Turkish Journal of Mathematics
http://journals.tubitak.gov.tr/math/
Research Article

Turk J Math
(2023) 47: 1129 – 1143
© TÜBİTAK
doi:10.55730/1300-0098.3416

On k-generalized Lucas sequence with its triangle

Abdullah AÇIKEL[1], **Said AMROUCHE**[2], **Hacène BELBACHIR**[2,3], **Nurettin IRMAK**[4,*]
[1]Hassa Vocational School, Hatay Mustafa Kemal University, Hatay, Turkey
[2]RECITS Laboratory, Faculty of Mathematics, USTHB, Po. Box 32, El Alia, Bab Ezzouar Algiers, Algeria
[3]Scientific and Technical Information Research Center, Ben Aknoun, Algiers, Algeria
[4]Department of Engineering Basic Science, Engineering and Natural Science Faculty,
Konya Technical University, Konya, Turkey

Received: 26.06.2021 • **Accepted/Published Online:** 03.02.2023 • **Final Version:** 16.05.2023

Abstract: In this paper, we investigate several identities of k-generalized Lucas numbers with k-generalized Fibonacci numbers. We also establish a link between generalized s-Lucas triangle and bi^{s}nomial coefficients given by the coefficients of the development of a power of $(1+x+x^2+\cdots+x^s)$, with $s \in \mathbb{N}$.

Key words: $k-$generalized Lucas sequence, arithmetic triangle, recurrence relation, bi^{s}nomial coefficient

1. Introduction

Let $\{G_n\}$ be a sequence defined by second-order linear recurrence relation $G_n = AG_{n-1} + BG_{n-2}$, $n \geq 2$ where A, B, G_0 and G_1 are given numbers. Assume that the sequence $\{H_n\}$ is defined by the same recurrence relation of $\{G_n\}$ with $H_0 = 2G_1 - AG_0$ and $H_1 = AG_1 + 2BG_0$. $\{H_n\}$ is called the associate sequence of $\{G_n\}$ (see [18]). Table 1 presents several well-known sequences with their associate sequences and A-numbers in Sloane's Encyclopedia of Integer Sequences[†].

Table 1. second-order well-known sequences

A	B	G_0	G_1	H_0	H_1	Sequence	Associate sequence	A-numbers
1	1	0	1	2	1	Fibonacci	Lucas	$A000045, A000032$
2	1	0	1	2	2	Pell	Pell-Lucas	$A000129, A002203$
1	2	0	1	2	1	Jacobsthal	Jacobsthal-Lucas	$A001045, A014551$
6	-1	0	1	2	6	Balancing	Balancing-Lucas	$A001109, A003499$

There are several generalizations of the Fibonacci sequence. One of the generalizations relating to order is k-generalized Fibonacci sequence. For $k \geq 2, k-$generalized Fibonacci sequence $\left\{F_n^{(k)}\right\}$ is defined by the

*Corresponding author: irmaknurettin@gmail.com, nirmak@ktun.edu.tr
2010 *AMS Mathematics Subject Classification:* 05A10, 11B65, 11B39.
[†]Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at http://oeis.org/.

following recurrence relation

$$F_n^{(k)} = \begin{cases} 0, & \text{if } n =, -1, \ldots, -k+1; \\ 1, & \text{if } n = 0; \\ F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} & \text{if } n > 1. \end{cases}$$

For the few values of k, we give in Table 2 of these numbers containing the A-number, according to the On-Line Encyclopedia of Integer Sequences (OEIS)[‡].

Table 2. k-order well-known sequences

k	Sequence name	Terms of the sequence	$A-numbers$
2	Fibonacci	$1, 1, 2, 3, 5, 8, 13, \ldots$	$A000045$
3	Tribonacci	$1, 1, 2, 4, 7, 13, 24, \ldots$	$A000073$
4	Tetranacci	$1, 2, 4, 8, 15, 29, 56, \ldots$	$A000078$
5	Pentanacci	$1, 1, 2, 4, 8, 16, 31, \ldots$	$A001591$

The Binet form of the k-generalized Fibonacci sequence is given by Dresden and Du [13] as follows

Theorem 1.1 *For $F_n^{(k)}$ the n^{th} k-generalized Fibonacci number, then*

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha_i - 1}{2 + (k+1)(\alpha_i - 2)} \alpha_i^{n-1}$$

for $\alpha_1, \alpha_2, \ldots, \alpha_k$ the roots of $x^k - x^{k-1} - \cdots - 1 = 0$.

There are also many other ways to represent the terms of k-generalized Fibonacci numbers (see [16], [14], [15], [17]).

