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Three-dimensional homogeneous contact metric manifolds with weakly η-Einstein structures

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Abstract: In this paper, we determine the geometric structures of 3-dimensional weakly η-Einstein almost contact metric manifolds and classify 3-dimensional weakly η-Einstein simply connected homogeneous contact metric manifolds based on Perrone's classification.

Key words: Weakly η-Einstein structure, homogeneous contact metric manifold

1. Introduction

Let $M = (M, g)$ be an m-dimensional Riemannian manifold. We consider a symmetric (0,2)-tensor field $\overline{R}$ on M defined by

$$\overline{R}(X,Y) = \sum_{i,j,k=1}^{m} R(e_i, e_j, e_k, X)R(e_i, e_j, e_k, Y)$$

for X, $Y \in \mathfrak{X}(M)$ and a local orthonormal frame field $\{e_i\}$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on M. Here, the $(0,4)$-type curvature tensor R is defined by $R(X,Y,Z,W) = g(R(X,Y)Z), W)$ for X, Y, Z, $W \in \mathfrak{X}(M)$, where $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$ with respect to the Levi-Civita connection ∇ of g.

In [9], Euh, Park, and Sekigawa defined a *weakly Einstein manifold* which is an m-dimensional Riemannian manifold (M, g) satisfying the following condition:

$$\overline{R}(X,Y) = \frac{||R||^2}{m} g(X,Y). \tag{1.1}$$

They showed that a 4-dimensional Einstein manifold necessarily satisfies (1.1), but the converse does not hold. They provided interesting examples of 4-dimensional weakly Einstein not Einstein manifolds [10]. A weakly Einstein manifold can be regarded as a generalization of an Einstein manifold in dimension 4. Weakly Einstein manifolds have been studied by many authors [1, 2, 5, 8, 14]. In particular, Arias-Marco and Kowalski [1] classified 4-dimensional homogeneous weakly Einstein manifolds and showed that there are just two spaces illustrated in [10]. On the other hand, the η-*Einstein structure* is one of the most important geometric structures

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in almost contact geometry, that is, the Ricci tensor ρ is of the form $\rho(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$ with α and β being smooth functions. Cho, Chun, and Euh [6] defined a weakly η-Einstein structure as analogues of a weakly Einstein structure on almost contact metric manifolds. An almost contact metric manifold M with dimension $m = 2n+1$ is said to be *weakly η-Einstein* if the symmetric $(0,2)$-tensor $\overline{R}$ satisfied

$$\overline{R}(X,Y) = \overline{\alpha}g(X,Y) + \overline{\beta}\eta(X)\eta(Y)$$

for smooth functions $\overline{\alpha}$ and $\overline{\beta}$ on M. They showed that a 3-dimensional η-Einstein almost contact metric manifold is necessarily weakly η-Einstein. In this paper, we shall classify a 3-dimensional weakly η-Einstein almost contact metric manifold. In section 2, we prepare for some preliminaries on almost contact metric manifolds. In section 3, we determine the geometric structures of weakly η-Einstein almost contact metric manifolds with dimension 3. In section 4, we recall Perrone's classification [13] of 3-dimensional simply connected homogeneous contact metric manifolds and classify such homogeneous spaces with weakly η-Einstein structures based on his classification.

2. Preliminaries

All manifolds in this paper are assumed to be connected and of class C^∞. We refer to [3] for some preliminaries on contact metric manifolds. Let M be a $(2n+1)$-dimensional differentiable manifold. Let φ, ξ, and η be a tensor field of type $(1,1)$, a vector field and a 1-form on M, respectively. If φ, ξ, and η satisfy the conditions

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad \eta(\xi) = 1$$

for any vector field $X \in \mathfrak{X}(M)$, then it is said that M has an *almost contact structure* (η,φ,ξ) and $M = (M,\eta,\varphi,\xi)$ is called an *almost contact manifold*. If an almost contact manifold (M,η,φ,ξ) admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any X and $Y \in \mathfrak{X}(M)$, then $M = (M,\eta,\varphi,\xi,g)$ is said to be an *almost contact metric manifold*. We define the fundamental 2-form Φ on M by $\Phi(\bar{X},\bar{Y}) = \bar{g}(\bar{X},\phi\bar{Y})$. An almost contact metric manifold $\bar{M}$ with $\Phi = d\eta$ is called *a contact metric manifold*, where d is the exterior differential operator. Given a contact metric manifold $M = (M,\eta,\varphi,\xi,g)$, we define the tensor fields h and τ by $h = \frac{1}{2}(\mathcal{L}_\xi\varphi)$ and $\tau = \mathcal{L}_\xi g$, where $\mathcal{L}_\xi$ is the Lie derivative in the direction of ξ. It is easily checked that h and τ are symmetric operators and satisfy the following conditions:

