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A study on conformable fractional version of Bullen-type inequalities

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Abstract: In this paper, we give an equality for the case of differentiable convex functions involving conformable fractional integrals. Bullen-type inequalities for the conformable fractional integrals are established by using this equality. Some important inequalities are obtained by taking advantage of the convexity, the Hölder inequality and the power mean inequality. By using special choices, we present some known results in the literature. Furthermore, we give an example using a graph in order to show that our main results are correct.

Key words: Bullen type inequality, fractional conformable integrals, fractional conformable derivatives, fractional calculus, convex function

1. Introduction

One of the most famous inequalities in the case of convex functions is the Hermite-Hadamard-type inequality in the literature. Hence, many mathematicians have considered the Hermite- Hadamard inequality and related inequalities such as trapezoid-type, midpoint-type, Simpson-type, and Bullen-type inequalities have contributed to science. In addition to this, Riemann-Liouville fractional integrals, conformable fractional integrals and many types of fractional integrals have been investigated to these types of inequalities.

Fractional integral operators in a variety of scientific disciplines have been investigated widely. Using the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative is improved in paper [19]. Some significant requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. However, in paper [2] the author proved that the conformable approach in [19] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [13, 24].

Many mathematicians have considered midpoint and trapezoid-type inequalities that give bounds via the left-hand side and right-hand side of the Hermite–Hadamard inequality, respectively. Dragomir and Agarwal first presented trapezoid-type inequalities for convex functions in [7], while Kirmacı first investigated midpoint-type inequalities for convex functions in [18]. Sarikaya et al. and Iqbal et al. presented some fractional midpoint and trapezoid-type inequalities for the case of convex mappings in papers [14] and [23], respectively. In addition to these, Bullen [5] proved the well-known Bullen-type inequalities in the literature. In paper [22], Sarikaya et al. established generalized Bullen inequality for generalized convex function. Erden and Sarikaya [9] investigated

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the generalized Bullen-type inequalities including local fractional integrals on fractal sets. In paper [8], Du et al. used the generalized fractional integrals to obtain Bullen-type inequalities. For some recent results connected with these types of inequalities, see [11, 12] and the references cited therein.

Twice differentiable functions have been studied extensively in order to get significant inequalities by many researchers. For example, Barani et al. [3] give inequalities for twice differentiable convex mappings which are connected with Hermite-Hadamard-type inequality. Some generalized fractional integral inequalities of midpoint-type and trapezoid-type for the case of twice differentiable convex functions are obtained in paper [20]. In [21], authors obtained some new inequalities of the Simpson and the Hermite–Hadamard-type for functions whose absolute values of derivatives are convex. Furthermore, several generalizations of integral inequalities of Bullen-type inequalities for the case of twice differentiable functions including Riemann-Liouville fractional integrals were obtained in paper [6]. For further information connected with fractional integral inequalities, see [4, 10] and the references cited therein.

We will present some Bullen-type inequalities for differentiable convex functions involving conformable fractional integrals with the help of ongoing studies and the articles mentioned above. The whole study consists of four parts including the introduction. In Section 2, the fundamental definitions of Riemann-Liouville integrals and conformable integrals will be explained for building our main results. In Section 3, an identity will be proved for the case of differentiable convex functions including the conformable fractional integrals. By using this equality, we prove some Bullen-type inequalities by convex functions with the aid of the conformable fractional integrals. Furthermore, we also give some corollaries and remarks. Finally, the summary and concluding remarks are presented in Section 4.

2. Preliminaries

This section discusses the fundamental definitions of Riemann-Liouville integrals and conformable integrals are given as follows:

Definition 2.1 *The gamma function, beta function, and incomplete beta function are defined by*

$$\Gamma(x) := \int_0^\infty \mu^{x-1} e^{-\mu} d\mu,$$

$$\mathcal{B}(x,y) := \int_0^1 \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

and

$$\mathscr{B}(x,y,r) := \int_0^r \mu^{x-1} (1-\mu)^{y-1} d\mu,$$

respectively for $0 < x, y < \infty$ *and* $r \in [0,1]$.

