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A generalization of the notion of helix

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Abstract: In this paper we generalize the notion of helix in the three-dimensional Euclidean space, which we define as that curve α for which there is an F-constant vector field W along α that forms a constant angle with a fixed direction V (called an axis of the helix). We find the natural equation and the geometric integration of helices α where the F-constant vector field W is orthogonal to its axis.

Key words: Helix, osculating helix, normal helix, rectifying helix, Darboux vector, F-constant vector field

1. Introduction

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable unit speed curve and let $\mathfrak{X}(\alpha)$ denote the set of differentiable vector fields along the curve. Let $F(s) = \{F_1(s), F_2(s), F_3(s)\}$ be a moving orthonormal frame along α, so we can see F as a differentiable map $F : I \to SO(3)$. It is easy to show (see, e.g., [3, pp. 43–45]) that there exists a unique vector field $D_F(s)$ along α satisfying the equations

$$F_1' = D_F \times F_1, \quad F_2' = D_F \times F_2, \quad F_3' = D_F \times F_3, \tag{1.1}$$

where $()'$ is the usual derivative in $\mathbb{R}^3$ and $\times$ stands for the cross product. The vector field D_F is called the Darboux vector associated to the frame F.

From a physical point of view, along the curve α, we have two coordinate systems associated to two references: (a) One is associated to the frame F, that may be imagined as being fixed on the curve. This system rotates and is thus accelerating, it is a noninertial frame. (b) The other is the canonical rectangular coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$ associated to the usual canonical basis $\{e_1, e_2, e_3\}$ that is fixed to the space and is an inertial frame. Obviously, given a vector field W along α, the variation in s of its coordinate functions in the two coordinate systems is not the same. In fact, they are related by the following equation:

$$W' = \frac{d_r}{ds}(W) + D_F \times W,$$

where $\frac{d_r}{ds}(W)$ denotes the rate of change of W as observed in the rotating coordinate system (i.e. in the frame F). The above equation is usually called the Transport Theorem in analytical dynamics (see [6, p. 11]). If we think of W as a curve in the 3-space, then the above formula tells us that its (absolute) velocity in the inertial

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frame $\{e_1, e_2, e_3\}$ is equal to the velocity relative to the moving (rotating) frame $\{F_1, F_2, F_3\}$ plus the velocity of the rotating coordinate system itself.

Motivated by the equations (1.1) and the Transport Theorem, we introduce the following definition.

Definition 1.1 *A vector field W along α is said to be constant with respect to the frame F (or F-*constant vector field*) if $\frac{d_r}{ds}(W) = 0$ (or, equivalently, $W' = D_F \times W$).*

The set of F-constant vector fields along α will be denoted by $\mathfrak{X}_F(\alpha)$. In [5], this kind of vector fields are said to be invariably attached to every point of the curve. If we think of the curve α as the trajectory of a rigid body, then an F-constant vector can be imagined as a body-fixed vector [6]. The following result is a straightforward computation.

Proposition 1.2 *The following properties of F-constant vector fields hold:*

1) *If W is an F-constant vector field, then W has constant length.*
2) *If W_1 and W_2 are F-constant vector fields, then $W_1 + W_2$ is an F-constant vector field.*
3) *Let W be a nowhere zero F-constant vector field and h a differentiable function, then hW is an F-constant vector field if and only if h is a constant function.*
4) *W is an F-constant vector field if and only if $W = a_1F_1 + a_2F_2 + a_3F_3$, for certain constants $a_i \in \mathbb{R}$. Hence, $\mathfrak{X}_F(\alpha)$ is a 3-dimensional real vector space.*

