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On orthogonally additive band operators and orthogonally additive disjointness preserving operators

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Abstract: Let M and N be Archimedean vector lattices. We introduce orthogonally additive band operators and orthogonally additive inverse band operators from M to N and examine their properties. We investigate the relationship between orthogonally additive band operators and orthogonally additive disjointness preserving operators and show that under some assumptions on vector lattices M or N, these two classes are the same. By using this relation, we show that if μ is a bijective orthogonally additive band operator (resp. orthogonally additive disjointness preserving operator) from M into N then $\mu^{-1}: N \to M$ is an orthogonally additive band operator (resp. orthogonally additive disjointness preserving operator).

Key words: Vector lattice, orthogonally additive band operator, orthogonally additive inverse band operator, orthogonally additive disjointness preserving operator

1. Introduction

Let M be a vector lattice. The order closed ideals of M are called bands. The smallest band including a given nonempty subset D of M is called the band generated by D and is denoted by B_D. If D consists of only one element, say m, the band generated by m is denoted by B_m. The elements m and k of a vector lattice M are called disjoint if $|m| \wedge |k| = 0$ and denoted by $m \perp k$. If D is a nonempty subset of a vector lattice M, the set of all elements disjoint to each element of D is called the disjoint complement of D, and denoted by D^d. If M is an Archimedean vector lattice then $B_A = A^{dd}$ for each nonempty subset A of M [13, Theorem 22.3]. Let k be an element of a vector lattice M. An element $m \in M$ is said to be a component (or fragment) of k whenever $m \perp k - m$ and is written $m \sqsubseteq k$. The set of all fragments of an element $k \in M$ is denoted by $\mathcal{F}_k$.

Let M and N be a vector lattice and a real linear space, respectively. A function $\mu : M \to N$ is said to be an *orthogonally additive operator* (*oa-operator in short*) if $\mu(m + k) = \mu(m) + \mu(k)$ for all $m, k \in M$ with $m \perp k$. From the definition one can easily show that if μ is an oa-operator, then $\mu(0) = 0$ holds. For any function μ from $\mathbb{R}$ to $\mathbb{R}$ to be an oa-operator, a necessary and sufficient condition is that $\mu(0) = 0$. Let M and N be vector lattices. A function $\mu : M \to N$ is called a disjointness preserving function if $\mu(m) \perp \mu(k)$ for all $m, k \in M$ satisfying $m \perp k$. If a disjointness preserving function is also orthogonally additive, it is called an *orthogonally additive disjointness preserving operator* (*oadp-operator in short*).

In the problem section of [6], Abramovich and Kitover gave the following open problem: if M and N

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2010 *AMS Mathematics Subject Classification:* 47H30, 47H99, 47B60, 46B99 are vector lattices, and $\mu : M \to N$ is (linear) invertible and disjointness preserving operator, is $\mu^{-1} : N \to M$ disjointness preserving operator as well? In [5,6], Abramovich and Kitover constructed an example and show that the answer to this question is negative for invertible disjointness preserving operators that are defined between arbitrary vector lattices. Under some conditions on vector lattices M or N, the positive answers of the above question were given in [5,6,10,12,17,18,19,20]. Then, the same question have been raised for other operators. A similar question was tried to be answered for band preserving operators in [11]. Turan and Özcan researched the answer to the following question: Let $\mu : M \to N$ be a (linear) bijective band operator between vector lattices. Is $\mu^{-1} : N \to M$ also a band operator? It was shown that this question has an affirmative answer under some conditions on vector lattices M or N [18]. In recent years, effective studies have been done on orthogonally additive and in general on nonlinear operators [1,2,3,4,8,9,14,15]. Orthogonally additive disjointness preserving operators between vector lattices were introduced and studied in [2,8,14]. In this study, we introduce the definitions of orthogonally additive band operator and orthogonally additive inverse band operator then we study their properties. Then, we obtain the relationship of an orthogonally additive band operator with an orthogonally additive disjointness preserving operator. Additionally, we investigate that the inverse of an invertible orthogonally additive disjointness preserving operator (resp. orthogonally additive band operator) is also an orthogonally additive disjointness preserving operator (resp. orthogonally additive band operator).

We refer to [7,13] for unexplained concepts and terminologies of vector lattices and operators which are not explained here. We assume that all vector lattices in this paper are Archimedean.

2. Orthogonally additive band operators and orthogonally additive disjointness preserving operators

The band operator and inverse band operator (linear cases) were studied in [16,18]. Now, we will give the definitions of the orthogonally additive band operator and orthogonally additive inverse band operator and we will search their properties.