Definition 2.2 *[17] The Riemann-Liouville integrals* $J_{\sigma+}^{\beta}\mathcal{F}(x)$ *and* $J_{\delta-}^{\beta}\mathcal{F}(x)$ *of order* $\beta > 0$ *are given by*

$$J_{\sigma+}^{\beta}\mathcal{F}(x) = \frac{1}{\Gamma(\beta)} \int_\sigma^x (x-\mu)^{\beta-1} \mathcal{F}(\mu) d\mu, \quad x > \sigma \tag{2.1}$$

and

$$J_{\delta-}^{\beta}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\delta}(\mu-x)^{\beta-1}\mathcal{F}(\mu)d\mu,\quad x<\delta, \tag{2.2}$$

respectively for $\mathcal{F}\in L_1[\sigma,\delta]$. *The Riemann-Liouville integrals reduce to the classical integrals for the case of* $\beta=1$.

The fractional conformable integral operators are defined as follows:

Definition 2.3 *[16] The fractional conformable integral operator* ${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x)$ *and* ${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(x)$ *of order* $\beta\in C$, $Re(\beta)>0$ *and* $\alpha\in(0,1]$ *are given by*

$${}^{\beta}\mathcal{J}_{\sigma+}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_{\sigma}^{x}\left(\frac{(x-\sigma)^{\alpha}-(\mu-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\mu-\sigma)^{1-\alpha}}d\mu,\qquad \mu>\sigma \tag{2.3}$$

and

$${}^{\beta}\mathcal{J}_{\delta-}^{\alpha}\mathcal{F}(x)=\frac{1}{\Gamma(\beta)}\int_{x}^{\delta}\left(\frac{(\delta-x)^{\alpha}-(\delta-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(\mu)}{(\delta-\mu)^{1-\alpha}}d\mu,\qquad \mu<\delta, \tag{2.4}$$

respectively for $\mathcal{F}\in L_1[\sigma,\delta]$.

Let us consider $\alpha=1$. Then, the fractional integral in (2.3) becomes to the Riemann-Liouville fractional integral in (2.1). Moreover, the fractional integral in (2.4) reduces to the Riemann-Liouville fractional integral in (2.2) if $\alpha=1$. For more information about fractional integral inequalities, see [1, 15] and the references cited therein.

3. Principal outcomes

Lemma 3.1 *Let* $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ *be a differentiable function on* (σ,δ) *such that* $\mathcal{F}'\in L_1[\sigma,\delta]$. *Then, the following equality holds:*

$$\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}-}^{\alpha}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}_{\frac{\sigma+\delta}{2}+}^{\alpha}\mathcal{F}(\delta)\right] \tag{3.1}$$

$$=\frac{(\delta-\sigma)\alpha^{\beta}}{4}\int_{0}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\left[\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)-\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right]d\mu.$$

Proof By using the integrating by parts, we get

$$I_1=\int_{0}^{1}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\left[\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)-\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right]d\mu \tag{3.2}$$

$$=\frac{2}{\delta-\sigma}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\Bigg|_{0}^{1}$$

$$-\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu$$

$$+\frac{2}{\delta-\sigma}\left[\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right]\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\Bigg|_0^1$$

$$-\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu$$

$$=\frac{2}{(\delta-\sigma)\,\alpha^{\beta}}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]$$

$$-\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)d\mu$$

$$-\frac{2\beta}{\delta-\sigma}\int_0^1\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\mu)^{\alpha-1}\mathcal{F}\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)d\mu.$$