By the motivation of the definition "associate sequence", we give the definition of the associate sequence of $\{F_n^{(k)}\}_n$ which we call it as k-order Lucas sequence $\{L_n^{(k)}\}_n$.

Definition 1.2 *Let $k \geq 2$ is an integer. The k-generalized Lucas sequence $\{L_n^{(k)}\}_n$ by the following recurrence relation*

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)}, \qquad (n \geq -k+2) \tag{1.1}$$

with the initials $L_0^{(k)} = k$, $L_1^{(k)} = 1$, $L_2^{(k)} = 3$, $\ldots$, $L_{k-1}^{(k)} = 2^{k-1} - 1$.

The explicit formulas of the k-generalized Fibonacci and Lucas sequences are given by Belbachir and Bencherif [7] as follows:

$$F_n^{(k)} = \sum_{j_1 + 2j_2 + \cdots + kj_k = n} \binom{j_1 + j_2 + \cdots + j_k}{j_1, j_2, \ldots, j_k}$$

[‡]Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at http://oeis.org/.

and

$$L_n^{(k)} = \lambda_0 y_{n-k+1} + \lambda_1 y_{n-k+2} + \cdots + \lambda_{k-1} y_n,$$

with $\lambda_j = -\sum_{i=j}^{k-j} L_j^{(k)}$ for $0 \le j \le k-j$, and $y_n = \sum_{j_1+2j_2+\cdots+kj_k=n} \binom{j_1+j_2+\cdots+j_k}{j_1,j_2,\cdots,j_k}$ for $n > -k$.

We note that the case $k = 2$ gives Lucas numbers and the case $k = 3$ gives Tribonacci-Lucas numbers (see [20]). There are two parts in the present paper; the first one gives combinatorial identities for k-generalized Lucas numbers and extends identities between Fibonacci and Lucas numbers. In the second part, we give several relations between k-generalized Lucas numbers and bi^{s} nomial coefficients.

2. Connections with k-generalized Fibonacci and Lucas numbers

Before giving our results of this section, we recall that the Binet formula of k-generalized Lucas numbers is given by the following result

Lemma 2.1 *Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct roots of $x^k - x^{k-1} - \cdots - 1 = 0$. Then, we have*

$$L_n^{(k)} = \alpha_1^n + \alpha_2^n + \cdots + \alpha_k^n.$$

This result is well-known. We give the proof for convenience.

Proof It is known that the term $L_n^{(k)}$ can be written by

$$L_n^{(k)} = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_k\alpha_k^n$$

where A_i are real numbers. Our aim to find the numbers A_i for $i = 1, 2, \ldots, k$. To find these values, we get the following system of equations

$$\begin{aligned}
L_0^{(k)} &= A_1 + A_2 + \cdots + A_k \\
L_1^{(k)} &= A_1\alpha_1 + A_2\alpha_2 + \cdots + A_k\alpha_k \\
L_2^{(k)} &= A_1\alpha_1^2 + A_2\alpha_2^2 + \cdots + A_k\alpha_k^2 \\
&\ \ \vdots \\
L_{n-1}^{(k)} &= A_1\alpha_1^{n-1} + A_2\alpha_2^{n-1} + \cdots + A_k\alpha_k^{n-1}.
\end{aligned}$$

By using the Cramer's rule, we obtain $A_1 = A_2 = \cdots = A_k = 1$. □

From now on, we generalize several well-known identities between Fibonacci and Lucas numbers. To prove these identities, we will use the Binet type formulas for k-generalized Fibonacci and Lucas numbers. We know that the following identity is given by Ramirez and Sirvent [19]. Here, we give its proof by using Binet formulas.

Theorem 2.2 *Let k and n nonnegative integers with $k \ge 2$, then we have*

$$\sum_{i=1}^{k} iF_{n-i+1}^{(k)} = L_n^{(k)}. \tag{2.1}$$

Proof By using the Binet Formula of k-generalized Fibonacci numbers, we get

$$\begin{aligned}
&\sum_{i=1}^{k} iF_{n-i+1}^{(k)} = F_n^{(k)} + 2F_{n-1}^{(k)} + 3F_{n-2}^{(k)} + \cdots + kF_{n-k+1}^{(k)} \\
&= \sum_{i=1}^{k} \frac{(\alpha_i - 1)\alpha_i^{n-1}}{2+(k+1)(\alpha_i - 2)} + 2\frac{(\alpha_i - 1)\alpha_i^{n-2}}{2+(k+1)(\alpha_i - 2)} + \cdots + k\frac{(\alpha_i - 1)\alpha_i^{n-k}}{2+(k+1)(\alpha_i - 2)} \\
&= \sum_{i=1}^{k} \frac{(\alpha_i - 1)\alpha_i^{n-k}[k + (k-1)\alpha_i + (k-2)\alpha_i^2 + \cdots + 2\alpha_i^{k-2} + \alpha_i^{k-1}]}{2+(k+1)(\alpha_i - 2)} \\
&= \sum_{i=1}^{k} \frac{\alpha_i^n(\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k)}{\alpha_i^k[(k+1)\alpha_i - 2k]}
\end{aligned}$$