Definition 2.1 *Let M and N be vector lattices and let $\mu : M \to N$ be an oa-operator.*
(i) μ is called an orthogonally additive band operator (oab-operator in short) whenever $\mu(B)$ is a band in N for each band B in M.
(ii) μ is called an orthogonally additive inverse band operator (oaib-operator in short) whenever $\mu^{-1}(D)$ is a band in M for each band D in N.

The following two lemmas are easily obtained from the definitions of *oab*-operator and *oaib*-operator.

Lemma 2.2 *Let μ be an oa-operator between vector lattices M and N. Then the following assertions hold:*
(i) μ is an oab-operator iff $B_{\mu(A)} \subseteq \mu(B_A)$ for every set $A \subseteq M$.
(ii) μ is an oaib-operator iff $\mu(B_A) \subseteq B_{\mu(A)}$ for every set $A \subseteq M$.

Lemma 2.3 *Let μ be an oa-operator between vector lattices M and N. If μ^{-1} is an oa-operator, then the following assertions hold:*
(i) μ is an oab-operator iff μ^{-1} is an oaib-operator.
(ii) μ is an oaib-operator iff μ^{-1} is an oab-operator.

Definition 2.4 *Let μ be an oa-operator between vector lattices M and N.*
(i) If $B_{\mu(m)} \subseteq \mu(B_m)$ for each $m \in M$, then μ is called an orthogonally additive principal band operator (oapb-operator in short).
(ii) If $\mu(B_m) \subseteq B_{\mu(m)}$ for each $m \in M$, then μ is called an orthogonally additive inverse principal band operator (oaipb-operator in short).

If $A = \{m\}$ is taken in the Lemma 2.2 , the following result can be obtained.

Corollary 2.5 *Let μ be an oa-operator between vector lattices M and N. Then the following assertions hold:*
(i) If μ is an oab-operator, then μ is an oapb-operator.
(ii) If μ is an oaib-operator, then μ is an oaipb-operator.

In general, orthogonally additive band operators and orthogonally additive disjointness preserving operators are two distinct classes of operators.

Example 2.6 *Let $\mu : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\mu(x, y) = (0, x^3 + y^3)$. Then, μ is an oab-operator. But μ is not an oadp-operator as $\mu(1, 0) \perp \mu(0, 1)$ is not true whenever $(1, 0) \perp (0, 1)$.*

Example 2.7 *Let M be a vector lattice. The function $\mu : M \to M$ defined by $m \to |m|$ is an oadp-operator. However, μ is not an oab-operator.*

Since $B_m = B_{|m|}$ and $B_{|m|} \cap B_{|k|} = B_{|m| \wedge |k|}$ for arbitrary elements m and k in a vector lattice M, we have the following Lemma.

Lemma 2.8 *Let M be a vector lattice and $m, k \in M$. Then, $m \perp k$ if and only if $B_m \cap B_k = \{0\}$.*

Proposition 2.9 *Let M, N be vector lattices and $\mu : M \to N$ be an oa-operator. It follows that*
(i) if μ is an injective oapb-operator, then μ is an oadp-operator,
(ii) if μ is a bijective oaipb-operator, then μ^{-1} is an oadp-operator.

Proof (i) Let $m \perp k$ in M. Then, $B_m \cap B_k = \{0\}$ and $B_{\mu(m)} \subseteq \mu(B_m)$, $B_{\mu(k)} \subseteq \mu(B_k)$ as μ is an orthogonally additive principal band operator. Hence, $B_{\mu(m)} \cap B_{\mu(k)} \subseteq \mu(B_m) \cap \mu(B_k) = \mu(B_m \cap B_k) = \{0\}$ which yields $\mu(m) \perp \mu(k)$. (ii) Since μ^{-1} is a principal band operator, similar to the proof of (i), it is obtained that μ^{-1} is disjointness preserving function. Since μ is orthogonally additive and μ^{-1} is disjointness preserving, for every $m, k \in N$ satisfying $m \perp k$, we get $\mu[\mu^{-1}(m) + \mu^{-1}(k)] = m + k$ and so $\mu^{-1}(m + k) = \mu^{-1}(m) + \mu^{-1}(k)$. Thus, μ^{-1} *is an oadp-operator.* □

Let us now recall some definitions that we will use later. A vector lattice has a cofinal family of projection bands if for each nonzero band B there is a nonzero projection band $D \subseteq B$. The following implications hold in any vector lattice M:

$$\begin{aligned} \text{Dedekind complete} \quad &\Rightarrow \quad \text{projection property} \Rightarrow \text{principal projection property} \\ &\Rightarrow \quad \text{cofinal family of projection bands.} \end{aligned}$$

Let μ be a function defined between vector lattices M and N. It is said that μ satisfies the condition $\vdash$ if for each band B in M and each m in M with $\mu(m) \perp \mu(B)$ implying that $m \perp B$ [5].