If we use change of variables in (3.2), then the equality (3.2) is converted as follows:

$$I_1=\frac{2}{(\delta-\sigma)\,\alpha^{\beta}}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]\tag{3.3}$$

$$-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\sigma}^{\frac{\sigma+\delta}{2}}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(\frac{\sigma+\delta}{2}-x\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(\frac{\sigma+\delta}{2}-x\right)^{1-\alpha}}\mathcal{F}(x)\,dx$$

$$-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{\Gamma(\beta)}\int_{\frac{\sigma+\delta}{2}}^{\delta}\left(\frac{\left(\frac{\delta-\sigma}{2}\right)^{\alpha}-\left(x-\frac{\sigma+\delta}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathcal{F}(x)}{\left(x-\frac{\sigma+\delta}{2}\right)^{1-\alpha}}\mathcal{F}(x)\,dx$$

$$=\frac{2}{(\delta-\sigma)\,\alpha^{\beta}}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\left(\frac{2}{\delta-\sigma}\right)^{\alpha\beta+1}\Gamma(\beta+1)\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right].$$

When the equality (3.3) is multiplied by $\frac{(\delta-\sigma)\alpha^{\beta}}{4}$, the proof of Lemma 3.1 is finished. □

Theorem 3.2 *Let us consider that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) so that $\mathcal{F}'\in L_1[\sigma,\delta]$ and $|\mathcal{F}'|$ are convex on $[\sigma,\delta]$. Then, the following inequality holds:*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{3.4}$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\varphi_1(\alpha,\beta)\left[|\mathcal{F}'(\sigma)|+|\mathcal{F}'(\delta)|\right].$$

Here,

$$\varphi_1(\alpha,\beta)=\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right] \tag{3.5}$$

$$+\frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-2\mathscr{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

where $\mathcal{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof Let us first take the absolute value of both sides of (3.1). Then, we have

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right| \tag{3.6}$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left\{\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|d\mu\right.$$

$$\left.+\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|d\mu\right\}.$$

Since $|\mathcal{F}'|$ is convex on $[\sigma,\delta]$, it follows

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left\{\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left[\frac{2-\mu}{2}|\mathcal{F}'(\sigma)|+\frac{\mu}{2}|\mathcal{F}'(\delta)|\right]d\mu\right.$$

$$+\int_0^1 \left| \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta - \frac{1}{2\alpha^\beta} \right| \left[\frac{\mu}{2} \left| \mathcal{F}'(\sigma) \right| + \frac{2-\mu}{2} \left| \mathcal{F}'(\delta) \right| \right] d\mu \Bigg\}$$

$$= \frac{(\delta-\sigma)\alpha^\beta}{4} \int_0^1 \left| \left(\frac{1-(1-\mu)^\alpha}{\alpha} \right)^\beta - \frac{1}{2\alpha^\beta} \right| d\mu \left[\left| \mathcal{F}'(\sigma) \right| + \left| \mathcal{F}'(\delta) \right| \right].$$

□

Corollary 3.3 *If we set $\alpha = 1$ in Theorem 3.2, then the following inequality holds:*

$$\left| \frac{1}{2} \left[\mathcal{F}\left(\frac{\sigma+\delta}{2} \right) + \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} \right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^\beta} \left[J^\beta_{\frac{\sigma+\delta}{2}-} \mathcal{F}(\sigma) + J^\beta_{\frac{\sigma+\delta}{2}+} \mathcal{F}(\delta) \right] \right|$$

$$\leq \frac{\delta-\sigma}{4} \varphi_1(1,\beta) \left[\left| \mathcal{F}'(\sigma) \right| + \left| \mathcal{F}'(\delta) \right| \right],$$

where

$$\varphi_1(1,\beta) = \int_0^1 \left| \mu^\beta - \frac{1}{2} \right| d\mu = \frac{\beta}{\beta+1} \left(\frac{1}{2} \right)^{\frac{1}{\beta}} + \frac{1}{\beta+1} - \frac{1}{2}. \tag{3.7}$$

Remark 3.4 *Let us consider $\alpha = 1$ and $\beta = 1$ in Theorem 3.2. Then, we have the following Bullen-type inequality*

$$\left| \frac{1}{2} \left[\mathcal{F}\left(\frac{\sigma+\delta}{2} \right) + \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} \right] - \frac{1}{\delta-\sigma} \int_\sigma^\delta \mathcal{F}(x)\, dx \right| \leq \frac{\delta-\sigma}{16} \left[\left| \mathcal{F}'(\sigma) \right| + \left| \mathcal{F}'(\delta) \right| \right]$$

which is given in [11, Remark 4.2].