A special moving frame is the Frenet frame $F = \{T_\alpha, N_\alpha, B_\alpha\}$, satisfying the well-known Frenet-Serret equations

$$T'_\alpha(s) = \kappa_\alpha(s)\, N_\alpha(s), \quad N'_\alpha(s) = -\kappa_\alpha(s)\, T_\alpha(s) + \tau_\alpha(s)\, B_\alpha(s), \quad B'_\alpha(s) = -\tau_\alpha(s)\, N_\alpha(s), \tag{1.2}$$

where κ_α and τ_α stand for the curvature and torsion functions, respectively. The vector fields $T_\alpha, N_\alpha, B_\alpha$ are trivial examples of F-constant vector fields. Throughout this paper, we will assume that our curves are nonplanar, i.e. with nonzero torsion. For this frame, the Darboux vector is simply denoted by D_α and is given by $D_\alpha = \tau_\alpha T_\alpha + \kappa_\alpha B_\alpha$. This vector can be interpreted as the angular velocity of the Frenet frame as a whole. In fact, the rate of change of the frame $\{T_\alpha, N_\alpha, B_\alpha\}$ with s can be characterized as an instantaneous rotation about the vector D_α, with angular velocity equal to the total curvature specified by

$$\omega = |D_\alpha| = \sqrt{\kappa_\alpha^2 + \tau_\alpha^2}.$$

Let W be a nonzero differentiable vector field along the curve α. W is said to be a normal vector field if $W(s)$ belongs to the normal plane for every s; similarly, we have rectifying or osculating vector fields, depending on whether $W(s)$ belongs to the rectifying or osculating plane, respectively.

The term helix (or general helix, or cylindrical helix, or curve of constant slope) has traditionally been used to define curves whose tangent vector field T_α forms a constant angle with a fixed direction in the 3-space, [7, p. 33]. The concept of helix has been extended by considering vector fields other than the tangent vector field T_α, such as the principal normal vector field N_α (thus giving rise to slant helices, [2, 4]). In this paper, we propose a new extension of the concept by considering F-constant vector fields.

Definition 1.3 *A curve α is said to be a* helix *if there exists an F-constant vector field W along α that forms a constant angle with a fixed direction V, called an* axis *of the helix.*

Without loss of generality, we can assume that W is a unit vector field. In the particular case that W is a normal (osculating or rectifying, resp.) vector field then α is called a normal (osculating or rectifying, resp.) helix. Note that we recover the notion of cylindrical helix or slant helix when the F-constant vector field W is given by T_α or N_α, respectively. Note also that the term osculating helix has been previously used to refer to the circular helix passing through a point of a curve, having the same tangent, curvature vector and torsion, [7, p. 42].

In this paper, we solve the following problem:

How are the helices α characterized when the F-constant vector field W is orthogonal to its axis V?

Note that when the vector field W is T_α, N_α, or B_α, then the curve α is nothing but a plane curve or a cylindrical helix. Therefore, in the following sections, we will address the question when the vector field W is expressed as a linear combination of at least two vector fields of the Frenet frame.

2. Normal helices

2.1. Natural equation of normal helices

Let α be a nonplanar curve in $\mathbb{R}^3$ with Frenet apparatus $\{\kappa_\alpha, \tau_\alpha; T_\alpha, N_\alpha, B_\alpha\}$, and assume that α is a normal helix with axis V, V being a constant vector. Suppose that there is a nonzero constant angle $\theta \in (-\pi/2, \pi/2)$ such that $W = \cos\theta\, N_\alpha + \sin\theta\, B_\alpha$ is orthogonal to V. Hence, we can write

$$V = \lambda\, T_\alpha + \mu(\sin\theta\, N_\alpha - \cos\theta\, B_\alpha), \tag{2.1}$$

for certain differentiable functions λ and μ. By taking derivative in (2.1), we get

$$\lambda' - \mu \sin\theta\, \kappa_\alpha = 0, \tag{2.2}$$

$$\sin\theta\, \mu' + \lambda\, \kappa_\alpha + \mu\cos\theta\, \tau_\alpha = 0, \tag{2.3}$$

$$-\cos\theta\, \mu' + \mu\sin\theta\, \tau_\alpha = 0. \tag{2.4}$$

From equation (2.4), we have

$$\mu = e^{\tan\theta \int \tau_\alpha}, \tag{2.5}$$

that jointly with (2.3) leads to

$$\lambda = -\sec\theta\, \rho\, e^{\tan\theta \int \tau_\alpha}, \tag{2.6}$$

where $\rho = \tau_\alpha/\kappa_\alpha$ is called the Lancret curvature. Finally, putting equations (2.5) and (2.6) in (2.2) yields

$$-\sec\theta(\rho' + \tan\theta\, \tau_\alpha\, \rho) = \sin\theta\, \kappa_\alpha, \tag{2.7}$$

and then

$$-\rho' = \sin\theta\cos\theta\, \kappa_\alpha + \tan\theta\, \tau_\alpha\, \rho. \tag{2.8}$$

