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Struwe compactness results for a critical $p-$Laplacian equation involving critical and subcritical Hardy potential on compact Riemannian manifolds

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Abstract: Let (M,g) be a compact Riemannian manifold. In this paper, we prove Struwe-type decomposition formulas for Palais-Smale sequences of functional energies corresponding to the equation:

$$\Delta_{g,p}u - \frac{h(x)}{(\rho_{x_o}(x))^s}|u|^{p-2}u = f(x)|u|^{p^*-2}u,$$

where $\Delta_{g,p}$ is the $p-$Laplacian operator, $p^* = \frac{np}{n-p}$, $0 < s \le p$, and $\rho_{x_o}(x)$ is a distance function to a fixed point x_o in M.

Key words: Riemannian manifolds, Yamabe equation, P-Laplacian, Sobolev exponent, Hardy potential, blow up analysis, bubbles

1. Introduction

Let (M,g) be a compact $n-$dimensional Riemannian manifold. Denote by $\mathrm{Inj_g}$ the injectivity radius of (M,g). Let x_o be a fixed point in M and define on M a distance function as follows:

$$\rho_{x_o}(x) = \begin{cases} \mathrm{dist_g}(x_o,x), & x \in B(x_o,\mathrm{Inj_g}), \\ \mathrm{Inj_g}, & x \in M \setminus B(x_o,\mathrm{Inj_g}). \end{cases} \tag{1.1}$$

For a real p such that $1 < p < n$, let us consider the Sobolev space $H_1^p(M)$ defined as the completion of $C^\infty(M)$ with respect to the norm:

$$||u||_{H_1^p(M)} = \int_M (|\nabla_g u|^p + |u|^p)dv_g,$$

where $|\nabla_g u|^2 = g(\nabla_g u, \nabla_g u)$. Let us also consider the $p-$Laplacian operator $\Delta_{g,p}$ that acts on functions $u \in H_1^p(M)$ as:

$$\Delta_{g,p}u = -\mathrm{div}(|\nabla_g u|^{p-2}\nabla_g u).$$

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Let h and f be two smooth functions on M. For a real s such that $0 < s \leq p$, let us consider the following singular elliptic quasilinear equation:

$$\Delta_{g,p}u - \frac{h(x)}{(\rho_{x_o}(x))^s}|u|^{p-2}u = f(x)|u|^{p^*-2}u, \qquad (E_s)$$

where $p^* = \frac{np}{n-p}$ is the critical exponent in the Sobolev inclusion $H_1^p(M) \subset L_{P^*}(M)$.
Equation (E_s), as one can immediately notice, is a generalization to the well-known geometric prescribed scalar curvature which corresponds to $s = 0$ and $p = 2$ and which has been largely studied starting from the middle of the last century. For a compendium on this equation and the related topic, the reader may refer to the books in [3] and [13].
For $s = 0$, we fall on the generalized prescribed scalar curvature equation which has been studied on compact manifolds in [9] and on complete noncompact manifolds in [5]. For $p = 2$, $0 < s \leq 2$, and $f \equiv 1$, we meet a singular Yamabe type equation to which existence of weak solutions has been studied in [16].
Now, define on $H_1^p(M)$ the energy functional:

$$J_{f,h,s}(u) = \frac{1}{p}\left(\int_M \left(|\nabla_g u|^p - \frac{h}{(\rho_{x_o})^s}|u|^p\right)dv_g\right) - \frac{1}{p^*}\int_M f|u|^{p^*}dv_g. \qquad (1.2)$$

This functional is of class C^2 on $H_1^p(M)$. Its Gâteau derivative at a point $v \in H_1^p(M)$ is given by:

$$\begin{aligned}(DJ_{f,h,s}u).v &= \int_M \left(|\nabla_g u|^{p-2}g(\nabla_g u, \nabla_g v) - \frac{h}{(\rho_{x_o})^s}|u|^{p-2}u.v\right)dv_g \\ &- \int_M f|u|^{p^*-2}u.v dv_g.\end{aligned}$$

A Palais-Smale sequence (P.S in short) of the functional $J_{f,h,s}$ at a level $\beta_s \in \mathbb{R}$, $0 < s \leq p$, is defined to be the sequence $u_m \in H_1^p(M)$ that satisfies $J_{f,h,s}(u_m) \to \beta_s$ and $(DJ_{f,h,s}u_m).v \to 0, \forall v \in H_1^p(M)$ as $m \to \infty$. To abbreviate, we denote β_p by β. A weak solution of (E_s), $0 < s \leq p$, is a function $u \in H_1^p(M)$ that satisfies $(DJ_{f,h,s}u).v = 0, \forall v \in H_1^p(M)$.
In this work, we aim at proving that a P.S sequence u_m of the functional $J_{f,h,s}$ is submitted to the well-known Struwe decomposition formulas [24]. Note that similar decomposition results, on Riemannian manifolds, are obtained in [10] in the case $s = 0$ and $p = 2$, in [21] in the case $s = 0$ and $1 < p < n$, and in [18] in the case $s = p = 2$. In the present work, we generalize those results to the case $s \in (0, p]$.
In proving the decomposition result, we distinguish the subcritical case $s \in (0, p)$ from the critical case $s = p$. More explicitly, we will prove that in case $s \in (0, p)$, a P.S sequence of the functional $J_{f,h,s}$ decomposes into the sum of a weak solution of (E_s), rescaled weak solutions of the Euclidean equation:

$$\Delta_{\xi,p}u = |u|^{p^*-2}u, \qquad (1.3)$$

where ξ is the Euclidean metric on $\mathbb{R}^n$, and a zero-converging term in $H_1^p(M)$. Note that existence and classification of positive solutions of (1.3) are studied in [7, 22, 27].
However, in case $s = p$, the singular term enters into the decomposition and leads to another term to be added.

This term is a rescaled solution of:

$$\Delta_{\xi,p}u-\frac{h(x_o)}{|x|^p}|u|^{p-2}u=f(x_o)|u|^{p^*-2}u, \tag{1.4}$$

whose existence of solutions is studied in [1]. Let $\delta>0$ be a constant and denote by η_δ a smooth cut-off function in $\mathbb{R}^n$ such that $0\le\eta_\delta\le 1$, $\eta_\delta(x)=1$ for $x\in B(\delta)$ and $\eta_\delta(x)=0$ for $x\in\mathbb{R}^n\setminus B(2\delta)$, $B(r)$ denotes the ball of center 0 and radius r. Let $y\in M$ and $0<\delta<\frac{\mathrm{Inj_g}}{2}$, we define the cut-off function $\eta_{\delta,y}$ by

$$\eta_{\delta,y}(x)=\eta_\delta(\exp_y^{-1}(x)),$$

where $\exp_y: B(\delta)\subset\mathbb{R}^n\to B(y,\delta)\subset M$ is the exponential map at point $y\in M$ which defines a diffeomorphism from $B(\delta)\subset\mathbb{R}^n$ to $B(y,\delta)$.

Let $D^{1,p}(\mathbb{R}^n)$ denote the Sobolev space defined as the completion of $C_o^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support in $\mathbb{R}^n$, with respect to the norm:

$$||u||^p_{D^{1,p}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}|\nabla u|^p dx.$$

Define on $D^{1,p}(\mathbb{R}^n)$ the following functionals:

$$\begin{aligned} E(u) &= \frac{1}{p}\int_{\mathbb{R}^n}|\nabla u|^p dx-\frac{1}{p^*}\int_{\mathbb{R}^n}|u|^{p^*}dx,\\ E_{f,h}(u) &= \frac{1}{p}\int_{\mathbb{R}^n}|\nabla u|^p dx-\frac{h(x_o)}{p}\int_{\mathbb{R}^n}\frac{|u|^p}{|x|^p}dx-\frac{f(x_o)}{p^*}\int_{\mathbb{R}^n}|u|^{p^*}dx. \end{aligned}$$

According to whether the exponent s is critical or subcritical, we state the following two main results

Theorem 1.1 *: Let (M,g) be a compact Riemannian manifold of dimension $n\ge 3$. Let f and h be two smooth functions on M. Let x_o be a point of M as defined in (1.1). Suppose that f satisfies $f(x_o)=\sup_M f(x), f(x)>0,\ x\in M$.*

Let u_m be a Palais-Smale sequence of the functional $J_{f,h,s}$ at level β_s, $0<s<p$. Then, there exist $k\in\mathbb{N}$, sequences $R_m^i\ge 0$, $R_m^i\underset{m\to\infty}{\to}0$, $k\in\mathbb{N}$, converging sequences of points in M, $x_m^i\underset{m\to\infty}{\to}x_o^i$, a solution $u\in H_1^p(M)$ of (E_s), $0<s<p$, nontrivial weak solutions $v_i\in D^{1,p}(\mathbb{R}^n)$ of (1.3) such that up to subsequence, for $0<s<p$, we have

$$u_m=u+\sum_{i=1}^k (R_m^i)^{\frac{p-n}{p}}\eta_\delta(\exp^{-1}_{x_m^i}(x))f(x_o^i)^{\frac{p-n}{p^2}}v_i((R_m^i)^{-1}\exp^{-1}_{x_m^i}(x))+\mathcal{W}_m,$$

$$\textit{with } \mathcal{W}_m\to 0 \textit{ in } H_1^p(M),$$

and

$$J_{f,h,s}(u_m)=J_{f,h,s}(u)+\sum_{i=1}^k f(x_o^i)^{\frac{p-n}{p}}E(v_i)+o(1).$$

Theorem 1.2 *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Let f and h be two smooth functions on M. Let x_o be a point of M as defined in (1.1). Suppose that f and h satisfy the following conditions*

1. $f(x_o) = \sup_M f(x), f(x) > 0, x \in M$,

2. $h(x_o) = \sup_M h(x)$ *and* $0 < h(x_o) < (\frac{n-p}{p})^p$.

Let u_m be a Palais-Smale sequence of the functional $J_{f,h,p}$ at level β. Then, there exist $k \in \mathbb{N}$, sequences $\mathcal{T}_m^i \geq 0$, $\mathcal{T}_m^i \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$ sequences $\tau_m^j \geq 0$, $\tau_m^j \underset{m\to\infty}{\to} 0$, $l \in \mathbb{N}$, converging sequences of points in M, $y_m^j \underset{m\to\infty}{\to} y_o^j \neq x_o$, a weak solution $u \in H_1^p(M)$ of (E_s), $s = p$, nontrivial weak solutions $v_i \in D^{1,p}(\mathbb{R}^n)$ of (1.4) and weak solutions $\nu_j \in D^{1,p}(\mathbb{R}^n)$ of (1.3) such that up to subsequence, we have:

$$\begin{aligned} u_m &= u + \sum_{i=1}^{k} (\mathcal{T}_m^i)^{\frac{p-n}{p}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_i((\mathcal{T}_m^i)^{-1} \exp_{x_o}^{-1}(x)) \\ &+ \sum_{j=1}^{l} (\tau_m^j)^{\frac{p-n}{p}} f(y_o^j)^{\frac{p-n}{p^2}} \eta_\delta(\exp_{y_m^j}^{-1}(x)) \nu_j((\tau_m^j)^{-1} \exp_{y_m^j}^{-1}(x)) + \mathcal{W}_m \end{aligned}$$

$$\text{with } \mathcal{W}_m \to 0 \text{ in } H_1^p(M)$$

and

$$J_{f,h,p}(u_m) = J_{f,h,p}(u) + \sum_{i=1}^{k} E_{f,h}(v_i) + \sum_{j=1}^{l} f(y_o^j)^{\frac{p-n}{p}} E(\nu_j) + o(1).$$

2. Preliminary results

In this section, we recall some known results that we need to achieve the proof of our main theorems.

2.1. Sobolev inequality

Denote by $K(n,p)$ the best constant in the Euclidean Sobolev inequality, that is for $u \in D^{1,p}(\mathbb{R}^n)$, there holds:

$$\int_{\mathbb{R}^n} |u|^{p^*} dx \leq K(n,p)^{p^*} \left(\int_{\mathbb{R}^n} |\nabla u|^p dx \right)^{\frac{p^*}{p}},$$

The value of $K(n,p)$ is calculated in Aubin [2] and Talenti [25] and is given by:

$$K(n,p) = \frac{p-1}{n-p} \left(\frac{n-p}{n(p-1)} \right)^{\frac{1}{p}} \left(\frac{\Gamma(n+1)}{\Gamma(\frac{n}{p})\Gamma(n+1-\frac{n}{p}) w_{n-1}} \right)^{\frac{1}{n}}.$$

On compact Riemannian manifold (M,g), in [2] the following Sobolev inequality is proven: for every $\varepsilon > 0$, there exists a positive constant $A_\varepsilon > 0$ such that for every $u \in H_1^p(M)$,

$$\int_M |u|^{p^*} dv_g \leq (K(n,p)^{p^*} + \varepsilon) \left(\int_M |\nabla_g u|^p dv_g \right)^{\frac{p^*}{p}} + A_\varepsilon \left(\int_M |u|^p dv_g \right)^{\frac{p^*}{p}}. \tag{2.1}$$

It is commonly known (see for example [13, 14]) that the inclusion $H_1^p(M) \subset L_q(M)$ is compact for $q < p^*$ and continuous for $q = p^*$.