After using the facts $\alpha_i^k = \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + 1$ and $\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k = 2\alpha^k - 1 - k,$ we have

$$\begin{aligned}
&\sum_{i=1}^{k} \frac{\alpha_i^n(\alpha_i^k + \alpha_i^{k-1} + \alpha_i^{k-2} + \cdots + \alpha_i - k)}{\alpha_i^k[(k+1)\alpha_i - 2k]} \\
&= \sum_{i=1}^{k} \frac{\alpha_i^n(2\alpha_i^k - 1 - k)}{(k+1)\alpha_i^k + (1-k)\alpha_i^k - 1 - k} \\
&= \sum_{i=1}^{k} \alpha_i^n = L_n^{(k)}.
\end{aligned}$$

For $k = 2$,

$$F_n + 2F_{n-1} = L_n.$$

□

We have also the following identity.

Theorem 2.3 *Let k and n be nonnegative integers with $k \geq 2$, then we have*

$$L_{n-1}^{(k)} + L_{n+1}^{(k)} = 2kF_n^{(k)} + \sum_{i=0}^{k-2}(3 - k + 2i)F_{n-i}^{(k)}. \tag{2.2}$$

Proof Together with the Binet Formula for k-generalized Fibonacci number, we get the followings

$$\begin{aligned}
& 2kF_n^{(k)} + \sum_{i=0}^{k-2}(3-k+2i)F_{n-i}^{(k)} \\
= \quad & (k+3)F_n^{(k)} + (5-k)F_{n-1}^{(k)} + (7-k)F_{n-2}^{(k)} + \cdots + (k-3)F_{n-k+3}^{(k)} + (k-1)F_{n-k+2}^{(k)} \\
= \quad & \sum_{i=1}^{k} \frac{(k+3)(\alpha_i-1)\alpha_i^{n-1}}{2+(k+1)(\alpha_i-2)} + \frac{(5-k)(\alpha_i-1)\alpha_i^{n-2}}{2+(k+1)(\alpha_i-2)} + \frac{(7-k)(\alpha_i-1)\alpha_i^{n-3}}{2+(k+1)(\alpha_i-2)} + \\
& \cdots + \frac{(k-3)(\alpha_i-1)\alpha_i^{n-k+2}}{2+(k+1)(\alpha_i-2)} + \frac{(k-1)(\alpha_i-1)\alpha_i^{n-k+1}}{2+(k+1)(\alpha_i-2)} \\
= \quad & \sum_{i=1}^{k} \frac{(k+3)\alpha_i^n + (2-2k)\alpha_i^{n-1} + 2\alpha_i^{n-2} + 2\alpha_i^{n-3} + \cdots + 2\alpha_i^{n-k+2} + (1-k)\alpha_i^{n-k+1}}{(k+1)\alpha_i - 2k} \\
= \quad & \sum_{i=1}^{k} \frac{\alpha_i^{n-k+1}(\alpha_i^{k-1} + 2\alpha_i^{k-2} + \alpha_i^{k-3} + \alpha_i^{k-4} + \cdots + \alpha_i + 1)[(k+1)\alpha_i - 2k]}{(k+1)\alpha_i - 2k} \\
= \quad & \sum_{i=1}^{k} \alpha_i^{n-k+1}(\alpha_i^{k-1} + 2\alpha_i^{k-2} + \alpha_i^{k-3} + \alpha_i^{k-4} + \cdots + \alpha_i + 1) \\
= \quad & \sum_{i=1}^{k} \alpha_i^{n+1} + \alpha_i^{n-1} = L_{n+1}^{(k)} + L_{n-1}^{(k)}.
\end{aligned}$$

□

This generalizes the identity

$$L_{n-1} + L_{n-1} = 5F_n.$$

Since one can prove the following theorem as before, we do not give the proof.

Theorem 2.4 *Assume that k and n are nonnegative integers, with $k \geq 2$, we have*

$$L_{n+k-2}^{(k)} = kF_{n+k-1}^{(k)} - \sum_{i=1}^{k-1} iF_{n+i-1}^{(k)}, \tag{2.3}$$

$$L_{n-2}^{(k)} = (2k-1)F_{n-1}^{(k)} - F_{n+k-2}^{(k)} + \sum_{i=1}^{k-3}(k-i-2)F_{n+i-1}^{(k)}. \tag{2.4}$$

These generalize the identities $L_n = 2F_n - F_{n-1}$ and $L_{n-2} = 3F_{n-1} - F_n$.

