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On the geometry of nearly trans-Sasakian manifolds

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Abstract: The geometry of nearly trans-Sasakian manifolds is researched in this paper. The complete group of structural equations and the components of the Lee vector on the space of the associated G-structure are obtained for such manifolds. Conditions are found under which a nearly trans-Sasakian structure is a trans-Sasakian, a cosymplectic, a closely cosymplectic, a Sasakian structure or a Kenmotsu structure. The conditions are obtained when the nearly trans-Sasakian structure is a special generalized Kenmotsu structure of the second kind. A complete classification of nearly trans-Sasakian manifolds is obtained, i.e. it is proved that a nearly trans-Sasakian manifold is either a trans-Sasakian manifold or has a closed contact form. It is proved that the nearly trans-Sasakian structure with a nonclosed contact form is homothetic to the Sasakian structure. The criterion of ownership of a nearly trans-Sasakian structure is obtained. It is proved that the class of nearly trans-Sasakian manifolds with a closed contact form and a closed Lee form coincides with the class of almost contact metric manifolds with a closed contact form, which are locally conformal to the closely cosymplectic manifolds. Examples of such manifolds are given. The necessary and sufficient conditions for the complete integrability of the first fundamental distribution of a nearly trans-Sasakian manifold are obtained. It is proved that a nearly Kähler structure on the leaves of the first fundamental distribution of a nearly trans-Sasakian manifold is induced.

Key words: Trans-Sasakian structure, nearly trans-Sasakian manifold, closely cosymplectic structure, linear extension of almost Hermitian structure, Lee form

1. Introduction

The geometric properties of almost Hermitian and almost contact metric structures have a number of peculiar interconnections. For example, it is well known [23, 24, 32, 34] that if M is an almost contact metric manifold then on the manifold $M \times \mathbf{R}$ an almost Hermitian structure is canonically induced (it is called linear extension of the original almost contact metric structure [16]). The question of the connection between these structures was studied many times. The classical result in the field is the famous result of Nakayama [26], which declares that an almost contact metric structure is normal if and only if its linear extension is a Hermitian structure. On the other hand, Gray and Hervella [9] have naturally in a certain way singled out the complete system containing 16 classes of almost Hermitian structures. That motivated us to classify almost contact metric

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2010 *AMS Mathematics Subject Classification:* 53D15 structures corresponding to the classification of their linear extensions. This problem was considered by such researchers as Nakayama [26], Kanemaki [10], Chinea, Gonzalez [6]. They have formulated a classification of almost contact metric structures. On this way Oubiña [28] singled out the classes of trans-Sasakian and almost trans-Sasakian structures, linear extensions of which belong to the classes W_4 and $W_2 \oplus W_4$ of almost Hermitian structures in Gray-Hervella classification [9] correspondingly. Kirichenko and Rodina in [16] got a number of insightful results related to the geometry of trans-Sasakian and almost trans-Sasakian manifolds. This paper encouraged us to describe the geometry of almost contact metric manifolds, whose linear extension falls in the Gray-Hervella class $W_1 \oplus W_4$. Such structures are called nearly trans-Sasakian structures [20]. Also, we use the methods of [16] in this research.

In [33] a nearly trans-Sasakian manifold was defined as an almost contact metric manifold, satisfying some identity, the normality properties of such manifolds was studied. Note that the nearly trans-Sasakian manifolds defined in [33] are a special case of the nearly trans-Sasakian manifolds we have defined.

The semiinvariant submanifolds of a nearly trans-Sasakian manifold are researched; the Nijenhuis tensor of a nearly trans-Sasakian manifold is calculated in [3, 7, 12, 13]. The integrability conditions of some distributions on an invariant submanifold of a nearly trans-Sasakian manifold are researched. The articles also deal with totally umbilical, totally contact umbilical, totally geodesic and totally contact geodesic submanifolds. It is supposed that the considered spaces, i.e. their connection is smooth sufficiently enough, otherwise, for geodesic, there exists geodesic bifurcation when at a certain point in a given direction geodesic splits into two, see [25, 29–31]. There is a classification of totally umbilical semiinvariant submanifolds of a nearly trans-Sasakian manifold in [12].

This paper has the following structure. Section 2 describes the method of associated G-structure and presents the general information necessary for the following research. In Section 3, we recall the definition of the nearly trans-Sasakian structure and characterize Sasakian, closely cosymplectic and special generalized Kenmotsu [1, 2]. Section 4 deals with the contact distribution of nearly trans-Sasakian manifolds.

2. Research methods

We recall that an almost contact metric (AC- for short) structure on the manifold M is a quadruple $(\xi, \eta, \Phi, g = \langle \cdot, \cdot \rangle)$ of tensor fields on M, where ξ is a vector field, which is called ***characteristic***, η is a differential 1-form called a contact form, Φ is an endomorphism of module $\mathcal{X}(M)$ of smooth vector fields on M and it is called a structural endomorphism, $g = \langle \cdot, \cdot \rangle$ – Riemannian metric. In addition,

$$1)\ \eta(\xi) = 1; \quad 2)\ \Phi(\xi) = 0; \quad 3)\ \eta \circ \Phi = 0; \quad 4)\ \Phi^2 = -id + \xi \otimes \eta;$$
$$5)\ \langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y); X, Y \in \mathcal{X}(M).$$

Such structures naturally appear on hypersurfaces of almost Hermitian manifolds [8], on the spaces of main T^1-bundles on symplectic manifolds with an even-dimensional fundamental forms (Boothby-Wang bundle [22]) and generally on almost Hermitian manifolds [17] and are natural generalizations of the so-called contact metric manifolds appearing on odd-dimensional manifolds with a fixed 1-form of the maximal rank (a contact structure).

