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Perfect fluid spacetimes and k-almost Yamabe solitons

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Abstract: In this article, we presumed that a perfect fluid is the source of the gravitational field while analyzing the solutions to the Einstein field equations. With this new and creative approach, here we study k-almost Yamabe solitons and gradient k-almost Yamabe solitons. First, two examples are constructed to ensure the existence of gradient k-almost Yamabe solitons. Then we show that if a perfect fluid spacetime admits a k-almost Yamabe soliton, then its potential vector field is Killing if and only if the divergence of the potential vector field vanishes. Besides, we prove that if a perfect fluid spacetime permits a k-almost Yamabe soliton (g, k, ρ, λ), then the integral curves of the vector field ρ are geodesics, the spacetime becomes stationary and the isotopic pressure and energy density remain invariant under the velocity vector field ρ. Also, we establish that if the potential vector field is pointwise collinear with the velocity vector field and $\rho(a) = 0$ where a is a scalar, then either the perfect fluid spacetime represents a phantom era, or the potential function Φ is invariant under the velocity vector field ρ. Finally, we prove that if a perfect fluid spacetime permits a gradient k-almost Yamabe soliton $(g, k, D\Phi, \lambda)$ and R, λ, k are invariant under ρ, then the vorticity of the fluid vanishes.

Key words: Perfect fluids, k-almost Yamabe solitons, Robertson-Walker spacetimes

1. Introduction

In this article, we will deal with the spacetimes that obey the Einstein field equations (in short, EFEs) when a perfect fluid (in short, PF) serves as the source of the gravitational field. If a Lorentzian manifold's nonvanishing Ricci tensor S fulfills the conditions

$$S = \alpha g + \beta C \otimes C, \tag{1.1}$$

it is referred to as a PF spacetime, in which α, β (not simultaneously zero) are scalars, and for any X_1, $g(X_1, \rho) = C(X_1)$, ρ stands for the velocity vector field.

Let (M^n, g) be a Lorentzian manifold whose metric g is of signature $(-, +, +, \ldots, +)$, that is, g is of index 1. In [1], L. Alias et al. proposed the notion of generalized Robertson-Walker (in short, GRW) spacetimes. A GRW spacetime is a Lorentzian manifold M^n $(n \geq 3)$ that may be expressed as $M = -I \times f^2 M^*$, in which the open interval I contained in $\mathbb{R}$, $M^{*(n-1)}$ denotes the Riemannian manifold and $f > 0$ is a smooth function,

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named as scale factor or warping function. The above-stated spacetime turns into Robertson-Walker (in short, RW) spacetime when the dimension of M^* is three and the sectional curvature is constant.

R. Hamilton [13] initially proposed the fascinating concept of Yamabe flow while simultaneously introducing the Ricci flow to address Yamabe's conjecture. According to Yamabe's conjecture, a compact connected Riemannian manifold and a manifold with constant scalar curvature are conformally equivalent. This theory was described by H. Yamabe in 1960 [17]. A partial differential equation of this kind is described by

$$\frac{\partial}{\partial t} g = -Rg, \quad g(0) = g_\circ,$$

in which R stands for the scalar curvature. The foregoing equation is called a Yamabe flow on a Riemannian manifold.

If a Riemannian metric g of a Riemannian manifold fulfills

$$\pounds_Z g = (R - \lambda) g, \tag{1.2}$$

it is referred to as a Yamabe soliton (in short, YS), in which λ, Z and $\pounds$ stand for real numbers, smooth vector fields, and the Lie derivative, respectively.

Similar to the Riemannian context, YSs have been characterized in semi-Riemannian manifolds in [5].

In the latest study, B.Y. Chen and S. Deshmukh [6] introduced the novel idea of quasi-YS which is a generalization of YS on a Riemannian manifold as

$$(\pounds_Z g)(X_1, Y_1) = (R - \lambda) g(X_1, Y_1) + \ \Psi Z^*(X_1)\ Z^*(Y_1), \tag{1.3}$$

where Z^*, Ψ, and λ denote, in that order, the dual 1-form of Z, the smooth function, and a real integer. It is referred to as a proper quasi-YS if $\Psi \neq 0$.