Therefore,

$$\frac{\kappa_\alpha}{\cos^2\theta\,\kappa_\alpha^2+\tau_\alpha^2}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = -\tan\theta. \tag{2.9}$$

Conversely, let α be a curve in $\mathbb{R}^3$ satisfying equation (2.9), for a nonzero constant $\theta \in (-\pi/2, \pi/2)$. Let V be the vector field given in (2.1), where μ and λ are defined by equations (2.5) and (2.6), respectively. Then equations (2.3) and (2.4) are satisfied. On the other hand, from equation (2.9), we easily get (2.7), and then we deduce (2.2) since

$$\mu\sin\theta\;\kappa_\alpha = -\sec\theta\big(\rho'\mu + \tan\theta\,\tau_\alpha\,\rho\,\mu\big) = \lambda'.$$

Hence, there is a constant θ such that the vector field $\cos\theta\,N_\alpha + \sin\theta\,B_\alpha$ is orthogonal to a constant direction V, so α is a normal helix.

Summing up, we have shown the following result (we also include the case $\theta = 0$).

Theorem 2.1 *Let α be a nonplanar arclength parametrized curve in $\mathbb{R}^3$, with curvature $\kappa_\alpha > 0$ and torsion τ_α. Then α is a normal helix (with W orthogonal to V) if and only if the following equation holds*

$$\frac{\kappa_\alpha}{\cos^2\theta\,\kappa_\alpha^2+\tau_\alpha^2}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = -\tan\theta,$$

for a certain constant $\theta \in (-\pi/2, \pi/2)$.

2.2. Geometric interpretation of normal helices

Let α be a normal helix with axis V, and assume that V is orthogonal to the F-constant vector field $W = \cos\theta\,N_\alpha + \sin\theta\,B_\alpha$, θ being a constant. Let us consider $C_{\alpha,V}$ the cylinder parametrized by $X(t,z) = \alpha(t) + z\,V$, then we have $N = X_t \times X_z = T_\alpha \times V = \cos\theta\,N_\alpha + \sin\theta\,B_\alpha$, up to a sign. This shows that the principal normal vector field N_α of the curve α makes a constant angle θ with the unit vector field N normal to the cylinder $C_{\alpha,V}$. It is not difficult to see that this condition characterizes the normal helices.

Let $M = C_{\beta,V}$ be a general cylinder parametrized by $X(t,z) = \beta(t) + zV$, where β is a unit planar curve and V is a unit vector orthogonal to that plane. If $\{T_\beta, N_\beta\}$ is the Frenet frame of β, assume that the unit normal vector to the cylinder is given by $N(t,z) = T_\beta(t) \times V = N_\beta(t)$. Let us assume that $\alpha(s) = X(t(s), z(s))$, $s \in I$, is an arclength parametrized curve in M such that N_α makes a constant angle θ with N. A straightforward computation yields

$$\begin{aligned}
T_\alpha(s) &= \cos\varphi(s)\,T_\beta(t(s)) + \sin\varphi(s)\,V, && \text{(a)}\\
N_\alpha(s) &= \sin\theta\,\big(-\sin\varphi(s)\,T_\beta(t(s)) + \cos\varphi(s)\,V\big) + \cos\theta\,N, && \text{(b)}\\
B_\alpha(s) &= -\cos\theta\,\big(-\sin\varphi(s)\,T_\beta(t(s)) + \cos\varphi(s)\,V\big) + \sin\theta\,N, && \text{(c)}
\end{aligned} \tag{2.10}$$

where $\varphi \in \mathcal{C}^\infty(I)$ is a differentiable function with $t'(s) = \cos\varphi(s)$ and $z'(s) = \sin\varphi(s)$. It is easy to see that

$$V = \sin\varphi(s)\,T_\alpha(s) + \cos\varphi(s)\big(\sin\theta\,N_\alpha(s) - \cos\theta\,B_\alpha(s)\big), \tag{2.11}$$

and then we can define the F-constant vector field $W = \cos\theta\,N_\alpha + \sin\theta\,B_\alpha$ satisfying $\langle W, V\rangle = 0$, showing that α is a normal helix. Therefore, we have proven the following result.