Proposition 2.10 *Let M be a vector lattice with a cofinal family of projection bands, and N be an arbitrary vector lattice. If $\mu : M \to N$ is an injective oadp-operator, then μ satisfies the condition $\vdash$.*

Proof Let $\mu(m) \perp \mu(B)$ for some $m \in M$ and some band B in M. Assume that m is not disjoint to B. In this case, there exists an element b in B_+ such that $|m| \wedge b > 0$. Let $k = |m| \wedge b > 0$. From the hypothesis, there is a projection band D in M such that $D \subseteq B_k$ and $D \neq \{0\}$. Let $Q_D(m) = n$, with the band projection Q_D defined by D. From the definition of n, clearly $m = n + (m - n)$, $n \perp (m - n)$, and $n \neq 0$. If $n = 0$, we have

$$\begin{aligned} n = 0 \quad &\Rightarrow \quad Q_D(m) = 0 \\ &\Rightarrow \quad m \in D^d \\ &\Rightarrow \quad B_m \subseteq D^d \\ &\Rightarrow \quad D \subseteq B_k \subseteq B_m \subseteq D^d \\ &\Rightarrow \quad D = \{0\}. \end{aligned}$$

Since this is a contradiction, $n \neq 0$. Also, $\mu(n) \in \mu(B)$ because $n \in D \subseteq B_k \subseteq B_b \subseteq B$. On the other hand, considering that μ is an *oadp*-operator, we see that

$$\begin{aligned} n \perp (m - n) \quad &\Rightarrow \quad \mu(n) \perp \mu(m - n) \\ &\Rightarrow \quad \mu(n) \perp \mu(m) - \mu(n) \\ &\Rightarrow \quad |\mu(m)| = |\mu(n) + \mu(m) - \mu(n)| = |\mu(n)| + |\mu(m) - \mu(n)| \\ &\Rightarrow \quad |\mu(n)| \leq |\mu(m)| \\ &\Rightarrow \quad \mu(n) \perp \mu(B) \\ &\Rightarrow \quad \mu(n) \in \mu(B)^d, \end{aligned}$$

and so $\mu(n) = 0$. Since μ is injective and $\mu(0) = 0$, we get $n = 0$, which is a contradiction. □

Proposition 2.11 *Let $\mu : M \to N$ is a surjective oadp-operator from a vector lattice M to a vector lattice N and μ satisfies the condition $\vdash$. Then μ is an oab-operator.*

Proof It is enough to show that $\mu(B)^{dd} = \mu(B)$ for every B band in M. Since μ is a disjointness preserving function, it is easy to see that $\mu(B^d) \subseteq \mu(B)^d$. On the other hand, if $n \in \mu(B)^d$ holds, by the surjectivity of μ, there exists some $u \in M$ with $\mu(u) = n$. It follows that $u \perp B$ because μ satisfies the condition $\vdash$. Then $\mu(u) = n \in \mu(B^d)$, which proves that $\mu(B)^d \subseteq \mu(B^d)$. Thus, we get $\mu(B^d) = \mu(B)^d$ implying that

$$\mu(B)^{dd} = \mu(B^d)^d = \mu(B^{dd}) = \mu(B).$$

□

Considering Proposition 2.10 and Proposition 2.11, the following result is obtained.

Corollary 2.12 *Let M be a vector lattice with a cofinal family of projection bands and let N be an arbitrary vector lattice. If $\mu : M \to N$ is a bijective oadp-operator, then μ is an oab-operator.*

Corollary 2.13 *Let M and N be vector lattices and $\mu : M \to N$ be a bijective function. Then, μ and μ^{-1} are oadp-operators if and only if μ and μ^{-1} are oab-operators.*

Proof Let μ and μ^{-1} be *oadp*-operators. Since μ^{-1} is disjointness preserving, it can be easily seen that μ satisfies the condition $\vdash$. From Proposition 2.11, we get μ is an *oab*-operator. Similarly, it can be shown that μ^{-1} is an *oab*-operator. If μ and μ^{-1} are *oab*-operator, then μ and μ^{-1} are *oadp*-operators from Proposition 2.9. $\square$

By using Theorem 4.9 in [14], we get the following corollary.