2.2. Hardy inequality

Let ρ_{x_o} be the distance function defined by (1.1). Denote by $L_p(M, (\rho_{x_o})^s)$ the space of functions u such that $\frac{|u|^p}{(\rho_{x_o})^s}$ is integrable. This space, endowed with the norm $\int_M \frac{|u|^p}{(\rho_{x_o})^s} dv_g$, is a Banach space.
Now, for $u \in D^{1,p}(\mathbb{R}^n)$, the following Hardy inequality holds:

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq \left(\frac{p}{n-p}\right)^p \int_{\mathbb{R}^n} |\nabla u|^p dx. \tag{2.2}$$

This inequality has been extended to compact Riemannian manifolds in [16] as follows: for every $\varepsilon > 0$, there exists a positive constant $B_\varepsilon > 0$ such that for every $u \in H_1^p(M)$,

$$\int_M \frac{|u|^p}{(\rho_{x_o})^p} dv_g \leq \left((\frac{p}{n-p})^p + \varepsilon \right) \int_M |\nabla_g u|^p dv_g + B_\varepsilon \int_M |u|^p dv_g. \tag{2.3}$$

For a function $u \in H_1^p(M)$ with support included in $B(x_o, \delta)$, where $\delta < \mathrm{Inj_g}$, there holds:

$$\int_M \frac{|u|^p}{(\rho_{x_o})^p} dv_g \leq (K_\delta(n,p,-p))^p \int_M |\nabla_g u|^p dv_g, \tag{2.4}$$

with $K_\delta(n,p,-p) \to \frac{p}{n-p}$ as $\delta \to 0$.
In [16], it has been proven that the inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^p)$ is continuous and the inclusion $H_1^p(M) \subset L_p(M, (\rho_{x_o})^s)$, with $0 < s < p$, is compact.

3. Proof of the main theorems

In this section, we prove theorems 1.1 and 1.2. The proof goes through a series of lemmas:

Lemma 3.1 *Let u_m be a P.S sequence for $J_{f,h,s}$, $0 < s \leq p$, at level β_s. Suppose that the sequence u_m converges to a function u weakly in $H_1^p(M)$ and $L_p(M, \rho_{x_o}^p)$, strongly in $L_q(M), 1 \leq q < p^*$ and almost everywhere in M. Then, the function u is a weak solution of (E_s) and the sequence $v_m = u_m - u$ is a P.S sequence of $J_{f,h,s}$ such that $J_{f,h,s}(v_m) = \beta_s - J_{f,h,s}(u) + o(1)$.*

Proof Let u_m be a P.S. sequence for $J_{f,h,s}$ at a level β_s. As a first step in the proof of the lemma, we prove that the sequence u_m is bounded in $H_1^p(M)$.
First, on the one hand, we have:

$$J_{f,h,s}(u_m) - \frac{1}{p^*} DJ_{s,f,h}(u_m)u_m = \beta_s + o(1) + o(||u_m||_{H_1^p(M)}).$$

On the other hand, we have:

$$\begin{aligned} J_{f,h,s}(u_m) - \frac{1}{p^*} DJ_{f,h,s}(u_m)u_m &= \frac{1}{n} \int_M \left(|\nabla_g u_m|^p - \frac{h}{(\rho_{x_o})^s} |u_m|^p \right) dv_g \\ &= \frac{1}{n} \left(J_{f,h,s}(u_m) + \frac{1}{p^*} \int_M f\, |u_m|^{p^*} dv_g \right), \end{aligned}$$

then

$$\frac{1}{np^*}\int_M f\,|u_m|^{p^*}\,dv_g = \left(1-\frac{1}{n}\right)\beta_s + o(1) + o(||u_m||_{H_1^p(M)}).$$

Since f is supposed strictly positive on the compact manifold, we deduce that u_m is bounded in $L_{p^*}(M)$ and so in $L_p(M)$.
Moreover, we have:

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &= nJ_{f,h,s}(u_m) + \int_M \frac{h}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(\|u_m\|_{H_1^p(M)})\\ &= n\beta_s + \int_M \frac{h}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}).\end{aligned}$$

Let $\delta > 0$ be a small constant. Then we have:

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &= n\beta_s + \int_{B(x_o,\delta)} (\rho_{x_o})^{p-s}\frac{h}{(\rho_{x_o})^p}\,|u_m|^p\,dv_g\\ &+ \int_{M\setminus B(x_o,\delta)} \frac{h(x)}{(\rho_{x_o})^s}\,|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}),\end{aligned}$$

Since $p \geq s$, we get:

$$\begin{aligned}\int_M |\nabla_g u_m|^p\,dv_g &\leq n\beta_s + \delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|\int_{B(x_o,\delta)}\frac{|u_m|^p}{(\rho_{x_o})^p}dv_g\\ &+ \delta^{-s}\max_{x\in M}|h(x)|\int_{M\setminus B(x_o,\delta)}|u_m|^p\,dv_g + o(1) + o(\|u_m\|_{H_1^p(M)}).\end{aligned}$$

By Hardy inequality (2.4), since u_m is bounded in $L_p(M)$, we get that there is a positive constant C such that

$$\begin{aligned}&\left(1-\delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p\right)\int_M |\nabla_g u_m|^p\,dv_g \leq n\beta_s + C\\ &+ o(1) + o(\|u_m\|_{H_1^p(M)}).\end{aligned}$$

Now, for $p > s$, by choosing δ as small as

$$1-\delta^{p-s}\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p > 0,$$

we get that $\int_M |\nabla_g u_m|^p\,dv_g$ is bounded.
For $p = s$, since $\max_{B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)$ tends to $h(x_o)(\frac{p}{n-p})^p$ as $\delta \to 0$ and since by assumption $1-h(x_o)(\frac{p}{n-p})^p > 0$, then there exists $\delta_o > 0$ such that for all $\delta < \delta_o$, we have:

$$1-\max_{x\in B(x_o,\delta)}|h(x)|K_\delta(n,p,-p)^p > 0,$$

and hence $\int_M |\nabla_g u_m|^p\,dv_g$ is bounded which ends the proof of the fact that u_m is bounded in $H_1^p(M)$.
Now, suppose that the sequence u_m converges to a function u weakly in $H_1^p(M)$. We prove that for $\varphi \in H_1^p(M)$,

$(DJ_{f,h,s}(u_m)).\varphi$ converges to $(DJ_{f,h,s}(u).\varphi)$, that is, u is a weak solution of (E_s). First, since the sequence u_m converges to u almost everywhere in M, by basic integration theory (see for example [15] Lemma 4.8), we can conclude that the sequence $f|u_m|^{p^*-2}u_m$ converges to $f|u|^{p^*-2}u$ weakly in $L_{\frac{p^*}{p^*-1}}(M)$ and the sequence $h|u_m|^{p-2}u_m$ converges weakly to $h|u|^{p-2}u$ in $L_{\frac{p}{p-1}}(M,(\rho_{x_o})^s)$.

On the other hand, the same arguments as in the proof of Step 1.2 in [21] give that $\nabla_g u_m$ converges almost everywhere to ∇u in M and then $\int_M |\nabla_g u_m|^{p-2} g(\nabla_g u_m, \nabla_g \varphi) dv_g$ converges to $\int_M |\nabla_g u|^{p-2} g(\nabla_g u, \nabla_g \varphi) dv_g$. We conclude that u is a weak solution of (E_s).

Now, we prove that the sequence $v_m = u_m - u$ is a P.S sequence for $J_{f,h,s}$ at level $\beta_s - J_{f,h,s}(u)$. For $\varphi \in H_1^p(M)$, we write

$$
\begin{aligned}
& D(J_{s,f,h}(v_m)).\varphi = D(J_{s,f,h}(u_m)).\varphi - D(J_{s,f,h}(u)).\varphi \qquad (3.1)\\
+ & \int_M g(|\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v_m + \nabla_g u|^{p-2}(\nabla_g v_m + \nabla_g u) + \nabla_g u|^{p-2}\nabla_g u, \nabla_g \varphi) dv_g \\
- & \int_M \frac{h}{(\rho_{x_o})^s}(|v_m|^{p-2}v_m - |v_m + u|^{p-2}(v_m+u) + |u|^{p-2}u)\varphi dv_g \\
- & \int_M f(|v_m|^{p^*-2}v_m - |v_m + u|^{p^*-2}(v_m+u) + |u|^{p^*-2}u)\varphi dv_g
\end{aligned}
$$

We should recall the following inequality: for any vectors x and y in normed vector space and $p > 1$

$$
\| \|x+y\|^{p-2}(x+y) - \|x\|^{p-2}x - \|y\|^{p-2}y\| \leq C(||x||^{p-1-\theta}||y||^{\theta} + ||y||^{p-1-\theta}||x||^{\theta}), \qquad (3.2)
$$

where θ is a small constant that depends on p. We deduce from this inequality that:

$$
\begin{aligned}
& \int_M g(|\nabla_g v_m|^{p-2}\nabla_g v_m - |\nabla_g v_m + \nabla_g u|^{p-2}(\nabla_g v_m + \nabla_g u) + \nabla_g u|^{p-2}\nabla_g u, \nabla_g \varphi) dv_g \\
\leq\ & C \int_M \left(|\nabla_g v_m|^{p-1-\theta} |\nabla_g u|^{\theta} + |\nabla_g v_m|^{\theta} |\nabla_g u|^{p-1-\theta} \right) |\nabla_g \varphi| dv_g \\
\leq\ & C \|\nabla_g \varphi\|_{L_p(M)} \left[\left(\int_M |\nabla_g v_m|^{\frac{p(p-1-\theta)}{p-1}} |\nabla_g u|^{\frac{p\theta}{p-1}} dv_g \right)^{\frac{p-1}{p}} \right. \\
+ & \left. \left(\int_M |\nabla_g v_m|^{\frac{p\theta}{p-1}} |\nabla_g u|^{\frac{p(p-1-\theta)}{p-1}} dv_g \right)^{\frac{p-1}{p}} \right].
\end{aligned}
$$

Now, the sequence $|\nabla_g v_m|^{p\frac{p-1-\theta}{p-1}}$ is bounded in $L_{\frac{p-1}{p-1-\theta}}(M)$ and converges almost everywhere to 0 in M. Then, it converges weakly to 0 in $L_{\frac{p-1}{p-1-\theta}}(M)$, that is $\int_M |\nabla_g v_m|^{p\frac{p-1-\theta}{p-1}} \varphi dv_g \to 0, \forall \varphi \in L_{\frac{p-1}{\theta}}(M)$. Since $|\nabla_g u|^{p\frac{\theta}{p-1}}$ belongs to $L_{\frac{p-1}{\theta}}(M)$, we get:

$$
\int_M |\nabla_g v_m|^{\frac{p(p-1-\theta)}{p-1}} |\nabla_g u|^{\frac{p\theta}{p-1}} dv_g \to 0.
$$

By similar arguments, we get also:

$$
\int_M |\nabla_g v_m|^{\frac{p\theta}{p-1}} |\nabla_g u|^{\frac{p(p-1-\theta)}{p-1}} dv_g \to 0,
$$

together with the second and the third integral in (3.1) tend to zero. Hence, $(DJ_{f,h,s}(v_m)).\varphi \to 0, \forall \varphi \in H_1^p(M)$. Finally, to prove that $J_{f,h,s}(v_m)$ tends to $\beta_s - J_{f,h,s}(u)$, we just apply the Brezis-Lieb lemma (see for example [15], lemma 4.6) to the sequences u_m and $\nabla_g u_m$. In fact, since u_m converges to u a.e and $\nabla_g u_m$ converges to $\nabla_g u$ a.e. in M, and since ∇_g is bounded in $L_p(M)$, u_m is bounded in $L_{p^*}(M)$, we get by the Brezis-Lieb lemma that:

$$\int_M |\nabla_g u|^p dv_g = \lim_{m\to\infty}\left(\int_M |\nabla_g u_m|^p dv_g - \int_M |\nabla_g (u_m-u)|^p dv_g\right),$$

and

$$\int_M f|u|^{p^*} dv_g = \lim_{m\to\infty}\left(\int_M f|u_m|^{p^*} dv_g - \int_M f|u_m-u|^{p^*} dv_g\right)$$