It is well known that a manifold admitting an AC-structure is odd-dimensional and orientable. In $C^\infty(M)$-module $\mathcal{X}(M)$ of smooth vector fields on such manifold there are two internally determined mutually complementing projections $l = id - m = -\Phi^2$ and $m = \xi \otimes \eta$. These are the projections to the distributions

$\mathcal{L} = Im\Phi = ker\eta$ and $\mathcal{M} = ker\Phi$, correspondingly, which we will name the first and the second fundamental distributions of an AC-structure. Therefore, for module $\mathcal{X}(M)$ of smooth vector fields it is correct that $\mathcal{X}(M) = \mathcal{M} \oplus \mathcal{L}$, where $dim\mathcal{L} = 2n$, and $dim\mathcal{M} = 1$. More than that in case of introducing $\mathcal{X}(M)^C$ is the complexification of the module $\mathcal{X}(M)$, then $\mathcal{X}(M)^C = D_\Phi^{\sqrt{-1}} \oplus D_\Phi^{-\sqrt{-1}} \oplus D_\Phi^0$, where $D_\Phi^{\sqrt{-1}}$, $D_\Phi^{-\sqrt{-1}}$, D_Φ^0 are eigen distributions of the structural endomorphism Φ corresponding to their own values $\sqrt{-1}$, $-\sqrt{-1}$ and 0. Moreover the projections on the addends of this direct sum are endomorphisms $\pi = -\frac{1}{2}(\Phi^2 + \sqrt{-1}\Phi)$, $\overline{\pi} = \frac{1}{2}(-\Phi^2 + \sqrt{-1}\Phi)$, $m = \eta \otimes \xi$ correspondingly [19].

The assignment of an AC-structure on the manifold M^{2n+1} is equivalent to assigning G-structure $\mathcal{G}$ on M with the structural group $G = U(n) \times \{1\}$. The total space elements of this G-structure are complex frames of manifold M of the form $p = (p, \xi_p, \epsilon_1, , \epsilon_n, \epsilon_{\hat{1}}, , \epsilon_{\hat{n}})$. These frames are characterized by the fact that the matrices of tensors Φ and g in them are as follows:

$$(\Phi_i^j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

correspondingly, where I_n is an identity matrix of the order n. Here and subsequently throughout the paper indexes $i, j, k, \ldots$ run from 0 to $2n$, and indexes $a, b, c, d, \ldots$ run from 1 to n. Let $\hat{a} = a + n$.

It is well known [19] that the first group of structural equations of the G-structure $\mathcal{G}$ is as follows:

$$\begin{aligned} d\theta^a &= -\theta_b^a \wedge \theta^b + C^{ab}{}_c\theta^c \wedge \theta_b + C^{abc}\theta_b \wedge \theta_c + C^a{}_b\theta^b \wedge \theta + C^{ab}\theta_b \wedge \theta; \\ d\theta_a &= \theta_a^b \wedge \theta_b + C_{ab}{}^c\theta_c \wedge \theta^b + C_{abc}\theta^b \wedge \theta^c + C_a{}^b\theta_b \wedge \theta + C_{ab}\theta^b \wedge \theta; \\ d\theta &= D_{ab}\theta^a \wedge \theta^b + D^{ab}\theta_a \wedge \theta_b + D_a^b\theta^a \wedge \theta_b + D_a\theta \wedge \theta^a + D^a\theta \wedge \theta_a, \end{aligned} \tag{2.1}$$

where $\{\theta_j^i\}$ are the components of the Riemannian connection ∇ form of metrics g, $\{\theta^i\}$ are the components of the solder form, $\theta = \theta^0 = \pi^*\eta$, π is a natural projection of the G-structure total space to the manifold M and

$$\begin{gathered} \Phi^a_{b,k} = 0,\ \Phi^{\hat{a}}_{\hat{b},k} = 0,\ \Phi^0_{0,k} = 0,\ C^{abc} = \tfrac{\sqrt{-1}}{2}\Phi^a_{[\hat{b},\hat{c}]},\ C_{abc} = -\tfrac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{[b,c]}, \\ C^{ab}{}_c = -\tfrac{\sqrt{-1}}{2}\Phi^a_{\hat{b},c},\ C_{ab}{}^c = \tfrac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,\hat{c}},\ C^{ab} = \sqrt{-1}(\tfrac{1}{2}\Phi^a_{\hat{b},0} - \Phi^a_{0,\hat{b}}), \\ C^a{}_b = -\sqrt{-1}\Phi^a_{0,b},\ C_{ab} = -\sqrt{-1}(\tfrac{1}{2}\Phi^{\hat{a}}_{b,0} - \Phi^{\hat{a}}_{0,b}),\ C_a{}^b = \sqrt{-1}\Phi^{\hat{a}}_{0,\hat{b}}, \\ D^{ab} = \sqrt{-1}\Phi^0_{[\hat{a},\hat{b}]},\ D_{ab} = -\sqrt{-1}\Phi^0_{[a,b]},\ D_a^b = -\sqrt{-1}(\Phi^0_{a,\hat{b}} + \Phi^0_{\hat{b},a}), \\ D^a = -\sqrt{-1}\Phi^0_{\hat{a},0},\ D_a = \sqrt{-1}\Phi^0_{a,0}. \end{gathered} \tag{2.2}$$

In addition $C^{abc} = -C^{acb}$, $C^{ab}{}_c = -C^{ba}{}_c$; $D^{ab} = -D^{ba}$; $C_{abc} = -C_{acb}$; $C_{ab}{}^c = C_{ba}{}^c$; $D_{ab} = -D_{ba}$; $D_a^b = C_a{}^b - C^b{}_a$.

We remind that an almost Hermitian structure (AH- for short) on the manifold M is a couple $(J, g = \langle\cdot,\cdot\rangle)$ of tensor fields on M, where J is an almost complex structure, $J^2 = -id$, g is a Riemannian metric such that $\langle JX, JY\rangle = \langle X, Y\rangle$; $X, Y \in \mathcal{X}(M)$.

The assignment of an AH-structure on M^{2n} is equivalent to the assignment of a G-structure on M with a structural group $U(n)$. The total space elements of this G-structure are complex frames of the manifold M, characterized by the fact that, the matrices of tensors J and g are as follows:

$$(J_i^j) = \begin{pmatrix} \sqrt{-1}I_n & 0 \\ 0 & -\sqrt{-1}I_n \end{pmatrix}, \quad (g_{ij}) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

correspondingly.

