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The inequalities on dual numbers and their topological structures

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Abstract: Inequalities are frequently used in various fields of mathematics to prove theorems. The existence of inequalities contributes significantly to the foundations of such branches. In this paper, we study the properties of order relations in the system of dual numbers, which is inspired by order relations defined on real numbers. Besides, some special inequalities that are used in various fields of mathematics, such as Cauchy-Schwarz, Minkowski, and Chebyshev are studied in this framework. An example is also provided to validate our research findings.

Key words: Dual numbers, dual absolute value, dual inequalities, dual norm

1. Introduction

As an area of study, inequalities do not have a long history. As a mathematical concept, however, they were used by ancient mathematicians. For example, Euclid used the words "falls short" or "is in excess of" to mean that one area is larger than another ([5, 8]). A number of studies on inequality and the history of inequality have been written ([2, 9, 15]). Inequalities are basic tools in many areas of mathematics including algebra, geometry, trigonometry, and modern calculus [5]. More specifically, in order to constitute the expression of metric, which is one of the most basic structures of a space, the presence of inequalities is required. The ordering of numbers that are used by way of inequalities finds applications in various theoretical and practical fields [10].

Dual numbers were defined by W. K. Clifford (1845-1879) as a tool for his geometrical studies and their first applications were presented by Kotelnikov [12]. Eduard Study [17] used dual numbers and dual vectors in his research on line geometry and kinematics. He proved that there exists a one-to-one correspondence between the points of the dual unit sphere in $\mathbb{D}^3$ and the directed lines of Euclidean 3-space. These numbers play an important role in field theory as well [7]. The most interesting use of dual numbers in field theory can be shown in a series of articles by Wald et al. [19]. Dual numbers have modern application fields such as computer modelling of rigid body, mechanism design, kinematics, modelling of human body, and dynamics([6, 13]).

This paper investigates the order relation on dual numbers. It is natural to ask how important the order relation on dual numbers is. As it is well known, while the absolute value of a complex number and the norm of a complex vector are real numbers, the absolute value of a dual number and the norm of a dual vector are dual numbers. It is clear that the order relation on dual numbers is needed so as to carry out mathematical studies in dual space. This paper is fundamental of the order relation on dual numbers and shows that dual numbers

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2010 *AMS Mathematics Subject Classification:* 97H30, 47L50 have their own structural system.

In this paper, the properties of the order relation on dual numbers are examined in detail. Then Cauchy-Schwarz, Minkowski, Chebyshev, and arithmetic-geometric inequalities are investigated in the system of dual numbers. In the last section, using the order relation $<_{\mathrm{D}}$, we obtain the topologies on D^n denoted by $\overline{\tau}_{\overline{d}}$ and $\overline{\tau}$ such that the spaces $\left(\mathrm{D}^n, \overline{\tau}_{\overline{d}}\right)$ and $(\mathrm{D}^n, \overline{\tau})$ are Hausdorff spaces. Also, the general solution of the set

$$U = \left\{\overline{x} \in \mathrm{D}^2 \mid \|\overline{x}\|_{\mathrm{D}} <_{\mathrm{D}} \overline{r},\ \overline{r} \in \mathrm{D}^+\right\}$$

is obtained. A specific solution of this set was investigated in [1]. In this paper, an easier method is given in order to find this specific solution.

2. Preliminaries

Let the set of the pair (γ, γ^*) be $\mathrm{D} = \mathbb{R} \times \mathbb{R} = \{\overline{\gamma} = (\gamma, \gamma^*) \mid \gamma, \gamma^* \in \mathbb{R}\}$. For $\overline{\gamma} = (\gamma, \gamma^*)$, $\overline{\delta} = (\delta, \delta^*) \in \mathrm{D}$, the equality and the two inner operations on D are defined as follows:

$$\begin{aligned} \text{Equality} &: \quad \overline{\gamma} = \overline{\delta} \Leftrightarrow \gamma = \delta \text{ and } \gamma^* = \delta^*, \\ \text{Addition} &: \quad \overline{\gamma} \oplus \overline{\delta} = (\gamma + \delta, \gamma^* + \delta^*), \\ \text{Multiplication} &: \quad \overline{\gamma} \odot \overline{\delta} = (\gamma\delta, \gamma\delta^* + \delta\gamma^*). \end{aligned}$$

If the equality and the two operators on D with a set of real numbers $\mathbb{R}$ are defined as above, the set D is called the dual numbers system and the element $\overline{\gamma} = (\gamma, \gamma^*)$ is called a dual number. For $\overline{\gamma} = (\gamma, \gamma^*)$, the real number γ is called the real part of $\overline{\gamma}$, and the real number γ^* is called the dual part of $\overline{\gamma}$. The dual number $(1, 0) = \overline{1}$ is called the unit element of the multiplication operation in D. The dual number $(0, 1) = \varepsilon$ is to be called dual unit that satisfies the conditions that

$$\varepsilon \neq 0,\ \varepsilon^2 = 0,\ \varepsilon \odot 1 = 1 \odot \varepsilon = \varepsilon.$$

Let us consider the element $\overline{\gamma}$ of the form $\overline{\gamma} = (\gamma, 0)$. Then, the mapping $\xi : \mathrm{D} \to \mathbb{R}$, $\xi(\gamma, 0) = \gamma$ is an isomorphism. In this case, we can write

$$\begin{aligned} \overline{\gamma} &= (\gamma, \gamma^*) \\ &= (\gamma, 0) \oplus (0, \gamma^*) \\ &= (\gamma, 0) \oplus (0, 1) \odot (\gamma^*, 0) \\ &= \gamma \oplus \varepsilon \odot \gamma^*. \end{aligned}$$

For convenience, throughout this paper, we will use $+$ and $\cdot$ instead of $\oplus$ and $\odot$, respectively. Thus, the set of all dual numbers is given by

$$\mathrm{D} = \left\{\overline{\gamma} = \gamma + \varepsilon\gamma^* \mid \gamma, \gamma^* \in \mathbb{R},\ \varepsilon^2 = 0\right\}.$$