On the other hand, by Hardy inequality (2.3), we have:

$$\int_M \frac{|u_m|^p}{(\rho_{x_o})^s} dv_g \le Diam(M)^{p-s}\int_M \frac{|u_m|^p}{(\rho_{x_o})^p} dv_g \le C\|u_m\|_{H_1^p(M)},$$

which means that the sequence u_m is also bounded in $L_p(M,(\rho_{x_o})^s)$ and then we get by the Brezis-Lieb lemma that:

$$\int_M \frac{h}{(\rho_{x_o})^s}|u|^p dv_g = \lim_{m\to\infty}\left(\int_M \frac{h}{(\rho_{x_o})^s}|u_m|^p dv_g - \int_M \frac{h}{(\rho_{x_o})^s}|u_m-u|^p dv_g\right)$$

which gives that:

$$J_{f,h,s}(v_m) = \beta_s - J_{f,h,s}(u) + o(1).$$

□

Lemma 3.2 *Suppose that* $\sup_M f > 0$ *and* $1 - h(x_o)(\frac{p}{n-p})^p > 0$. *Let* v_m *be a P.S sequence of* $J_{f,h,s}$ *at level* β_s, $0 < s \le p$, *that converges weakly to* 0 *in* $H_1^p(M)$. *If*

$$\beta_s < \beta^* = \begin{cases} \dfrac{1}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, & \text{if } s < p \\ \dfrac{(1-h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K(n,p)^n}, & \text{if } s = p, \end{cases}$$

then v_m *converges strongly to* 0 *in* $H_1^p(M)$.

Proof First, we write:

$$\begin{aligned} DJ_{f,h,s}(v_m).v_m &= o(\|v_m\|_{H_1^p(M)}) \\ &= \int_M (|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p) dv_g - \int_M f\,|v_m|^{p^*} dv_g, \end{aligned}$$

then

$$\beta_s = \frac{1}{n}\int_M \left(|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p\right) dv_g + o(1) = \frac{1}{n}\int_M f\,|v_m|^{p^*} dv_g + o(1) \tag{3.3}$$

This implies that $\beta_s \geq 0$. Moreover, let $\delta > 0$ be a small constant, we have:

$$\int_M (|\nabla_g v_m|^p - \frac{h}{(\rho_{x_o})^s}|v_m|^p)dv_g = \int_M |\nabla_g v_m|^p \, dv_g - \int_{B(x_o,\delta)} \frac{h}{(\rho_{x_o})^s}|v_m|^p \, dv_g$$
$$- \int_{M\setminus B(x_o,\delta)} \frac{h}{(\rho_{x_o})^s}|v_m|^p \, dv_g$$
$$\geq \int_M |\nabla_g v_m|^p \, dv_g - \max_{x\in B(x_o,\delta)} |h(x)|\delta^{p-s} \int_{B(x_o,\delta)} \frac{|v_m|^p}{(\rho_{x_o})^p} dv_g$$
$$- \delta^{-s} \max_{x\in M} |h(x)| \int_{M\setminus B(x_o,\delta)} |v_m|^p \, dv_g$$

Now, the sequence v_m is bounded in $L_p(M)$ and $L_p(M, (\rho_{x_o})^p)$, we have then:
For $0 < s < p$, by letting δ go to 0, we get from (3.3):

$$\int_M |\nabla_g v_m|^p \, dv_g \leq n\beta_s + o(1). \tag{3.4}$$

For $s = p$, by letting δ go to 0, we get from (3.3) together with Hardy inequality (2.4):

$$\int_M |\nabla_g v_m|^p \, dv_g \leq \frac{n\beta_s}{1 - h(x_o)(\frac{p}{n-p})^p} + o(1), \tag{3.5}$$

On the other hand, by Sobolev inequality, we get also by (3.3) that for $0 < s \leq p$,

$$\int_M |\nabla_g v_m|^p \, dv_g \geq \Big(\frac{n\beta_s}{\sup_M f(K(n,p)+\varepsilon)^{p^*}}\Big)^{\frac{p}{p^*}} + o(1) \tag{3.6}$$

Now, suppose by contradiction that $\beta_s > 0$. Then, after letting m go to ∞, inequalities (3.4), (3.5), and (3.6) give:

$$\beta_s \geq \frac{1}{n(\sup_M f)^{\frac{n-p}{p}}(K(n,p)+\varepsilon)^n}, \text{ for } 0 < s < p,$$

and

$$\beta_s \geq \frac{(1-(h(x_o)K^p(n,p,-p))^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}}(K(n,p))^n}, \text{ for } s = p.$$

Both cases present a patent contradiction with the hypothesis of the lemma. Hence, under the assumption of the lemma, $\beta_s = 0$ and thus $v_m \to 0$ in $H_1^p(M)$. □

Now, we divide the proof of the main theorems into two parts according to whether $0 < s < p$ or $s = p$.

3.1. The subcritical Hardy potential

Lemma 3.3 *Let v_m be a P.S sequence of $J_{f,h,s}$, with $0 < s < p$, at level β_s that converges weakly and not strongly to 0 in $H_1^p(M)$. Then, there exists a converging sequence of points $x_m \to x^o$ in M, a sequence of positive reals $R_m \to 0$ as $m \to \infty$ and nontrivial weak solution $v \in D^{1,p}(\mathbb{R}^n)$ of*

$$\Delta_{\xi,p} v = f(x^o)|v|^{p^*-2}v, \tag{3.7}$$

such that the subsequence

$$w_m(x) = v_m(x) - R_m^{\frac{p-n}{p}} \eta_\delta(\exp_{x_m}^{-1}(x)) v(R_m^{-1} \exp_{x_m}^{-1}(x)),$$

where $0 < \delta < \frac{\mathrm{Inj_g}}{2}$, *admits a subsequence* w_m *that is a P-S sequence of* $J_{f,h,s}$, *with* $0 < s < p$, *at level* $J_{f,h,s}(w_m) = \beta_s - (f(x^o))^{\frac{p-n}{p}} E(u)$, *with* u *is a nontrivial weak solution of* (1.3), *and that converges to* 0 *weakly in* $H_1^p(M)$.

Proof Let v_m be a P.S sequence of $J_{f,h,s}$ at level β_s that converges to 0 weakly and not strongly in $H_1^p(M)$. Then, up to a subsequence, we can assume that v_m converges strongly to 0 in $L_p(M)$. For $t > 0$, we let

$$F_m(t) = \max_{x \in M} \int_{B(x,t)} |\nabla_g v_m| dv_g$$

For t_o small, by (3.6), there exists $z_o \in M$ and $\gamma_o > 0$ such that

$$\int_{B(z_o,t_o)} |\nabla_g v_m| dv_g \geq \gamma_o.$$

Since F_m is continuous in t, we get that for each $\gamma \in (0, \gamma_o)$ and for each $m > 0$, we can find a point x_m and a constant $r_m \in (0, t_o)$ such that

$$\int_{B(x_m,r_m)} |\nabla_g v_m|^p \, dv_g = \gamma \tag{3.8}$$

Let $0 < r_o < \frac{Inj_g}{2}$ be such that there exists a positive constant $C_o \in [1,2]$ such that for all $x \in M$ and $y, z \in B(r_o) \subset \mathbb{R}^n$ the following inequality holds

$$dist_g(\exp_x(y), \exp_x(z)) \leq C_o |y - z|. \tag{3.9}$$

Let $0 < R_m < 1$ and $x \in B(R_m^{-1} \delta_g)$. Define

$$\begin{aligned} \hat{v}_m(x) &= R_m^{\frac{n-p}{p}} v_m(\exp_{x_m}(R_m x)), \; x \in \mathbb{R}^n \\ \hat{g}_m(x) &= \exp_{x_m}^* g(R_m x) \end{aligned}$$

Then, we have

$$|\nabla_{\hat{g}_m} \hat{v}_m|^p(x) = R_m^n |\nabla_g v_m|^p(\exp_{x_m}(R_m x)). \tag{3.10}$$

Thus, it follows that if $z \in \mathbb{R}^n$ is such that $|z| + r < \mathrm{Inj_g} R_m^{-1}$, then we have:

$$\int_{B(z,r)} |\nabla_{\hat{g}_m} \hat{v}|^p dv_{\hat{g}_m} = \int_{\exp_{x_m}(R_m B(z,r))} |\nabla_g v_m|^p dv_g. \tag{3.11}$$

Moreover, for $|z| + r < r_o R_m^{-1}$, by using (3.9) we have:

$$\exp_{x_m}(R_m B(z,r)) \subset B_{\exp_{x_m}(R_m z)}(r C_o R_m). \tag{3.12}$$

Since for $y \in B(rC_oR_m) \subset B(\mathrm{Inj_g})$, we have $dist_g(x_m, \exp_{x_m}(R_m y)) = R_m|y|$, and thus

$$\exp_{x_m}(B(rC_oR_m)) = B(x_m, rC_oR_m). \tag{3.13}$$

Now, for $r \in (0, r_o)$ take $R_m = \frac{r_m}{rC_o}$, where r_m is as defined above. By (3.10), (3.11), and (3.12), we get:

$$\int_{B(z,r)} |\nabla_{\hat{g}_m} \hat{v}|^p dv_{\hat{g}_m} \le \gamma, \tag{3.14}$$

and

$$\int_{B(rC_o)} |\nabla_{\hat{g}_m} \hat{v}|^p dv_{\hat{g}_m} = \gamma, \tag{3.15}$$

Let $\delta \in (0, \mathrm{Inj_g})$ and $u \in D^{1,p}(\mathbb{R}^n)$ with support included in $B(\delta R^{-1})$, where $0 < R \le 1$ is a constant. There exists a constant C_1 such that if $\hat{g}(x) = \exp_y^*(g(Rx))$, then

$$\frac{1}{C_1}\int_{\mathbb{R}^n} |\nabla u|^p \, dx \le \int_{\mathbb{R}^n} |\nabla_{\hat{g}} u|^p \, dv_{\hat{g}} \le C_1 \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{3.16}$$

Without loss of generality, we can also assume that for all $u \in L_p(\mathbb{R}^n)$ with support included in $B(\delta R^{-1})$, we have:

$$\frac{1}{C_1}\int_{\mathbb{R}^n} |u|^p \, dx \le \int_{\mathbb{R}^n} |u|^p \, dv_{\hat{g}} \le C_1 \int_{\mathbb{R}^n} |u|^p \, dx. \tag{3.17}$$

Now, consider a cut-off function $\eta \in C_o(\mathbb{R}^n)$ such that

$$0 \le \eta \le 1, \eta(x) = 1, x \in B(\frac{1}{4}) \text{ and } \eta(x) = 0, x \in \mathbb{R}^n \setminus B(\frac{3}{4}). \tag{3.18}$$

Put $\hat{\eta}_m(x) = \eta(\delta^{-1}R_m x)$, where $\delta \in (0, In_g)$. We get that there exists a positive constant C such that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^p \, dv_{\hat{g}_m} = \int_{B(\frac{3\delta R_m^{-1}}{4})} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^p \, dv_{\hat{g}_m} \\
\le\ & 2^{p-1} \int_{B(\frac{3\delta R_m^{-1}}{4}))} \left(|\eta(\delta^{-1}R_m x)|^p \, |\nabla_{\hat{g}_m} \hat{v}_m|^p + \delta^{-p} R_m^p |(\nabla_{\hat{g}_m}\eta)(\delta^{-1}R_m x)|^p \, |\hat{v}_m|^p\right) dv_{\hat{g}_m} \\
=\ & 2^{p-1} \int_{B(x_m, \frac{3\delta}{4})} \left(|\eta(\delta^{-1}\exp_{x_m}^{-1}(x))|^p \, |\nabla_g v_m|^p + |(\nabla_g \eta)(\delta^{-1}\exp_{x_m}^{-1}(x))|^p \, |v_m|^p\right) dv_g \\
\le\ & C \int_{B(x_m, \frac{3\delta}{4})} (|\nabla_g v_m|^p + |v_m|^p) \, dv_g,
\end{aligned}$$

since the sequence is bounded in $H_1^p(M)$, this implies by (3.16) that the sequence $\hat{\eta}_m \hat{v}_m$ is bounded in $D^{1,p}(\mathbb{R}^n)$ and thus, it converges weakly in $D^{1,p}(\mathbb{R}^n)$ and almost everywhere in $\mathbb{R}^n$ to some function $v \in D^{1,p}(\mathbb{R}^n)$.
Now, we divide the remaining of proof of the lemma into several steps.