It is well known that the first group of structural equations of this G-structure is as follows [4]:

$$d\omega^a = -\omega_b^a \wedge \omega^b + B^{ab}{}_c\omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c; \tag{2.3}$$

$$d\omega_a = \omega_a^b \wedge \omega_b + B_{ab}{}^c\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c. \tag{2.4}$$

Here $\{\omega_j^i\}$ are the components of the Riemannian connection form of metrics g, $\{\omega^i\}$ are the components of the solder form, $B^{abc} = \overline{B_{abc}}$ and $B^{ab}{}_c = \overline{B_{ab}{}^c}$ are the components of the so-called structural and virtual tensors in the space of the associated G-structure. With that: $B^{abc} = -B^{acb}$, $B_{abc} = -B_{acb}$, $B^{ab}{}_c = -B^{ba}{}_c$, $B_{ab}{}^c = -B_{ba}{}^c$ (for example, see [4]).

We remind [9], that an AH-structure of class $W_1 \oplus W_4$ in Gray-Hervella classification (Vaisman-Gray structure) on the manifold M^{2n} is defined by the identity $\nabla_X(\Psi)(X,Y) = -\frac{1}{2(n-1)}\{\langle X,X\rangle\,\delta\Psi(Y) - \langle X,Y\rangle\,\delta\Psi(X) - \langle JX,Y\rangle\,\delta\Psi(JX)\}$, where $\Psi(X,Y) = \langle X, JY\rangle$ is the fundamental form of the AH-structure, δ is the codifferentiation operator. Direct calculation proves that this identity is equivalent to the following relation in the space of the associated G-structure: $B^{[abc]} = B^{abc}$; $B^{ab}{}_c = \beta^{[a}\delta_c^{b]}$; $B_{[abc]} = B_{abc}$; $B_{ab}{}^c = \beta_{[a}\delta_{b]}^c$, where $\{\beta_i\}$ are functions in the space of the associated G-structure, which are the components of the so-called Lee form (see Definition 2.1).

We will briefly remind the linear extension construction of the AC-manifold M (or its AC-structure linear extension, which is the same). It should be noted that on the manifold $M \times \mathbf{R}$ the two-dimensional distribution Δ is internally determined such that $\Delta_{(p,t)} = \mathcal{M}_p \oplus \mathbf{R}$. It is obvious, that this distribution is provided with a canonical almost Hermitian structure (J_0, g_0), where J_0 is an operator of rotation by $\frac{\pi}{2}$ angle in the positive direction. It is also obvious that the couple $(J, \tilde{g})$ is an almost Hermitian structure on the manifold $M \times \mathbf{R}$, where $J_{(p,t)} = \Phi|_{\mathcal{L}_p} \oplus J_0, \tilde{g}$ is a metric of Cartesian product. We will notice that the distribution $\Delta^{\perp}$ is invariant in relation to the endomorphism J. The triple $(M \times \mathbf{R}, J, \tilde{g})$ is called the linear extension of the initial AC-manifold [16]. On the manifold $(M \times \mathbf{R})$ there is the internal definition of the vector field ν, generated by the unit vector of the number axis $\mathbf{R}$, the closed 1-form ζ, dual to it and determining the Pfaffian completely integrable equation $\zeta = 0$, the maximal integral manifolds of which are naturally identified with the manifold M, and the vector field ξ and the covector field η, correspondingly, with the eigen vector and the contact form of the manifold M. Due to them the frames of the type $p = (p, \xi_p, \epsilon_1, \ldots, \epsilon_n, \epsilon_{\hat{1}}, \ldots, \epsilon_{\hat{n}})$ of the manifold M are naturally complemented up to the frames of the type $\tilde{p} = (p, \xi_p, \epsilon_1, \ldots, \epsilon_n, \nu_p, \epsilon_{\hat{1}}, \ldots, \epsilon_{\hat{n}})$ of the manifold $(M \times \mathbf{R})$. This manifold is naturally associated with the G-structure $\mathbb{G}$ with the structural group $G = U(n) \times U(1)$, whose first group structural equations is as follows:

$$d\omega^\alpha = -\omega_\beta^\alpha \wedge \omega^\beta + B_\gamma^{\alpha\beta}\omega^\gamma \wedge \omega_\beta + B^{\alpha\beta\gamma}\omega_\beta \wedge \omega_\gamma\ ;$$

$$d\omega_\alpha = \omega_\alpha^\beta \wedge \omega_\beta + B_{\alpha\beta}^\gamma\omega_\gamma \wedge \omega^\beta + B_{\alpha\beta\gamma}\omega^\beta \wedge \omega^\gamma$$

(indexes α, β, γ, ... run from 0 to n). The elements of the total space of this G-structure are the complex frames of the type $r = (p, \epsilon_0, \epsilon_1, , \epsilon_n, \epsilon_{\hat{0}}, \dots, \epsilon_{\hat{n}})$, where $\epsilon_p = \frac{1}{\sqrt{2}}\left(\xi_p - \sqrt{-1}\nu_p\right)$; $\epsilon_{\hat{p}} = \frac{1}{\sqrt{2}}\left(\xi_p + \sqrt{-1}\nu_p\right)$. Completing the system (2.1) with the equations $d\theta_0 = 0$, where $\theta_0 = \pi^*\zeta$, and the transition matrix from frame $\tilde{p}$ to frame r, it is easy to determine the fundamental connection between the structural objects of G-structure $\mathcal{G}$ and $\mathbb{G}$:

$$1.\ B^{ab}{}_c = C^{ab}{}_c;\quad 2.\ B^{ab}{}_0 = \frac{1}{\sqrt{2}}\left(D^{ab} - C^{[ab]}\right);\quad 3.\ B^{a0}{}_b = \frac{1}{\sqrt{2}}C^a{}_b;$$
$$4.\ B^{a0}{}_0 = \frac{1}{2}D^a;\quad 5.\ B^{abc} = C^{abc};\quad 6.\ B^{ab0} = \frac{1}{2\sqrt{2}}C^{ab};$$
$$7.\ B^{0ab} = \frac{1}{\sqrt{2}}D^{ab};\quad 8.\ B^{00a} = -\frac{1}{4}D^a \tag{2.5}$$