The set D forms a commutative ring with unity according to the operations

$$(\gamma + \varepsilon\gamma^*) + (\delta + \varepsilon\delta^*) = (\gamma + \delta) + \varepsilon(\gamma^* + \delta^*),$$

and

$$(\gamma + \varepsilon\gamma^*) \cdot (\delta + \varepsilon\delta^*) = \gamma\delta + \varepsilon(\gamma\delta^* + \delta\gamma^*).$$

For the dual numbers $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ and $\overline{\delta} = \delta + \varepsilon\delta^*$, if $\delta \neq 0$, then the division $\dfrac{\overline{\gamma}}{\overline{\delta}}$ is defined as follows [11]:

$$\frac{\overline{\gamma}}{\overline{\delta}} = \frac{\gamma}{\delta} + \varepsilon\left(\frac{\gamma^*\delta - \gamma\delta^*}{\delta^2}\right).$$

The set of dual vectors on D^n is represented by

$$\mathrm{D}^n = \left\{\overrightarrow{\overline{\gamma}} = (\overline{\gamma}_1, ..., \overline{\gamma}_n) \mid \overline{\gamma}_i \in \mathrm{D},\ i = 1, ..., n\right\}.$$

These vectors can be given in the form $\overrightarrow{\overline{\gamma}} = \left(\overrightarrow{\gamma}, \overrightarrow{\gamma^*}\right) = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$, where $\overrightarrow{\gamma}$ and $\overrightarrow{\gamma^*}$ are the vectors of $\mathbb{R}^n$.

Let $\overrightarrow{\overline{\gamma}} = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$ and $\overrightarrow{\overline{\delta}} = \overrightarrow{\delta} + \varepsilon\overrightarrow{\delta^*}$ be dual vectors of D^n, and let $\overline{\lambda} = \lambda + \varepsilon\lambda^*$ be a dual number. Then, the set D^n is a module over the ring D which is called dual space D^n according to the operations

$$\overrightarrow{\overline{\gamma}} + \overrightarrow{\overline{\delta}} = \left(\overrightarrow{\gamma} + \overrightarrow{\delta}\right) + \varepsilon\left(\overrightarrow{\gamma^*} + \overrightarrow{\delta^*}\right),$$

and

$$\overline{\lambda} \cdot \overrightarrow{\overline{\gamma}} = \lambda\overrightarrow{\gamma} + \varepsilon\left(\lambda\overrightarrow{\gamma^*} + \lambda^*\overrightarrow{\gamma}\right).$$

For any $\overrightarrow{\overline{\gamma}} = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$ and $\overrightarrow{\overline{\delta}} = \overrightarrow{\delta} + \varepsilon\overrightarrow{\delta^*} \in \mathrm{D}^n$, the dual inner product of $\overrightarrow{\overline{\gamma}}$ and $\overrightarrow{\overline{\delta}}$ is defined by

$$\begin{aligned}\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}} &= \overline{\gamma}_1\overline{\delta}_1 + ... + \overline{\gamma}_n\overline{\delta}_n \\ &= \left\langle \overrightarrow{\gamma}, \overrightarrow{\delta} \right\rangle + \varepsilon\left(\left\langle \overrightarrow{\gamma}, \overrightarrow{\delta^*} \right\rangle + \left\langle \overrightarrow{\gamma^*}, \overrightarrow{\delta} \right\rangle\right),\end{aligned}$$

where $\langle , \rangle$ is Euclid inner product on $\mathbb{R}^n$. The dual norm $\left\|\overrightarrow{\overline{\gamma}}\right\|_{\mathrm{D}}$ of $\overrightarrow{\overline{\gamma}} = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$ is given by

$$\left\|\overrightarrow{\overline{\gamma}}\right\|_{\mathrm{D}} = \sqrt{\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}}} = \begin{cases} \overline{0} & , \overrightarrow{\gamma} = \overrightarrow{0} \\ \|\overrightarrow{\gamma}\| + \varepsilon\dfrac{\left\langle \overrightarrow{\gamma}, \overrightarrow{\gamma^*} \right\rangle}{\|\overrightarrow{\gamma}\|} & , \overrightarrow{\gamma} \neq \overrightarrow{0}. \end{cases}$$

Let $\overline{x} = x + \varepsilon x^*$ be a dual variable. A dual variable function $\overline{\xi} : \mathrm{D} \to \mathrm{D}$ is defined as follows:

$$\overline{\xi}(\overline{x}) = \xi(x, x^*) + \varepsilon\xi^0(x, x^*),$$

where ξ and ξ^0 are real functions with two real variables x and x^*. Dimentberg [3] investigated the properties of dual functions. He showed that the analytic (differentiable) conditions of the dual functions are

$$\frac{\partial\xi}{\partial x^*} = 0 \text{ and } \frac{\partial\xi^0}{\partial x^*} = \frac{\partial\xi}{\partial x}.$$

In this case, the general notation of dual analytic functions is as follows:

$$\overline{\xi}(\overline{x}) = \xi(x) + \varepsilon\left(x^*\xi'(x) + \widetilde{\xi}(x)\right),$$

where $\widetilde{\xi}$ is an arbitrary function of the real part of a dual variable. The derivative of the dual analytic function $\overline{\xi}$ with respect to $\overline{x}$ is

$$\frac{d\overline{\xi}}{d\overline{x}} = \xi'(x) + \varepsilon\left(x^*\xi''(x) + \widetilde{\xi}'(x)\right).$$

This definition allows to write some well-known dual functions as follows ([3, 14]):

$$\begin{aligned}
\sin(x+\varepsilon x^*) &= \sin x + \varepsilon x^* \cos x. \\
\cos(x+\varepsilon x^*) &= \cos x - \varepsilon x^* \sin x. \\
\sqrt[n]{x+\varepsilon x^*} &= \sqrt[n]{x} + \varepsilon \frac{x^*}{n\sqrt[n]{x^{n-1}}},\ x \neq 0.
\end{aligned}$$