Step 1

For γ small and $s \in (0,p)$, the sequence $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(C_o r))$.

Proof Let $a \in \mathbb{R}^n$ and $\mu \in [r, 2r]$. Set $\mathcal{A} = B(a,3r) \setminus B(a,\mu)$. In [21] (see also [10]), it has been proven that there exists a sequence $z_m \in H_1^p(\mathcal{A})$ that converges strongly to 0 in $H_1^p(\mathcal{A})$ and that z_m is solution of

$$\begin{cases} \Delta_{\xi,p} z_m = 0 \text{ in } \mathcal{A}, \\ z_m - \varphi_m - \varphi_m^o \in D^{1,p}(\mathcal{A}), \end{cases} \tag{3.19}$$

where $\varphi_m = \hat{\eta}_m \hat{v}_m - v$ in $B(a, \mu+\varepsilon)$, $\varphi_m = 0$ in $\mathbb{R}^n \setminus B(a, 3\mu - \varepsilon)$ and φ_m^o is such that $\|\varphi_m + \varphi_m^o\|_{H_1^p(\mathcal{A})} \leq C\|\varphi_m\|_{H^p_{\frac{p-1}{p}}(\partial\mathcal{A})}$. We let $\hat{\psi}_m \in D^{1,p}(\mathbb{R}^n)$ be the sequence

$$\begin{cases} \hat{\psi}_m = \hat{\eta}_m \hat{v}_m - v & \text{in } \overline{B}(a,\mu), \\ \hat{\psi}_m = z_m & \text{in } \overline{B}(a,3r) \setminus B(a,\mu), \\ \hat{\psi}_m = 0 & \text{in } \mathbb{R}^n \setminus B(a,3r). \end{cases}$$

For $r < \frac{\delta}{24}$, consider the rescaling sequence ψ_m of $\hat{\psi}_m$

$$\begin{cases} \psi_m(x) = R_m^{\frac{p-n}{p}} \hat{\psi}_m(R_m^{-1} \exp_{x_m}^{-1}(x)), & \text{if } x < d_g(x_m, 6r), \\ \psi_m(x) = 0, & \text{otherwise}. \end{cases}$$

Let η be the cut-off function considered above. Then, $\eta(\delta^{-1}\exp_{x_m}^{-1}(x)) = 1$ for x such that $d_g(x_m, x) < 6r$. Put $\hat{\eta}(x) = \eta(\delta^{-1}\exp_{x_m}^{-1}(x)) = 1$, then if $|a| < 3r$, we have:

$$\begin{aligned} DJ_{f,h,s}(v_m).\psi_m &= DJ_{f,h,s}(\eta(\delta^{-1}\exp_{x_m}^{-1}(x))v_m).\psi_m \\ &= \int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^{p-2} \hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m), \nabla_{\hat{g}_m}\hat{\psi}_m\right) dv_{\hat{g}_m} \\ &- R_m^{p-s} \int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m(x))}{|x|_\xi^s} |\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ &- \int_{B(a,3r)} f(\exp_{x_m}(R_m(x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}. \end{aligned}$$

It is clear that the sequence $\hat{\psi}_m$ is bounded in $D^{1,p}(\mathbb{R}^n)$ and we have that $||\psi_m||_{H_1^p(M)} \leq C||\hat{\psi}_m||_{D^{1,p}(\mathbb{R}^n)}$. Then, the sequence ψ_m is bounded in $H_1^p(M)$ and since v_m is a P-S sequence of $J_{f,h,s}$, we get:

$$\begin{aligned} o(1) = &\int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^{p-2} \hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m), \nabla_{\hat{g}_m}\hat{\psi}_m\right) dv_{\hat{g}_m} \\ &- R_m^{p-s} \int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m(x))}{|x|_\xi^s} |\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m} \\ &- \int_{B(a,3r)} f(\exp_{x_m}(R_m(x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}. \end{aligned} \tag{3.20}$$

By the same arguments as in [21], we can have:

$$\int_{B(a,3r)} |\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m)|^{p-2}\hat{g}\left(\nabla_{\hat{g}_m}(\hat{\eta}_m\hat{v}_m),\nabla_{\hat{g}_m}\hat{\psi}_m\right)dv_{\hat{g}_m}=\int_{\mathbb{R}^n}|\nabla_{\hat{g}_m}\hat{\psi}_m|^p dv_{\hat{g}_m}+o(1),$$

and

$$\begin{aligned}&\int_{B(a,3r)} f(\exp_{x_m}(R_m x))|\hat{\eta}_m\hat{v}_m|^{p^*-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ =\;&\int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x))|\hat{\psi}_m|^{p^*}dv_{\hat{g}_m}+o(1).\end{aligned}$$

Rather, we prove that:

$$\begin{aligned}&\int_{B(a,3r)}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ =\;&\int_{\mathbb{R}^n}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}|\hat{\psi}_m|^p dv_{\hat{g}_m}+o(1)\end{aligned}\tag{3.21}$$

We distinguish two cases, $0\in B(a,\mu)$ and $0\notin B(a,\mu)$. If $0\notin B(a,\mu)$, then there exists $\varrho>$ such that $B(\varrho)\cap B(a,\mu)=\emptyset$. Then, by using convexity, Hölder inequality and inequality (3.2), we get:

$$\begin{aligned}&|\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m+v|^{p-2}(\hat{\psi}_m+v)-|\hat{\psi}_m|^{p-2}\hat{\psi}_m-|v|^{p-2}v\right]\hat{\psi}_m dv_{\hat{g}_m}|\\ \le\;& C\varrho^{-s}\sup|h|\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\\ &\left(\int_{B(a,\mu)}\left[|\;|\hat{\psi}_m+v|^{p-2}(\hat{\psi}_m+v)-|\hat{\psi}_m|^{p-2}\hat{\psi}_m-|v|^{p-2}v\;|\right]^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\\ \le\;& C'\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\left(\int_{B(a,\mu)}\left[|\;|\hat{\psi}_m|^{p-1-\theta}|v|^\theta-\hat{\psi}_m|^\theta|v|^{p-1-\theta}\;|\right]^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\\ \le\;& C''\|\hat{\psi}_m\|_{L_p(\mathbb{R}^n)}\,[\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p(p-1-\theta)}{p-1}}|v|^{\frac{p\theta}{p-1}}dx\right)^{\frac{p-1}{p}}\\ +\;&\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p\theta}{p-1}}|v|^{\frac{p(p-1-\theta)}{p-1}}dx\right)^{\frac{p-1}{p}}\;]\end{aligned}$$

Since $\hat{\psi}_m$ converges to 0 almost everywhere and is bounded in $L_p(\mathbb{R}^n)$, we get that $|\hat{\psi}_m|^{\frac{p(p-\theta-1)}{p-1}}$ and $|\hat{\psi}_m|^{\frac{p\theta}{p-1}}$ converge almost everywhere to 0 and are bounded respectively in $L_{\frac{p-1}{p-1-\theta}}(\mathbb{R}^n)$ and $L_{\frac{p-1}{\theta}}(\mathbb{R}^n)$. We get then

$$\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p(p-1-\theta)}{p-1}}|v|^{\frac{p\theta}{p-1}}dx\right)^{\frac{p-1}{p}}+\left(\int_{B(a,\mu)}|\hat{\psi}_m|^{\frac{p\theta}{p-1}}|v|^{\frac{p(p-1-\theta)}{p-1}}dx\right)^{\frac{p-1}{p}}=o(1).$$

Hence, we get:

$$\begin{aligned}&\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ =&\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m|^p+|v|^{p-2}v\hat{\psi}_m\right]dv_{\hat{g}_m}+o(1).\end{aligned}$$

Now, if $0\in B(a,\mu)$, let $\varrho'>0$ be such that $B(\varrho')\subset B(a,\mu)$. Then, as above, we have:

$$\begin{aligned}&\int_{B(a,\mu)\backslash B(\varrho')}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ =&\int_{B(a,\mu)\backslash B(\varrho')}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m|^p+|v|^{p-2}v\hat{\psi}_m\right]dv_{\hat{g}_m}+o(1).\end{aligned}$$

Moreover, by Hölder inequality, we have:

$$\begin{aligned}&\int_{B(\varrho')}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ \leq\;& C\sup|h|\left(\int_{B(\varrho')}\frac{|\hat{\eta}_m\hat{v}_m|^p}{|x|_\xi^s}dx\right)^{\frac{1}{p}}\left(\int_{B(\varrho')}\frac{|\hat{\psi}_m|^p}{|x|_\xi^s}dx\right)^{1-\frac{1}{p}}\\ \leq\;& C\sup|h|{\varrho'}^{\frac{p-s}{p}}\left(\int_{B(\varrho')}\frac{|\hat{\eta}_m\hat{v}_m|^p}{|x|_\xi^p}dx\right)^{\frac{1}{p}}\left(\int_{B(\varrho')}\frac{|\hat{\psi}_m|^p}{|x|_\xi^s}dx\right)^{1-\frac{1}{p}}.\end{aligned}$$

Now, by Hardy inequality (2.2), $\left(\int_{B(\varrho')}\frac{|\hat{\eta}_m\hat{v}_m|^p}{|x|_\xi^p}dx\right)^{\frac{1}{p}}$ is bounded. Since $\hat{\psi}_m$ converges to 0 strongly in $L_p(B(\varrho'),|x|^s)$, $0<s<p$, then

$$\int_{B(\varrho')}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}=o(1).$$

Thus, in both cases, we have:

$$\begin{aligned}&\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s}|\hat{\eta}_m\hat{v}_m|^{p-2}(\hat{\eta}_m\hat{v}_m)\hat{\psi}_m dv_{\hat{g}_m}\\ =&\int_{B(a,\mu)}\frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s}\left[|\hat{\psi}_m|^p+|v|^{p-2}v\hat{\psi}_m\right]dv_{\hat{g}_m}+o(1).\end{aligned}$$

Now, using the fact that $\hat{\psi}_m$ converges to 0 strongly in $D^{1,p}(\mathcal{A})$ and weakly to 0 in $D^{1,p}(\mathbb{R}^n)$, we get:

$$\begin{aligned}
&\int_{B(a,3r)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
= &\int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\eta}_m \hat{v}_m|^{p-2} (\hat{\eta}_m \hat{v}_m) \hat{\psi}_m dv_{\hat{g}_m} \\
= &\int_{B(a,\mu)} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} \left[|\hat{\psi}_m|^p + |v|^{p-2} v \hat{\psi}_m \right] dv_{\hat{g}_m} + o(1) \\
= &\int_{\mathbb{R}^n} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\psi}_m|^p dv_{\hat{g}_m} + o(1).
\end{aligned}$$

We deduce that:

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} - R_m^{p-s} \int_{\mathbb{R}^n} \frac{h(\exp_{x_m}(R_m x))}{|x|_\xi^s} |\hat{\psi}_m|^p dv_{\hat{g}_m} \\
= &\int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x)) |\hat{\psi}_m|^{p^*} dv_{\hat{g}_m} + o(1).
\end{aligned}$$

Since the sequence $\hat{\psi}_m$ converges strongly to 0 in $L_p(B(a, 3\mu), |x|^s)$, $s < p$ and since $R_m \leq 1$, we get that:

$$R_m^{p-s} \left| \int_{\mathbb{R}^n} \frac{h(\exp_{x_m}(R_m(x)))}{|x|_\xi^s} |\hat{\psi}_m|^p dv_{\hat{g}_m} \right| \leq \sup hC \int_{\mathbb{R}^n} \frac{|\hat{\psi}_m|^p}{|x|_\xi^s} dx = o(1).$$

We get then:

$$\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} = \int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x)) |\hat{\psi}_m|^{p^*} dv_{\hat{g}_m} + o(1). \tag{3.22}$$