Theorem 2.2 *A curve α in $\mathbb{R}^3$ is a normal helix with axis V if and only if α lies on a cylinder C and its principal normal vector field makes a constant angle with the normal vector field to the cylinder.*

Note that when $\theta = 0$, then α is a geodesic of the cylinder, and so it is a cylindrical helix. Hence, Theorem 2.2 is an extension of the well-known theorem of Lancret that characterizes the cylindrical helices as the geodesics of the cylinders.

We finish this section with the following result about curves $\alpha(s)$ in a cylinder $C_{\beta,V}$. By taking derivative in equations (a) and (c) of (2.10), we get

$$\begin{aligned}\kappa_\alpha(s)N_\alpha(s) &= \varphi'(s)(-\sin\varphi(s)\,T_\beta(t(s)) + \cos\varphi(s)\,V) + \cos^2\varphi(s)\kappa_\beta(t(s))\,N_\beta(t(s)),\\ -\tau_\alpha(s)N_\alpha(s) &= \varphi'(s)\cos\theta(\cos\varphi(s)\,T_\beta(t(s)) + \sin\varphi(s)\,V)+\\ &\quad + \cos\varphi\kappa_\beta(-\sin\theta\,T_\beta(t(s)) + \cos\theta\sin\varphi\,N_\beta(t(s))).\end{aligned}$$

These two equations lead to the following result.

Proposition 2.3 *Let $\alpha(s) = X(t(s), z(s))$ be an arclength parametrized curve in a cylinder $C_{\beta,V}$. The principal normal vector field N_α makes a constant angle θ with the normal to the cylinder if and only if there is a differentiable function φ such that the following equations hold:*

$$t'(s) = \cos\varphi(s), \tag{2.12}$$

$$z'(s) = \sin\varphi(s), \tag{2.13}$$

$$\varphi'(s) = \tan\theta\,\cos^2(\varphi(s))\,\kappa_\beta(t(s)). \tag{2.14}$$

Moreover, the curvature and torsion of α are given by

$$\kappa_\alpha(s) = \frac{\cos^2\varphi(s)}{\cos\theta}\kappa_\beta(t(s)), \quad \tau_\alpha(s) = -\sin\varphi(s)\cos\varphi(s)\,\kappa_\beta(t(s)). \tag{2.15}$$

On the other hand, from (2.15), we get

$$\frac{\tau_\alpha}{\kappa_\alpha}(s) = -\cos\theta\tan\varphi(s),$$

and since $(\tan\varphi)'(s) = \tan\theta\,\kappa_\beta(t(s))$, we have that only in circular cylinders there exist normal helices that are also rectifying curves, see [1].

2.3. An example: normal helices in circular cylinders

Let $C_{\beta,V}$ be a cylinder over a circle β of radius one. Then from Proposition 2.3 we get

$$\begin{aligned}\varphi(s) &= \arctan(\tan(\theta)s),\\ t(s) &= \cot(\theta)\sinh^{-1}(\tan(\theta)s) + t_0,\\ z(s) &= \cot(\theta)\sqrt{1+\tan^2(\theta)s^2} + z_0,\end{aligned}$$

where θ, t_0, z_0 are constants. Hence, a family of normal helices in the cylinder is given by

$$\alpha(s) = \Big(\cos(\cot(\theta)\sinh^{-1}(\tan(\theta)s) + t_0), \\ \sin(\cot(\theta)\sinh^{-1}(\tan(\theta)s) + t_0), \\ \cot(\theta)\sqrt{1+\tan^2(\theta)s^2} + z_0 \Big).$$