$$h\xi = 0, \qquad h\varphi = -\varphi h, \tag{2.1}$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \qquad \nabla_\xi\varphi = 0, \tag{2.2}$$

$$\tau(\xi,X) = 0, \qquad \tau(X,Y) = 2g(\varphi X, hY).$$

If the vector field ξ on a contact metric manifold (M,η,φ,ξ,g) is a Killing vector field (i.e. $\tau = 0$), then M is called *a K-contact manifold*. This is the case if and only if $h = 0$. For an almost contact manifold $(M^{2n+1},\eta,\varphi,\xi)$, we consider the manifold $M^{2n+1}\times\mathbb{R}$. We define a vector field on $M^{2n+1}\times\mathbb{R}$ by $(X, f\frac{d}{dt})$, where X is tangent to M^{2n+1}, t the coordinate on $\mathbb{R}$ and f a smooth function on $M^{2n+1}\times\mathbb{R}$. Define an almost complex structure J on $M^{2n+1}\times\mathbb{R}$ by $J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$. If J is integrable, we say that an almost contact structure (η, φ, ξ) is *normal*. A normal contact metric manifold is called a *Sasakian manifold*. It is well-known that a Sasakian manifold is necessarily a K-contact manifold. In dimension 3, the converse is true.

3. Three-dimensional almost contact metric manifolds

Let (M, g) be a 3-dimensional almost contact metric manifold. Then we see that the following equation is satisfied on M:

$$\begin{aligned} R(X,Y,Z,W) =& \rho(Y,Z)g(X,W) - \rho(X,Z)g(Y,W) \\ &+ g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W) \\ &- \frac{r}{2}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \end{aligned} \tag{3.1}$$

for X, Y, Z, $W \in \mathfrak{X}(M)$, where ρ is the Ricci tensor on M and r is the scalar curvature of M. From (3.1), we have the symmetric (0,2)-tensor $\overline{R}$ as follows:

$$\begin{aligned} \overline{R}(X,Y) &= \sum_{i,j,k=1}^{3} R(e_i, e_j, e_k, X)R(e_i, e_j, e_k, Y) \\ &= (2||\rho||^2 - r^2)g(X,Y) + 2r\rho(X,Y) - 2\sum_{i=1}^{3}\rho(X, e_i)\rho(Y, e_i) \end{aligned}$$

for any orthonormal frame field $\{e_i\}$ on M. Now, we suppose that M is weakly η-Einstein. We define the Ricci operator Q of M by $g(QX,Y) = \rho(X,Y)$ and consider the orthonormal frame field $\{e_i\} = \{e_1, e_2, e_3 = \xi\}$ as eigenvectors of Q, that is, $Qe_i = \lambda_i e_i$ $(i = 1, 2)$ and $Q\xi = \lambda_3 \xi$. Then we have

$$2||\rho||^2 - r^2 + 2\lambda_1(r - \lambda_1) = \overline{\alpha}, \tag{3.2}$$

$$2||\rho||^2 - r^2 + 2\lambda_2(r - \lambda_2) = \overline{\alpha}, \tag{3.3}$$

$$2||\rho||^2 - r^2 + 2\lambda_3(r - \lambda_3) = \overline{\alpha} + \overline{\beta}. \tag{3.4}$$

From (3.2) and (3.3), we have

$$(\lambda_1 - \lambda_2)(r - (\lambda_1 + \lambda_2)) = 0. \tag{3.5}$$

From (3.2) and (3.4), we have

$$(\lambda_3 - \lambda_1)(r - (\lambda_1 + \lambda_3)) = \frac{\overline{\beta}}{2}. \tag{3.6}$$

From (3.3) and (3.4), we have

$$(\lambda_3 - \lambda_2)(r - (\lambda_2 + \lambda_3)) = \frac{\overline{\beta}}{2}. \tag{3.7}$$