3. The generalized s-Lucas triangle

In this section, we propose a generalization of Lucas and Tribonacci-Lucas triangles, such that the sum of elements located along the direction $(1,1)$ (see [8] for the details about the notion of direction) in the generalized s-Lucas triangle gives the terms of $(s+1)$-generalized Lucas sequence, the explicit formula is given. We establish a link between generalized s-Lucas triangle and bi^{s}nomial coefficients. We also give the recurrence relation for the sum of elements lying over the finite direction of the generalized s-Lucas triangle.

Alladi and Hoggat [1] have defined the Tribonacci triangle, (this triangle is a generalization of Pascal triangle) and proved that the sum of elements lying over the principal diagonal rays in the Tribonacci triangle gives the Tribonacci sequence

$$T_{n+1} = T_n + T_{n-1} + T_{n-2},$$

with $T_0 = 0, T_1 = 1, T_2 = 1$.
Denote by $\binom{n}{k}_{[2]}$ the element in the n^{th} row and k^{th} column of the Tribonacci triangle, the triangle is produced by the recurrence relation,

$$\binom{n}{k}_{[2]} = \binom{n-1}{k}_{[2]} + \binom{n-1}{k-1}_{[2]} + \binom{n-2}{k-1}_{[2]},$$

where $\binom{n}{0}_{[2]} = \binom{n}{n}_{[2]} = 1$. We use the convention $\binom{n}{k}_{[2]} = 0$ for $k \notin \{0, \dots, n\}$. We present several values of $\binom{n}{n}_{[2]}$ in Table 3.

Table 3. Tribonacci triangle.

n\k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1									
1	1	1								
2	1	3	1							
3	1	5	5	1						
4	1	7	13	7	1					
5	1	9	25	25	9	1				
6	1	11	41	63	41	11	1			
7	1	13	61	129	129	61	13	1		
8	1	15	85	231	321	231	85	15	1	
9	1	17	113	377	681	681	377	113	17	1

Moreover, Barry [6] has shown that for $0 \leq k \leq n$ these coefficients satisfy the relation

$$\binom{n}{k}_{[2]} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k}, \tag{3.1}$$

we recall that the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and we use the convention $\binom{n}{k} = 0$ for $k > n$, $k < 0$ or $n < 0$.
Recently, Yilmaz and Taskara [20] have defined the Tribonacci-Lucas triangle which is a generalization of Lucas triangle and they have shown that the sum of elements lying over the principal diagonal rays in this triangle gives the Tribonacci-Lucas sequence.

$$K_n = K_{n-1} + K_{n-2} + K_{n-3},$$

with $K_0 = 3, K_1 = 1, K_2 = 3$.
Denote by $\begin{bmatrix} n \\ k \end{bmatrix}_{[2]}$ the element in the n^{th} row and k^{th} column of the Tribonacci-Lucas triangle, the triangle is

produced by the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = \begin{bmatrix} n-1 \\ k \end{bmatrix}_{[2]} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{[2]} + \begin{bmatrix} n-2 \\ k-1 \end{bmatrix}_{[2]},$$

where $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_{[2]} = 3$, $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right]_{[2]} = 1$ and $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]_{[2]} = 2$ for $n \geq 1$. We use the convention $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{[2]} = 0$ for $k \notin \{0, \ldots, n\}$. Table 4 shows the values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{[2]}$ for special cases k and n.

Table 4. Tribonacci-Lucas triangle.

n\k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	3									
1	1	2								
2	1	6	2							
3	1	8	10	2						
4	1	10	24	14	2					
5	1	12	42	48	18	2				
6	1	14	64	114	80	22	2			
7	1	16	90	220	242	120	26	2		
8	1	18	120	374	576	442	168	30	2	
9	1	20	154	584	1170	1260	730	224	34	2

The explicit formula of the coefficients of the Tribonacci-Lucas triangle is given by, see [20], for $n \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[2]} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-j}{k} \frac{n+k}{n-j}.$$

3.1. The s-Pascal triangle

The bi^{s}nomial coefficient $\binom{n}{k}_s$ is the element in the n^{th} row and k^{th} column of s-Pascal triangle. The s-Pascal triangle is constructed by the following recurrence relation, see [3, 5, 10],

$$\binom{n}{k}_s = \binom{n-1}{k}_s + \binom{n-1}{k-1}_s + \cdots + \binom{n-1}{k-s}_s. \tag{3.2}$$