By the same way as in [21], we can prove that for $|a| + 3r < r_o$:

$$\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} \leq N\gamma + o(1), \tag{3.23}$$

where $N \in \mathbb{N}$ is such that $B(a,\mu) \subset B(a, 2r) \subset \bigcup_{1 \leq i \leq N} B(x_i, r)$, with $x_i \in B(a, 2r)$. We get then by the Sobolev inequality that:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x)) |\hat{\psi}_m|^{p^*} dv_{\hat{g}_m} &\leq \sup_M f C_1 \int_{\mathbb{R}^n} |\hat{\psi}_m|^{p^*} dx \\
&\leq \sup_M f C_1^{\frac{p^*}{p}+1} K(n,p)^{p^*} \left(\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} \right)^{\frac{p^*}{p}}.
\end{aligned}$$

Then, by (3.22) and (3.23), we get:

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} &\leq \sup_M f C_1 \int_{\mathbb{R}^n} |\hat{\psi}_m|^{p^*} dx \\
&\leq \sup_M f C_1^{\frac{p^*}{p}+1} K(n,p)^{p^*} (N\gamma + o(1))^{\frac{p^*}{p}-1} \int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m}.
\end{aligned}$$

By taking γ such that

$$\sup_M f C_1^{\frac{p^*}{p}+1} K(n,p)^{p^*} (N\gamma)^{\frac{p^*}{p}-1} < 1, \tag{3.24}$$

we get:

$$\int_{\mathbb{R}^n} |\nabla_{\hat{g}_m} \hat{\psi}_m|^p dv_{\hat{g}_m} = o(1),$$

which means that $\hat{\psi}_m$ converges strongly in $D^{1,p}(\mathbb{R}^n)$. Thus, since $r \le \mu$, we get that $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$. This strong convergence holds as soon as μ and r are small enough, $|a| < 3r$ and $|a| + 3r < \min(r_o, \delta)$. Then, let μ be small enough such that condition (3.24), then $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$ for all $|a| < 2r$. Since $C_o \le 2$, $B(C_o r)$ can be covered by N balls $B(a,r)$, with $a \in B(2r)$ and thus $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(C_o r))$. □

Step 2

For any $R > 0$ and $s \in (0,p)$, the sequence $\hat{v}_m$ converges strongly to v in $H_1^p(B(R))$ and v is a nontrivial solution of (3.7).

Proof First, to prove that $v \neq 0$, we use step 1 above. Take r small enough so that $\hat{\eta}_m = 1$ on $B(C_o r)$, we then obtain

$$\begin{aligned} \gamma &= \int_{B(C_o r)} |\nabla_{\hat{g}_m} (\hat{\eta}_m \hat{v}_m)|^p dv_{\hat{g}_m} \\ &\le \int_{B(C_o r)} |\nabla v|^p dx + o(1). \end{aligned}$$

Hence, $v \neq 0$. As consequence, we get that $R_m \to 0$. In fact, if $R_m \to R > 0$. Since v_m converges weakly to 0, we get that $\hat{v}_m$ converges weakly to 0 in $H_1^p(B(C_o r))$ since $v \neq 0$ and $(\hat{\eta}_m \hat{v}_m)$ converges strongly to v in $H_1^p(B(C_o r))$, we get a contradiction. Thus, $R_m \to 0$.

Now, let $R > 1$. For m is large, $R < R_m^{-1}$ and (3.14) and (3.15) are satisfied for $z + r < R r_o$. Thus, as one can easily check from the proof of Step 1, $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$ for $|a| + 3r < rR$ and $|a| \le 3r(2R-1)$. In particular, $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(a,r))$ for $|a| < 2rR$. Hence, $\hat{\eta}_m \hat{v}_m$ converges strongly to v in $H_1^p(B(2rR))$. Since for m is large, $\hat{\eta}_m = 1$ and R is arbitrary chosen, we get that $\hat{v}_m$ converges strongly to v in $H_1^p(B(R))$.

Now, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ with compact support included in a ball $B(R), R > 0$. For m is large, define on M the sequence φ_m as:

$$\varphi_m(x) = R_m^{\frac{p-n}{p}} \varphi(R_m^{-1}(\exp_{x_m}^{-1}(x))).$$

Then, we have:

$$\int_M |\nabla_g v_m|^{p-2} g(\nabla_g v_m, \nabla_g \varphi_m) dv_g = \int_{\mathbb{R}^n} |\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m)|^{p-2} \hat{g}(\nabla_{\hat{g}_m}(\hat{\eta}_m \hat{v}_m), \nabla_{\hat{g}_m}\varphi) dv_{\hat{g}_m}. \tag{3.25}$$

Knowing that $d_g(y, \exp_y(R_m x)) = R_m |x|$, we have:

$$d_g(x_o, x_m) - R_m|x| \le d_g(x_o, \exp_{x_m}(R_m x)) \le d_g(x_o, x_m) + R_m|x|. \tag{3.26}$$

Suppose that $x_m \to x_o$ as $m \to \infty$. Then, either $\frac{R_m}{d_g(x_o,x_m)} \to 0$ as $m \to \infty$, then $\frac{d_g(x_o,\exp_{x_m}(R_m x))}{d_g(x_o,x_m)} \to 1$ as $m \to \infty$ and consequently,

$$\frac{R_m}{d_g\left(x_o, \exp_{x_m}(R_m x)\right)} \to 0 \text{ as } m \to \infty,$$

or $\frac{R_m}{d_g(x_o,x_m)} \to A > 0$ as $m \to \infty$. Then, always by (3.26), we get:

$$\frac{1}{\frac{1}{A} + |x|} \leq \lim_{m\to\infty} \frac{R_m}{d_g\left(x_o, \exp_{x_m}(R_m x)\right)} \leq \frac{1}{\frac{1}{A} - |x|}.$$

Hence, by writing

$$\int_M \frac{h}{\rho_{x_o}^s} |v_m|^{p-2} v_m \varphi_m dv_g \quad = \quad R_m^{p-s} \int_{\mathbb{R}^n} \frac{R_m^s}{d_g(x_o, \exp_{x_m}(R_m x))^s} h(\exp_{x_o}(R_m x)) |(\hat{\eta}_m \hat{v}_m)|^{p-2} (\hat{\eta}_m \hat{v}_m) \varphi dv_{\hat{g}_m},$$

and

$$\int_M f|v_m|^{p^*-2} v_m \varphi_m dv_g = \int_{\mathbb{R}^n} f(\exp_{x_m}(R_m x)) |(\hat{\eta}_m \hat{v}_m)|^{p^*-2} (\hat{\eta}_m \hat{v}_m) \varphi dv_{\hat{g}_m}. \tag{3.27}$$

Since $\hat{g}_m \to \xi$ in $C^1(B(R))$ for any $R > 0$, the sequence φ_m is bounded in $H_1^p(M)$, the sequence v_m is a P-S sequence of $J_{f,h,s}$ and the sequence $\hat{\eta}_m \hat{v}_m$ converges strongly to $v \neq 0$ in $D^{1,p}(\mathbb{R}^n)$, by passing to the limit, we get that v is a weak solution of

$$\Delta_{\xi,p} v = f(x^o)|v|^{p^*-2} v.$$

□

Step 3

Let $w_m = v_m - \mathcal{B}_m$, with

$$\mathcal{B}_m(x) = R_m^{\frac{p-n}{p}} \eta_{\delta,x_m}(x) v(R_m^{-1} \exp_{x_m}^{-1}(x)), \tag{3.28}$$

where $\eta_{\delta,x_m}(x) = \eta_\delta(\exp_{x_m}^{-1}(x))$. Then, the following statements hold:

$$\mathcal{B}_m \text{ converges weakly to 0 in } H_1^p(M), \tag{3.29}$$

$$DJ_{f,h,s}(\mathcal{B}_m) \to 0,\ DJ_{f,h,s}(w_m) \to 0 \text{ strongly}, \tag{3.30}$$

and

$$J_{f,h,s}(w_m) = J_{f,h,s}(v_m) - (f(x^o))^{\frac{p-n}{p}} E(u), \tag{3.31}$$

with u is a nontrivial weak solution of (1.3).

Proof The proof of (3.29) is identical to that of statement (14) of Step 2.4 in [21] and thus we omit it. We prove (3.30). Let $\varphi \in H_1^p(M)$. For $x \in B(\delta R_m^{-1})$ put $\varphi_m(x) = R_m^{\frac{n-p}{p}} \varphi(\exp_{x_m}(R_m x))$ and $\bar{\varphi}_m = \eta_\delta(R_m x)\varphi_m(x)$. Let $R > 0$ be a constant, we have:

$$\int_M |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g = \int_{B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g$$

$$+ \quad \int_{B(x_m, 2\delta) \setminus B(x_m, R_m R)} |\nabla_g \mathcal{B}_m|^{p-2} g(\nabla_g \mathcal{B}_m, \nabla_g \varphi) dv_g.$$

Direct computations give:

$$\int_{B(x_m,2\delta)\setminus B(x_m,R_mR)} |\nabla_g\mathcal{B}_m|^{p-2}g(\nabla_g\mathcal{B}_m,\nabla_g\varphi)dv_g = O(||\varphi||_{H_1^p(M)})\varepsilon(R),$$

where $\varepsilon(R)\to 0$ as $R\to\infty$.
For m is large, we have:

$$\int_{B(x_m,R_mR)} |\nabla_g\mathcal{B}_m|^{p-2}g(\nabla_g\mathcal{B}_m,\nabla_g\varphi)dv_g = \int_{B(R)} |\nabla_{\hat{g}_m}v|^{p-2}\hat{g}(\nabla_{\hat{g}_m}v,\nabla_{\hat{g}_m}\overline{\varphi}_m)dv_{\hat{g}_m}$$

knowing that

$$\int_{B(x_m,R_mR)} |\nabla_g\varphi|^p dv_g = \int_{B(R)} |\nabla_{\hat{g}_m}\varphi_m|^p dv_{\hat{g}_m},$$

and that the sequence of metrics $\hat{g}_m$ converges in $C^1(B(R')), R'>R$, we get that:

$$\begin{aligned}&\int_{B(x_m,R_mR)} |\nabla_g\mathcal{B}_m|^{p-2}g(\nabla_g\mathcal{B}_m,\nabla_g\varphi)dv_g\\ =\ &\int_{B(R)} |\nabla_{\hat{g}_m}v|^{p-2}\hat{g}(\nabla_{\hat{g}_m}v,\nabla_{\hat{g}_m}\overline{\varphi}_m)dx + o(||\varphi||_{H_1^p(M)}).\\ =\ &\int_{\mathbb{R}^n} |\nabla v|_\xi^{p-2}\nabla v.\nabla\overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R),\end{aligned}$$

where $\varepsilon(R)\to 0$ as $R\to\infty$.Thus,

$$\begin{aligned}&\int_M |\nabla_g\mathcal{B}_m|^{p-2}g(\nabla_g\mathcal{B}_m,\nabla_g\varphi)dv_g \qquad (3.32)\\ =\ &\int_{\mathbb{R}^n} |\nabla v|_\xi^{p-2}\nabla v.\nabla\overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R),\end{aligned}$$

By the same way, we get that:

$$\begin{aligned}&\int_M f(x)|\mathcal{B}_m|^{p^*-2}\mathcal{B}_m\varphi dv_g \qquad (3.33)\\ =\ & f(x^o)\int_{\mathbb{R}^n} |v|^{p^*-2}v\overline{\varphi}_m dx + o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R).\end{aligned}$$

Since the sequence $\mathcal{B}_m$ converges to 0 weakly in $H_1^p(M)$ and the inclusion $H_1^p(M)\subset L_p(M,(\rho_{x_o})^s)$ is compact for $s\in(0,p)$, we can assume that $\mathcal{B}_m\to 0$ in $L_p(M,(\rho_{x_o})^s)$. Then, using the fact that v is a weak solution of $\Delta_{\xi,p}v = f(x^o)|v|^{p^*-2}v$, we get

$$DJ_{f,h,s}(\mathcal{B}_m).\varphi = o(||\varphi||_{H_1^p(M)}) + O(||\varphi||_{H_1^p(M)})\varepsilon(R).$$

Since R is arbitrary, we get that $DJ_{f,h,s}(\mathcal{B}_m)\to 0$. This proves the first part of (3.30). For the proof of the second part of (3.30), we write:

$$DJ_{f,h,s}(w_m) = DJ_{f,h,s}(v_m) - DJ_{f,h,s}(\mathcal{B}_m) + \mathcal{A}_m.\varphi + \mathcal{C}_m\varphi + \mathcal{D}_m\varphi,$$

where

$$\begin{aligned}\mathcal{A}_m.\varphi &= \int_M g(|\nabla_g w_m|^{p-2}\nabla_g w_m - |\nabla_g v_m|^{p-2}\nabla_g v_m + |\nabla_g \mathcal{B}_m|^{p-2}\nabla_g \mathcal{B}_m, \nabla_g\varphi)dv_g,\\ \mathcal{C}_m\varphi &= \int_M \frac{h}{(\rho_{x_o})^s}\left(|w_m|^{p-2}w_m + |v_m|^{p-2}v_m - |\mathcal{B}_m|^{p-2}\mathcal{B}_m\right).\varphi dv_g,\end{aligned}$$

and

$$\mathcal{D}_m\varphi = \int_M f\left(|w_m|^{p^*-2}w_m + |v_m|^{p^*-2}v_m - |\mathcal{B}_m|^{p^*-2}\mathcal{B}_m\right).\varphi dv_g.$$

We repeat the same arguments as in (3.1), we get that $\mathcal{A}_m.\varphi \to 0, \mathcal{C}_m.\varphi \to 0$ and $\mathcal{D}_m.\varphi \to 0$ which ends the proof of (3.30). Now, we prove (3.31). First, we repeat the same calculation in [21], we get:

$$\int_M |\nabla_g w_m|_g^p dv_g = \int_M |\nabla_g v_m|^p dv_g - \int_{\mathbb{R}^n} |\nabla v|^p dx + B_m(R) + o(1), \tag{3.34}$$

and

$$\int_M f|w_m|^{p^*} dv_g = \int_M f|v_m|^{p^*} dv_g - f(x^o)\int_{\mathbb{R}^n} |v|^{p^*} dx + B_m(R) + o(1), \tag{3.35}$$

with $\lim\limits_{R\to\infty}\limsup\limits_{m\to\infty} B_m(R) = 0$.