Then from (3.5) we obtain $\lambda_1 = \lambda_2$ or $\lambda_3 = 0$. (Similarly, from (3.6) and (3.7), we have the same result.) If $\lambda_1 = \lambda_2$, the Ricci operator Q of M has two eigenvalues of multiplicities $(2, 1)$. Then, we see that M has an η-Einstein structure [7]. If $\lambda_3 = 0$, M satisfies $Q\xi = 0$ and hence $\overline{R}$ is given by $\overline{R} = (\lambda_1^2 + \lambda_2^2)g - 2\lambda_1\lambda_2\eta \otimes \eta$. Therefore, we have the following theorem.

Theorem 3.1 *Let M be a 3-dimensional almost contact metric manifold. If M is weakly η-Einstein then either it is η-Einstein or it satisfies $Q\xi = 0$.*

Remark 1 (**[6]**) *A 3-dimensional contact (0,2)-space satisfies $Q\xi = 0$ and it is an example which is weakly η-Einstein but not η-Einstein.*

Let $M = (M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold. Now, let U be the open subset of M on which $h \neq 0$, and V be the open subset of M on which h is identically zero. Then $U \cup V$ is open and dense in M. If U is not empty for any point $p \in U$ we can choose a local orthonormal frame field $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ on a neighborhood of p in such a way that

$$he_1 = \mu e_1, \qquad he_2 = -\mu e_2, \tag{3.8}$$

where μ is a smooth positive function on U. We note that if V is not empty, then V is a Sasakian manifold. Now, we assume that U is not empty. Then by making use of (2.1), (2.2), (3.1), and (3.8), we have the Ricci operator Q on U as following formulas [13]:

$$\begin{aligned} Qe_1 &= \Big(\frac{r}{2} - 1 + \mu^2 + 2\mu\nu\Big)e_1 + \xi(\mu)e_2 + \rho_{13}\xi, \\ Qe_2 &= \xi(\mu)e_1 + \Big(\frac{r}{2} - 1 + \mu^2 - 2\mu\nu\Big)e_2 + \rho_{23}\xi, \\ Q\xi &= \rho_{13}e_1 + \rho_{23}e_2 + 2(1-\mu^2)\xi, \end{aligned} \tag{3.9}$$

where $\nu = -g(\nabla_\xi e_1, e_2)$. We suppose that a 3-dimensional contact metric manifold $(M, \varphi, \xi, \eta, g)$ has a weakly η-Einstein structure. From Theorem 3.1, taking account of (3.9), we get $\nu = 0$ if it is η-Einstein or we have the positive smooth function $\mu = 1$ if $Q\xi = 0$. Then, we have

Corollary 3.2 *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold. If M is weakly η-Einstein, then either $\nu = 0$ or h has eigenvalues 1, -1, and 0.*

4. Three-dimensional weakly η-Einstein homogeneous contact metric manifolds

In this section, we consider the weakly η-Einstein structure on 3-dimensional homogeneous contact metric manifolds. A contact manifold is said to be *homogeneous* if there exists a connected Lie group G acting transitively as a group of diffeomorphisms on it which preserves the contact form η. If g is a metric associated to η and G is a group acting transitively as isometries which leave η invariant, then (η, g) is said to be a *homogeneous contact metric structure* on M. Perrone [13] showed that 3-dimensional simply connected homogeneous contact metric manifolds are Lie groups with left invariant contact metric structure. Furthermore, he classified such homogeneous spaces using the result of Milnor [12] and taking account of the Webster scalar curvature W and torsion invariant $||\tau||$ introduced by Chern and Hamilton (see [4], p. 284). Here, the Webster scalar curvature W is given by

$$W = \frac{1}{8}(r - \rho(\xi,\xi) + 4) = \frac{1}{8}\Big(r + 2 + \frac{||\tau||^2}{4}\Big).$$

Proposition 4.1 *[13] Let $(M, \eta, \varphi, \xi, g)$ be a 3-dimensional simply connected homogeneous contact metric manifold. Then M is a Lie group G together with a left invariant contact metric structure (η, φ, ξ, g).*