Using the classical binomial coefficient, one has

$$\binom{n}{k}_s = \sum_{j_1+j_2+\cdots+j_s=k} \binom{n}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-1}}{j_s}. \tag{3.3}$$

Some other readily well known established properties are:
the symmetry relation

$$\binom{n}{k}_s = \binom{n}{sn-k}_s, \tag{3.4}$$

the diagonal recurrence relation

$$\binom{n}{k}_s = \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k-j}_{s-1}, \tag{3.5}$$

and de Moivre's expression (see [11, 12])

$$\binom{n}{k}_s = \sum_{j=0} (-1)^j \binom{n}{j}\binom{k-j(s+1)+n-1}{n-1}. \tag{3.6}$$

For $s=2$, we have bitrinomial triangle illustrated in Table 5, see Sloane as $A027907$[§].

Table 5. Bitrinomial triangle ($s=2$).

n\k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0	1												
1	1	1	1										
2	1	2	3	2	1								
3	1	3	6	7	6	3	1						
4	1	4	10	16	19	16	10	4	1				
5	1	5	15	30	45	51	45	30	15	5	1		
6	1	6	21	50	90	126	141	126	90	50	21	6	1

4. Quasi s-Pascal triangle

Recently, Amrouche and Belbachir [2–4] have defined a generalization of Pascal and Delannoy triangles, called quasi s-Pascal triangle. They denoted by $\binom{n}{k}_{[s]}$ the coefficient in the n^{th} row and k^{th} column of this triangle such that the coefficient $\binom{n}{k}_{[s]}$ satisfies,

$$\binom{n}{k}_{[s]} = \binom{n-1}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + \binom{n-2}{k-1}_{[s]} + \cdots + \binom{n-s}{k-1}_{[s]}. \tag{4.1}$$

The following result gives the explicit formula of the coefficients of the quasi s-Pascal triangle in terms of binomial coefficients.

Theorem 4.1 *[3] The quasi-bi^{s}nomial coefficient $\binom{n}{k}_{[s]}$ satisfies*

$$\binom{n}{k}_{[s]} = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1} j_i}{k}. \tag{4.2}$$

The sum of elements located along the direction $(1,1)$ in the quasi s-Pascal triangle gives the terms of $(s+1)$-generalized Fibonacci sequence.

Let $(T_{n,s})_n$ be the terms of the $(s+1)$-generalized Fibonacci sequence, for $n \geq 0$

$$T_{n+1,s} := \sum_{k} \binom{n-k}{k}_{[s]},$$

§Sloane NJA, The Online Encyclopedia of Integer Sequences. Available online at http://oeis.org/.

with $T_{0,s} = 0$.

Theorem 4.2 *[3] For $n \geq 0$, $(T_{n,s})_n$ satisfies the following recurrence relation*

$$T_{n+1,s} = T_{n,s} + T_{n-1,s} + \cdots + T_{n-s,s},$$

with $T_{1,s} = 1, T_{i,s} = 0$ for $i \in \{0, -1, \ldots, -(s-1)\}$.

Amrouche and Belbachir [3] have extended the last result, they considered the sum of elements located along the finite direction (α, r) ($r + \alpha > 0$, $r \in \mathbb{N}$, $0 \leq p < r$ and $\alpha \in \mathbb{Z}$) in the quasi s-Pascal triangle, (for the details about the direction in arithmetic triangles one can see [2, 4, 8, 9]).
Let $T_{n,s}^{(\alpha,\beta,r)}$ be the terms of the sequence obtained by this sum, for $n \geq 0$

$$T_{n+1,s}^{(\alpha,\beta,r)} := \sum_k \binom{n-rk}{\beta+\alpha k}_{[s]}, \text{ with } T_{0,s}^{(\alpha,\beta,r)} = 0.$$

Theorem 4.3 *[3] For $n \geq \alpha s + r$, $(T_{n,s}^{(\alpha,\beta,r)})_n$ satisfies the following linear recurrence relation*

$$\sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} T_{n-i,s}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha(s-1)} \binom{\alpha}{i}_{s-1} T_{n-\alpha-r-i,s}^{(\alpha,\beta,r)}. \tag{4.3}$$

5. The quasi s-Lucas triangle

We propose an extension of Lucas and Tribonacci-Lucas triangle called quasi s-Lucas triangle such that the sum of elements located along the direction $(1,1)$ gives the terms of $(s+1)$-generalized Lucas sequence.

Definition 5.1 *Let $\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}$ the element of the n^{th} line and k^{th} column of s-generalized Lucas triangle*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \binom{n}{k}_{[s]} + \binom{n-1}{k-1}_{[s]} + 2\binom{n-2}{k-1}_{[s]} + \cdots + s\binom{n-s}{k-1}_{[s]}, \tag{5.1}$$

with $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_{[s]} = s + 1$.