Since $w_m \to 0$ weakly in $H_1^p(M)$ which is compactly embedded in $L_p(M,(\rho_{x_o})^s)$ for $s \in (0,p)$, we may assume that $w_m \to 0$ strongly in $L_p(M,(\rho_{x_o})^s)$. Therefore, since R is arbitrarily chosen, by combining (3.34), (3.35), we get:

$$J_{f,h,s}(w_m) = J_{f,h,s}(v_m) - (f(x^o))^{\frac{p-n}{p}} E(u) + o(1),$$

with u is a weak solution of (1.3). □

□

3.2. The critical Hardy potential

Lemma 3.4 *Let v_m be a P.S sequence of $J_{f,h,p}$ at a level β that converges weakly and not strongly to 0 in $H_1^p(M)$. Then, there exists a sequence of positive reals $\mathcal{T}_m \to 0$ as $m \to \infty$ such that the sequence $\tilde{\eta}_m\tilde{v}_m$ with*

$$\tilde{v}_m(x) = \mathcal{T}_m^{\frac{n-p}{p}} v_m(\exp_{x_o}(\mathcal{T}_m x)),$$

and $\tilde{\eta}_m(x) = \eta(\delta^{-1}\mathcal{T}_m x)$, $0 < \delta \le \frac{Inj_g}{2}$ and η is defined by (3.18), converges up to subsequence to a weak solution $v \in D^{1,p}(\mathbb{R}^n)$ of

$$\Delta_{\xi,p} v + \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

Moreover, the sequence

$$w_m(x) = v_m(x) - \mathcal{T}_m^{\frac{p-n}{p}}\eta_\delta(\exp_{x_o}^{-1}(x))v(\mathcal{T}_m^{-1}\exp_{x_o}^{-1}(x)),$$

where $0<\delta<\frac{Inj_g}{2}$, admits a subsequence w_m that is a P-S sequence of $J_{f,h,p}$, at level $\beta - E_{f,h}(v)$ that converges to 0 weakly in $H_1^p(M)$.

Proof Let v_m be a P.S sequence of $J_{f,h,p}$ at level β that converges to 0 weakly and not strongly in $H_1^p(M)$. Then, up to a subsequence, we can assume that v_m converges strongly to 0 in $L_p(M)$ and that, by (3.6) there exists a small positive constant $\tilde{\gamma}$, such that

$$\limsup_{m\to\infty}\int_M |\nabla_g v_m|^p\, dv_g > \tilde{\gamma} > 0.$$

Up to a subsequence, for each $m>0$, there exists a constant $\tilde{r}_m > 0$ such that

$$\int_{B(x_o,\tilde{r}_m)} |\nabla_g v_m|^p\, dv_g = \tilde{\gamma} \tag{3.36}$$

For $0<r_o<\frac{\mathrm{Inj_g}}{2}$ and C_o as in (3.9). For $0<r<r_o$, put $\mathcal{T}_m = \frac{\tilde{r}_m}{rC_o}$ and for $x\in B(\mathcal{T}_m^{-1}\delta_g)$ and define

$$\begin{aligned}\tilde{v}_m(x) &= \mathcal{T}_m^{\frac{n-p}{p}} v_m(\exp_{x_o}(\mathcal{T}_m x)),\ x\in\mathbb{R}^n\\ \tilde{g}_m(x) &= \exp_{x_o}^* g(\mathcal{T}_m x)\end{aligned}$$

We let the sequence $\tilde{\eta}_m\tilde{v}_m$ such that $\tilde{\eta}_m = \eta(\delta^{-1}\mathcal{T}_m x)$, $\delta\in(0,\frac{\mathrm{Inj_g}}{2})$ and $\eta\in C_o(\mathbb{R}^n)$ is the cut-off function such that $0\le\eta\le1$, $\eta(x)=1$, $x\in B(\frac{1}{4})$ and $\eta(x)=0, x\in\mathbb{R}^n\setminus\ B(\frac{3}{4})$. Going through the same way in the proof of Lemma 3.3, we get then that the sequence $\tilde{\eta}_m\tilde{v}_m$ is bounded in $D^{1,p}(\mathbb{R}^n)$ and then it converges weakly in $D^{1,p}(\mathbb{R}^n)$ to a function $v\in D^{1,p}(\mathbb{R}^n)$.

Suppose that $v\neq0$, we get then that $\mathcal{T}_m\to0$. To prove that v solves (1.4), we let $\varphi\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ with compact support included in a ball $B(R)$, $R>0$. For m is large, define on M the sequence φ_m as

$$\varphi_m(x) = \mathcal{T}_m^{\frac{p-n}{p}}\varphi(\mathcal{T}_m^{-1}(\exp_{x_o}^{-1}(x)))$$

Identities (3.25) and (3.27) still hold and we have:

$$\int_M \frac{h}{\rho_{x_o}^p}|v_m|^{p-2}v_m\varphi_m dv_g \quad = \quad \int_{\mathbb{R}^n}\frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p}|(\tilde{\eta}_m\tilde{v}_m)|^{p-2}(\tilde{\eta}_m\tilde{v}_m)\varphi dv_{\tilde{g}_m}.$$

Since $\mathcal{T}_m\to0$, $\tilde{g}_m\to\xi$ in $C^1(B(R))$ and thus we can write $dv_{\tilde{g}_m}=\varepsilon_m dx$, with $\varepsilon\to1$ uniformly in $B(R)$. In addition, we can prove, as in [21] (proof of step 2.1), that $\nabla(\tilde{\eta}_m\tilde{v}_m)\to\nabla v$ a.e. Since we have also $\tilde{\eta}_m\tilde{v}_m\to v$ a.e, and the sequence $\tilde{\eta}_m\tilde{v}_m$ is bounded in $L_p(\mathbb{R}^n,|x|^p)$ we get by basic integration theory together with the fact that the sequence φ_m is bounded in $H_1^p(M)$ and the sequence v_m is a P-S sequence of $J_{f,h,p}$, that v is a weak solution of

$$\Delta_{\xi,p}v - \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

Now, that the sequence w_n converges weakly to 0 in $H_1^p(M)$ follows in the same manner as in the proof of Step 3 above. To prove that $DJ_{f,h,p}(w_m)\to0$, we consider the sequence $\mathcal{B}_m$ defined by (3.28). Let $\varphi\in H_1^p(M)$.

For $x \in B(\delta\mathcal{T}_m^{-1})$, put $\varphi_m(x) = \mathcal{T}_m^{\frac{n-p}{p}} \varphi(\exp_{x_o}(\mathcal{T}_m x))$ and $\bar{\varphi}_m = \eta_\delta(\mathcal{T}_m x)\varphi_m(x)$. Then, identities (3.32) and (3.33) still hold. Let $R > 0$ be a constant, we have:

$$\int_M \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g = \int_{B(x_o,\mathcal{T}_m R)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g + \int_{B(x_o,\delta)\setminus B(x_o,\mathcal{T}_m R)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g.$$

By Hölder and Hardy inequalities, we have:

$$\begin{aligned} \int_{B(x_o,\delta)\setminus B(x_o,\mathcal{T}_m R)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g &\leq \sup_M |h| \|\varphi\|_{H_1^p(M)} \int_{B(x_o,\delta)\setminus B(x_o,\mathcal{T}_m R)} |\nabla_g \mathcal{B}_m|^p dv_g + o(1) \\ &= \sup_M |h| \|\varphi\|_{H_1^p(M)} \int_{B(\delta\mathcal{T}_m^{-1}))\setminus B(R)} |\nabla v|^p dx + o(1) \\ &= O(\|\varphi\|_{H_1^p(M)})\varepsilon(R) + o(1), \end{aligned}$$

with $\varepsilon \to 0$ as $R \to \infty$.
Put

$$\overline{\varphi}(x) = \mathcal{T}_m^{\frac{n-p}{p}} \varphi(\exp_{x_o}(\mathcal{T}_m x)).$$

Then, for m is large

$$\int_{B(x_o,\mathcal{T}_m R)} \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g = \int_{B(R)} \frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dv_{\tilde{g}_m}$$

Since $\tilde{g} \to \xi$ in $C^1(B(R'))$, $R' > R$, we get

$$\begin{aligned} \int_{B(R)} \frac{h(\exp_{x_o}(\mathcal{T}_m x))}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dv_{\tilde{g}_m} &= h(x_o)\int_{B(R)} \frac{1}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) \\ &= h(x_o)\int_{\mathbb{R}^n} \frac{1}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) + O(\|\varphi\|_{H_1^p(M)})\varepsilon(R). \end{aligned}$$

Therefore,

$$\int_M \frac{h}{(\rho_{x_o})^p}|\mathcal{B}_m|^{p-2}\mathcal{B}_m\varphi dv_g = h(x_o)\int_{\mathbb{R}^n} \frac{1}{|x|^p}|v|^{p-2}v\overline{\varphi}_m dx + o(\|\varphi\|_{H_1^p(M)}) + O(\|\varphi\|_{H_1^p(M)})\varepsilon(R) + o(1). \quad (3.37)$$

Since v is a weak solution of (1.4), we get by (3.32), (3.33), and (3.37) that $DJ_{f,h,p}(\mathcal{B}_m) \to 0$. This implies, as in the proof of (3.30) of Step 3, that $DJ_{f,h,p}(w_m) \to 0$.
Now, we prove the last statement of the lemma. Put

$$\hat{w}_m(x) = \mathcal{T}_m^{\frac{n-p}{p}} w_m(\exp_{x_o}(\mathcal{T}_m x)) = \tilde{v}_m - \eta_\delta(\mathcal{T}_m x)v(x)$$

By convexity, we have:

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla(v(\eta_\delta(\mathcal{T}_m x)-1))|^p dx \\
= \;& \int_{\mathbb{R}^n\setminus B(\delta\mathcal{T}_m^{-1})} |\nabla v|^p dx + \int_{B(2\delta\mathcal{T}_m^{-1})\setminus B(\delta\mathcal{T}_m^{-1})} |\nabla(v(\eta_\delta(\mathcal{T}_m x)-1))|^p dx \\
\le \;& 2^{p-1}\left(\int_{B(2\delta\mathcal{T}_m^{-1})\setminus B(\delta\mathcal{T}_m^{-1})} |\eta_\delta(\mathcal{T}_m x)-1)|^p|\nabla v|^p dx + \mathcal{T}_m^p\int_{B(2\delta\mathcal{T}_m^{-1})\setminus B(\delta\mathcal{T}_m^{-1})} |v|^p|(\nabla\eta_\delta)(\mathcal{T}_m x)|^p dx\right) \\
+ \;& \int_{\mathbb{R}^n\setminus B(\delta\mathcal{T}_m^{-1})} |\nabla v|^p dx \\
\le \;& 2^{p-1}\left(\int_{B(2\delta\mathcal{T}_m^{-1})\setminus B(\delta\mathcal{T}_m^{-1})} |\nabla v|^p dx + C\mathcal{T}_m^p\int_{B(2\delta\mathcal{T}_m^{-1})\setminus B(\delta\mathcal{T}_m^{-1})} |v|^p dx\right) + \int_{\mathbb{R}^n\setminus B(\delta\mathcal{T}_m^{-1})} |\nabla v|^p dx \\
= \;& o(1).
\end{aligned}$$