V. Pirhadi and A. Razavi looked into a gradient almost quasi-YS in [15] several years ago, treating λ as a smooth function. They received a few intriguing formulae. X.M. Chen [7] has recently investigated almost quasi-YSs in relation to almost Cosymplectic manifolds. Additionally, A.M. Blaga has investigated this kind of soliton on warped products [2].

X.M. Chen [8] introduced a novel idea known as the k-almost Yamabe soliton (in short, k-AYS) in a current paper. X.M. Chen claims that if a nonzero function k, a smooth vector field Z, and a smooth function λ exist such that

$$\frac{k}{2} \pounds_Z g - (R - \lambda) g = 0 \tag{1.4}$$

holds, then a Riemannian metric is a k-AYS. The k-AYS is designated by the symbol (g, Z, k, λ). The foregoing equation turns into a gradient k-AYS (g, Φ, k, λ) for some smooth function Φ if $Z = D\Phi$. Then the previous equation takes into the following form

$$Hess\Phi = \frac{1}{k}(R - \lambda) g, \tag{1.5}$$

in which $Hess$ denotes the Hessian. If $Z = 0$, the k-AYS is trivial; otherwise, nontrivial. Additionally, the preceding equation yields the k-YS when $\lambda =$ constant.

Recently, in PF spacetimes, several researchers studied numerous types of solitons like YSs [11], gradient YSs[9], Ricci solitons ([3],[10]), gradient Ricci solitons ([9], [10]), Ricci-Yamabe solitons [16], gradient η-Einstein solitons ([10]), gradient m-quasi Einstein solitons [9], gradient Schouten solitons ([10]), respectively.

The above-mentioned studies tell us that many recent investigations on gradient solitons have been published on Lorentzian manifolds. In [11], U.C. De et al. have studied YSs in PF-spacetimes and in this paper, we investigate k-AYSs in PF-spacetimes which is a natural generalization of YSs. Specifically, we prove the following theorems:

Theorem 1.1 *If a PF spacetime admits a k-AYS, then its potential vector field is Killing if and only if $divZ = 0$.*

Theorem 1.2 *If a PF spacetime permits a k-AYS (g, k, ρ, λ), then*

(i) the integral curves of the vector field ρ are geodesics,

(ii) the spacetime becomes stationary and

(iii) the energy density σ and isotopic pressure p remain invariant under the velocity vector field ρ.

Theorem 1.3 *Let a PF spacetime permit a k-AYS (g, k, Z, λ). If the potential vector field Z is pointwise collinear with the velocity vector field ρ and for a scalar a, $\rho(a) = 0$, then either the PF spacetime represents a phantom era, or potential function Φ is invariant under the velocity vector field ρ.*

Theorem 1.4 *Let a PF spacetime permit a gradient k-AYS $(g, k, D\Phi, \lambda)$. If R, λ, k are invariant under ρ, then the vorticity of the fluid vanishes.*

Remark 1.1 *Since every RW-spacetime is a PF-spacetime [14], all the results of PF-spacetimes which have been established in this paper are also true in RW-spacetime.*

2. Perfect fluid spacetimes

The equation (1.1) yields

$$QX_1 = \alpha X_1 + \beta C(X_1)\rho, \tag{2.1}$$

in which Q is the Ricci operator described by $g(QX_1, Y_1) = S(X_1, Y_1)$.

The covariant derivative of (2.1) provides

$$(\nabla_{X_1} Q)(Y_1) = X_1(\alpha)Y_1 + X_1(\beta)C(Y_1)\rho + \beta(\nabla_{X_1} C)(Y_1)\rho + \beta C(Y_1)\nabla_{X_1}\rho. \tag{2.2}$$

In the absence of the cosmological constant, Einstein's field equations have the following structure

$$S - \frac{R}{2}g = \kappa T, \tag{2.3}$$

κ stands for the gravitational constant and T for the energy momentum tensor.