Theorem 5.2

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{j_1} \sum_{j_2} \cdots \sum_{j_{s-1}} \binom{k}{j_1} \binom{j_1}{j_2} \cdots \binom{j_{s-2}}{j_{s-1}} \binom{n - \sum_{i=1}^{s-1} j_i}{k} \frac{n+k}{n - j_1 - \cdots - j_{s-1}}.$$

Proof By the relation (5.1) and (4.2), we have

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} &= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
&+ \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&+ 2\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-2}{k-1} \\
&\vdots \\
&+ (s-1)\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-(s-1)}{k-1} \\
&+ s\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-s}{k-1} \\
&= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k} \\
&+ \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&+ 2\sum_{j_1'}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}} \binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&\vdots \\
&+ (s-1)\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}} \binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}-1}{k-1} \\
&+ s\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}'} \binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}'-1}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}'-1}{k-1},
\end{aligned}$$

as $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, then

$$\begin{aligned}
&\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1} \\
&= \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k-1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1}
\end{aligned}$$

$$+\sum_{j_1'}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}-j_{s-1}-1}{k-1}$$

$\vdots$

$$+\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}-1}{k-1}$$

$$+\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}'}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}'-1}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}'-1}{k-1},$$

then we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}=\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}$$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}-1}{k-1}$$

$$+\sum_{j_1'}\sum_{j_2}\cdots\sum_{j_{s-2}}\sum_{j_{s-1}}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}-j_{s-1}-1}{k-1}$$

$\vdots$

$$+(s-2)\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}-1}{k-1}$$

$$+(s-1)\sum_{j_1'}\sum_{j_2'}\cdots\sum_{j_{s-2}'}\sum_{j_{s-1}'}\binom{k-1}{j_1'-1}\binom{j_1'-1}{j_2'-1}\cdots\binom{j_{s-2}'-1}{j_{s-1}'-1}\binom{n-j_1'-\cdots-j_{s-2}'-j_{s-1}'-1}{k-1}$$

$$=\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}$$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}\frac{k}{n-j_1-\cdots-j_{s-2}-j_{s-1}}$$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}\frac{j_1-j_2}{n-j_1-\cdots-j_{s-2}-j_{s-1}}$$

$\vdots$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}\frac{(s-2)(j_{s-2}-j_{s-1})}{n-j_1-\cdots-j_{s-2}-j_{s-1}}$$

$$+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-j_1-\cdots-j_{s-2}-j_{s-1}}{k}\frac{(s-1)j_{s-1}}{n-j_1-\cdots-j_{s-2}-j_{s-1}}$$

$$=\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1}j_i}{k}\frac{n+k}{n-j_1-\cdots-j_{s-1}}.$$

□

For $s=1$ and $s=2$, we obtain the Lucas and Tribonacci-Lucas triangles respectively.

The sum of elements located along the direction $(1,1)$ in the generalized s-Lucas triangle gives the $(s+1)$-generalized Lucas sequence $\{L_n^{(s)}\}_{n\geq 0}$.

Let the sequence $\{L_n^{(s)}\}_{n\geq 0}$ given by

$$L_n^{(s+1)}=\sum_k \begin{bmatrix} n-k \\ k \end{bmatrix}_{[s]}. \tag{5.2}$$

We present the following theorem in Theorem 2.2. Here, we give another proof of the theorem by using (5.2).

Theorem 5.3 *The sequence* $\{L_n^{(s)}\}_{n\geq 0}$ *satisfies the following recurrence relation*

$$L_n^{(s+1)}=F_{n+1}^{(s+1)}+F_{n-1}^{(s+1)}+2F_{n-2}^{(s+1)}+\cdots+sF_{n-s}^{(s+1)}. \tag{5.3}$$

Proof From (5.1) and (5.2) we have

$$L_n^{(s+1)}=\sum_k\binom{n-k}{k}_{[s]}+\sum_k\binom{n-k-1}{k-1}_{[s]}+2\sum_k\binom{n-k-2}{k-1}_{[s]}+\cdots$$
$$\cdots+s\sum_k\binom{n-k-s}{k-1}_{[s]}$$
$$=\sum_k\binom{n-k}{k}_{[s]}+\sum_{k'}\binom{n-k'-2}{k'}_{[s]}+2\sum_{k'}\binom{n-k'-3}{k'}_{[s]}+\cdots$$
$$\cdots+s\sum_{k'}\binom{n-k'-s-1}{k'}_{[s]}$$
$$=F_{n+1}^{(s+1)}+F_{n-1}^{(s+1)}+2F_{n-2}^{(s+1)}+\cdots+F_{n-s}^{(s+1)}.$$

□

5.1. Link between generalized s-Lucas triangle and bi^{s}nomial coefficients

The following result establishes the relation between the generalized s-Lucas triangle and s-Pascal triangle.

