let F be a field with $char(F) = p$, and G be a finite group such that $G = C_2 \ltimes G_p$ where G_p is a p-group and C_2 is not normal in G. Then $R = F[G]$ is a nilary ring.*

Proof If $p = 2$ then G is a p-group. By Theorem 4.5, R is a nilary ring. Assume $p > 2$. Then G is solvable. By using the notation of Theorem 4.12, $O_p(G) = H_0 = H_{k-1} = G_p$, $H_1 = H_k = G$, $q = 2$. Then,

$$\frac{F[H_k]}{\Delta(H_k, H_{k-1})} = \frac{F[G]}{\Delta(G, Gp)} \cong F[C_2].$$

Hence, $u.dim(F(C_2)) \leq dim(F[C_2]) = 2$. So $F[G]$ has a complete set $\{e, 1-e\}$, of orthogonal primitive idempotents, by Theorem 4.12. Now, we claim that $F[G]$ is indecomposable. Indeed, since G has order $2p^n$ and G has a normal p-Sylow subgroup, then G is p-constrained and it is clear that $O_{p'}(G) = 1$. Therefore, by using [8, p.112, Proposition 1.12], we have that $F[G]$ is indecomposable. Since $F[G]$ is QF-ring, by our claim and by using [1, Corollary 2.3], we get that $F[G]$ is nilary. □

Note that for $p \neq 2$, R is neither local nor prime.

Corollary 4.17 *Let F be a field with $char(F) = p$, and G be the dihedral group $G = D_{p^n}$ for a positive integer n and a prime p. Then $F[G]$ is a nilary ring. In particular, the ring $F[S_3]$ is nilary, for any field F with $char(F) = 3$.*

Open Problem:

Characterize the nilary group algebras, $F[G]$, where $char(F) = p$ (a prime) in terms of properties of G and F. Note that the case for $char(F) = 0$ is included in Proposition 4.2.

Funding

No funding was received for conducting this study.

Conflict of interest

The authors have no relevant financial or nonfinancial interests to disclose.

Data

This paper has no associated data.

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