Theorem 2.1 *([3]) Let $\overline{x} = x + \varepsilon x^*$ be a dual variable. For $n \in \mathbb{N}$,*

$$\overline{x}^n = x^n + \varepsilon x^* n x^{n-1}.$$

Definition 2.2 *([16]) A relation C on the set A is called an order relation if it has the following properties:*

1) *For every γ and δ in A for which $\gamma \neq \delta$, either $\gamma C \delta$ or $\delta C \gamma$.*
2) *For no γ in A does the relation $\gamma C \gamma$ hold.*
3) *If $\gamma C \delta$ and $\delta C \omega$, then $\gamma C \omega$.*

Definition 2.3 (*Dictionary order relation*) *Given two words, one compares their first letters and orders the words according to the order in which their first letters appear in the alphabet. If the first letters are the same, one compares their second letters and orders accordingly [16].*

Definition 2.4 *Assume that A and B are two sets with order relations $<_A$ and $<_B$, respectively. For (γ_1, δ_1) and $(\gamma_2, \delta_2) \in A \times B = \{(\gamma, \delta) \mid \gamma \in A,\ \delta \in B\}$, the relation $(\gamma_1, \delta_1) < (\gamma_2, \delta_2)$ is defined as follows:*

1) *One compares the first components of these expressions and they must be $\gamma_1 <_A \gamma_2$,*
2) *If their first components are the same, then one compares their second components and they must be $\delta_1 <_B \delta_2$ [16].*

3. Dual absolute value and dual inequalities

Definition 3.1 *([18]) Let $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ be a dual number. The absolute value of dual number $\overline{\gamma}$ is*

$$|\overline{\gamma}|_{\mathrm{D}} = \sqrt{\overline{\gamma}^2} = \begin{cases} \overline{0} & ,\gamma = 0 \\ |\gamma| + \varepsilon\gamma^* \dfrac{\gamma}{|\gamma|} & ,\gamma \neq 0. \end{cases}$$

Theorem 3.2 *For $\overline{\gamma} = \gamma + \varepsilon\gamma^*, \overline{\delta} = \delta + \varepsilon\delta^* \in \mathrm{D}$ and $n \in \mathbb{N}$, the following properties are satisfied.*

1) $|-\overline{\gamma}|_{\mathrm{D}} = |\overline{\gamma}|_{\mathrm{D}}$.
2) $\left|\overline{\gamma} \cdot \overline{\delta}\right|_{\mathrm{D}} = |\overline{\gamma}|_{\mathrm{D}} \left|\overline{\delta}\right|_{\mathrm{D}}$.
3) $\left|\dfrac{\overline{\gamma}}{\overline{\delta}}\right|_{\mathrm{D}} = \dfrac{|\overline{\gamma}|_{\mathrm{D}}}{\left|\overline{\delta}\right|_{\mathrm{D}}}$, *for $\delta \neq 0$.*

4) $\left|\overline{\gamma}^n\right|_{\mathrm{D}} = \left|\overline{\gamma}\right|_{\mathrm{D}}^n$.

5) $\left|\overline{\gamma}^{-n}\right|_{\mathrm{D}} = \left|\overline{\gamma}\right|_{\mathrm{D}}^{-n}$, *for* $\gamma \neq 0$.

6) *For* $\gamma \neq 0$ *and* $\delta \neq 0$, *if* $\left|\overline{\gamma}\right|_{\mathrm{D}} = \left|\overline{\delta}\right|_{\mathrm{D}}$, *then either* $\overline{\gamma} = \overline{\delta}$ *or* $\overline{\gamma} = -\overline{\delta}$.

Proof Using the definition 3.1, these equalities can be shown. □

Definition 3.3 *Let* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ *and* $\overline{\delta} = \delta + \varepsilon\delta^*$ *be dual numbers. The relation* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ $\left(\text{resp. } \overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}\right)$ *between these dual numbers is as follows:*

1) *Firstly, one compares the real parts of these dual numbers and they must be* $\gamma < \delta$ $(\text{resp. } \gamma < \delta)$.

2) *If the real parts of these dual numbers are the same, one compares their dual parts and they must be* $\gamma^* < \delta^*$ $(\text{resp. } \gamma^* \leq \delta^*)$.

Thus, considering the above definition, the following corollary can be given.

Corollary 3.4 *Let* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ *and* $\overline{\delta} = \delta + \varepsilon\delta^*$ *be dual numbers. The following statements hold.*

1) $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ *if and only if* $\gamma < \delta$ *or* $(\gamma = \delta$ *and* $\gamma^* < \delta^*)$.

2) $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$ *if and only if* $\gamma < \delta$ *or* $(\gamma = \delta$ *and* $\gamma^* \leq \delta^*)$.

Theorem 3.5 *Let* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$, $\overline{\delta} = \delta + \varepsilon\delta^*$ *and* $\overline{\omega} = \omega + \varepsilon\omega^*$ *be dual numbers. The relations* $<_{\mathrm{D}}$ *and* $\leq_{\mathrm{D}}$ *provide the following expressions.*

1) *If* $\overline{\gamma} \neq \overline{\delta}$, *then either* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ *or* $\overline{\delta} <_{\mathrm{D}} \overline{\gamma}$.

2) *If* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \neq \overline{\delta}$.

3) *If* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ *and* $\overline{\delta} <_{\mathrm{D}} \overline{\omega}$, *then* $\overline{\gamma} <_{\mathrm{D}} \overline{\omega}$.

4) $\overline{\gamma} \leq_{\mathrm{D}} \overline{\gamma}$.

5) *If* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$ *and* $\overline{\delta} \leq_{\mathrm{D}} \overline{\gamma}$, *then* $\overline{\gamma} = \overline{\delta}$.

6) *If* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$ *and* $\overline{\delta} \leq_{\mathrm{D}} \overline{\omega}$, *then* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\omega}$.