In a PF spacetime T is described by

$$T = (p + \sigma)C \otimes C + pg, \tag{2.4}$$

in which p is the isotropic pressure and σ denotes the energy density. The equations (1.1), (2.3), and (2.4) together yield

$$\alpha = \frac{\kappa(p - \sigma)}{2 - n}, \quad \beta = \kappa(p + \sigma). \tag{2.5}$$

Additionally, an equation of state (briefly, EOS) with the shape $p = p(\sigma)$ connects p and σ, and the PF-spacetime is known as isentropic. Furthermore, if $p = \sigma$, the PF-spacetime is referred to as stiff matter. The PF- spacetime is referred to as the dark matter era if $p + \sigma = 0$, the dust matter fluid if $p = 0$, and the radiation era if $p = \frac{\sigma}{3}$ [4]. The universe is represented as accelerating phase when $\frac{p}{\sigma} < -\frac{1}{3}$. It covers the quintessence phase if $-1 < \frac{p}{\sigma} < 0$ and phantom era if $\frac{p}{\sigma} < -1$.

3. Examples of gradient k-almost Yamabe solitons

In order to establish the validity of gradient k-AYSs, we set two examples of spacetime. While the second example has a nonconstant potential function, the first one is of constant potential function.

If the function Φ is smooth, we obtain

$$\Phi_{,i} = \frac{\partial \Phi}{\partial w_i} \quad \Phi_{,ij} = \frac{\partial^2 \Phi}{\partial w_i \partial w_j} - \Gamma^k_{ij}\Phi_{,i}\,. \tag{3.1}$$

The equation (1.5) can be described as follows (in local coordinate system):

$$\Phi_{,ij} = \frac{1}{k}(R - \lambda)g_{ij}. \tag{3.2}$$

Example 1:

We choose a Lorentzian metric g, expressed by

$$ds^2 = g_{ij}dw^i dw^j = (dw_1)^2 + (w_1)^2(dw_2)^2 + (w_2)^2(dw_3)^2 - (dw_4)^2, \tag{3.3}$$

in a 4-dimensional Lorentzian manifold $\mathbb{R}^4$, in which $i, j = 1, 2, 3, 4$.

Using the equation (3.3), the Lorentzian metric's nonvanishing components are stated by

$$g_{11} = 1, \quad g_{22} = (w_1)^2, \quad g_{33} = (w_2)^2, \quad g_{44} = -1. \tag{3.4}$$

Using the equation (3.4), the components of the nonvanishing Christoffel symbols are described by:

$$\Gamma^1_{22} = -w_1, \quad \Gamma^2_{12} = \frac{1}{w_1}, \quad \Gamma^2_{33} = -\frac{w_2}{(w_1)^2}, \quad \Gamma^3_{23} = \frac{1}{w_2}.$$

Let us choose $\Phi(w_1, w_2, w_3, w_4)$ on M, an arbitrary smooth function. Hence, the followings are calculated

$$\Phi_{,11} = \frac{\partial^2 \Phi}{(\partial w_1)^2} \quad ,$$

$$\Phi_{,12} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_2} - \frac{1}{w_1}\frac{\partial \Phi}{\partial w_2} = \Phi_{,21} \quad ,$$

$$\Phi_{,13} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_3} = \Phi_{,31} \quad ,$$

$$\Phi_{,14} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_4} = \Phi_{,41} \quad ,$$

$$\Phi_{,22} = \frac{\partial^2 \Phi}{(\partial w_2)^2} - w_1 \frac{\partial \Phi}{\partial w_1} \quad ,$$

$$\Phi_{,23} = \frac{\partial^2 \Phi}{\partial w_2 \partial w_3} - \frac{1}{w_2} \frac{\partial \Phi}{\partial w_3} = \Phi_{,32} \quad ,$$