Moreover, from (2.15), we obtain that the curvature and torsion of α are given by

$$\kappa_\alpha(s) = \frac{\cos\theta}{\cos^2\theta + \sin^2(\theta)\, s^2}, \quad \tau_\alpha(s) = \frac{-\sin\theta\cos(\theta)\, s}{\cos^2\theta + \sin^2(\theta)\, s^2}. \tag{2.16}$$

Note that these curves verify that $\tau_\alpha/\kappa_\alpha(s) = -\sin(\theta)s$, so they are rectifying curves.

We can reparametrize the curves α to obtain a simpler expression. Indeed, let us consider the change of parameter $\tan(\theta)s = \sinh(\tan(\theta)t)$, then

$$\alpha(t) = \big(\cos(t+t_0), \sin(t+t_0), \cot(\theta)\cosh(\tan(\theta)t) + z_0\big),$$

and the curvature and torsion can be computed as follows:

$$\kappa_\alpha(t) = \frac{\sec\theta}{\cosh^2(\tan(\theta)t)}, \quad \tau_\alpha(t) = \frac{-\sinh(\tan(\theta)t)}{\cosh^2(\tan(\theta)t)}. \tag{2.17}$$

A picture of a normal helix is shown in Figure.

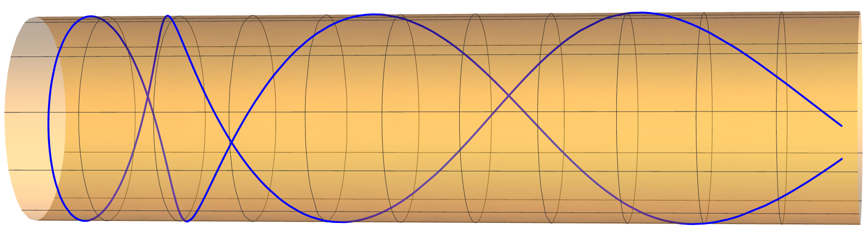

Figure. A normal helix with $\theta = \pi/36$ in a circular cylinder.

3. Osculating helices

3.1. Natural equation of osculating helices

Let α be a nonplanar osculating helix with axis V. Then there is an osculating vector field $W = \cos\theta\, T_\alpha + \sin\theta\, N_\alpha$, for a nonzero constant angle $\theta \in (-\pi/2, \pi/2)$, which is orthogonal to V. Hence, we can write

$$V = \mu(-\sin\theta\, T_\alpha + \cos\theta\, N_\alpha) + \lambda\, B_\alpha, \tag{3.1}$$

for certain differentiable functions λ and μ. By derivating here we obtain the following equations:

$$-\sin\theta\, \mu' - \mu\cos\theta\, \kappa_\alpha = 0, \tag{3.2}$$

$$\cos\theta\, \mu' - \lambda\, \tau_\alpha - \mu\sin\theta\, \kappa_\alpha = 0, \tag{3.3}$$

$$\lambda' + \mu\cos\theta\, \tau_\alpha = 0. \tag{3.4}$$

From equation (3.2), we get

$$\mu = e^{-\cot\theta \int \kappa_\alpha}, \tag{3.5}$$

that jointly with (3.3) yields

$$\lambda = \frac{-1}{\sin\theta\rho}\, e^{-\cot\theta \int \kappa_\alpha}, \tag{3.6}$$

where ρ is the Lancret curvature. On the other hand, by putting equations (3.5) and (3.6) in (3.4), we get

$$\frac{1}{\sin\theta}\left(\frac{\rho'}{\rho^2} + \cot\theta\,\kappa_\alpha\,\frac{\rho}{\rho^2}\right) = -\cos\theta\,\tau_\alpha,$$

and then

$$-\rho' = \sin\theta\cos\theta\,\tau_\alpha\rho^2 + \cot\theta\,\kappa_\alpha\,\rho,$$

which can be rewritten as

$$\frac{\kappa_\alpha^2}{\kappa_\alpha^2 + \sin^2\theta\,\tau_\alpha^2}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = -\cot\theta\,\tau_\alpha. \tag{3.7}$$