Proof Using the corollary 3.4, it can be shown that each dual inequality exists. □

Definition 3.6 *Let* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ *be a dual number. The sets*

$$\begin{aligned}
\mathrm{D}^{+} &= \left\{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma > 0,\ \gamma^* \in \mathbb{R}\right\}, \\
\mathrm{D}^{-} &= \left\{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma < 0,\ \gamma^* \in \mathbb{R}\right\}, \\
\mathrm{D}^{0+} &= \left\{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma = 0,\ \gamma^* > 0\right\}, \\
\mathrm{D}^{0-} &= \left\{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma = 0,\ \gamma^* < 0\right\}
\end{aligned}$$

are called dual positive, dual negative, pure dual positive, and pure dual negative numbers, respectively.

Theorem 3.7 *Assume that* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ $\left(\text{resp. } \overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}\right)$. *For* $\overline{\omega} \in \mathrm{D}$, *we have* $\overline{\gamma} \pm \overline{\omega} <_{\mathrm{D}} \overline{\delta} \pm \overline{\omega}$ $\left(\text{resp. } \overline{\gamma} \pm \overline{\omega} \leq_{\mathrm{D}} \overline{\delta} \pm \overline{\omega}\right)$.

Theorem 3.8 *Let* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}$ *and* $\overline{\omega} \in \mathrm{D}^{+}$. *Then, the following expressions hold.*

1) *If* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \cdot \overline{\omega} <_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ *and* $\dfrac{\overline{\gamma}}{\overline{\omega}} <_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$.

2) *If* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \cdot \overline{\omega} \leq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ *and* $\dfrac{\overline{\gamma}}{\overline{\omega}} \leq_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$.

Proof 1) Suppose that $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$ and $\overline{\omega} = \omega + \varepsilon\omega^* \in \mathrm{D}^{+}$. If $\gamma < \delta$, then we have $\gamma\omega < \delta\omega$ and $\dfrac{\gamma}{\omega} < \dfrac{\delta}{\omega}$ such that $\overline{\gamma} \cdot \overline{\omega} <_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ and $\dfrac{\overline{\gamma}}{\overline{\omega}} <_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$. Now, assume that $\gamma = \delta$. Under the condition stated in $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$, we have $\gamma^* < \delta^*$. From the hypothesis, the real parts of $\overline{\gamma} \cdot \overline{\omega}$ and $\overline{\delta} \cdot \overline{\omega}$ $\left(\text{resp. } \dfrac{\overline{\gamma}}{\overline{\omega}} \text{ and } \dfrac{\overline{\delta}}{\overline{\omega}}\right)$ are equal to each other and there is the relationship $\gamma\omega^* + \gamma^*\omega < \delta\omega^* + \delta^*\omega$ $\left(\text{resp. } \dfrac{\gamma^*}{\omega} - \dfrac{\gamma\omega^*}{\omega^2} < \dfrac{\delta^*}{\omega} - \dfrac{\delta\omega^*}{\omega^2}\right)$ between the dual parts of these numbers, too. Considering the order relation $<_{\mathrm{D}}$, we get $\overline{\gamma} \cdot \overline{\omega} <_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ and $\dfrac{\overline{\gamma}}{\overline{\omega}} <_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$.

2) The proof for this case is easily made as in case 1. □

Theorem 3.9 *Let* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}$ *and* $\overline{\omega} \in \mathrm{D}^{-}$. *Then, the following statements are obtained.*

1) *If* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \cdot \overline{\omega} >_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ *and* $\dfrac{\overline{\gamma}}{\overline{\omega}} >_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$.

2) *If* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \cdot \overline{\omega} \geq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ *and* $\dfrac{\overline{\gamma}}{\overline{\omega}} \geq_{\mathrm{D}} \dfrac{\overline{\delta}}{\overline{\omega}}$.

Theorem 3.10 *Let* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}$, $\overline{\omega} \in \mathrm{D}^{0+}$ *and* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$.

1) *Assume that* $\gamma < \delta$. *In this case, we have* $\overline{\gamma} \cdot \overline{\omega} <_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$.

2) *Assume that* $\gamma = \delta$ *and* $\gamma^* < \delta^*$. *In this case, we have* $\overline{\gamma} \cdot \overline{\omega} \leq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$.

Proof 1) Let us take $\gamma < \delta$ and $\overline{\omega} \in \mathrm{D}^{0+}$. Considering the expressions $\overline{\gamma} \cdot \overline{\omega}$ and $\overline{\delta} \cdot \overline{\omega}$ together with the hypothesis, we can write $\gamma\omega = \delta\omega = 0$ and $\gamma\omega^* + \gamma^*\omega < \delta\omega^* + \delta^*\omega$ such that $\overline{\gamma} \cdot \overline{\omega} <_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$.
2) Suppose that $\gamma = \delta$. Since $\overline{\omega} \in \mathrm{D}^{0+}$, this allows us to write $\gamma\omega = \delta\omega = 0$ and $\gamma\omega^* + \gamma^*\omega \leq \delta\omega^* + \delta^*\omega$. From the partial order relation on dual numbers, we have $\overline{\gamma} \cdot \overline{\omega} \leq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$. □

Theorem 3.11 *Let* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}$, $\overline{\omega} \in \mathrm{D}^{0-}$ *and* $\overline{\gamma} <_{\mathrm{D}} \overline{\delta}$.

1) *Assume that* $\gamma < \delta$. *In this case, we have* $\overline{\gamma} \cdot \overline{\omega} >_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$.

2) *Assume that* $\gamma = \delta$ *and* $\gamma^* < \delta^*$. *In this case, we have* $\overline{\gamma} \cdot \overline{\omega} \geq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$.

Theorem 3.12 *Let* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}$ *and* $\overline{\omega} \in \mathrm{D}^{0+}$ $\left(\textit{resp. } \overline{\omega} \in \mathrm{D}^{0-}\right)$. *If* $\overline{\gamma} \leq_{\mathrm{D}} \overline{\delta}$, *then* $\overline{\gamma} \cdot \overline{\omega} \leq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}$ $\left(\textit{resp. } \overline{\gamma} \cdot \overline{\omega} \geq_{\mathrm{D}} \overline{\delta} \cdot \overline{\omega}\right)$.