Corollary 2.14 *Let μ be a bijective oa-operator between vector lattices M and N, and M has cofinal family of projection bands. Then the following statements are equivalent:*
(i) μ is an oab-operator.
(ii) μ is an oapb-operator.
(iii) μ is an oadp-operator.
(iv) For every $m \in M$ there exists an $n \in N$ such that $\mu(\mathcal{F}_m) \subseteq \mathcal{F}_n$ (that is, μ is laterally bounded).
(v) $\mu(\mathcal{F}_m) \subseteq \mathcal{F}\mu(m)$ for every $m \in M$.
(vi) If $m \sqsubseteq n$ then $\mu(m) \sqsubseteq \mu(n)$ for every $m, n \in M$ (that is, μ is lateral order preserving).

Proof (i) $\Rightarrow$(ii) $\Rightarrow$(iii) $\Rightarrow$(i) The implications are obtained from Corollary 2.5 and Proposition 2.9 and Corollary 2.12.

(iii) $\Rightarrow$(iv) $\Rightarrow$(v) $\Rightarrow$(vi)$\Rightarrow$(iii) follow from Theorem 4.9 in [14]. This completes the proof. $\square$

Let M be a vector lattice and $L_r(M)$ be the vector lattice of the regular operators defined on M. The ideal generated by the identity operator I in $L_r(M)$ is called the ideal center of M and denoted by $Z(M)$ (i.e. $Z(M) = \{\pi \in L_r(M) : \exists\lambda \in \mathbb{R}^+,\ |\pi| \leq \lambda I\}$). The Boolean algebra of order projections of M is denoted by $\wp(M)$. By definition, $\wp(M) \subseteq Z(M)$. Let $T : M \to N$ be a linear operator. If there exists a (linear) lattice homomorphism (or disjointness preserving operator) $\rho : Z(M) \to Z(N)$ which satisfies $\rho(\gamma)T = T\gamma$ for each $\gamma \in Z(M)$, with the help of this operator, under some conditions on M, Turan obtained that T and T^{-1} are disjointness preserving operators in [17, Proposition 3.7, Proposition 3.8]. Let $\mu : M \to N$ be a function. Abasov and Pliev said that μ is a ρ-operator whenever there exists a Boolean algebra homomorphism $\rho : \wp(M) \to \wp(N)$ which satisfies $\rho(Q)\mu = \mu Q$ for each $Q \in \wp(M)$. If μ is a ρ-operator, then they obtained some properties of μ with the help of the Boolean algebra homomorphism ρ [1]. Let B be a Boolean algebra. It is known that it is possible to define in B an addition and a multiplication such that with respect to these operations, B becomes a commutative ring (in the ordinary algebraic sense) [13, p.8]. Therefore, the orthogonally additive function between two Boolean algebras can be defined as between vector lattices. It can be easily shown that the Boolean algebra homomorphism between two Boolean algebras is an orthogonally additive function. We will consider the case where the function ρ is only an orthogonally additive function. If there exists an orthogonally additive function $\rho : \wp(M) \to \wp(N)$ which satisfies $\rho(Q)\mu = \mu Q$ for each $Q \in \wp(M)$, then we will obtain the properties of μ with the help of orthogonally additive function ρ. Next, for the bijective case of μ, we will investigate the necessary conditions for the existence of an orthogonally additive function ρ. Using this, we will show that μ^{-1} satisfies the same properties with μ.

Proposition 2.15 *Let M be a vector lattice with the principal projection property, let N be a vector lattice, and $\mu : M \to N$ be a function. If there exists an orthogonally additive function $\rho : \wp(M) \to \wp(N)$ which* *satisfies* $\rho(Q)\mu = \mu Q$ *for each* $Q \in \wp(M)$*, then* μ *is an oadp-operator.*

Proof Take $m, k \in M$, with $m \perp k$. Let Q_m and Q_k be the band projections on B_m and B_k, respectively. From $m \perp k$ we see that $Q_m \perp Q_k$ in $\wp(M)$. Also, since ρ is an orthogonally additive function, we have

$$\begin{aligned}
\mu(m+k) &= \mu[(Q_m + Q_k)(m+k)] \\
&= [\mu(Q_m + Q_k)](m+k) \\
&= [\rho(Q_m + Q_k)\mu](m+k) \\
&= [\rho(Q_m) + \rho(Q_k)]\mu(m+k) \\
&= \rho(Q_m)\mu(m+k) + \rho(Q_k)\mu(m+k) \\
&= \mu Q_m(m+k) + \mu Q_k(m+k) \\
&= \mu(m) + \mu(k).
\end{aligned}$$