Theorem 5.4 *For fixed nonnegative integers* n,k *and* s, *we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]}=\sum_i\binom{n-i}{k}\binom{k}{i}_{s-1}\frac{n+k}{n-i}.$$

Proof We have

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_{s-1}} \binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}}\binom{n-\sum_{i=1}^{s-1} j_i}{k}\frac{n+k}{n-j_1-\cdots-j_{s-1}}.$$

Considering the summations by blocks $j_1+j_2+\cdots+j_{s-1}=i$, we get

$$\begin{bmatrix} n \\ k \end{bmatrix}_{[s]} = \sum_{i}\binom{n-i}{k}\frac{n+k}{n-i}\sum_{j_1+j_2+\cdots+j_{s-1}=i}\binom{k}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{s-2}}{j_{s-1}} = \sum_{i}\binom{n-i}{k}\binom{k}{i}_{s-1}\frac{n+k}{n-i}.$$

□

5.2. Recurrence relations

Consider the sum of elements located along the direction (α, r), with $r+\alpha>0$, $r\in\mathbb{N}$, $0\le p<r$ and $\alpha\in\mathbb{Z}$ in the generalized s-Lucas triangle.

Let $(L_{n,s}^{(\alpha,\beta,r)})_n$ be the sequence obtain by this sum

$$L_{n,s}^{(\alpha,\beta,r)} := \sum_{k}\begin{bmatrix} n-rk \\ \beta+\alpha k \end{bmatrix}_{[s]}.$$

Theorem 5.5 *For $n\ge \alpha s+r$, $(L_{n+1,s}^{(\alpha,\beta,r)})_n$ satisfies the following linear recurrence relation*

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}L_{n-i,s}^{(\alpha,\beta,r)} = \sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}L_{n-\alpha-r-i,s}^{(\alpha,\beta,r)}. \tag{5.4}$$

Proof

$$\begin{aligned}
&\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}L_{n-i,s}^{(\alpha,\beta,r)}\\
&=\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_{k}\begin{bmatrix} n-rk-i \\ \beta+\alpha k \end{bmatrix}_{[s]}\\
&=\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_{k}\binom{n-rk-i}{\beta+\alpha k}_{[s]}+\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_{k}\binom{n-rk-i-1}{\beta+\alpha k-1}_{[s]}\\
&+2\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_{k}\binom{n-rk-i-2}{\beta+\alpha k-1}_{[s]}+\cdots+s\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\sum_{k}\binom{n-rk-i-s}{\beta+\alpha k-1}_{[s]}.
\end{aligned}$$

By Theorem 4.3 we obtain

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}L_{n-i,s}^{(\alpha,\beta,r)}$$

$$=\sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}\sum_k\binom{n-rk-i-r-\alpha}{\beta+\alpha k}_{[s]}+\sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}\sum_k\binom{n-rk-i-r-\alpha-1}{\beta+\alpha k-1}_{[s]}$$

$$+2\sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}\sum_k\binom{n-rk-i-r-\alpha-2}{\beta+\alpha k-1}_{[s]}+\cdots+s\sum_{i=0}^{\alpha(s-1)}\binom{\alpha}{i}_{s-1}\sum_k\binom{n-rk-i-r-\alpha-s}{\beta+\alpha k-1}_{[s]}.$$

Finally by the relation (5.1) we get the result. □

Corollary 5.6 *The sum of elements located along the direction* (r,α) *in the Lucas and Tribonacci-Lucas triangles are given respectively by*

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}L_{n-i,1}^{(\alpha,\beta,r)}=L_{n-\alpha-r,1}^{(\alpha,\beta,r)},\tag{5.5}$$

$$\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}L_{n-i,2}^{(\alpha,\beta,r)}=\sum_{i=0}^{\alpha}\binom{\alpha}{i}L_{n-\alpha-r-i,2}^{(\alpha,\beta,r)}.\tag{5.6}$$