Theorem 3.13 *For all* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}^{+}$, *we assume* $\overline{0} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{\delta}$. *For* $1 \leq n \in \mathbb{N}$, *the following statements hold.*

1) $\overline{0} <_{\mathrm{D}} \overline{\gamma}^{2n} <_{\mathrm{D}} \overline{\delta}^{2n}$.

2) $\overline{0} <_{\mathrm{D}} \overline{\gamma}^{2n+1} <_{\mathrm{D}} \overline{\delta}^{2n+1}$.

3) $\frac{1}{\overline{\gamma}} >_{\mathrm{D}} \frac{1}{\overline{\delta}} >_{\mathrm{D}} \overline{0}$.

Proof Let $\overline{\gamma}, \overline{\delta} \in \mathrm{D}^{+}$ and $\overline{0} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{\delta}$. For each dual inequality, there exist two situations due to the order relation on dual numbers.

1) If $0 < \gamma < \delta$, we say that $0 < \gamma^{2n} < \delta^{2n}$. This immediately implies that $\overline{0} <_{\mathrm{D}} \overline{\gamma}^{2n} <_{\mathrm{D}} \overline{\delta}^{2n}$. Assume that $0 < \gamma = \delta$. Since $\gamma^* < \delta^*$ and $2n\gamma^{2n-1} = 2n\delta^{2n-1} > 0$, it is obvious that $\gamma^* 2n\gamma^{2n-1} < \delta^* 2n\delta^{2n-1}$. Considering the order relation on dual numbers, we have $\overline{0} <_{\mathrm{D}} \overline{\gamma}^{2n} <_{\mathrm{D}} \overline{\delta}^{2n}$.

2) The proof for this case is as in the case 1.

3) Assume that $0 < \gamma < \delta$. Since $\frac{1}{\gamma} > \frac{1}{\delta} > 0$, this gives $\frac{1}{\overline{\gamma}} >_{\mathrm{D}} \frac{1}{\overline{\delta}} >_{\mathrm{D}} \overline{0}$. Now, suppose that $0 < \gamma = \delta$. Since $\gamma^* < \delta^*$ and $\frac{1}{\gamma} = \frac{1}{\delta} > 0$, we can write $-\frac{\gamma^*}{\gamma^2} > -\frac{\delta^*}{\delta^2}$ such that $\frac{1}{\overline{\gamma}} >_{\mathrm{D}} \frac{1}{\overline{\delta}} >_{\mathrm{D}} \overline{0}$. □

Theorem 3.14 *For all $\overline{\gamma}, \overline{\delta} \in \mathrm{D}^{-}$, we assume $\overline{\gamma} <_{\mathrm{D}} \overline{\delta} <_{\mathrm{D}} \overline{0}$. For $1 \leq n \in \mathbb{N}$, the following statements hold.*

1) $\overline{\gamma}^{2n} >_{\mathrm{D}} \overline{\delta}^{2n} >_{\mathrm{D}} \overline{0}$.

2) $\overline{\gamma}^{2n+1} <_{\mathrm{D}} \overline{\delta}^{2n+1} <_{\mathrm{D}} \overline{0}$.

3) $\overline{0} >_{\mathrm{D}} \frac{1}{\overline{\gamma}} >_{\mathrm{D}} \frac{1}{\overline{\delta}}$.

Theorem 3.15 *Let $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ be a dual number. In this case, $\overline{0} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{1}$ if and only if $\overline{\gamma} >_{\mathrm{D}} \overline{\gamma}^2$.*

Proof Let $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ and $\overline{0} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{1}$. The solution set of this dual inequality is as below:

$$\overline{S} = S_1 \cup S_2 \cup S_3,$$

where

$$\begin{aligned}
S_1 &= \{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid 0 < \gamma < 1,\ \gamma^* \in \mathbb{R}\}, \\
S_2 &= \{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma = 0,\ \gamma^* > 0\}, \\
S_3 &= \{\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D} \mid \gamma = 1,\ \gamma^* < 0\}.
\end{aligned}$$

i) For $\overline{\gamma} \in S_1$, since $\gamma > \gamma^2$, we have $\overline{\gamma} >_{\mathrm{D}} \overline{\gamma}^2$.

ii) For $\overline{\gamma} \in S_2$, since $\overline{\gamma}^2 = 0 + \varepsilon 0$ and $\gamma^* > 0$, we have $\overline{\gamma} >_{\mathrm{D}} \overline{\gamma}^2$.

iii) For $\overline{\gamma} \in S_3$, since $\overline{\gamma}^2 = 1 + 2\varepsilon\gamma^*$ and $\gamma^* < 0$, we have $\overline{\gamma} >_{\mathrm{D}} \overline{\gamma}^2$.

Conversely, suppose that $\overline{\gamma} >_{\mathrm{D}} \overline{\gamma}^2$. From the order relation on dual numbers, it is clear that $\overline{0} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{1}$. Thus, the proof is completed. □

Corollary 3.16 *Let $\overline{\gamma}$ and $\overline{\delta}$ be dual numbers. Using the order relation on dual numbers, there exist the following situations:*

1) *If $\overline{\gamma}, \overline{\delta} \in \mathrm{D}^{+}$, then $\overline{\gamma} \cdot \overline{\delta} >_{\mathrm{D}} \overline{0}$.*

2) *If $\overline{\gamma} \in \mathrm{D}^{+}$ and $\overline{\delta} \in \mathrm{D}^{0+}$, then $\overline{\gamma} \cdot \overline{\delta} >_{\mathrm{D}} \overline{0}$.*

3) *If $\overline{\gamma} \in \mathrm{D}^{0+}$ and $\overline{\delta} \in \mathrm{D}^{0+}$, then $\overline{\gamma} \cdot \overline{\delta} = \overline{0}$.*

4) *If* $\overline{\gamma}, \overline{\delta} \in \mathrm{D}^-$ *, then* $\overline{\gamma} \cdot \overline{\delta} >_{\mathrm{D}} \overline{0}$.