and the complex conjugate formulae. Let M be a $(2n+1)$-dimensional almost contact metric manifold, provided with the AC-structure $\{\Phi, \xi, \eta, g = \langle\cdot,\cdot\rangle\}$. We will denote by $\Omega(X,Y) = \langle X, \Phi Y\rangle$ the structure fundamental form; $\Omega(X,Y) = -\Omega(Y,X)$. We remind [5], that an AC-structure is called contact metric or almost Sasakian, if $d\eta = \Omega$, normal, if $2N + d\eta \otimes \xi = 0$, where $N(X,Y) = \frac{1}{4}\left\{\Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y]\right\}$ is a structure operator Nijenhuis tensor. We remind [19] that an AC-structure is normal if and only if in the space of the associated G-structure one has

$$C^{abc} = C^{ab}{}_c = C^{ab} = D^{ab} = D^a = 0;$$
$$C_{abc} = C_{ab}{}^c = C_{ab} = D_{ab} = D_a = 0. \tag{2.6}$$

A normal contact metric structure is called Sasakian.

We remind, that an AC-structure is called almost cosymplectic, if its contact and fundamental forms are closed. A normal almost cosymplectic structure is called cosymplectic. An AC-structure, for which $\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0$, is called nearly cosymplectic. A nearly cosymplectic structure with a closed contact form is called closely cosymplectic. It is known [14] that any closely cosymplectic manifold is locally equivalent to the product of the nearly Kähler manifold by the real straight line.

Definition 2.1 *The Lee form of the almost Hermitian structure* $(J, \tilde{g})$ *on the manifold* M^{2n+2} *is the form* $\alpha = \frac{1}{n}\delta\Psi \circ J$, *where* $\Psi(X,Y) = \tilde{g}(X, JY)$ *is a fundamental structure form,* δ *is the codifferentiation operator. Vector* β *dual to the Lee form is called the Lee vector.*

By AC-structures Lee form in this article we mean the Lee form of its linear extension.

It is easy to verify, that in the space of the associated G-structure $\mathbb{G}$ Lee vector (or form) components are found from the formula $\beta^\alpha = \frac{2}{n}B^{\alpha\gamma}{}_\gamma$ or considering (2.5): $\beta^a = \frac{2}{n}C^{ah}{}_h + \frac{1}{n}D^a$; $\beta^0 = -\frac{\sqrt{2}}{n}C^h{}_h$; $\beta_a = \frac{2}{n}C_{ah}{}^h + \frac{1}{n}D_a$; $\beta_0 = -\frac{\sqrt{2}}{n}C_h{}^h$.

3. The definition of a nearly trans-Sasakian structure and its structural equations

Definition 3.1 *[20]. An AC-structure is called nearly trans-Sasakian (NTS - for short) structure if its linear extension belongs to the class* $W_1 \oplus W_4$ *of almost Hermitian structures in Gray-Hervella classification. An AC-manifold, provided with an NTS-structure, is called an NTS-manifold.*

The following theorem is valid.

Theorem 3.2 *An AC-structure is an NTS-structure if and only if in the space of the associated G-structure one has:*

$$1)\ C^{ab}{}_{c}=C^{ab}=D^{ab}=D^{a}=0;\quad 2)\ C^{a}{}_{b}=-\tfrac{1}{\sqrt{2}}\beta^{0}\delta^{a}_{b};$$

$$3)\ C_{ab}{}^{c}=C_{ab}=D_{ab}=D_{a}=0;\quad 4)\ C_{a}{}^{b}=-\tfrac{1}{\sqrt{2}}\beta_{0}\delta^{b}_{a};$$

$$5)\ D^{a}_{b}=\tfrac{1}{\sqrt{2}}\left(\beta^{0}-\beta_{0}\right)\delta^{a}_{b};\quad 6)\ C^{[abc]}=C^{abc};\quad 7)\ C_{[abc]}=C_{abc}. \tag{3.1}$$

Proof Let M be an NTS-manifold. According to the definition it means that the linear extension of its AC-structure belongs to the Gray-Hervella class $W_1\oplus W_4$. As it has been mentioned, it matches the relations: $B^{\alpha\beta}{}_{\gamma}=\beta^{[\alpha}\delta^{\beta]}_{\gamma}$; $B^{[\alpha\beta\gamma]}=B^{\alpha\beta\gamma}$; $B_{\alpha\beta}{}^{\gamma}=\beta_{[\alpha}\delta^{\gamma}_{\beta]}$; $B_{[\alpha\beta\gamma]}=B_{\alpha\beta\gamma}$. Describing (2.5) while considering these relations we get:

1) $B^{ab}{}_{c}=\beta^{[a}\delta^{b]}_{c}$, i.e. $C^{ab}{}_{c}=\beta^{[a}\delta^{b]}_{c}$;

2) $B^{ab}{}_{0}=\beta^{[a}\delta^{b]}_{0}=0$, i.e. $C^{[ab]}=D^{ab}$;

3) $B^{a0}{}_{b}=\beta^{[a}\delta^{0]}_{b}$, i.e. $C^{a}{}_{b}=-\frac{1}{\sqrt{2}}\beta^{0}\delta^{a}_{b}$;

4) $B^{a0}{}_{0}=\beta^{[a}\delta^{0]}_{0}=\frac{1}{2}\beta^{a}$, i.e. $\beta^{a}=D^{a}$;

5) $B^{[abc]}=B^{abc}$, i.e. $C^{[abc]}=C^{abc}$;

6) $B^{ab0}=-B^{ba0}$, i.e. $C^{ab}=-C^{ba}$, and it means that $D^{ab}=C^{ab}$;

7) $B^{ab0}=B^{0ab}$, i.e. $C^{ab}=2D^{ab}$, and it means that $C^{ab}=D^{ab}=0$;

8) $B^{00a}=0$, i.e. $D^{a}=0$, and it means that $\beta^{a}=0$, $C^{ab}{}_{c}=0$.