Example 5.7 *The sum of elements located along the direction* $(3,2)$ *in the Tribonacci-Lucas triangle (*$s=2$*,* $r=3$*,* $\alpha=2$ *and* $\beta=0$*) satisfies the following recurrence relation*

$$L_{n,2}^{(2,0,3)}=3L_{n-1,2}^{(2,0,3)}-3L_{n-2,2}^{(2,0,3)}+L_{n-3,2}^{(2,0,3)}+L_{n-5,2}^{(2,0,3)}+3L_{n-6,2}^{(2,0,3)}+3L_{n-7,2}^{(2,0,3)}+L_{n-8,2}^{(2,0,3)},$$

with $L_{0,2}^{(2,0,3)}=3, L_{1,2}^{(2,0,3)}=1, L_{2,2}^{(2,0,3)}=1, L_{3,2}^{(2,0,3)}=1, L_{4,2}^{(2,0,3)}=1, L_{5,2}^{(2,0,3)}=3, L_{6,2}^{(2,0,3)}=15, L_{7,2}^{(2,0,3)}=49$.
The first terms of this sequence are $(3,1,1,1,1,3,15,49,115,221,377,611,1027,1935,\ldots)$

Acknowledgment

The authors express their gratitude to the anonymous reviewer for the instructive suggestions.

References

[1] Alladi K, Hoggatt VE. On Tribonacci numbers and related functions. Fibonacci Quarterly 1997; 15 (1): 42-45.

[2] Amrouche S, Belbachir H. Unimodality and Linear Recurrences Associated with Rays in the Delannoy triangle. Turkish Journal of Mathematics 2020; 44.1: 118-130. https://doi.org/10.3906/mat-1811-109.

[3] Amrouche S, Belbachir H. Asymmetric Extension of Pascal-Delannoy triangles. Applicable Analysis and Discrete Mathematics 2022; 16 (2): 328-349. https://doi.org/10.2298/AADM200411028A.

[4] Amrouche S, Belbachir H, Ramirez JL. Unimodality, Linear Recurrences and Combinatorial Properties Associated to rays in the Generalized Delannoy matrix. Journal of Difference Equations and Applications 2019; 25 (8): 1200-1215. https://doi.org/10.1080/10236198.2019.1662413.

[5] Andrews G, Baxter J. Lattice Gas Generalization of the Hard Hexagon Model III q-trinomials coefficients. Journal of Statistical Physics 1987; 47: 297-330. https://doi.org/10.1007/BF01007513

[6] Barry P, On integer-Sequence-Based Constructions of Generalized Pascal triangles. Journal of Integer Sequences 2006; 9: Article 06.2.4.

[7] Belbachir H, Bencherif F, Linear Recurrence Sequences and Powers of Square matrix. Integers 2006; 6 (12): 1-17.

[8] Belbachir H, Komatsu T, Szalay L. Linear Recurrences Associated to rays in Pascal's Triangle and Combinatorial Identities. Mathematica Slovaca 2014; 64 (2): 287-300. https://doi.org/10.2478/s12175-014-0203-0.

[9] Belbachir H, Szalay L, On the Arithmetic Triangles. Siauliai Mathematics Seminars 2014; 9 (17): 15-26.

[10] Bollinger R, A note on Pascal T-triangles, Multinomial coefficients and Pascal pyramids. Fibonacci Quarterly 1986; 24: 140-144.

[11] de Moivre A, The doctrine of chances, Third edition 1756 (first ed. 1718 and second ed. 1738), reprinted by Chelsea, N. Y., 1967.

[12] de Moivre A, Miscellanca Analytica de Scrichus et Quadraturis, London, 1731.

[13] Dresden GP, Du Z. A Simplified Binet Formula for $k-$ generalized Fibonacci numbers. Journal of Integer Sequence 2014; 17: Article 14.4.7.

[14] Ferguson DE. An expression for generalized Fibonacci numbers. Fibonacci Quarterly 1966; 4: 270-273.

[15] Flores I. Direct calculation of k-generalized Fibonacci numbers. Fibonacci Quarterly 1967; 5: 259-266.

[16] Gabai H. Generalized Fibonacci k-sequences. Fibonacci Quarterly 1970; 8: 31-38.

[17] Lee GY, Lee SG. Jin-Soo Kim, and Hang-Kyun Shin. The Binet formula and representations of k-generalized Fibonacci numbers. Fibonacci Quarterly 2001; 39: 158-164.

[18] Liptai K, Panda GK, Szalay L. A balancing problem on a binary recurrence and its associate. Fibonacci Quarterly 2016; 54 (3): 235-240.

[19] Ramirez JL, Sirvent VF. A Generalization of the k-bonacci Sequence from Riordan Arrays. The electronic Journal of Combinatorics 2015; 22 (1): 1-38. https://doi.org/10.37236/4618.

[20] Yilmaz N, Taskara N. Incomplete Tribonacci-Lucas Numbers and Polynomials. Advances in Applied Clifford Algebras 2015; 25 (3): 741-753. https://doi.org/10.1007/s00006-014-0523-8.