Theorem 3.5 *If $\mathcal{F} : [\sigma,\delta] \to \mathbb{R}$ is a differentiable function on (σ,δ) so that $\mathcal{F}' \in L_1[\sigma,\delta]$ and $|\mathcal{F}'|^q$ are convex on $[\sigma,\delta]$ with $q > 1$, then the following double inequality holds:*

$$\left| \frac{1}{2} \left[\mathcal{F}\left(\frac{\sigma+\delta}{2} \right) + \frac{\mathcal{F}(\sigma) + \mathcal{F}(\delta)}{2} \right] - \frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}} \left[{}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}-} \mathcal{F}(\sigma) + {}^\beta\mathcal{J}^\alpha_{\frac{\sigma+\delta}{2}+} \mathcal{F}(\delta) \right] \right| \tag{3.8}$$

$$\leq \frac{(\delta-\sigma)\alpha^\beta}{4} \left(\psi_1^{\alpha,\beta}(p) \right)^{\frac{1}{p}} \left[\left(\frac{3\left| \mathcal{F}'(\sigma) \right|^q + \left| \mathcal{F}'(\delta) \right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left| \mathcal{F}'(\sigma) \right|^q + 3\left| \mathcal{F}'(\delta) \right|^q}{4} \right)^{\frac{1}{q}} \right].$$

$$\leq \frac{(\delta-\sigma)\alpha^\beta}{4} \left(4\psi_1^{\alpha,\beta}(p) \right)^{\frac{1}{p}} \left[\left| \mathcal{F}'(\sigma) \right| + \left| \mathcal{F}'(\delta) \right| \right].$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\psi_1^{\alpha,\beta}(p)=\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^{p}d\mu.$$

Proof Let us apply Hölder inequality in (3.6). Then, we obtain

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right\}.$$

It is known that $|\mathcal{F}'|^{q}$ is convex on $[\sigma,\delta]$. Then, it yields

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{2-\mu}{2}|\mathcal{F}'(\sigma)|^{q}+\frac{\mu}{2}|\mathcal{F}'(\delta)|^{q}\right)d\mu\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|^{p}d\mu\right)^{\frac{1}{p}}\left(\int_0^1\left(\frac{\mu}{2}|\mathcal{F}'(\sigma)|^{q}+\frac{2-\mu}{2}|\mathcal{F}'(\delta)|^{q}\right)d\mu\right)^{\frac{1}{q}}\right\}$$

$$=\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left(\psi_1^{\alpha,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathcal{F}'(\sigma)|^{q}+|\mathcal{F}'(\delta)|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}'(\sigma)|^{q}+3|\mathcal{F}'(\delta)|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

The second inequality of Theorem 3.5 can be obtained immediately by letting $\eta_1=3|\mathcal{F}'(\sigma)|^{q}$, $\theta_1=|\mathcal{F}'(\delta)|^{q}$, $\eta_2=|\mathcal{F}'(\sigma)|^{q}$ and $\theta_2=3|\mathcal{F}'(\delta)|^{q}$ and applying the inequality:

$$\sum_{k=1}^{n}(\eta_k+\theta_k)^{s}\leq\sum_{k=1}^{n}\eta_k^{s}+\sum_{k=1}^{n}\theta_k^{s},\qquad 0\leq s<1.$$