Finally, $D^{a}_{b}=C_{b}{}^{a}-C^{a}{}_{b}=\frac{1}{\sqrt{2}}\left(\beta^{0}-\beta_{0}\right)\delta^{a}_{b}$. The remaining relations are verified in the same way. □

Corollary 3.3 *The first group of structural equations of an NTS-structure in the space of the associated G-structure is as follows:*

$$1)\ d\theta^{a}=-\theta^{a}_{b}\wedge\theta^{b}+C^{abc}\theta_{b}\wedge\theta_{c}+\tfrac{\beta^{0}}{\sqrt{2}}\theta\wedge\theta^{a};$$

$$2)\ d\theta_{a}=\theta^{b}_{a}\wedge\theta_{b}+C_{abc}\theta^{b}\wedge\theta^{c}+\tfrac{\beta_{0}}{\sqrt{2}}\theta\wedge\theta_{a};$$

$$3)\ d\theta=\tfrac{1}{\sqrt{2}}\left(\beta^{0}-\beta_{0}\right)\delta^{b}_{a}\theta^{a}\wedge\theta_{b}, \tag{3.2}$$

where $C^{[abc]}=C^{abc}$, $C_{[abc]}=C_{abc}$.

Corollary 3.4 *The Lee vector (or form) components are as follows:* $\beta^{a}=\beta_{a}=0$; $\beta^{0}=-\frac{\sqrt{2}}{n}C^{h}{}_{h}$; $\beta_{0}=-\frac{\sqrt{2}}{n}C_{h}{}^{h}$ *for an NTS-structure in the space of the associated G-structure.*

Corollary 3.5 *For the covariant differential components of the structural endomorphism of an NTS-structure in the space of the associated G-structure we have:*

$$1)\ \Phi^{a}_{0,b}=\tfrac{1}{\sqrt{2}}\sqrt{-1}\beta^{0}\delta^{a}_{b};\quad 2)\ \Phi^{\hat{a}}_{0,\hat{b}}=-\tfrac{1}{\sqrt{2}}\sqrt{-1}\beta_{0}\delta^{b}_{a};$$

$$3)\ \Phi^{0}_{\hat{a},b}=-\tfrac{1}{\sqrt{2}}\sqrt{-1}\beta^{0}\delta^{a}_{b};\quad 4)\ \Phi^{0}_{a,\hat{b}}=\tfrac{1}{\sqrt{2}}\sqrt{-1}\beta_{0}\delta^{b}_{a};$$

$$5)\ \Phi^{\hat{c}}_{a,b}=-\Phi^{\hat{c}}_{b,a};\quad 6)\Phi^{c}_{\hat{a},\hat{b}}=-\Phi^{c}_{\hat{b},\hat{a}}\ .$$

Corollary 3.6 *An NTS-structure is a trans-Sasakian structure if and only if in the space of the associated G-structure one has* $C^{abc} = C_{abc} = 0$. *In addition, it is: Sasakian* $\Leftrightarrow$ $\beta^0 = -\beta_0 = \sqrt{-2}$*; cosymplectic* $\Leftrightarrow$ $\beta^0 = \beta_0 = 0$*; Kenmotsu* $\Leftrightarrow$ $\beta^0 = \beta_0 = \sqrt{2}$.

Corollary 3.7 *An NTS-structure is a closely cosymplectic structure if and only if in the space of the associated G-structure one has* $\beta^0 = \beta_0 = 0$.

Corollary 3.8 *An NTS-structure is a special generalized Kenmotsu structure of the second kind if and only if in the space of the associated G-structure one has* $\beta^0 = \beta_0$.

The standard procedure of a differential extension of relations (3.2) allows to get the second group of structural equations of an NTS-structure:

$$d\theta^a_b + \theta^a_c \wedge \theta^c_b = \left(A^{ad}_{bc} - 2C^{adh}C_{hbc}\right)\theta^c \wedge \theta_d, \tag{3.3}$$

where $\{A^{ad}_{bc}\}$ is a class of functions in the field of the associated G-structure, serving as the components of the so-called curvature tensor of the associated Q-algebra [2], or the structural tensor of the second type, besides

$$\begin{gathered} 1)\ A^{ad}_{[bc]} = \tfrac{1}{2}\beta^0\left(\beta^0 - \beta_0\right)\delta^a_{[b}\delta^d_{c]}; \\ 2)\ A^{[bd]}_{ac} = -\tfrac{1}{2}\beta_0\left(\beta^0 - \beta_0\right)\delta^{[b}_a\delta^{d]}_c; \\ 3)\ \overline{A^{ad}_{bc}} = A^{bc}_{ad}. \end{gathered} \tag{3.4}$$

Moreover,

$$\begin{gathered} 1)\ dC^{abc} + C^{dbc}\theta^a_d + C^{adc}\theta^b_d + C^{abd}\theta^c_d = C^{abcd}\theta_d + \tfrac{1}{\sqrt{2}}\beta^0 C^{abc}\theta; \\ 2)\ dC_{abc} - C_{dbc}\theta^d_a - C_{adc}\theta^d_b - C_{abd}\theta^d_c = C_{abcd}\theta^d + \tfrac{1}{\sqrt{2}}\beta_0 C_{abc}\theta; \\ 3)\ d\beta^0 = \beta^{00}\theta; \quad 4)\ d\beta_0 = \beta_{00}\theta, \end{gathered} \tag{3.5}$$

where C^{abcd}, C_{abcd}, $C^{abc}{}_d$, $C_{abc}{}^d$, β^{00}, β_{00} are suitable functions in the field of the associated G-structure, besides

$$\begin{gathered} 1)\ \left(\beta^0 - \beta_0\right)C_{abc} = 0; \quad 2)\ \left(\beta^0 - \beta_0\right)C^{abc} = 0; \\ 3)\ \left(\beta^{00} - \beta_{00}\right) = \tfrac{1}{\sqrt{2}}\left\{\left(\beta^0\right)^2 - \left(\beta_0\right)^2\right\}; \\ 4)\ C^{a[bcd]} = 0; \quad 5)\ C_{a[bcd]} = 0. \end{gathered} \tag{3.6}$$

Theorem 3.9 *An NTS-manifold is either a trans-Sasakian manifold, or it has a closed contact form.*