Note that this equation is equivalent to

$$\frac{\tau_\alpha}{\kappa_\alpha^2 + \sin^2\theta\,\tau_\alpha^2}\left(\frac{\kappa_\alpha}{\tau_\alpha}\right)' = \cot\theta. \tag{3.8}$$

Now, we will see that this equation characterizes the osculating helices. Indeed, let α be a curve in $\mathbb{R}^3$ satisfying (3.8) for a nonzero constant $\theta \in (-\pi/2, \pi/2)$. Let us define a vector field V as in (3.1), where λ and μ are given by (3.6) and (3.5), respectively. Then equations (3.2) and (3.3) are satisfied. Finally, it is straightforward to see that equation (3.8) leads to (3.4). Then there exists a constant θ such that $\cos\theta\, T_\alpha + \sin\theta\, N_\alpha$ is orthogonal to the constant direction V, that is, α is an osculating helix.

We have proven the following characterization of the osculating helices (we also include the cases $\theta = \pm\pi/2$).

Theorem 3.1 *Let α be a nonplanar arclength parametrized curve in $\mathbb{R}^3$, with curvature $\kappa_\alpha > 0$ and torsion τ_α. Then α is an osculating helix (with W orthogonal to V) if and only if the following equation holds*

$$\frac{\tau_\alpha}{\kappa_\alpha^2 + \sin^2\theta\,\tau_\alpha^2}\left(\frac{\kappa_\alpha}{\tau_\alpha}\right)' = \cot\theta,$$

for a nonzero constant angle $\theta \in [-\pi/2, \pi/2]$.

3.2. Normal helices and osculating helices

Let $\alpha(s)$ be a normal helix with Frenet apparatous $\{\kappa_\alpha, \tau_\alpha; T_\alpha, N_\alpha, B_\alpha\}$. Then the curve

$$\overline{\alpha}(s) = \int_{s_0}^{s} B_\alpha(t)\,dt \tag{3.9}$$

is an arclength parametrized curve, and without loss of generality, its Frenet frame is given by

$$T_{\overline{\alpha}} = B_\alpha, \quad N_{\overline{\alpha}} = -N_\alpha, \quad B_{\overline{\alpha}} = T_\alpha.$$

Hence, their curvature and torsion are given by $\kappa_{\overline{\alpha}} = \tau_\alpha$ and $\tau_{\overline{\alpha}} = \kappa_\alpha$. It is straightforward to see that $\overline{\alpha}$ satisfies (3.8), and so it is an osculating helix.

On the other hand, and following a similar reasoning, it can be proven that if $\overline{\alpha}$ is an osculating helix, then the curve

$$\alpha(s) = \int_{s_0}^{s} B_{\overline{\alpha}}(t)\, dt \tag{3.10}$$

is a normal helix. Therefore, and in a certain sense, normal helices and osculating helices can be considered duals of each other.

4. Rectifying helices

Let α be a nonplanar curve in $\mathbb{R}^3$ with Frenet apparatous $\{\kappa_\alpha, \tau_\alpha; T_\alpha, N_\alpha, B_\alpha\}$, and assume that α is a rectifying helix with axis V, V being a constant vector. Let us suppose there is a constant angle θ such that the rectifying vector field $W = \cos\theta\, B_\alpha + \sin\theta\, T_\alpha$ is orthogonal to V. Hence, we can write

$$V = \lambda\, N_\alpha + \mu(\sin\theta\, B_\alpha - \cos\theta\, T_\alpha),$$

for certain differentiable functions λ and μ. By taking derivative there, we get

$$-\lambda\, \kappa_\alpha - \mu' \cos\theta = 0,$$

$$\lambda' - \mu \cos\theta\, \kappa_\alpha - \mu \sin\theta\, \tau_\alpha = 0,$$

$$\lambda\, \tau_\alpha + \mu' \sin\theta = 0.$$

From these equations, we easily deduce that $\tau_\alpha/\kappa_\alpha$ is constant, and so α is a cylindrical helix. Since every cylindrical helix is also a rectifying helix, then we have proven the following result.