(1) *If G is unimodular, then G is one of the following:*

(1.a) *the Heisenberg group H_3 when $W = ||\tau|| = 0$;*

(1.b) *the 3-sphere group $SU(2)$ when $4\sqrt{2}W > ||\tau||$;*

(1.c) *the group $\widetilde{E}(2)$, universal covering of the group of rigid motions of Euclidean 2-space, when $4\sqrt{2}W = ||\tau|| > 0$;*

(1.d) *the group $\widetilde{SL}(2,\mathbb{R})$ when $-||\tau|| \neq 4\sqrt{2}W < ||\tau||$;*

(1.e) *the group $E(1,1)$ of rigid motions of Minkowski 2-space when $4\sqrt{2}W = -||\tau|| < 0$.*

The Lie algebra $\mathfrak{g}$ of G is generated by an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ with commutation relation:

$$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = ae_1, \qquad [e_3, e_1] = be_2. \tag{4.1}$$

(2) *If G is nonunimodular, then the Lie algebra $\mathfrak{g}$ of G is given by*

$$[e_1, e_2] = ce_2 + 2e_3, \qquad [e_2, e_3] = 0, \qquad [e_3, e_1] = de_2, \tag{4.2}$$

where $c \neq 0$, $e_1, e_2 = \varphi e_1 \in ker\ \eta$ and $4\sqrt{2}W < ||\tau||$. If $d = 0$, then the structure is Sasakian and $W = -\dfrac{c^2}{4}$.

First, we consider the weakly η-Einstein unimodular Lie group G with a left invariant contact metric structure. Then by Proposition 4.1, we can choose an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ which satisfies (4.1).

We set $\nabla_{e_i} e_j = \sum_{k=1}^{3} \Gamma_{ijk} e_k$ $1 \leqq i, j \leqq 3$. Then we get $\Gamma_{ijk} = -\Gamma_{ikj}$ and further from (4.1) we obtain the coefficients $\{\Gamma_{ijk}\}$ as follows:

$$\Gamma_{123} = \frac{1}{2}(2 - a + b), \qquad \Gamma_{213} = \frac{1}{2}(-2 - a + b), \qquad \Gamma_{312} = \frac{1}{2}(-2 + a + b) \tag{4.3}$$

and otherwise being zero up to sign. From (4.3), by direct calculations, we have

$$\begin{aligned} &R(e_1, e_2)e_1 = -Ae_2, \qquad R(e_1, e_2)e_2 = Ae_1, \qquad R(e_1, e_2)e_3 = 0, \\ &R(e_1, e_3)e_1 = Be_3, \qquad R(e_1, e_3)e_2 = 0, \qquad R(e_1, e_3)e_3 = -Be_1, \\ &R(e_2, e_3)e_1 = 0, \qquad R(e_2, e_3)e_2 = Ce_3, \qquad R(e_2, e_3)e_3 = -Ce_2, \end{aligned} \tag{4.4}$$

where the coefficients are as follows:

$$\begin{aligned} A &= \frac{1}{4}(a - b)^2 + (a + b) - 3, \\ B &= \frac{1}{4}(a - b)^2 - \frac{1}{2}(a^2 - b^2) + (a - b) - 1, \\ C &= \frac{1}{4}(a - b)^2 + \frac{1}{2}(a^2 - b^2) - (a - b) - 1. \end{aligned}$$

By using (4.4), we have the following Ricci operators:

$$\begin{aligned}
Qe_1 &= \Big(-\frac{1}{2}(b^2-a^2)-2+2b\Big)e_1,\\
Qe_2 &= \Big(\frac{1}{2}(b^2-a^2)-2+2a\Big)e_2,\\
Qe_3 &= \Big(-\frac{1}{2}(b-a)^2+2\Big)e_3.
\end{aligned} \tag{4.5}$$

From (4.1) and by the definition of the tensor field h, we have

$$he_1 = -\frac{1}{2}(a-b)e_1, \qquad he_2 = \frac{1}{2}(a-b)e_2, \qquad he_3 = h\xi = 0. \tag{4.6}$$