5) *If* $\overline{\gamma} \in \mathrm{D}^-$ *and* $\overline{\delta} \in \mathrm{D}^{0-}$ *, then* $\overline{\gamma} \cdot \overline{\delta} >_{\mathrm{D}} \overline{0}$.

6) *If* $\overline{\gamma} \in \mathrm{D}^{0-}$ *and* $\overline{\delta} \in \mathrm{D}^{0-}$ *, then* $\overline{\gamma} \cdot \overline{\delta} = \overline{0}$.

7) *If* $\overline{\gamma} \in \mathrm{D}^+$ *and* $\overline{\delta} \in \mathrm{D}^-$ *, then* $\overline{\gamma} \cdot \overline{\delta} <_{\mathrm{D}} \overline{0}$.

8) *If* $\overline{\gamma} \in \mathrm{D}^{0+}$ *and* $\overline{\delta} \in \mathrm{D}^-$ *, then* $\overline{\gamma} \cdot \overline{\delta} <_{\mathrm{D}} \overline{0}$.

9) *If* $\overline{\gamma} \in \mathrm{D}^{0-}$ *and* $\overline{\delta} \in \mathrm{D}^+$ *, then* $\overline{\gamma} \cdot \overline{\delta} <_{\mathrm{D}} \overline{0}$.

10) *If* $\overline{\gamma} \in \mathrm{D}^{0-}$ *and* $\overline{\delta} \in \mathrm{D}^{0+}$ *, then* $\overline{\gamma} \cdot \overline{\delta} = \overline{0}$.

Corollary 3.17 1) *For any dual number* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ *and* $\overline{\varsigma} = \varsigma + \varepsilon\varsigma^* \in \mathrm{D}^+$ *, we have*

$$|\overline{\gamma}|_{\mathrm{D}} <_{\mathrm{D}} \overline{\varsigma} \text{ if and only if } -\overline{\varsigma} <_{\mathrm{D}} \overline{\gamma} <_{\mathrm{D}} \overline{\varsigma}.$$

2) *For any dual number* $\overline{\gamma} = \gamma + \varepsilon\gamma^*$ *with the exception of* $\gamma = 0$ *and* $\overline{\varsigma} = \varsigma + \varepsilon\varsigma^* \in \mathrm{D}^+$*, we have*

$$|\overline{\gamma}|_{\mathrm{D}} >_{\mathrm{D}} \overline{\varsigma} \text{ if and only if } \overline{\gamma} >_{\mathrm{D}} \overline{\varsigma} \text{ or } \overline{\gamma} <_{\mathrm{D}} -\overline{\varsigma}.$$

Theorem 3.18 (*Dual Cauchy-Schwarz inequality*) *Let* $\{\overline{\gamma}_1, ..., \overline{\gamma}_n\}$ *and* $\{\overline{\delta}_1, ..., \overline{\delta}_n\}$ *be any two sets of dual numbers, where* $\overline{\gamma}_i = \gamma_i + \varepsilon\gamma_i^*$, $\overline{\delta}_i = \delta_i + \varepsilon\delta_i^*$ *and* $1 \leq i \leq n$*. Then, we have*

$$\left(\overline{\gamma}_1\overline{\delta}_1 + ... + \overline{\gamma}_n\overline{\delta}_n\right)^2 \leq_{\mathrm{D}} \left(\overline{\gamma}_1^2 + ... + \overline{\gamma}_n^2\right)\left(\overline{\delta}_1^2 + ... + \overline{\delta}_n^2\right),$$

or, equivalently,

$$\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 \leq_{\mathrm{D}} \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}},$$

where $\overrightarrow{\overline{\gamma}} = (\overline{\gamma}_1, ..., \overline{\gamma}_n) = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$ *and* $\overrightarrow{\overline{\delta}} = \left(\overline{\delta}_1, ..., \overline{\delta}_n\right) = \overrightarrow{\delta} + \varepsilon\overrightarrow{\delta^*}$.

Proof For $\overline{\lambda} \in \mathrm{D}$, assume that $\overrightarrow{\overline{\gamma}} = \overline{\lambda}\overrightarrow{\overline{\delta}}$. In this case, using the definition of dual inner product, we easily obtain $\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 = \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}$ such that $\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 \leq_{\mathrm{D}} \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}$. If at least one of the vectors $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ is zero vector, then we get $\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 = \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}} = \overline{0}$ such that $\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 \leq_{\mathrm{D}} \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}$. Now, assume that $\overrightarrow{\gamma} \neq \overrightarrow{0}$ and $\overrightarrow{\delta} \neq \overrightarrow{0}$. If $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ are linearly independent, since $\left\langle \overrightarrow{\gamma}, \overrightarrow{\delta} \right\rangle^2 < \langle \overrightarrow{\gamma}, \overrightarrow{\gamma} \rangle \left\langle \overrightarrow{\delta}, \overrightarrow{\delta} \right\rangle$, from the order relation on dual numbers, it is clear that $\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 \leq_{\mathrm{D}} \left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \overrightarrow{\overline{\delta}}, \overrightarrow{\overline{\delta}} \right\rangle_{\mathrm{D}}$. If $\overrightarrow{\gamma}$ and $\overrightarrow{\delta}$ are linearly dependent, i.e. $\overrightarrow{\gamma} = \mu\overrightarrow{\delta}$, for $\mu \in \mathbb{R} - \{0\}$, then we get

$$\left\langle \overrightarrow{\gamma}, \overrightarrow{\delta} \right\rangle^2 = \langle \overrightarrow{\gamma}, \overrightarrow{\gamma} \rangle \left\langle \overrightarrow{\delta}, \overrightarrow{\delta} \right\rangle$$

and

$$\left\langle \vec{\gamma}, \vec{\delta} \right\rangle \left(\left\langle \vec{\gamma}, \vec{\delta^*} \right\rangle + \left\langle \vec{\gamma^*}, \vec{\delta} \right\rangle \right) \leq \left\langle \vec{\gamma}, \vec{\gamma} \right\rangle \left\langle \vec{\delta}, \vec{\delta^*} \right\rangle + \left\langle \vec{\delta}, \vec{\delta} \right\rangle \left\langle \vec{\gamma}, \vec{\gamma^*} \right\rangle .$$