Theorem 4.1 *Let α be an arclength parametrized curve in $\mathbb{R}^3$ with curvature $\kappa_\alpha > 0$. Then α is a rectifying helix (with W orthogonal to V) if and only if it is a cylindrical helix.*

5. The general case

Let us assume, in this section, that the unit F-constant vector field W along α is given by $W = aT_\alpha + bN_\alpha + cB_\alpha$, where a, b, c are nonzero constants with $a^2 + b^2 + c^2 = 1$ (the other cases have already been analyzed in the preceding sections). Since we are assuming that $\langle W, V\rangle = 0$, then there exists two differentiable functions λ, μ such that

$$V = \lambda(-cT_\alpha + aB_\alpha) + \mu(-cN_\alpha + bB_\alpha), \tag{5.1}$$

and by derivating here we obtain the following equations:

$$0 = \lambda' - \mu\kappa_\alpha, \tag{5.2}$$

$$0 = c\mu' + \lambda(c\kappa_\alpha + a\tau_\alpha) + b\mu\tau_\alpha, \tag{5.3}$$

$$0 = a\lambda' + b\mu' - c\mu\tau_\alpha. \tag{5.4}$$

From (5.2) and (5.4), we get

$$\mu' = \frac{1}{b}(c\tau_\alpha - a\kappa_\alpha)\mu, \tag{5.5}$$

that jointly with (5.3) leads to

$$c\mu(c\tau_\alpha - a\kappa_\alpha) + \lambda b(c\kappa_\alpha + a\tau_\alpha) + b^2\mu\tau_\alpha = 0.$$

Then

$$\lambda = g\mu, \quad \text{with } g = \frac{ac\kappa_\alpha - (b^2+c^2)\tau_\alpha}{b(c\kappa_\alpha + a\tau_\alpha)}. \tag{5.6}$$

Now, by using (5.2) and (5.5), we obtain

$$b\kappa_\alpha = bg' + g(c\tau_\alpha - a\kappa_\alpha). \tag{5.7}$$

Since g' is given by

$$g' = \frac{-c\kappa_\alpha^2}{b(c\kappa_\alpha + a\tau_\alpha)^2}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)',$$

a straightforward computation from (5.7) yields

$$\frac{\kappa_\alpha^2}{c\kappa_\alpha\big((1-c^2)\kappa_\alpha^2 + (1-3a^2)\tau_\alpha^2\big) + a\tau_\alpha\big((1-a^2)\tau_\alpha^2 + (1-3c^2)\kappa_\alpha^2\big)}\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = -\frac{1}{b}. \tag{5.8}$$

Conversely, let α be a curve satisfying (5.8) for certain nonzero constants a, b, c. Let V be the nonzero vector field given in (5.1), where μ and λ are given by (5.5) and (5.6), respectively. As in the preceding sections, it is a straightforward (if somewhat laborious) calculation to check that equations (5.2)–(5.4) are satisfied. Therefore, we have found an F-constant vector field $W = aT_\alpha + bN_\alpha + cB_\alpha$ which is orthogonal to the fixed direction V.

To finish this section, and in order to consider also the cases in which any of the constants a, b, or c could be zero, let us note that equation (5.8) can be rewritten as

$$-b\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = c\kappa_\alpha\big((1-c^2)\kappa_\alpha^2 + (1-3a^2)\tau_\alpha^2\big) + a\tau_\alpha\big((1-a^2)\tau_\alpha^2 + (1-3c^2)\kappa_\alpha^2\big), \tag{5.9}$$

and then we have all the cases previously analyzed:

W	α is a	Eq. (5.9) reduces to
T_α	plane curve	$\rho = 0$
N_α	cylindrical helix	$\rho' = 0$
B_α	plane curve	$\rho = 0$
$bN_\alpha + cB_\alpha$	normal helix	Eq. (2.9)
$aT_\alpha + bN_\alpha$	osculating helix	Eq. (3.8)
$aT_\alpha + cB_\alpha$	rectifying helix	$\rho' = 0$
$aT_\alpha + bN_\alpha + cB_\alpha$	helix	Eq. (5.8)

In conclusion, we have shown the following result.