Proof It follows from the identity (3.6:1), that $\beta^0 = \beta_0$ or $C_{abc} = 0$. In the first case an NTS-manifold has a closed contact form. In the second case an NTS-manifold, according to Corollary 3.6 of the Theorem 3.2, is a trans-Sasakian manifold. □

Definition 3.10 *[16, 25]. A transformation* $(\xi,\eta,\Phi,g)\to\left(f^{-1}\xi,f\eta,\Phi,f^2g\right)$ *is called a conformal transformation of an AC-structure, here* f *is a positive function. The function* $\sigma=-\ln f$ *is called the determining function of the conformal transformation. If* $f=const$*, the conformal transformation is called a homothety with the coefficient* f*.*

Definition 3.11 *[16] A point of an* AC*-manifold is called nonspecial, if the 2-form* $d\theta$ *in this point is other than zero. An open submanifold of an* AC*-manifold, which consists of nonspecial points, is called a nonspecial submanifold.*

Theorem 3.12 *An NTS-structure on* M^{2n+1} $(n>1)$ *with a nonclosed contact form is homothetic to a Sasakian structure.*

Proof Let $\Omega(X,Y)=\langle X,\Phi Y\rangle$; $X,Y\in\mathcal{X}(M)$ be a fundamental form of an AC-structure. It is obvious that in the space of the associated G-structure one has

$$\pi^*\Omega=-2\sqrt{-1}\theta^a\wedge\theta_a\ . \tag{3.7}$$

Therefore, the relation (3.2:3) could be put in the form $d\theta=\sqrt{-\frac{1}{8}}\left(\beta^0-\beta_0\right)\pi^*\Omega$. Let us call $\sqrt{-\frac{1}{8}}\left(\beta^0-\beta_0\right)=f$ and differentiate the relation $d\theta=f(\pi^*\Omega)$ externally taking into account relations (3.2) and (3.7), supposing $df=f_a\theta^a+f^a\theta_a+f_0\theta$. Comparing coefficients of the similar terms, because of the linear independence of the basic forms we get:

$$\begin{gathered}1)\ \delta^a_{[b}f_{c]}=0;\quad 2)\ \delta^{[a}_b f^{c]}=0;\quad 3)\ fC^{abc}=0;\\ 4)\ fC_{abc}=0;\quad 5)\ f_0=-\tfrac{f}{\sqrt{2}}\left(\beta^0+\beta_0\right).\end{gathered} \tag{3.8}$$

Convoluting the first two relations over the indexes a and b, we get that, if $n>1$ $f^a=f_a=0$, it means that, $df=f_0\theta$. Let us differentiate this identity externally: $df_0\wedge\theta-2\sqrt{-1}f_0f\theta^a\wedge\theta_a=0$. If at any point $f\neq0$, then at this point one has $f_0=0$, because of the linear independence of basic forms. It means that on the nonspecial submanifold $M_0\subset M$ we have: $df=0$, i.e. $f=const$. Among other facts, submanifold $M_0\subset M$ is open-closed, and it means that if M is connected, then either $M_0=M$ or $M_0=\varnothing$.

In the first case replacing, as necessary, η with $-\eta$, and ξ with $-\xi$, we can consider, that $f>0$. According to (3.8), in this case one has $C^{abc}=C_{abc}=0$, and, because of (3.1), an AC-structure is normal. Moreover, in this case we get $\pi^*\left(d\eta\right)=d\left(\pi^*\eta\right)=d\theta=f\pi^*\Omega=\pi^*\left(f\Omega\right)$. That is why, performing a homothetic transformation of the structure (obviously without interfering with its normality) with a coefficient f, we get contact metric and therefore a Sasakian structure. Indeed, in this case $\tilde\eta=f\eta$, $\tilde\Omega=f^2\Omega$, which means, $\eta=\frac{1}{f}\tilde\eta$, $\Omega=\frac{1}{f^2}\tilde\Omega$, and because $d\eta=\Omega$, then $d\tilde\eta=\tilde\Omega$. Hence in the first case an NTS-structure is homothetic to a Sasakian one. □

In the second case, when $M_0=\varnothing$, $d\eta=0$, meaning, $\beta^0=\beta_0$. In this case $\pi^*\alpha=\beta_i\theta^i=\beta_0\theta^0+\beta^0\theta_0=\beta^0\left(\theta^0+\theta_0\right)=\sqrt{2}\beta^0\theta=\sqrt{2}\beta^0\pi^*\eta$. It means that $\exists\tilde\beta^0\in C^\infty(M)$ and $\beta^0=\pi^*\tilde\beta^0$. Let us assume that $\chi=-\frac{1}{\sqrt{2}}\tilde\beta^0$. Then, because

$$\alpha=\sqrt{2}\tilde\beta^0\eta,\quad \alpha=-2\chi\eta, \tag{3.9}$$

and the first group of structural equations will have the form:

$$1)\ d\theta^a = -\theta^a_b \wedge \theta^b + C^{abc}\theta_b \wedge \theta_c + \chi\theta^a \wedge \theta;$$
$$2)\ d\theta_a = \theta^b_a \wedge \theta_b + C_{abc}\theta^b \wedge \theta^c + \chi\theta_a \wedge \theta;\quad 3) d\theta = 0.$$

Considering Theorem 3.12 it is natural to assume.

Definition 3.13 *[20] An NTS-structure with a closed contact form is called an eigen NTS-structure.*

Theorem 3.14 *An AC-structure with a closed contact form on the manifold M is an eigen NTS-structure if and only if the following identity is true*

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = \chi\left\{\eta(X)\Phi Y + \eta(Y)\Phi X\right\},\ X, Y \in \mathcal{X}(M). \tag{3.10}$$

Proof It is obvious that the identity (3.10) is equivalent to the following identities in the bundle of frames space above the manifold M:

$$\Phi^i_{j,k} + \Phi^i_{k,j} = \chi\left\{\delta^0_k\Phi^i_j + \delta^0_j\Phi^i_k\right\}. \tag{3.11}$$