Thus, the proof of Theorem 3.5 is finished. □

Corollary 3.6 *If we choose $\alpha = 1$ in Theorem 3.5, then we derive*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^{\beta}}\left[\ J^{\beta}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+J^{\beta}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\ \right]\right|$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{4}\left(\psi_1^{1,\beta}(p)\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\delta)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{4}\left(4\psi_1^{1,\beta}(p)\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Corollary 3.7 *If we take $\alpha = 1$ and $\beta = 1$ in Theorem 3.5, then we have*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{1}{\delta-\sigma}\int_{\sigma}^{\delta}\mathcal{F}(x)\,dx\right|$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{4}\left(\psi_1^{1,1}(p)\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\delta)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{F}'(\sigma)\right|^{q}+3\left|\mathcal{F}'(\delta)\right|^{q}}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{\delta-\sigma}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left|\mathcal{F}'(\sigma)\right|+\left|\mathcal{F}'(\delta)\right|\right].$$

Theorem 3.8 *Assume that $\mathcal{F}:[\sigma,\delta]\to\mathbb{R}$ is a differentiable function on (σ,δ) such that $\mathcal{F}'\in L_1([\sigma,\delta])$ and $|\mathcal{F}'|^{q}$ are convex on $[\sigma,\delta]$ with $q\geq 1$. Then, the following inequality*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[\ {}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+\ {}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{(\delta-\sigma)\,\alpha^{\beta}}{4}\left(\varphi_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\left(2\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta)\right)\left|\mathcal{F}'(\sigma)\right|^{q}+\varphi_2(\alpha,\beta)\left|\mathcal{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\varphi_2(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}+\left(2\varphi_1(\alpha,\beta)-\varphi_2(\alpha,\beta)\right)\left|\mathcal{F}'(\delta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]$$

is valid. Here, $\varphi_1(\alpha,\beta)$ is defined as in (3.5) and

$$\varphi_2(\alpha,\beta)=\int_0^1 \mu\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu=\frac{1}{\alpha^{\beta}}\left[\frac{1}{2}\left(1-\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right)^{2}-\frac{1}{4}\right]$$

$$-\frac{1}{\alpha^{\beta+1}}\left[2\mathscr{B}\left(\frac{1}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)-\mathcal{B}\left(\frac{1}{\alpha},\beta+1\right)+\mathcal{B}\left(\frac{2}{\alpha},\beta+1\right)-2\mathscr{B}\left(\frac{2}{\alpha},\beta+1,\left(\frac{1}{2}\right)^{\frac{1}{\beta}}\right)\right],$$

where $\mathcal{B}$ and $\mathscr{B}$ denote the beta function and incomplete beta function, respectively.

Proof If we apply power-mean inequality in (3.6), then we have

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left\{\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{2-\mu}{2}\sigma+\frac{\mu}{2}\delta\right)\right|^{q}d\mu\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{\mu}{2}\sigma+\frac{2-\mu}{2}\delta\right)\right|^{q}d\mu\right)^{\frac{1}{q}}\right\}.$$

Since $|\mathcal{F}'|^{q}$ is convex on $[\sigma,\delta]$, we obtain

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right)+\frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\delta-\sigma)^{\alpha\beta}}\left[{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma)+{}^{\beta}\mathcal{J}^{\alpha}_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq\frac{(\delta-\sigma)\alpha^{\beta}}{4}\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|d\mu\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1\left|\left(\frac{1-(1-\mu)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{2\alpha^{\beta}}\right|\left[\frac{2-\mu}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{\mu}{2}\left|\mathcal{F}'(\delta)\right|^{q}\right]d\mu\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \left|\left(\frac{1-(1-\mu)^\alpha}{\alpha}\right)^\beta - \frac{1}{2\alpha^\beta}\right| \left[\frac{\mu}{2}\left|\mathcal{F}'(\sigma)\right|^q + \frac{2-\mu}{2}\left|\mathcal{F}'(\delta)\right|^q\right] d\mu\right)^{\frac{1}{q}}\Bigg],$$