Let us show that μ is a disjointness preserving function. For this, from Theorem 4.9 in [14], it will be sufficient to prove that $m \sqsubseteq k$ implies $\mu(m) \sqsubseteq \mu(k)$ for every $m, k \in M$. Considering Lemma 3.12 in [14], we have

$$\begin{aligned}
m \sqsubseteq k &\Rightarrow \exists Q \in \wp(M),\ Q(k) = m \\
&\Rightarrow \exists Q \in \wp(M),\ \mu(Q(k)) = \mu(m) \\
&\Rightarrow \exists Q \in \wp(M),\ \rho(Q)(\mu(k)) = \mu(m),
\end{aligned}$$

this shows that $\mu(m) \sqsubseteq \mu(k)$. □

From Corollary 2.14 and Proposition 2.15, we get the following result.

Corollary 2.16 *Let* M *be a vector lattice with the principal projection property,* N *be a vector lattice, and* $\mu : M \to N$ *be a bijective function. If there exists an orthogonally additive function* $\rho : \wp(M) \to \wp(N)$ *which satisfies* $\rho(Q)\mu = \mu Q$ *for each* $Q \in \wp(M)$*, then* μ *is an oab-operator.*

Now, we will obtain the converse direction of Proposition 2.15.

Proposition 2.17 *Let* M *be a vector lattice with a cofinal family of projection bands, let* N *be a vector lattice with the projection property, and* $\mu : M \to N$ *be a bijective oab-operator. Then there exists a Boolean algebra homomorphism* $\rho : \wp(M) \to \wp(N)$ *satisfying* $\rho(Q)\mu = \mu Q$ *for every* $Q \in \wp(M)$*.*

Proof For an arbitrary element Q from $\wp(M)$, since $Q(M) = B$ is a projection band, we can take $Q = Q_B$. Since μ is *oabp*-operator and N is a vector lattice with the projection property, $\mu(B)$ is a projection band in N. Thus, we can define the function $\rho : \wp(M) \to \wp(N)$ as $Q_B \to \rho(Q_B) = Q_{\mu(B)}$. If D and B are projection bands, then $D + B$ is a projection band and the equations $D + B = (D \cap B^d) + B$, $Q_D \vee Q_B = Q_{D+B}$ and $Q_D \wedge Q_B = Q_{D \cap B}$ are hold. Also, from Proposition 2.9 μ is disjointness preserving, and from Proposition 2.10 μ satisfies the condition $\vdash$. Thus, we have $\mu(B^d) = \mu(B)^d$. From all these, we see that

$$
\begin{aligned}
\mu(D+B) &= \mu[(D\cap B^d)+B] \\
&= \mu(D\cap B^d)+\mu(B) \\
&= [\mu(D)\cap \mu(B^d)]+\mu(B) \\
&= [\mu(D)\cap \mu(B)^d]+\mu(B) \\
&= \mu(D)+\mu(B).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\rho(Q_D\vee Q_B) &= \rho(Q_{D+B}) \\
&= Q_{\mu(D+B)} \\
&= Q_{\mu(D)+\mu(B)} \\
&= Q_{\mu(D)}\vee Q_{\mu(B)} \\
&= \rho(Q_D)\vee \rho(Q_B)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(Q_D\wedge Q_B) &= \rho(Q_{D\cap B}) \\
&= Q_{\mu(D\cap B)} \\
&= Q_{\mu(D)\cap\mu(B)} \\
&= Q_{\mu(D)}\wedge Q_{\mu(B)} \\
&= \rho(Q_D)\wedge \rho(Q_B).
\end{aligned}
$$

Clearly, $\rho(\theta_M)=\theta_N$ and $\rho(I_M)=I_N$, and so ρ is a Boolean algebra homomorphism. It is easy to see that $\rho(Q)\mu=\mu Q$ for every $Q\in\wp(M)$, and the proof is completed. □

Since every Boolean algebra homomorphism is orthogonally additive, we get the following result.

Corollary 2.18 *Let M be a vector lattice with a cofinal family of projection bands, N be a vector lattice with the projection property, and $\mu: M\to N$ be a bijective oab-operator. Then there exists an orthogonally additive function $\rho:\wp(M)\to\wp(N)$ satisfying $\rho(Q)\mu=\mu Q$ for every $Q\in\wp(M)$.*

From Corollary 2.12, the following result is obtained.