On the other hand, a (0,2)-tensor $\overline{R}$ of G is given by

$$\begin{aligned}
&\overline{R}(X,Y)\\
&= \sum_{i,j,k=1}^{3} R(e_i,e_j,e_k,X)R(e_i,e_j,e_k,Y)\\
&=2\sum_{c=1}^{3} R(e_1,e_2,e_c,X)R(e_1,e_2,e_c,Y)\\
&\qquad + R(e_1,e_3,e_c,X)R(e_1,e_3,e_c,Y)\\
&\qquad + R(e_2,e_3,e_c,X)R(e_2,e_3,e_c,Y)\\
&=2\Big\{R(e_1,e_2,e_1,X)R(e_1,e_2,e_1,Y) + R(e_1,e_2,e_2,X)R(e_1,e_2,e_2,Y)\\
&\qquad + R(e_1,e_3,e_1,X)R(e_1,e_3,e_1,Y) + R(e_1,e_3,e_3,X)R(e_1,e_3,e_3,Y)\\
&\qquad + R(e_2,e_3,e_2,X)R(e_2,e_3,e_2,Y) + R(e_2,e_3,e_3,X)R(e_2,e_3,e_3,Y)\Big\}\\
&=2\Big\{A^2g(e_2,X)g(e_2,Y) + A^2g(e_1,X)g(e_1,Y)\\
&\qquad + B^2g(e_3,X)g(e_3,Y) + B^2g(e_1,X)g(e_1,Y)\\
&\qquad + C^2g(e_3,X)g(e_3,Y) + C^2g(e_2,X)g(e_2,Y)\Big\}\\
&=2\Big\{A^2\big(g(X,Y)-\eta(X)\eta(Y)\big) + B^2g(X,Y)\\
&\qquad + C^2\eta(X)\eta(Y) + (C^2-B^2)g(e_2,X)g(e_2,Y)\Big\}\\
&=2\Big\{(A^2+B^2)g(X,Y) + (C^2-A^2)\eta(X)\eta(Y)\\
&\qquad - (B^2-C^2)g(e_2,X)g(e_2,Y)\Big\}
\end{aligned}$$

If G is weakly η-Einstein, then $B^2-C^2=0$. Therefore in the case of $B=C$ we have $a=b$ or $a+b=2$ or in the case of $B=-C$ we have $b=a\pm 2$. Here, we note that if $a=b$, by (4.6), we get $h=0$ and hence we see that G is Sasakian. In addition, from (4.5), G has an η-Einstein structure. If $a+b=2$ $(a\neq b)$, G is non-Sasakian η-Einstein from (4.5). By Milnor's classification of 3-dimensional homogeneous spaces [12], we see that the following structures are admissible.

(1) If $a=b$, M is isometric to one of

$$\begin{cases} H_3 \text{ with an } \eta\text{-Einstein Sasakian structure} \\ SU(2) \text{ with an } \eta\text{-Einstein Sasakian structure} \end{cases}$$

(2) If $a+b=2$ $(a\neq b)$, M is isometric to one of

$$\begin{cases} SU(2) \text{ with a non-Sasakian } \eta\text{-Einstein structure} \\ \widetilde{SL}(2,\mathbb{R}) \text{ with a non-Sasakian } \eta\text{-Einstein structure} \\ \widetilde{E}(2) \text{ with a non-Sasakian } \eta\text{-Einstein structure} \end{cases}$$

(3) If $a-b=\pm 2$, M is isometric to one of

$$\begin{cases} SU(2) \text{ with a contact metric structure} \\ \widetilde{SL}(2,\mathbb{R}) \text{ with a contact metric structure} \\ E(1,1) \text{ with a contact metric structure} \\ \widetilde{E}(2) \text{ with a contact metric structure} \end{cases}$$

Now, if we consider the weakly η-Einstein nonunimodular Lie group G with contact left invariant metric structure, from Proposition 4.1, then there exists an orthonormal basis $\{e_1, e_2=\varphi e_1, e_3=\xi\}$ satisfying (4.2). By using the Koszul formula we have

$$\Gamma_{123}=\frac{d+2}{2}, \qquad \Gamma_{212}=-c, \qquad \Gamma_{213}=\frac{d-2}{2}, \qquad \Gamma_{312}=\frac{d-2}{2} \tag{4.7}$$