which completes the proof of Theorem 3.8. □

Corollary 3.9 *If we set $\alpha = 1$ in Theorem 3.8, then we acquire*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(\delta-\sigma)^\beta}\left[J^\beta_{\frac{\sigma+\delta}{2}-}\mathcal{F}(\sigma) + J^\beta_{\frac{\sigma+\delta}{2}+}\mathcal{F}(\delta)\right]\right|$$

$$\leq \frac{\delta-\sigma}{4}\left(\varphi_1(1,\beta)\right)^{1-\frac{1}{q}}\left[\left(\frac{\varphi(\beta)\left|\mathcal{F}'(\sigma)\right|^q + \varphi_2(1,\beta)\left|\mathcal{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{\varphi_2(1,\beta)\left|\mathcal{F}'(\sigma)\right|^q + \varphi(\beta)\left|\mathcal{F}'(\delta)\right|^q}{2}\right)^{\frac{1}{q}}\right].$$

Here,

$$\varphi(\beta) = 2\varphi_1(1,\beta) - \varphi_2(1,\beta)$$

$$= \frac{2\beta}{\beta+1}\left(\frac{1}{2}\right)^{\frac{1}{\beta}} - \frac{\beta-2}{2(\beta+2)}\left(\frac{1}{2}\right)^{\frac{2}{\beta}} + \frac{\beta+3}{(\beta+1)(\beta+2)} - \frac{3}{4},$$

where $\varphi_1(1,\beta)$ is defined as in (3.7) and

$$\varphi_2(1,\beta) = \frac{\beta-2}{2(\beta+2)}\left(\frac{1}{2}\right)^{\frac{2}{\beta}} + \frac{1}{\beta+2} - \frac{1}{4}.$$

Corollary 3.10 *If we take $\alpha = 1$ and $\beta = 1$ in Theorem 3.8, then we obtain*

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{\sigma+\delta}{2}\right) + \frac{\mathcal{F}(\sigma)+\mathcal{F}(\delta)}{2}\right] - \frac{1}{\delta-\sigma}\int_\sigma^\delta \mathcal{F}(x)\,dx\right|$$

$$\leq \frac{\delta-\sigma}{16}\left[\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^q + \left|\mathcal{F}'(\delta)\right|^q}{12}\right)^{\frac{1}{q}} + \left(\frac{\left|\mathcal{F}'(\sigma)\right|^q + 11\left|\mathcal{F}'(\delta)\right|^q}{12}\right)^{\frac{1}{q}}\right].$$

4. Summary & concluding remarks

In this manuscript, we derive an equality for the case of convex differentiable functions. By using this identity, we prove Bullen-type inequalities for the case of conformable fractional integrals. Furthermore, our results generalize known results from the literature. In future studies, the ideas and strategies for our results concerning Bullen-type inequalities via conformable fractional integrals may open new avenues for further research in this area. Moreover, mathematicians can obtain Bullen-type inequalities for convex functions by using quantum calculus. Furthermore, interested readers can apply these resulting inequalities to different types of fractional integrals.

Availability of data and material

Data sharing is not applicable to this paper as no data sets were generated or analysed during the current study.

Competing interests

The authors declare that they have no competing interests.

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Author contributions

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References

[1] Abdeljawad T. On conformable fractional calculus. Journal of Computational and Applied Mathematics 2015; 279: 57-66. https://doi.org/10.1016/j.cam.2014.10.016

[2] Abdelhakim AA. The flaw in the conformable calculus: It is conformable because it is not fractional. Fractional Calculus and Applied Analysis 2019; 22: 242-254. https://doi.org/10.1515/fca-2019-0016

[3] Barani A, Barani S, Dragomir SS. Refinements of Hermite–Hadamard inequalities for functions when a power of the absolute value of the second derivative is P-convex. Journal of Applied Mathematics 2012. https://doi.org/10.1155/2012/615737

[4] Budak H, Hezenci F, Kara H. On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co–ordinated convex functions via generalized fractional integrals. Advances in Difference Equations 2021; 2021: 1-32. https://doi.org/10.1186/s13662-021-03463-0

[5] Bullen PS. Error estimates for some elementary quadrature rules. Publikacije Elektrotehničkog fakulteta. Serija Matematika i fizika 1978; (602/633): 97-103.