$$\Phi_{,24} = \frac{\partial^2 \Phi}{\partial w_2 \partial w_4} = \Phi_{,42} \quad ,$$

$$\Phi_{,34} = \frac{\partial^2 \Phi}{\partial w_3 \partial w_4} = \Phi_{,43} \quad ,$$

$$\Phi_{,33} = \frac{\partial^2 \Phi}{(\partial w_3)^2} - \frac{w_2}{(w_1)^2} \frac{\partial \Phi}{\partial w_2} \quad ,$$

$$\Phi_{,44} = \frac{\partial^2 \Phi}{(\partial w_4)^2} \quad .$$

Using the aforementioned equations, we obtain from the equation (3.2)

$$\frac{\partial^2 \Phi}{(\partial w_1)^2} = \frac{1}{k}(R - \lambda) \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_1 \partial w_2} - \frac{1}{w_1} \frac{\partial \Phi}{\partial w_2} = 0 \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_1 \partial w_3} = 0 \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_1 \partial w_4} = 0 \quad ,$$

$$\frac{\partial^2 \Phi}{(\partial w_2)^2} - w_1 \frac{\partial \Phi}{\partial w_1} = \frac{1}{k}(w_1)^2 (R - \lambda) \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_2 \partial w_3} - \frac{1}{w_2} \frac{\partial \Phi}{\partial w_3} = 0 \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_2 \partial w_4} = 0 \quad ,$$

$$\frac{\partial^2 \Phi}{\partial w_3 \partial w_4} = 0 \quad ,$$

$$\Phi_{,33} = \frac{\partial^2 \Phi}{(\partial w_3)^2} - \frac{w_2}{(w_1)^2} \frac{\partial \Phi}{\partial w_2} = \frac{1}{k}(w_2)^2 (R - \lambda) \quad ,$$

$$\frac{\partial^2 \Phi}{(\partial w_4)^2} = -\frac{1}{k}(R - \lambda) \quad .$$

According to the aforementioned equations, $R = \lambda$ and Φ should be a constant function.

As a result, the metric is a gradient k-AYS with a constant potential function.

Example 2:

We choose a Lorentzian metric g, described by

$$ds^2 = g_{ij}dw^i dw^j = e^{w_1+1}(dw_1)^2 + e^{w_1}[(dw_2)^2 + (dw_3)^2 - (dw_4)^2], \tag{3.5}$$

in a Lorentzian manifold $\mathbb{R}^4$, in which $i, j = 1, 2, 3, 4$.

Using the equation (3.5), we find

$$g_{11} = e^{w_1+1}, \quad g_{22} = e^{w_1}, \quad g_{33} = e^{w_1}, \quad g_{44} = -e^{w_1}. \tag{3.6}$$

Using the equation (3.6), we acquire the components of the nonvanishing Christoffel symbols as:

$$\Gamma^1_{11} = \frac{1}{2}, \ \Gamma^1_{22} = \Gamma^1_{33} = -\frac{1}{2e}, \quad \Gamma^1_{44} = \frac{1}{2e}, \ \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{1}{2}.$$

Let us set an arbitrary smooth function $\Phi(w_1, w_2, w_3, w_4)$ on M, and calculate the followings

$$\Phi_{,11} = \frac{\partial^2 \Phi}{(\partial w_1)^2} - \frac{1}{2}\frac{\partial \Phi}{\partial w_1} \quad ,$$

$$\Phi_{,12} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_2} - \frac{1}{2}\frac{\partial \Phi}{\partial w_2} = \Phi_{,21} \quad ,$$

$$\Phi_{,13} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_3} - \frac{1}{2}\frac{\partial \Phi}{\partial w_3} = \Phi_{,31} \quad ,$$

$$\Phi_{,14} = \frac{\partial^2 \Phi}{\partial w_1 \partial w_4} - \frac{1}{2}\frac{\partial \Phi}{\partial w_4} = \Phi_{,41} \quad ,$$