Similarly, we get that $\tilde{\eta}_m v = v + o(1)$. Thus, we obtain:

$$\tilde{\eta}_m\hat{w}_m = \tilde{\eta}_m\tilde{v}_m - v + o(1).$$

Since $\tilde{\eta}_m\tilde{v}_m \to v$ a.e in $\mathbb{R}^n$ and $\nabla(\tilde{\eta}_m\tilde{v}_m) \to \nabla v$ a.e in $\mathbb{R}^n$, we get, as in the proof of Lemma 3.1, that

$$E_{f,h}(\tilde{\eta}_m\hat{w}_m) = E_{f,h}(\tilde{\eta}_m\tilde{v}_m) - E_{h,f}(v) + o(1).$$

By using rescaling invariance and the fact that $\tilde{g}_m \to \xi$ in $C^1(B(R))$ for any $R > 0$, we get that:

$$J_{f,h,p}(w_m) = J_{f,h,p}(v_m) - E_{h,f}(v) + o(1).$$

□

Lemma 3.5 *Suppose that the weak limit v in $D^{1,p}(\mathbb{R}^n)$ of the sequence $\tilde{\eta}_m\tilde{v}_m$ of the above lemma is null. Then, there exists a sequence of positive numbers $\tau_m \to 0$ and a sequence of points $y_i \in M\setminus\{x_o\}$, $y_i \to y_o \neq x_o$ such that up to a subsequence, the sequence*

$$\nu_m = \tau_m^{\frac{n-p}{p}} v_m(\exp_{y_i}(\tau_m x))$$

converges weakly to a nontrivial weak solution ν of the Euclidean equation

$$\Delta_{\xi,p}\nu = f(y_o)|\nu|^{p^\star-2}\nu$$

and the sequence

$$\mathcal{W}_m = v_m - \tau_m^{\frac{p-n}{p}}\eta_\delta(exp_{y_i}^{-1}(x))\nu(\tau_m^{-1}exp_{y_i}^{-1}(x))$$

is a Palais-Smale sequence for $J_{f,h,p}$ that converges weakly to 0 in $H_1^p(M)$ and

$$J_{f,h,p}(\mathcal{W}_m) = J_{f,h,p}(v_m) - f(y_o)^{\frac{p-n}{p}}E(u),$$

with u is a solution of(1.3).

Proof Take a function $\varphi \in \mathcal{C}_0^\infty(B(C_o r))$ and put $\varphi_m(x) = \varphi(\mathcal{T}_m^{-1} \exp_{x_o}^{-1}(x))$. We have:

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m} \tilde{v}_m|^{p-2} \tilde{g}(\nabla_{\tilde{g}_m} \tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m |\varphi|^p)) dv_{\tilde{g}_m} = \int_{\mathbb{R}^n} |\varphi|^p |\nabla_{\tilde{g}_m} \tilde{v}_m|^p dv_{\tilde{g}_m}$$
$$+ \quad \int_{\mathbb{R}^n} p|\varphi|^{p-1} |\tilde{v}_m| |\nabla_{\tilde{g}_m} \tilde{v}_m|^{p-2} \tilde{g}(\nabla_{\tilde{g}_m} \tilde{v}_m, \nabla_{\tilde{g}_m} |\varphi|) dv_{\tilde{g}_m}.$$

Since the sequence $\tilde{\eta}_m \tilde{v}_m$ is bounded in $D^{1,p}(\mathbb{R}^n)$ and it converges strongly to 0 in $L_{p,loc}(\mathbb{R}^n)$, we have:

$$\left| \int_{\mathbb{R}^n} p|\varphi|^{p-1} |\tilde{v}_m| |\nabla_{\tilde{g}_m} \tilde{v}_m|^{p-2} \tilde{g}(\nabla_{\tilde{g}_m} \tilde{v}_m, \nabla_{\tilde{g}_m} |\varphi|) dv_{\tilde{g}_m} \right|$$
$$\leq \quad C \int_{B(C_o r)} |\tilde{v}_m| |\nabla_{\tilde{g}_m} \tilde{v}_m|^{p-1} dv_{\tilde{g}_m}$$
$$\leq \quad C \left(\int_{B(C_o r)} |\tilde{v}_m|^p dv_{\tilde{g}_m} \right)^{\frac{1}{p}} \left(\int_{B(C_o r)} |\nabla_{\tilde{g}_m} \tilde{v}_m|^p_{\tilde{g}_m} dv_{\tilde{g}_m} \right)^{1-\frac{1}{p}} = o(1).$$

Then

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m} \tilde{v}_m|^{p-2} \tilde{g}(\nabla_{\tilde{g}_m} \tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m |\varphi|^p)) dv_{\tilde{g}_m} = \int_{\mathbb{R}^n} |\varphi|^p |\nabla_{\tilde{g}_m} \tilde{v}_m|^p dv_{\tilde{g}_m} + o(1).$$

Now, by lemma A.4 in [4], the following inequalities hold

1. If $1 < p < 2$, for a given $\gamma \in (1, p)$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + Ct^\gamma,$$

   for $t \geq 0$ uniformly in α.

2. If $2 \leq p \leq 3$, for a given $\gamma \in [p-1, 2]$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + Ct^\gamma,$$

   for $t \geq 0$ uniformly in α.

3. If $p \geq 3$, there exists a constant such that

$$(1 + t^2 + 2t\cos\alpha)^{\frac{p}{2}} \leq 1 + t^p + pt\cos\alpha + C(t^2 + t^{p-1}),$$

   for $t \geq 0$ uniformly in α.

Using these inequalities together with Hölder inequality and the strong convergence of $\tilde{\eta}_m \tilde{v}_m$ in $L_{p,loc}(\mathbb{R}^n)$, we get:

$$\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m \varphi)|^p dv_{\tilde{g}_m} \leq \int_{\mathbb{R}^n} |\varphi|^p |\nabla_{\tilde{g}_m} \tilde{v}_m|^p dv_{\tilde{g}_m} + o(1),$$

in such a way that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p dv_{\tilde{g}_m} \\ \leq &\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}\tilde{v}_m|^{p-2}\tilde{g}(\nabla_{\tilde{g}_m}\tilde{v}_m, \nabla_{\tilde{g}_m}(\tilde{v}_m|\varphi|^p))dv_{\tilde{g}_m} + o(1), \\ = &\int_M |\nabla v_m|^{p-2}\, g(\nabla_g v_m, \nabla_g(v_m|\varphi_m|^p))dv_g + o(1) \end{aligned}$$

Moving to and from rescaling, using Hölder, Hardy, and Sobolev inequalities and the fact that v_m is P-S sequence and that $v_m|\varphi_m|^p$ is bounded in $H_1^p(M)$, we get:

$$\begin{aligned} &\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p dv_{\tilde{g}_m} \\ \leq &\int_M |\nabla_g v_m|^{p-2}\, g(\nabla_g v_m, \nabla_g(v_m|\varphi_m|^p))dv_g + o(1) \\ = &(DJ_{f,h,p}(v_m)).(v_m|\varphi_m|^p) + \int_M \frac{h}{\rho_{x_o}^p}\,|v_m\varphi_m|^p\,dv_g + \int_M f\,|v_m|^{p^*-p}\,|v_m\varphi_m|^p\,dv_g + o(1) \\ \leq &(h(x_o)+\varepsilon)\left(\left(\frac{p}{n-p}\right)^p + \varepsilon\right)\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p\,dv_{\tilde{g}_m} \\ + &(K^{p^\star}(n,p)+\varepsilon)\sup f\Big(\int_{B(C_or)} |\nabla_{\tilde{g}_m}(\tilde{v}_m)|^p\,dv_{\tilde{g}_m}\Big)^{\frac{p}{n-p}}\int_{\mathbb{R}^n} |\nabla_{\tilde{g}_m}(\tilde{v}_m\varphi)|^p\,dv_{\tilde{g}_m} \\ + &o(1). \end{aligned}$$

Thus, since $1-h(x_o)(\frac{p}{n-p})^p>0$, for $\tilde{\gamma}$ in (3.36) chosen small enough, we get that for each t, $0<t<C_or$

$$\int_{B(x_o,t\mathcal{T}_m)} |\nabla_g v_m|^p\,dv_g = \int_{B(t)} |\nabla_{\tilde{g}_m}\tilde{v}_m|^p\,dv_{\tilde{g}_m} \to 0, m\to\infty \tag{3.38}$$

Now, the sequence v_m is a P.S sequence that converges to 0 weakly and not strongly in $H_1^p(M)$, we get as in lemma 3.2 that

$$\int_M |\nabla_g v_m|^p\,dv_g \geq \left(\frac{n\beta^*}{\sup_M f(K(n,p)+\varepsilon)^{p^*}}\right)^{\frac{p}{p^*}} + o(1). \tag{3.39}$$

Consider for $t>0$ the function

$$t\mapsto \mathcal{F}_m(t) = \max_{y\in M}\int_{B(y,t)} |\nabla_g v_m|^p\,dv_g$$

Given that t_o is small, it follows from (3.39) that there exists $y\in M$ and $\lambda_o>0$ such that up to a subsequence

$$\int_{B(y,t_o)} |\nabla_g v_m|^p\,dv_g \geq \lambda_o \tag{3.40}$$

Since $\mathcal{F}_m$ is continuous, it follows that for any $\lambda\in(0,\lambda_o)$, there exist $t_m\in(0,t_o)$ and $y_m\in M$ such that

$$\mathcal{F}_m(t_m) = \int_{B(y_m,t_m)} |\nabla_g v_m|^p\,dv_g = \lambda. \tag{3.41}$$

Since M is compact, up to a subsequence, we may assume that y_m converges to some point $y_o \in M$.
Note first that for all $m \geq 0$, $t_m < \tilde{r}_m = C_o r \mathcal{T}_m$, otherwise if there exists $m_o \geq 0$ such that $t_{m_o} \geq \tilde{r}_{m_o}$, we get:

$$\lambda = \int_{B(y_{m_o}, t_{m_o})} |\nabla_g v_{m_o}|^p \, dv_g \geq \int_{B(x_o, t_{m_o})} |\nabla_g v_{m_o}|^p \, dv_g \geq \int_{B(x_o, \tilde{r}_{m_o})} |\nabla v_{m_o}|^p \, dv_g = \gamma.$$

Hence, if we choose λ small enough such that $0 < \lambda < \gamma$, we get a contradiction.
Now, suppose that for all $\varepsilon > 0$, there exists $m_\varepsilon > 0$ such that for all $m \geq m_\varepsilon$ $dist_g(y_m, x_o) \leq \varepsilon$. Choose r'_m such that, $t_m < r'_m < \tilde{r}_m$ and take $\varepsilon' = r'_m - t_m$, we get that for some $m_{\varepsilon'} > 0$ and $m \geq m_{\varepsilon'}$

$$B(y_m, t_m) \subset B(x_o, r'_m)$$

which gives, by virtue of (3.38) and (3.41), a contradiction. We deduce then that $y_o \neq x_o$.
Now, take $0 < \tau_m < 1$ such that $C_o r \tau_m = t_m$, where $r \in (0, r_o)$ and C_o and r_o are as in (3.9). Then, for $x \in B(\tau_m^{-1} \delta_g) \subset \mathbb{R}^n$ consider the sequences

$$\begin{aligned} \check{\nu}_m(x) &= \tau_m^{\frac{n-p}{p}} v_m(\exp_{y_m}(\tau_m x)), \\ \check{g}_m(x) &= \exp_{y_m}^* g(\tau_m x) \end{aligned}$$