[6] Ciobotariu-Boer V. On Some Common Generalizations of two classes of integral inequalities for twice differentiable functions. Analele Universitatii Oradea Fasc. Matematica 2018; 1: 43-50.

[7] Dragomir SS, Agarwal RP. Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Applied Mathematics Letters 1998; 11 (5): 91-95. https://doi.org/10.1016/S0893-9659(98)00086-X

[8] Du T, Luo C, Cao Z. On the Bullen-type inequalities via generalized fractional integrals and their applications. Fractals 2021; 29 (07): 2150188. https://doi.org/10.1142/S0218348X21501887, 2021

[9] Erden S, Sarikaya MZ. Generalized Bullen-type inequalities for local fractional integrals and its applications. Palestine Journal of Mathematics 2020; 9 (2): 945-956.

[10] Hezenci F, Budak H, Kara H. New version of Fractional Simpson type inequalities for twice differentiable functions. Advances in Difference Equations 2021; 2021(460). https://doi.org/10.1186/s13662-021-03615-2

[11] Hwang HR, Tseng KL, Hsu KC. New inequalities for fractional integrals and their applications. Turkish Journal of Mathematics 2016; 40(3): 471-486. https://doi.org/10.3906/mat-1411-61

[12] Hwang S, Tseng K. New Hermite–Hadamard-type inequalities for fractional integrals and their applications. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 2018; 112: 1211-1223. https://doi.org/10.1007/s13398-017-0419-2

[13] Hyder A, Soliman AH. A new generalized θ-conformable calculus and its applications in mathematical physics. Physica Scripta 2020; 96: 015208. https://doi.org/10.1088/1402-4896/abc6d9

[14] Iqbal M, Qaisar S, Muddassar M. A short note on integral inequality of type Hermite–Hadamard through convexity. Journal of Computational Analysis and Applications 2016; 21 (5): 946-953.

[15] Jarad F, Abdeljawad T, Baleanu D. On the generalized fractional derivatives and their Caputo modification. Journal of Nonlinear Sciences and Applications 2017; 10 (5): 2607-2619. doi:10.22436/jnsa.010.05.27

[16] Jarad F, Uğurlu E, Abdeljawad T, Baleanu D. On a new class of fractional operators. Advances in Difference Equations 2017; (2017): 247. https://doi.org/10.1186/s13662-017-1306-z

[17] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.

[18] Kirmaci US. Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula. Applied Mathematics and Computation 2004; 147 (5): 137-146. https://doi.org/10.1016/S0096-3003(02)00657-4

[19] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. Journal of Computational and Applied Mathematics 2014; 264: 65-70. https://doi.org/10.1016/j.cam.2014.01.002

[20] Mohammed PO, Sarikaya MZ. On generalized fractional integral inequalities for twice differentiable convex functions. Journal of Computational and Applied Mathematics 2020; 372: 112740. https://doi.org/10.1016/j.cam.2020.112740

[21] Sarikaya MZ, Aktan N. On the generalization of some integral inequalities and their applications. Mathematical and computer Modelling 2011; 54 (9-10): 2175-2182. https://doi.org/10.1016/j.mcm.2011.05.026

[22] Sarikaya MZ, Budak H. Some integral inequalities for local fractional integrals. International Journal of Analysis and Applications 2017; 14 (1): 9-19.

[23] Sarikaya MZ, Set E, Yaldiz H, Basak N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Mathematical and Computer Modelling 2013; 57 (9-10): 2403-2407. https://doi.org/10.1016/j.mcm.2011.12.048

[24] Zhao D, Luo M. General conformable fractional derivative and its physical interpretation. Calcolo 2017; 54: 903-917. https://doi.org/10.1007/s10092-017-0213-8