Corollary 2.19 *Let M be a vector lattice with a cofinal family of projection bands, N be a vector lattice with the projection property, and $\mu: M\to N$ be a bijective oadp-operator. Then there exists an orthogonally additive function $\rho:\wp(M)\to\wp(N)$ satisfying $\rho(Q)\mu=\mu Q$ for every $Q\in\wp(M)$.*

Corollary 2.20 *Let M be a vector lattice with the principal projection property, N be a vector lattice with the projection property and $\mu: M\to N$ be a bijective function. Then, μ is an oadp-operator (or oab-operator) if and only if there exists an orthogonally additive function $\rho:\wp(M)\to\wp(N)$ satisfying $\rho(Q)\mu=\mu Q$ for every $Q\in\wp(M)$.*

Corollary 2.21 *Let M and N be vector lattices with the projection property and $\mu : M \to N$ be a bijective oadp-operator(resp. oab-operator). Then, the Boolean algebra homomorphism $\rho : \wp(M) \to \wp(N)$ given in Proposition 2.17 is surjective if and only if $\mu^{-1} : N \to M$ is an oadp-operator (resp. oab-operator).*

Proof Let ρ be surjective. It is easy to see that ρ is injective. Since the inverse of every bijective Boolean algebra homomorphism is also a Boolean algebra homomorphism, $\rho^{-1} : \wp(N) \to \wp(M)$ is a Boolean algebra homomorphism, and so ρ^{-1} is orthogonally additive. Moreover, $\rho^{-1}(P)\mu^{-1} = \mu^{-1}P$ holds for each $P \in \wp(N)$. By Proposition 2.15, μ^{-1} is an *oadp*-operator. Now, let μ^{-1} be an *oadp*-operator. For every $S \in \wp(N)$ there is a band D in N with $S = S_D$. By Corollary 2.12, $\mu^{-1}(D)$ is a band in M. Since M has projection property, then $Q_{\mu^{-1}(D)}$ is an element of $\wp(M)$, and $\rho(Q_{\mu^{-1}(D)}) = S$ holds. □

Remark 2.22 *Let M and N be vector lattices with the projection property and $\mu : M \to N$ be a bijective oa-operator. If μ satisfies one of the six conditions in Corollary 2.14 then, from Proposition 2.17, there exists an orthogonally additive function $\rho : \wp(M) \to \wp(N)$ satisfying $\rho(Q)\mu = \mu Q$ for every $Q \in \wp(M)$. Hence, we obtain that the $\rho : \wp(M) \to \wp(N)$ is surjective if and only if $\mu^{-1} : N \to M$ satisfies the same condition with μ.*

A function μ from a vector lattice M into itself is called band preserving if $\mu(B) \subseteq B$ holds for each band B of M. The properties of linear band preserving operators are well known. In [3], orthogonally additive band preserving operators are defined and their properties are examined. It has been shown that if M is a vector lattice with the projection property, and $\mu : M \to M$ is a function, then μ is an orthogonally additive band preserving operator iff μ commutes with projections (i.e. $\mu Q = Q\mu$ for all $Q \in \wp(M)$) [3, Proposition 2]. In general, orthogonally additive band operators and orthogonally additive band preserving operators are two distinct classes of functions.

Example 2.23 *The function μ given in Example 2.6 is an oab-operator, but μ is not an orthogonally additive band preserving operator.*

Example 2.24 *The function μ given in Example 2.7 is an orthogonally additive band preserving operator. However, μ is not an oab-operator.*

Corollary 2.25 *Let M be a vector lattice with a cofinal family of projection bands and $\mu : M \to M$ be a bijective orthogonally additive band preserving operator. Then, μ is an oab-operator.*

Proof Since each band preserving operator is the disjointness preserving operator, μ is a disjointness preserving operator. The proof is completed by using Corollary 2.14. □

Corollary 2.26 *Let M be a vector lattice with the projection property, and $\mu : M \to M$ be a bijective orthogonally additive band preserving operator. Then, $\mu^{-1} : M \to M$ is an orthogonally additive band preserving operator.*

Proof If we take ρ as the unit function from $\wp(M)$ to $\wp(M)$, then ρ is an orthogonally additive function, and $Q\mu^{-1} = \mu^{-1}Q$ holds for each $Q \in \wp(M)$. By Proposition 2.15 μ^{-1} is an orthogonally additive, and it is band preserving operator from Proposition 2 in [3].

□

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