$$\Phi_{,22} = \frac{\partial^2 \Phi}{(\partial w_2)^2} + \frac{1}{2e}\frac{\partial \Phi}{\partial w_1} \quad ,$$

$$\Phi_{,23} = \frac{\partial^2 \Phi}{\partial w_2 \partial w_3} = \Phi_{,32} \quad ,$$

$$\Phi_{,24} = \frac{\partial^2 \Phi}{\partial w_2 \partial w_4} = \Phi_{,42} \quad ,$$

$$\Phi_{,34} = \frac{\partial^2 \Phi}{\partial w_3 \partial w_4} = \Phi_{,43} \quad ,$$

$$\Phi_{,33} = \frac{\partial^2 \Phi}{(\partial w_3)^2} + \frac{1}{2e}\frac{\partial \Phi}{\partial w_1} \quad ,$$

$$\Phi_{,44} = \frac{\partial^2 \Phi}{(\partial w_4)^2} + \frac{1}{2e}\frac{\partial \Phi}{\partial w_1} \quad .$$

Using the previous formulae, we derive from (3.2)

$$\frac{\partial^2 \Phi}{(\partial w_1)^2} - \frac{1}{2}\frac{\partial \Phi}{\partial w_1} = \frac{1}{k}e^{w_1+1}(R - \lambda) \quad ,$$

$$\frac{\partial^2\Phi}{\partial w_1 \partial w_2} - \frac{1}{2}\frac{\partial\Phi}{\partial w_2} = 0 \ ,$$

$$\frac{\partial^2\Phi}{\partial w_1 \partial w_3} - \frac{1}{2}\frac{\partial\Phi}{\partial w_3} = 0 \ ,$$

$$\frac{\partial^2\Phi}{\partial w_1 \partial w_4} - \frac{1}{2}\frac{\partial\Phi}{\partial w_4} = 0 \ ,$$

$$\frac{\partial^2\Phi}{(\partial w_2)^2} + \frac{1}{2e}\frac{\partial\Phi}{\partial w_1} = \frac{1}{k}e^{w_1}(R-\lambda) \ ,$$

$$\frac{\partial^2\Phi}{\partial w_2 \partial w_3} = 0 \ ,$$

$$\frac{\partial^2\Phi}{\partial w_2 \partial w_4} = 0 \ ,$$

$$\frac{\partial^2\Phi}{\partial w_3 \partial w_4} = 0 \ ,$$

$$\frac{\partial^2\Phi}{(\partial w_3)^2} + \frac{1}{2e}\frac{\partial\Phi}{\partial w_1} = \frac{1}{k}e^{w_1}(R-\lambda) \ ,$$

$$\frac{\partial^2\Phi}{(\partial w_4)^2} + \frac{1}{2e}\frac{\partial\Phi}{\partial w_1} = \frac{1}{k}e^{w_1}(R-\lambda) \ .$$

When we take $\frac{R-\lambda}{k}$ = constant and solving the foregoing equations, we acquire $\Phi = 2ce^{w_1+1}$, $c \in \mathbb{R}$. Thus, the metric is a gradient k-AYS with a potential function $\Phi = 2ce^{w_1+1}$.

4. Proof of the main theorems

Proof of the Theorem 1.1:

Let the Lorentzian metric of the PF-spacetimes permit a k-AYS. Hence, we acquire

$$k \ \pounds_Z g = 2(R-\lambda)g. \tag{4.1}$$

Using the explicit form of the Lie derivative, we reveal

$$k[g(\nabla_{X_1} Z, Y_1) + g(X_1, \nabla_{Y_1} Z)] = 2(R-\lambda)g(X_1, Y_1). \tag{4.2}$$

Taking contraction of the foregoing equation yields

$$k \ divZ = 8(R-\lambda), \tag{4.3}$$

where div stands for divergence.