Put $\check{\eta}_m(x) = \eta(\delta^{-1} \tau_m x)$, where $\delta \in (0, Inj_g)$ and $x \in \mathbb{R}^n$. As in the proof of lemma 3.3, we can easily check that there is a subsequence of $\tilde{\eta}_m \tilde{\nu}_m$ that converges weakly in $\mathcal{D}^{1,p}(\mathbb{R}^n)$ to some function ν. We prove that actually the strong convergence holds in $H_1^p(B(R)), R > 0$. In fact, we go through the same proof of Step 1 above by just replacing x_m by y_m and R_m by τ_m. We let then $a \in \mathbb{R}^n$ and $\mu \in [r, 2r]$ and consider the sequence

$$\begin{cases} \check{\psi}_m = \check{\eta}_m \check{\nu}_m - \nu & \text{in } \overline{B}(a, \mu), \\ \check{\psi}_m = z_m & \text{in } \overline{B}(a, 3r) \setminus B(a, \mu), \\ \check{\psi}_m = 0 & \text{in } \mathbb{R}^n \setminus B(a, 3r). \end{cases}$$

where z_m are solutions of (3.19). For $r < \frac{\delta}{24}$, consider the rescaling sequence ψ_m of $\check{\psi}_m$

$$\begin{cases} \psi_m(x) = \tau_m^{\frac{p-n}{p}} \check{\psi}_m(\tau_m^{-1} \exp_{y_m}^{-1}(x)), & \text{if } x < d_g(y_m, 6r), \\ \psi_m(x) = 0, & \text{otherwise.} \end{cases}$$

As in (3.20), we have:

$$\begin{aligned} o(1) = & \int_{B(a,3r)} |\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m)|^{p-2} \check{g} \left(\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m), \nabla_{\check{g}_m} \check{\psi}_m \right) dv_{\check{g}_m} \\ - & \tau_m^p \int_{B(a,3r)} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p} |\check{\eta}_m \check{\nu}_m|^{p-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m} \\ - & \int_{B(a,3r)} f(\exp_{y_m}(\tau_m x)) |\check{\eta}_m \check{\nu}_m|^{p^*-2} (\check{\eta}_m \check{\nu}_m) \check{\psi}_m dv_{\check{g}_m}. \end{aligned} \tag{3.42}$$

As above, we have:

$$\int_{B(a,3r)} |\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m)|^{p-2} \check{g} \left(\nabla_{\check{g}_m}(\check{\eta}_m \check{\nu}_m), \nabla_{\check{g}_m} \check{\psi}_m \right) dv_{\check{g}_m} = \int_{\mathbb{R}^n} |\nabla_{\check{g}_m} \check{\psi}_m|^p dv_{\check{g}_m} + o(1),$$

and

$$\begin{aligned} &\int_{B(a,3r)} f(\exp_{y_m}(\tau_m x))|\check{\eta}_m\check{\nu}_m|^{p^*-2}(\check{\eta}_m\check{\nu}_m)\check{\psi}_m dv_{\check{g}_m} \\ = &\int_{\mathbb{R}^n} f(\exp_{y_m}(\tau_m x))|\check{\psi}_m|^{p^*} dv_{\check{g}_m} + o(1). \end{aligned}$$

Since $\tau_m \to 0$, we get that for all $\varepsilon > 0$, there exists m_o such that for all $m \geq m_o$, we have:

$$\rho_{x_o}(\exp_{y_m}(\tau_m x)) = dist_g(x_o, \exp_{y_m}(\tau_m x)) \geq dist_g(x_o, y_o) - \varepsilon = \varrho > 0.$$

Then, as in the proof of step 1, we get:

$$\begin{aligned} &\int_{B(a,3r)} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p}|\check{\eta}_m\check{\nu}_m|^{p-2}(\check{\eta}_m\check{\nu}_m)\check{\psi}_m dv_{\check{g}_m} \\ = &\int_{\mathbb{R}^n} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p}|\check{\psi}_m|^p dv_{\check{g}_m} + o(1). \end{aligned} \tag{3.43}$$

Since the sequence $\check{\psi}_m$ converges strongly to 0 in $L_{p,loc}(\mathbb{R}^n)$, we get:

$$\int_{\mathbb{R}^n} \frac{h(\exp_{y_m}(\tau_m x))}{\left(\rho_{x_o}(\exp_{y_m}(\tau_m x))\right)^p}|\check{\psi}_m|^p dv_{\check{g}_m} \leq C\int_{\mathbb{R}^n} |\check{\psi}_m|^p dv_{\check{g}_m} = o(1).$$

We deduce that:

$$\int_{\mathbb{R}^n} |\nabla_{\check{g}_m}\check{\psi}_m|^p dv_{\check{g}_m} = \int_{\mathbb{R}^n} f(\exp_{y_m}(\tau_m x))|\check{\psi}_m|^{p^*} dv_{\check{g}_m} + o(1).$$

The remaining of the proof goes in the same way as in the proof of step 1 and step 2. Thus, we get that $\nu \neq 0$ and ν is a weak solution of

$$\Delta_{p,\xi}\nu = f(y_o)|\nu|^{p^*-2}\nu.$$

□

Now, we are in position to prove Theorems 1.1 and 1.2

Proof [Proof of Theorem 1.1] Let us first note that if $u \in D^{1,p}(\mathbb{R}^n)$ is a nontrivial weak solution of (1.4), then

$$E_{f,h}(u) \geq \frac{(1 - h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p)}. \tag{3.44}$$

In fact, by Hardy and Sobolev inequalities, we have:

$$\begin{aligned} \left(1 - h(x_o)\left(\frac{p}{n-p}\right)^p\right)\int_{\mathbb{R}^n} |\nabla u|^p dx &\leq \int_{\mathbb{R}^n} |\nabla u|^p dx - h(x_o)\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx = f(x_o)\int_{\mathbb{R}^n} |u|^{p^*} dx \\ &\leq f(x_o)K^{p^*}(n,p)\left(\int_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{p^*}{p}} \end{aligned}$$

Since u cannot be a constant, we get:

$$\int_{\mathbb{R}^n} |\nabla u|^p dx \geq \frac{(1-h(x_o)(\frac{p}{n-p})^p)^{\frac{n-p}{p}}}{(f(x_o))^{\frac{n-p}{p}} K^n(n,p)}.$$

Hence,

$$\begin{aligned} E_{f,h}(u) = \frac{1}{n}\left(\int_{\mathbb{R}^n} |\nabla u|^p dx - h(x_o)\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx\right) &\geq \frac{(1-h(x_o)(\frac{p}{n-p})^p)(1-h(x_o)(\frac{p}{n-p})^p)^{\frac{n-p}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p)} \\ &\geq \frac{(1-h(x_o)(\frac{p}{n-p})^p)^{\frac{n}{p}}}{n(f(x_o))^{\frac{n-p}{p}} K^n(n,p)}. \end{aligned}$$

By the same way, we can also have that for a nontrivial solution $u \in D^{1,p}(\mathbb{R}^n)$ of (1.3),

$$E(u) \geq \frac{1}{nK^n(n,p))}. \tag{3.45}$$

Now, let u_m be a P-S sequence for $J_{f,h,s}$ at level β_s^u, $0<s<p$. Then, u_m is bounded in $H_1^p(M)$ and it converges, up to a subsequence, to a function u weakly in $H_1^p(M)$ and almost everywhere to u in M. Thus, by Lemma 3.1, the function u is a weak solution of (E_s), $0<s<p$ and the sequence $v_m = u_m - u$ is a Palais-Smale sequence for $J_{f,h,s}$ at level $\beta_s = \beta_s^u - J_{f,h,s}(u)$.

If v_m converges strongly to 0 in $H_1^p(M)$, then the theorem is proved with $k=0$. If not, by Lemma 3.2, $\beta_s \geq \beta^* = \frac{1}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p))}$. Then, by Lemma 3.3 and its proof, there exists a nontrivial weak solution $v_1 \in D^{1,p}(\mathbb{R}^n)$ of $\Delta_{p,\xi} v = f(x_1^o)|v|^{p^*-2}v$, a converging sequence of points $x_m^1 \to x_1^o$ and a sequence of reals $R_m^1 \to 0$ such that, the sequence

$$w_m(x) = v_m - (R_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{x_m^1}^{-1}(x)) v_1((R_m^1)^{-1}\exp_{x_m^1}^{-1}(x)), x \in M$$

admits a subsequence that is P-S sequence of $J_{f,h,s}$, $0<s<p$, at level $\beta^1 = \beta_s - (f(x_1^o))^{\frac{p-n}{p}} E(u_1)$, with u_1 is a nontrivial weak solution of (1.3). By (3.45), $\beta^1 \leq \beta_s - \beta^*$. Then, if $\beta_s < 2\beta^*$, we get $\beta^1 < \beta^*$ and the sequence w_m converges strongly to 0 in $H_1^p(M)$. Hence, the theorem is proved with $k=1$. If not, we repeat the procedure until we obtain a P-S sequence at level $\beta^k \leq \beta_s - k\beta^* < \beta^*$ and Theorem 1.1 is proved.

□

Proof [Proof of theorem 1.2] In the same way as above, we prove theorem 1.2. We let u_m be a P-S sequence for $J_{f,h,p}$ at a level β^u. Then, u_m is bounded in $H_1^p(M)$ and it converges, up to a subsequence, to a function u weakly in $H_1^p(M)$ and almost everywhere to u in M. Thus, by Lemma 3.1, the function u is a weak solution of (E_s), $s=p$, and the sequence $v_m = u_m - u$ is a Palais-Smale sequence for $J_{f,h,p}$ at level $\beta = \beta^u - J_{f,h,p}(u)$. If v_m converges strongly to 0 in $H_1^p(M)$, then the theorem is proved with $k=0$, $l=0$. If not, by Lemma 3.2, $\beta \geq \beta^* = \frac{(1-h(x_o)(\frac{n-p}{p})^p)^{\frac{n}{p}}}{n(\sup_M f)^{\frac{n-p}{p}} K^n(n,p)}$. By Lemma 3.4, there exist a sequence of positive reals $\mathcal{T}_m^1 \to 0$ such that the sequence $\tilde{\eta}_m \tilde{v}_m$ with

$$\tilde{v}_m(x) = \mathcal{T}^{\frac{n-p}{p}} v_m(\exp_{x_o}(\mathcal{T}_m^1 x)),$$

and $\tilde{\eta}_m(x) = \eta(\delta^{-1}\mathcal{T}_m^1 x)$, $0 < \delta \leq \frac{\text{Inj}_g}{2}$ and η is defined by (3.18), converges, up to subsequence, weakly to some function $v_1 \in D^{1,p}(\mathbb{R}^n)$ such that if $v_1 \neq 0$, then v_1 is solution of

$$\Delta_{\xi,p} v + \frac{h(x_o)}{|x|^p}|v|^{p-2}v = f(x_o)|v|^{p^*-2}v,$$

and the sequence

$$w_m(x) = v_m(x) - (\mathcal{T}_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{x_o}^{-1}(x)) v_1((\mathcal{T}_m^1)^{-1}\exp_{x_o}^{-1}(x)),$$

where $0 < \delta < \frac{\text{Inj}_g}{2}$, admits a subsequence w_m that is a P-S sequence of $J_{f,h,p}$, at level $\beta^1 = \beta - E_{f,h}(v_1)$ that converges to 0 weakly in $H_1^p(M)$. By (3.44), $\beta^1 \leq \beta - \beta^*$. Then, if $\beta < 2\beta^*$, we get $\beta^1 < \beta^*$ and the sequence w_m converges strongly to 0 in $H_1^p(M)$. If not, we repeat the procedure until we obtain a Palais-Smale sequence at level $\beta^k \leq \beta - k\beta^* < \beta^*$.
Now, if the weak limit v of the sequence $\tilde{v}$ is the zero function by lemma 3.5, there exists a nontrivial weak solution ν_1 of $\Delta_{p,\xi}\nu = f(y_o^1)|\nu|^{p^*-2}\nu$, a sequence of positive reals $\tau_m^1 \to 0$ and a sequence $y_i^1 \to y_o^1 \neq x_o$ such that the sequence

$$\mathcal{W}_m(x) = v_m - (\tau_m^1)^{\frac{p-n}{p}} \eta_\delta(\exp_{y_i^1}^{-1}(x)) \nu_1((\tau_m^1)^{-1}\exp_{y_i^1}^{-1}(x)),\ x \in M$$

admits a subsequence which is a P-S sequence of $J_{f,h,p}$ at level $\beta^1 = \beta - (f(y_o^1))^{\frac{p-n}{p}} E(u_1) \leq \beta - \beta^*$, with u_1 is a nontrivial weak solution of (1.3). If $\beta < 2\beta^*$, then $\beta^1 < \beta^*$ and the sequence $\mathcal{W}_m$ converges strongly to 0 in $H_1^p(M)$. The theorem is then proved with $k = 0$ and $l = 1$. If not, we repeat the procedure until we obtain a P-S sequence at level $\beta^k \leq \beta - k\beta^* < \beta^*$. □

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