all others are zero. Then, using (4.7), by a direct calculation we get

$$\begin{aligned} &R(e_1,e_2)e_1=-\overline{A}e_2-\overline{D}e_3, \qquad R(e_1,e_2)e_2=\overline{A}e_1, \qquad R(e_1,e_2)e_3=\overline{D}e_1, \\ &R(e_1,e_3)e_1=-\overline{D}e_2-\overline{B}e_3, \qquad R(e_1,e_3)e_2=\overline{D}e_1, \qquad R(e_1,e_3)e_3=\overline{B}e_1, \\ &R(e_2,e_3)e_1=0, \qquad R(e_2,e_3)e_2=-\overline{C}e_3, \qquad R(e_2,e_3)e_3=\overline{C}e_2, \end{aligned} \tag{4.8}$$

where the coefficients are as follows:

$$\begin{aligned} &\overline{A}=\frac{d^2+4d-12}{4}-c^2, \qquad \overline{B}=\frac{-3d^2+4d+4}{4}, \\ &\overline{C}=\frac{(d-2)^2}{4}, \qquad \overline{D}=cd. \end{aligned}$$

From (4.8), we obtain the Ricci operator as follows:

$$\begin{aligned} Qe_1 &= \left(-c^2 - 2 + 2d - \frac{d^2}{2}\right) e_1, \\ Qe_2 &= \left(-c^2 - 2 + \frac{d^2}{2}\right) e_2 + cde_3, \\ Qe_3 &= cde_2 + \left(2 - \frac{d^2}{2}\right) e_3. \end{aligned} \tag{4.9}$$

From (4.2) and by the definition of h we have

$$he_1 = \frac{1}{2}de_1, \qquad he_2 = -\frac{1}{2}de_2, \qquad he_3 = 0.$$

We see that G is Sasakian if and only if $d = 0$ (i.e. $h = 0$). If the nonunimodular group $(G, \varphi, \eta, \xi, g)$ is weakly η-Einstein, then we have the following:

$$\begin{aligned}
&\overline{R}(X,Y) \\
&= \sum_{a,b,c=1}^{3} R(e_a, e_b, e_c, X) R(e_a, e_b, e_c, Y) \\
&=2\Big\{ R(e_1,e_2,e_1,X)R(e_1,e_2,e_1,Y) + R(e_1,e_2,e_2,X)R(e_1,e_2,e_2,Y) + R(e_1,e_2,e_3,X)R(e_1,e_2,e_3,Y) \\
&\qquad + R(e_1,e_3,e_1,X)R(e_1,e_3,e_1,Y)R(e_1,e_3,e_2,X)R(e_1,e_3,e_2,Y) + R(e_1,e_3,e_3,X)R(e_1,e_3,e_3,Y) \\
&\qquad + R(e_2,e_3,e_2,X)R(e_2,e_3,e_2,Y) + R(e_2,e_3,e_3,X)R(e_2,e_3,e_3,Y) \Big\} \\
&=2\Big\{ \overline{A}^2 g(e_2,X)g(e_2,Y) + \overline{A}\,\overline{D} g(e_2,X)g(e_3,Y) + \overline{A}\,\overline{D} g(e_3,X)g(e_2,Y) \\
&\quad + \overline{D}^2 g(e_3,X)g(e_3,Y) + \overline{A}^2 g(e_1,X)g(e_1,Y) + \overline{D}^2 g(e_1,X)g(e_1,Y) \\
&\quad + \overline{D}^2 g(e_2,X)g(e_2,Y) + \overline{B}\,\overline{D} g(e_2,X)g(e_3,Y) + \overline{B}\,\overline{D} g(e_3,X)g(e_2,Y) \\
&\quad + \overline{B}^2 g(e_3,X)g(e_3,Y) + \overline{D}^2 g(e_1,X)g(e_1,Y) + \overline{B}^2 g(e_1,X)g(e_1,Y) \\
&\quad + \overline{C}^2 g(e_3,X)g(e_3,Y) + \overline{C}^2 g(e_2,X)g(e_2,Y) \Big\} \\
&=2\Big\{ \overline{A}^2 \big(g(X,Y) - \eta(X)\eta(Y)\big) + \overline{D}^2 \big(g(X,Y) + g(e_1,X)g(e_1,Y) \\
&\quad + \overline{B}^2 \big(g(X,Y) - g(e_2,X)g(e_2,Y)\big) + \overline{C}^2 \big(g(X,Y) - g(e_1,X)g(e_1,Y)\big) \\
&\quad + (\overline{A} + \overline{B})\,\overline{D}\, \big(g(e_2,X)g(e_3,Y) + g(e_3,X)g(e_2,Y)\big) \\
&=\overline{\alpha} g(X,Y) + \overline{\beta}\eta(X)\eta(Y).
\end{aligned} \tag{4.10}$$