Hence, the equation (4.1) provides

$$k \ \pounds_Z g = \frac{divZ}{4} g, \tag{4.4}$$

where equation (4.3) is used.

From the above, we conclude that Z is Killing if and only if $divZ = 0$. Thus the proof is completed.

Proof of the Theorem 1.2:

Let the k-AYS's potential vector field $Z=\rho$. Hence, the equation (4.2) can be rewritten as

$$k\ \pounds_{\rho}g = k[g(\nabla_{X_1}\rho, Y_1)+g(X_1,\nabla_{Y_1}\rho)] = 2(R-\lambda)g(X_1,Y_1). \tag{4.5}$$

Since ρ is unit timelike in a PF spacetime, hence, we acquire $g(\rho,\rho)=-1$. The covariant derivative of $g(\rho,\rho)=-1$ gives $g(\nabla_{X_1}\rho,\rho)=0$ for all X_1, where ∇ is the Levi-Civita connection.

Now putting $Y_1=\rho$ in the equation (4.5) and using the previous result, we obtain

$$k\ \pounds_{\rho}\rho = 2(R-\lambda)\rho. \tag{4.6}$$

Also replacing $X_1=Y_1=\rho$ in (4.5) provides

$$R=\lambda. \tag{4.7}$$

Hence, the equation (4.6) entails $\pounds_{\rho}\rho=0$. Also from (4.5), we obtain $\pounds_{\rho}g=0$. We know that a spacetime is stationary if it has a time-like Killing vector field (see [12], p.73). Therefore, the spacetime becomes stationary.

Also, if ρ is Killing, then we have $\pounds_{\rho}p=0$ and $\pounds_{\rho}\sigma=0$ (see [12], p.89). Therefore, the proof is finished.

Proof of the Theorem 1.3:

Let $Z=a\rho$, where a is a smooth function, that is, Z is pointwise collinear with ρ. Therefore, we have

$$\nabla_{X_1}Z = X_1(a)\rho + a\nabla_{X_1}\rho. \tag{4.8}$$

Using the previous equation in (4.2), we acquire

$$k[X_1(a)C(Y_1)+ag(\nabla_{X_1}\rho,Y_1)+Y_1(a)C(X_1)+ag(X_1,\nabla_{Y_1}\rho)] = 2(R-\lambda)g(X_1,Y_1). \tag{4.9}$$

Putting $Y_1=\rho$ in the preceding equation yields

$$k[-X_1(a)+\rho(a)C(X_1)+ag(X_1,\nabla_{\rho}\rho)] = 2(R-\lambda)C(X_1). \tag{4.10}$$

Also, replacing X_1 with ρ gives

$$k\rho(a) = (R-\lambda). \tag{4.11}$$

Also contracting (4.9), we get

$$k\ a\ div\rho = 3(R-\lambda). \tag{4.12}$$

Let $\rho(a)=0$, then the equation (4.11) implies $R=\lambda$. Using this in the equation (4.12) yields $div\rho=0$ which implies that the velocity vector field is conservative. The nature of a conservative vector field is always irrotational. Thus we conclude that the PF has zero vorticity. This ends the proof.

Proof of the Theorem 1.4 :

Consider a gradient k-AYS (g,Φ,k,λ) on a PF spacetime. Then, the equation (1.5) is given by

$$k\nabla_{X_1}D\Phi = (R-\ \lambda)Y_1. \tag{4.13}$$

Covariant derivative of (4.13) yields

$$\begin{aligned} k\nabla_{X_1}\nabla_{Y_1}D\Phi &= X_1(R-\ \lambda)Y_1+(R-\ \lambda)\nabla_{X_1}Y_1 \\ &\quad -\frac{1}{k}X_1(k)(R-\ \lambda)Y_1. \end{aligned} \tag{4.14}$$