Considering the partial order relation on dual numbers, we attain

$$\left\langle \vec{\overline{\gamma}}, \vec{\overline{\delta}} \right\rangle_{\mathrm{D}}^2 \leq_{\mathrm{D}} \left\langle \vec{\overline{\gamma}}, \vec{\overline{\gamma}} \right\rangle_{\mathrm{D}} \left\langle \vec{\overline{\delta}}, \vec{\overline{\delta}} \right\rangle_{\mathrm{D}} .$$

Thus, the proof is completed. □

Corollary 3.19 *Let $\vec{\overline{\gamma}}$ and $\vec{\overline{\delta}}$ be any two dual vectors of D^n. Then, the following dual inequality exists:*

$$\left| \left\langle \vec{\overline{\gamma}}, \vec{\overline{\delta}} \right\rangle_{\mathrm{D}} \right|_{\mathrm{D}} \leq_{\mathrm{D}} \left\| \vec{\overline{\gamma}} \right\|_{\mathrm{D}} \left\| \vec{\overline{\delta}} \right\|_{\mathrm{D}} .$$

Theorem 3.20 (*Dual Minkowski's inequality*) *Let $\{\overline{\gamma}_1, ..., \overline{\gamma}_n\}$ and $\{\overline{\delta}_1, ..., \overline{\delta}_n\}$ be any two sets of dual numbers, where $\overline{\gamma}_i = \gamma_i + \varepsilon\gamma_i^*$, $\overline{\delta}_i = \delta_i + \varepsilon\delta_i^*$ and $1 \leq i \leq n$. For $1 \leq p \in \mathbb{N}$, there are two situations:*

1) *If $\forall\gamma_i = 0$, $\exists\delta_i \neq 0$ and $\left\langle \left(\delta_{t_1} |\delta_{t_1}|^{p-2}, ..., \delta_{t_n} |\delta_{t_n}|^{p-2} \right), \left(\gamma_{t_1}^*, ..., \gamma_{t_n}^* \right) \right\rangle \geq 0$, then we get*

$$\left(\sum_{k=1}^{n} \left| \overline{\gamma}_k + \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} \geq_{\mathrm{D}} \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \left| \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} . \tag{3.1}$$

Furthermore, this dual inequality is also true if
$\forall\delta_i = 0$, $\exists\gamma_i \neq 0$ and $\left\langle \left(\gamma_{t_1} |\gamma_{t_1}|^{p-2}, \gamma_{t_2} |\gamma_{t_2}|^{p-2}, ..., \gamma_{t_n} |\gamma_{t_n}|^{p-2} \right), \left(\delta_{t_1}^*, ..., \delta_{t_n}^* \right) \right\rangle \geq 0$, where $1 \leq t_1, ..., t_n \leq n$, $\forall\delta_{t_i} \neq 0$ and $\forall\gamma_{t_i} \neq 0$.

2) *In all other cases, the following dual inequality holds.*

$$\left(\sum_{k=1}^{n} \left| \overline{\gamma}_k + \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} \leq_{\mathrm{D}} \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \left| \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} . \tag{3.2}$$

Proof Firstly, let $\forall\gamma_i = 0$ and $\exists\delta_i \neq 0$. For $1 \leq t_1, ..., t_n \leq n$, we find

$$\begin{aligned} \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k + \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} &= \left(|\delta_{t_1}|^p + ... + |\delta_{t_n}|^p \right)^{\frac{1}{p}} \\ &\quad + \varepsilon \left(\frac{\delta_{t_1} |\delta_{t_1}|^{p-2} \left(\gamma_{t_1}^* + \delta_{t_1}^* \right) + ... + \delta_{t_n} |\delta_{t_n}|^{p-2} \left(\gamma_{t_n}^* + \delta_{t_n}^* \right)}{\sqrt[p]{\left(|\delta_{t_1}|^p + ... + |\delta_{t_n}|^p \right)^{p-1}}} \right) \end{aligned}$$

and

$$\begin{aligned} \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \left| \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} &= \left(|\delta_{t_1}|^p + ... + |\delta_{t_n}|^p \right)^{\frac{1}{p}} \\ &\quad + \varepsilon \left(\frac{\delta_{t_1} \delta_{t_1}^* |\delta_{t_1}|^{p-2} + ... + \delta_{t_n} \delta_{t_n}^* |\delta_{t_n}|^{p-2}}{\sqrt[p]{\left(|\delta_{t_1}|^p + ... + |\delta_{t_n}|^p \right)^{p-1}}} \right), \end{aligned}$$

where $\forall \delta_{t_i} \neq 0$. If the condition

$$\left\langle \left(\delta_{t_1} \left| \delta_{t_1} \right|^{p-2}, ..., \delta_{t_n} \left| \delta_{t_n} \right|^{p-2} \right), \left(\gamma_{t_1}^*, ..., \gamma_{t_n}^* \right) \right\rangle \geq 0$$

is valid, we have the dual inequality (3.1). If there exists the inequality

$$\left\langle \left(\delta_{t_1} \left| \delta_{t_1} \right|^{p-2}, ..., \delta_{t_n} \left| \delta_{t_n} \right|^{p-2} \right), \left(\gamma_{t_1}^*, ..., \gamma_{t_n}^* \right) \right\rangle \leq 0,$$

then we get the dual inequality (3.2). Now, suppose that $\forall \delta_i = 0$, $\exists \gamma_i \neq 0$ and

$$\left\langle \left(\gamma_{t_1} \left| \gamma_{t_1} \right|^{p-2}, ..., \gamma_{t_n} \left| \gamma_{t_n} \right|^{p-2} \right), \left(\delta_{t_1}^*, ..., \delta_{t_n}^* \right) \right\rangle \geq 0.$$