From (4.10), we have the following equations:

$$\overline{R}(e_1,e_1) = 3(\overline{A}^2 + \overline{B}^2 + 2\overline{D}^2) = \overline{\alpha}, \qquad \overline{R}(e_2,e_2) = 2(\overline{A}^2 + \overline{C}^2 + \overline{D}^2) = \overline{\alpha}$$

$$\overline{R}(e_3,e_3) = 2(\overline{B}^2 + \overline{C}^2 + \overline{D}^2) = \overline{\alpha} + \overline{\beta}, \qquad \overline{R}(e_2,e_3) = 2((\overline{A} + \overline{B})\overline{D}) = 0.$$

Then, we have the relations:

$$(\overline{A}+\overline{B})\overline{D}=0, \qquad \overline{B}^2+\overline{D}^2=\overline{C}^2. \tag{4.11}$$

Therefore, from (4.11) we can consider the two cases:

Case I) $\overline{A}+\overline{B}=0$ and $\overline{B}^2+\overline{D}^2=\overline{C}^2$.

Since $\overline{A}+\overline{B}=-\frac{1}{2}(d-2)^2-c^2=0$, we have $c=0$ and $d=2$. It is a contradiction for the condition $c\neq 0$.

Case II) $\overline{D}=0$ and $\overline{B}^2+\overline{D}^2=\overline{C}^2$.

(II-1) $\overline{B}=\overline{C}$ and $\overline{D}=0$.
From $\overline{B}=\overline{C}$ we obtain $d=0$ (Sasakian) or $d=2$. Since $\overline{D}=cd=0$ and $c\neq 0$ by assumption, we have $d=0$.

(II-2) $\overline{B}=-\overline{C}$ and $\overline{D}=0$.
From $\overline{B}=-\overline{C}$ we get $d=\pm 2$. It is a contradiction for $\overline{D}=0$ and $c\neq 0$.

Then, from (4.8), we have the curvatures $R_{1331}=R_{2332}=1$, $R_{1212}=c^2+3$ and otherwise being zero up to sign. Furthermore, since d is identically zero, we easily check that G has an η-Einstein structure from (4.9).

Finally, we have the following theorem.

Theorem 4.2 *Let (M,η,φ,ξ,g) be a 3-dimensional simply connected homogeneous contact metric manifold. Then M is a Lie group G together with a left invariant contact metric structure (η,φ,ξ,g). Suppose that G is weakly η-Einstein.*

(1) *If G is unimodular, then M is isometric to one of the following Lie groups:*

(1.1) *Heisenberg group H_3 with an η-Einstein Sasakian structure;*

(1.2) *$SU(2)$ with either an η-Einstein Sasakian structure, a non-Sasakian η-Einstein structure, or a contact metric structure;*

(1.3) *$\widetilde{E}(2)$ with either a non-Sasakian η-Einstein structure or a contact metric structure;*

(1.4) *$\widetilde{SL}(2,\mathbb{R})$ with either a non-Sasakian η-Einstein structure or a contact metric structure;*

(1.5) *$E(1,1)$ with a contact metric structure*

(2) *If G is nonunimodular, then M is an η-Einstein Sasakian manifold whose sectional curvatures containing the direction ξ are the same as one.*

Remark 2 *We summarize the above characterization as the table. Let (M,η,φ,ξ,g) be a 3-dimensional simply connected homogeneous contact metric manifold with a weakly η-Einstein structure. Then M is isometric to one of Lie groups which can admit the following structures:*

Geometric structures	Sasakian	non-Sasakian
η -Einstein	H_3, $SU(2)$, nonunimodular	$SU(2)$, $\widetilde{E}(2)$ $\widetilde{SL}(2,\mathbb{R})$
not η-Einstein	none	$SU(2)$, $\widetilde{E}(2)$ $\widetilde{SL}(2,\mathbb{R})$, $E(1,1)$

Consequently, we see that $SU(2)$, $\widetilde{E}(2)$ $\widetilde{SL}(2,\mathbb{R})$, or $E(1,1)$ with only a contact metric structure can be weakly η-Einstein not η-Einstein.

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