Interchanging X_1 and Y_1 in (4.14) provides

$$\begin{aligned} k\nabla_{Y_1}\nabla_{X_1}D\Phi &= Y_1(R-\ \lambda)X_1+(R-\ \lambda)\nabla_{Y_1}X_1 \\ &\quad -\frac{1}{k}Y_1(k)(R-\ \lambda)X_1 \end{aligned} \tag{4.15}$$

and

$$k\nabla_{[X_1,\ Y_1]}\ D\Phi=(R-\lambda)[X_1,\ Y_1]. \tag{4.16}$$

Utilizing (4.13)-(4.16) and together with $R(X_1,\ Y_1)V_1=\nabla_{X_1}\nabla_{Y_1}V_1-\nabla_{Y_1}\ \nabla_{X_1}V_1-\nabla_{[X_1,Y_1]}V_1$, we acquire

$$\begin{aligned} k^2R(X_1,\ Y_1)D\Phi &= k[\{X_1(R-\lambda)\}Y_1]-k[\{Y_1(R-\lambda)\}X_1] \\ &\quad -X_1(k)(R-\lambda)Y_1+Y_1(k)(R-\lambda)X_1. \end{aligned} \tag{4.17}$$

Contracting the above equation, we get

$$k^2S(Y_1,D\Phi)=-3k[Y_1(R)-Y_1(\lambda)]-3Y_1(k)(R-\lambda). \tag{4.18}$$

Also, from (1.1), we have

$$S(Y_1,D\Phi)=\alpha Y_1(\Phi)+\beta\rho(\Phi)C(Y_1). \tag{4.19}$$

From the previous two equations, we infer

$$-\frac{3}{k}[Y_1(R)-Y_1(\lambda)]-\frac{3}{k^2}Y_1(k)(R-\lambda)=\alpha Y_1(\Phi)+\beta\rho(\Phi)C(Y_1). \tag{4.20}$$

Putting $Y_1=\rho$ in the previous equation yields

$$-\frac{3}{k}[\rho(R)-\rho(\lambda)]-\frac{3}{k^2}\rho(k)(R-\lambda)=(\alpha-\beta)\rho(\Phi). \tag{4.21}$$

If R,λ,k are invariant under ρ, then

$$(\alpha-\beta)\rho(\Phi)=0. \tag{4.22}$$

Hence, from the above, we say that either $\alpha=\beta$, or $\alpha\neq\beta$.

Case i: If $\alpha=\beta$, then using (2.5) we obtain

$$\frac{p}{\sigma}=-\frac{1}{3}, \tag{4.23}$$

which implies that the PF spacetime represents the phantom era.

Case ii: If $\alpha\neq\beta$, then $\rho(\Phi)=0$, that is, Φ is invariant under ρ.

Hence, the theorem is proved.

5. Discussion

A particular class of solutions on which the metric changes through diffeomorphisms and dilation performs a significant role in the investigation of flow singularities since they appear as probable singularity models. Solitons is a common term used to describe them.

GR is applied mathematics' greatest achievement. GR has long been thought as both the most difficult and elegant physics theory ever created. Understanding GR, which disregards quantum effects, is essential for comprehending cosmology. In GR theory, the universe's matter content is determined by selecting the appropriate EMT, which is acknowledged to behave as a PF-spacetime in cosmological models. Here, EFE completes the crucial step in the construction of the cosmological model. In a few fields, including plasma physics, astronomy, atomic physical science, and nuclear physics, PF-spacetime models in general relativity theory are of great importance.

In this current investigation, we construct two examples to ensure the existence of gradient k-AYSs. Then we demonstrate that a PF spacetime's potential vector field is killing if and only if the potential vector field's divergence vanishes if a k-AYS is admitted to it. Additionally, we demonstrate that under specific circumstances, the PF spacetime reflects a phantom era. Finally, we demonstrate that the fluid's vorticity vanishes when a PF spacetime allows a gradient k-AYS.

Declarations

Competing interests

The authors declare that they have no conflict of interest.

Authors' contributions

All authors contributed equally to this work.

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