Theorem 5.1 *Let α be a nonplanar arclength parametrized curve in $\mathbb{R}^3$ with curvature $\kappa_\alpha > 0$ and torsion τ_α. Then α is a helix (associated to the unit F-constant vector field $W = aT_\alpha + bN_\alpha + cB_\alpha$ orthogonal to the* *axis) if and only if the following equation holds*

$$-b\kappa_\alpha^2\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)' = c\kappa_\alpha\big((1-c^2)\kappa_\alpha^2 + (1-3a^2)\tau_\alpha^2\big) + a\tau_\alpha\big((1-a^2)\tau_\alpha^2 + (1-3c^2)\kappa_\alpha^2\big),$$

with $a^2+b^2+c^2=1$.

We finish this section with the geometric interpretation of helices in the general case.

Theorem 5.2 *Let* α *be a nonplanar arclength parametrized curve in* $\mathbb{R}^3$ *with curvature* $\kappa_\alpha > 0$ *and torsion* τ_α. *Then* α *is a helix (associated to the unit* F*-constant vector field* $W = aT_\alpha + bN_\alpha + cB_\alpha$ *orthogonal to the axis* V*) if and only if* α *is contained in a cylinder* $C_{\beta,V}$ *and satisfies the following equation:*

$$a\tan\varphi + b\sin\theta - c\cos\theta = 0, \tag{5.10}$$

where a, b, c *are real constants and the angles* φ *and* θ *are given by*

$$\sin\varphi = \langle T_\alpha, V\rangle\,, \quad \sin\theta\,\langle B_\alpha, V\rangle + \cos\theta\,\langle N_\alpha, V\rangle = 0.$$

Proof We will follow the same reasoning as for proving Theorem 2.2. Let us first prove the reciprocal part. Let $X(t,z) = \beta(t) + zV$ be the canonical parametrization of $C_{\beta,V}$ and suppose $\alpha(s) = X(t(s), z(s))$. Although equations (2.10) and (2.11) have been obtained for the case where θ is constant, they also remain valid in the nonconstant case, and so the F-constant vector field $W = aT_\alpha + bN_\alpha + cB_\alpha$ satisfies $\langle V, W\rangle = 0$. Hence, α is a helix with axis V and F-constant vector field W orthogonal to it.

Now, let us consider α a helix with axis V, and suppose there is an F-constant vector field $W = aT_\alpha + bN_\alpha + cB_\alpha$ satisfying $\langle V, W\rangle = 0$. Then α is contained in the cylinder $C_{\alpha,V}$, which can be locally parametrized by $X(t,z) = \beta(t) + zV$, where $\beta(t)$ is a plane curve in $C_{\alpha,V}$ orthogonal to V. Then equations (2.10) and (2.11) are satisfied, and the condition $\langle V, W\rangle = 0$ implies equation (5.10), since $\cos\varphi \neq 0$ (otherwise, α would be a plane curve). □

Following a reasoning similar to the one used to prove Proposition 2.3, we can obtain from Theorem 5.2 the following result (which generalizes Proposition 2.3 to the case where θ is not constant). We leave the proof to the reader.

Proposition 5.3 *Let* $\alpha(s) = X(t(s), z(s))$ *be an arclength parametrized curve in a cylinder* $C_{\beta,V}$. *Then* α *satisfies equation (5.10), for a nonconstant function* $\theta \equiv \theta(s)$*, if and only if*

$$t'(s) = \frac{a}{\sqrt{a^2 + (-b\sin\theta + c\cos\theta)^2}},$$

$$z'(s) = \frac{-b\sin\theta + c\cos\theta}{\sqrt{a^2 + (-b\sin\theta + c\cos\theta)^2}},$$

$$\kappa_\beta(t(s)) = -\frac{(b\cos\theta + c\sin\theta)\theta'}{a\tan\theta}.$$

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