Describing the restriction of these identities in the space of the associated G-structure, we get: $\Phi^a_{0,b} + \Phi^a_{b,0} = \chi\left(\delta^0_0\Phi^a_b + \delta^0_b\Phi^a_0\right) = \chi\sqrt{-1}\delta^a_b$, which leads to, considering (2.2),

$$C^a{}_b = \chi\delta^a_b \text{ and, similarly, } C_a{}^b = \chi\delta^b_a. \tag{3.12}$$

In the similar way it is verified that $C^{abc} = -C^{acb}$, $C_{abc} = -C_{acb}$, i.e. $C^{[abc]} = C^{abc}$, $C_{[abc]} = C_{abc}$, and also

$$C^{ab}{}_c = C_{ab}{}^c = 0, \tag{3.13}$$

and, considering the closed nature of the contact form, we have

$$C^{ab} = 0,\ D^{ab} = 0,\ D^a_b = C_b{}^a - C^a{}_b = 0,\ D^a = 0,\ C_{ab} = 0,\ D_{ab} = 0,\ D_a = 0. \tag{3.14}$$

But according to (3.1), identities (3.12)–(3.14) are just characteristic identities of the eigen NTS-structure. □

Remark 3.15 *By putting $\chi = 1$ in 3.10, one obtains the defining condition of the NTS-manifold introduced by Shukla in [33].*

Theorem 3.16 *The class of NTS-manifolds with a closed contact form and a closed Lee form coincides with the class of AC-manifolds, with a closed contact form, which are locally conformal to closely cosymplectic manifolds.*

Proof Let $\sigma \in C^\infty(M)$. Let us have a conformal transformation of the eigen NTS-structure with a defining function σ: $\tilde{g} = e^{-2\sigma}g$; $\tilde{\eta} = e^{-\sigma}\eta$; $\tilde{\xi} = e^{\sigma}\xi$.

Let $\tilde{\nabla}$ be a Riemannian connection of the transformed structure. Then, as it is well known (see [9], for example), tensor T of the affine deformation from connection ∇ to connection $\tilde{\nabla}$ has the following form: $T(X,Y) = \langle X, Y\rangle\,\zeta - d\sigma(X)Y - d\sigma(Y)X$, $X, Y \in \mathcal{X}(M)$, where $\zeta = grad\sigma$. Thus, $\tilde{\nabla}_X Y = \nabla_X Y + \langle X, Y\rangle\,\zeta - d\sigma(X)Y - d\sigma(Y)X$, and it means that $\tilde{\nabla}_X(\Phi)Y = \tilde{\nabla}_X(\Phi Y) - \Phi(\tilde{\nabla}_X Y) = \nabla_X(\Phi Y) + \langle X, \Phi Y\rangle\,\zeta - d\sigma(X)\Phi Y -$

$d\sigma(\Phi Y)X - \Phi(\nabla_X Y) - \langle X, Y\rangle \Phi\zeta + d\sigma(X)\Phi Y + d\sigma(Y)\Phi X = \nabla_X(\Phi)Y + \langle X, \Phi Y\rangle \zeta - d\sigma(\Phi Y)X - \langle X, Y\rangle \Phi\zeta + d\sigma(Y)\Phi X$, i.e.

$$\tilde{\nabla}_X(\Phi)Y = \nabla_X(\Phi)Y + \langle X, \Phi Y\rangle \zeta - d\sigma(\Phi Y)X - \langle X, Y\rangle \Phi\zeta + d\sigma(Y)\Phi X. \tag{3.15}$$

In particular, considering (3.10)

$$\tilde{\nabla}_X(\Phi)X = \nabla_X(\Phi)X - d\sigma(\Phi X)X - \|X\|^2 \Phi\zeta + d\sigma(X)\Phi X = \\ = \chi\eta(X)\Phi X - d\sigma(\Phi X)X + d\sigma(X)\Phi X - \|X\|^2 \Phi\zeta. \tag{3.16}$$

If the function σ may be chosen so that

$$d\sigma = -\chi\eta, \tag{3.17}$$

then, obviously, $\zeta = -\chi\xi$, and considering the axiom of an AC-structure, $\tilde{\nabla}_X(\Phi)X = 0$. Moreover, in this case, because of the closed nature of the contact form η, $d\tilde{\eta} = d(e^{-\sigma}\eta) = -e^{-\sigma}d\sigma \wedge \eta = e^{-\sigma}\chi\eta \wedge \eta = 0$, i.e. the transformed structure is closely cosymplectic, and the manifold M is conformal to the closely cosymplectic manifold.

Conversely, let M be an AC-manifold with a closed contact form η, conformal to the closely cosymplectic manifold, and let σ be the determining function of the corresponding conformal transformation of its AC-structure $(\xi, \eta, \Phi, g) \to (\tilde{\xi}, \tilde{\eta}, \Phi, \tilde{g})$. Then $\tilde{\eta} = e^{-\sigma}\eta$, $0 = d\tilde{\eta} = -e^{-\sigma}d\sigma \wedge \eta$, and it means that, $d\sigma \wedge \eta = 0$. Thus, $\exists\chi \in C^\infty(M)$ and $d\sigma = -\chi\eta$. In accordance, for vector ξ, dual to the form $d\sigma$, we have: $\langle \zeta, X\rangle = d\sigma(X) = -\chi\eta(X) = -\chi\langle \xi, X\rangle$, and because of the nonsingular metric, $\zeta = -\chi\xi$. Consequently, considering (3.16), $0 = \tilde{\nabla}_X(\Phi)X = \nabla_X(\Phi)X - d\sigma(\Phi X)X - \|X\|^2 \Phi\zeta + d\sigma(X)\Phi X = \nabla_X(\Phi)X - \chi\eta(X)\Phi X$. Polarizing this identity, we get the identity (3.10). Due to Theorem 3.14, the initial structure is an eigen NTS-structure. □

We notice that the integrability condition of (3.17) is equivalent to the closeness of Lee form of the linear extension. By (3.9) it follows that the function σ is the half of the Lee form integral.

Thus, combining Theorems 3.14 and 3.5 we formulate the following theorem.

Theorem 3.17 *The class of (2n+1)-dimensional ($n > 1$) NTS-manifolds with a nonclosed contact form coincides with the class of AC-manifolds, homothetic to the Sasakian manifolds. The class of NTS-manifolds with a closed contact form and a closed Lee form coincides with the class of AC-manifolds, with a closed contact form which are locally conformal to closely cosymplectic manifolds.*

Example 3.18 *An important example of an eigen NTS-manifold with a closed contact form and a closed Lee form is the Kenmotsu manifold, i.e. the AC-manifold, characterized by the identity* $\nabla_X(\Phi)Y = \langle \Phi X, Y\rangle \xi - \eta(Y)\Phi X$; $X, Y \in \mathcal{X}(M)$, *and the following identity* $\nabla_X(\eta)Y = \langle X, Y\rangle - \eta(X)\eta(Y)$; $X, Y \in \mathcal{X}(M)$. *It follows directly from Theorem 3.14, assuming that in (3.10)* $\chi = -1$, *and thus,* $\alpha = 2\eta$, *and because of the last identity* $d\eta = 0$. *The Kenmotsu manifolds were introduced in 1972 [11] and have a number of extraordinary qualities. Kenmotsu structures, for example, are defined on odd-dimensional Lobachevsky spaces with curvature* 1. *The complete description of Kenmotsu structures is given in papers [18, 27].*

Another important example of an eigen NTS*-manifold with a closed contact form and a closed Lee form is the special generalized Kenmotsu manifold of the second kind [1].*

4. The contact distribution of a nearly trans-Sasakian manifold

Let us consider the differential 1-form $\omega = \eta \circ \pi_*$, where π is a natural projection in the principal frame bundle above manifold M, π_* is a dragging of the connected vector fields on manifold M generated by the form π. Obviously, this form is a Pfaffian form of the first fundamental distribution, i.e. the basis of codistribution associated with the first fundamental distribution $\mathcal{L}$ [15].

According to the classical Frobenius theorem total integrability of the first fundamental distribution is equivalent to the existence condition of such form θ, that $d\omega = \theta \wedge \omega$, i.e. the exterior differential of form ω must belong to the ideal of Grassmann algebra of manifold M [35].

Let us consider the first group of the structural equations of the NTS-structure in the space of the associated G-structure:

$$1)\ d\theta^a = -\theta^a_b \wedge \theta^b + C^{abc}\theta_b \wedge \theta_c + \tfrac{\beta^0}{\sqrt{2}}\theta \wedge \theta^a;$$
$$2)\ d\theta_a = \theta^b_a \wedge \theta_b + C_{abc}\theta^b \wedge \theta^c + \tfrac{\beta^0}{\sqrt{2}}\theta \wedge \theta_a;$$
$$3)\ d\theta = \tfrac{1}{\sqrt{2}}\left(\beta^0 - \beta_0\right)\delta^b_a \theta^a \wedge \theta_b. \tag{4.1}$$

The right part of this third equation is inconsistent with the form: $d\omega = \theta \wedge \omega$. Then the first fundamental distribution $\mathcal{L}$ is totally integrable if and only if $\beta^0 = \beta_0$. In this case an NTS-structure has a closed contact form, i.e. is an eigen NTS-structure.

Because an eigen NTS-manifold has a totally integrable first fundamental distribution, we have the following theorem.

Theorem 4.1 *The first fundamental distribution of an NTS-manifold is totally integrable if and only if the NTS-manifold is eigen.*

Theorem 4.2 *The almost Hermitian structure induced on the leaves of the first fundamental distribution of an NTS-manifold is a nearly Kählerian structure.*

Proof Let M be an NTS-manifold with a totally integrable first fundamental distribution $\mathcal{L}$. From the above we get the first group of the structural equations of this manifold:

$$1) d\theta^a = -\theta^a_b \wedge \theta^b + C^{abc}\theta_b \wedge \theta_c + \tfrac{\beta^0}{\sqrt{2}}\theta \wedge \theta^a;$$
$$2)\ d\theta_a = \theta^b_a \wedge \theta_b + C_{abc}\theta^b \wedge \theta^c + \tfrac{\beta^0}{\sqrt{2}}\theta \wedge \theta_a;$$
$$3)\ d\theta = 0.$$

Let $N \subset M$ be an integral manifold of maximal dimension of the first fundamental distribution of the manifold M. Then on it almost Hermitian structure $\langle J, \tilde{g}\rangle$ is canonically inspired, where $J = \Phi|_{\mathcal{L}}$, $\tilde{g} = g|_{\mathcal{L}}$. Because form ω is a Pfaffian form of the first fundamental distribution, the first group of the structural equations of the almost Hermitian structure on N has the form:

$$1)\ d\omega^a = -\theta^a_b \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c;$$
$$2)\ d\omega_a = \theta^b_a \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c; \quad 3)\ d\omega = 0. \tag{4.2}$$

Keeping in view the Gray-Hervella classification of almost Hermitian structures, presented as a table ([21], page 450), we get that the almost Hermitian structure, inspired on the integral submanifolds of manifold M, is nearly Kählerian. Thus, the theorem is proved. □

5. Conclusion

In this paper, we have defined the class of nearly trans-Sasakian manifolds. The complete group of its structural equations on the space of the associated G-structure is obtained; also the components of the Lee vector are calculated. Examples of nearly trans-Sasakian manifolds are trans-Sasakian, cosymplectic, closely cosymplectic, Sasakian, Kenmotsu and a special generalized Kenmotsu manifolds of the second kind. A classification of nearly trans-Sasakian manifolds is obtained, more precisely it is proved that a nearly trans-Sasakian manifold is either a trans-Sasakian manifold or has a closed contact form. It is proved that the nearly trans-Sasakian structure with a non closed contact form is homothetic to the Sasakian structure. The criterion of ownership of a nearly trans-Sasakian structure is obtained. It is proved that the class of nearly trans-Sasakian manifolds with a closed contact form and a closed Lee form coincides with the class of almost contact metric manifolds with a closed contact form, which are locally conformal to closely cosymplectic manifolds. Examples of such manifolds are given. The necessary and sufficient conditions for the complete integrability of the first fundamental distribution of a nearly trans-Sasakian manifold are obtained. It is proved that the almost Hermitian structure induced on the leaves of the first fundamental distribution of an NTS-manifold is a nearly Kählerian structure.

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