In this case, it is clear that

$$\begin{aligned} \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k + \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} &= \left(\left| \gamma_{t_1} \right|^p + ... + \left| \gamma_{t_n} \right|^p \right)^{\frac{1}{p}} \\ &\quad + \varepsilon \left(\frac{\gamma_{t_1} \left| \gamma_{t_1} \right|^{p-2} \left(\gamma_{t_1}^* + \delta_{t_1}^* \right) + ... + \gamma_{t_n} \left| \gamma_{t_n} \right|^{p-2} \left(\gamma_{t_n}^* + \delta_{t_n}^* \right)}{\sqrt[p]{\left(\left| \gamma_{t_1} \right|^p + ... + \left| \gamma_{t_n} \right|^p \right)^{p-1}}} \right) \\ &\geq_{\mathrm{D}} \left(\left| \gamma_{t_1} \right|^p + ... + \left| \gamma_{t_n} \right|^p \right)^{\frac{1}{p}} \\ &\quad + \varepsilon \left(\frac{\gamma_{t_1} \gamma_{t_1}^* \left| \gamma_{t_1} \right|^{p-2} + ... + \gamma_{t_n} \gamma_{t_n}^* \left| \gamma_{t_n} \right|^{p-2}}{\sqrt[p]{\left(\left| \gamma_{t_1} \right|^p + ... + \left| \gamma_{t_n} \right|^p \right)^{p-1}}} \right) \\ &= \left(\sum_{k=1}^{n} \left| \overline{\gamma}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} \left| \overline{\delta}_k \right|_{\mathrm{D}}^p \right)^{\frac{1}{p}}, \end{aligned}$$

where $\forall \gamma_{t_i} \neq 0$. If $\left\langle \left(\gamma_{t_1} \left| \gamma_{t_1} \right|^{p-2}, ..., \gamma_{t_n} \left| \gamma_{t_n} \right|^{p-2} \right), \left(\delta_{t_1}^*, ..., \delta_{t_n}^* \right) \right\rangle \leq 0$, then we obtain the dual inequality (3.2). Considering all the other states of γ_i and δ_i, we attain the dual inequality (3.2). $\square$

Theorem 3.21 (*Dual Chebyshev's inequality*) *Let* $\{\overline{\gamma}_1, ..., \overline{\gamma}_n\}$ *and* $\{\overline{\delta}_1, ..., \overline{\delta}_n\}$ *be any two sets of dual numbers, where* $\overline{\gamma}_i = \gamma_i + \varepsilon \gamma_i^*$, $\overline{\delta}_i = \delta_i + \varepsilon \delta_i^*$ *and* $1 \leq i \leq n$, *such that either* $\overline{\gamma}_1 \geq_{\mathrm{D}} \overline{\gamma}_2 \geq_{\mathrm{D}} ... \geq_{\mathrm{D}} \overline{\gamma}_n$ *and* $\overline{\delta}_1 \geq_{\mathrm{D}} \overline{\delta}_2 \geq_{\mathrm{D}} ... \geq_{\mathrm{D}} \overline{\delta}_n$, *or* $\overline{\gamma}_1 \leq_{\mathrm{D}} \overline{\gamma}_2 \leq_{\mathrm{D}} ... \leq_{\mathrm{D}} \overline{\gamma}_n$ *and* $\overline{\delta}_1 \leq_{\mathrm{D}} \overline{\delta}_2 \leq_{\mathrm{D}} ... \leq_{\mathrm{D}} \overline{\delta}_n$; *then*

$$\left(\frac{\overline{\gamma}_1 + \overline{\gamma}_2 + ... + \overline{\gamma}_n}{n} \right) \left(\frac{\overline{\delta}_1 + \overline{\delta}_2 + ... + \overline{\delta}_n}{n} \right) \leq_{\mathrm{D}} \frac{1}{n} \sum_{k=1}^{n} \overline{\gamma}_k \overline{\delta}_k.$$

Theorem 3.22 (*Dual arithmetic-geometric inequality*) *Let* $\{\overline{\gamma}_1, ..., \overline{\gamma}_n\}$ *be any set of dual positive numbers, where* $\overline{\gamma}_i = \gamma_i + \varepsilon \gamma_i^*$ *and* $\gamma_i > 0$, *with* (*dual*) *arithmetic mean*

$$\overline{A}_n = \left(\frac{\overline{\gamma}_1 + \overline{\gamma}_n + ... + \overline{\gamma}_n}{n} \right)$$

and (*dual*) *geometric mean*

$$\overline{G}_n = (\overline{\gamma}_1 \overline{\gamma}_2 ... \overline{\gamma}_n)^{\frac{1}{n}};$$

then $\overline{A}_n \geq_{\mathrm{D}} \overline{G}_n$.

Proof Expanding the expressions $\overline{A}_n$ and $\overline{G}_n$, we have

$$\overline{A}_n = \left(\frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n} \right) + \varepsilon \left(\frac{\gamma_1^* + \gamma_2^* + ... + \gamma_n^*}{n} \right)$$

and

$$\overline{G}_n = \sqrt[n]{\gamma_1 \gamma_2 ... \gamma_n} + \varepsilon \left(\frac{\gamma_1^* \gamma_2 \gamma_3 ... \gamma_n + \gamma_2^* \gamma_1 \gamma_3 ... \gamma_n + ... + \gamma_n^* \gamma_1 \gamma_2 ... \gamma_{n-1}}{n \sqrt[n]{(\gamma_1 \gamma_2 ... \gamma_n)^{n-1}}} \right).$$

If at least two of the numbers γ_i are different from each other, then we get $\left(\frac{\gamma_1+\gamma_2+...+\gamma_n}{n}\right) > \sqrt[n]{\gamma_1 \gamma_2 ... \gamma_n}$ such that $\overline{A}_n \geq_{\mathrm{D}} \overline{G}_n$. If all of the numbers γ_i are equal, then we obtain

$$\left(\frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n} \right) = \sqrt[n]{\gamma_1 \gamma_2 ... \gamma_n}$$

and

$$\left(\frac{\gamma_1^* + \gamma_2^* + ... + \gamma_n^*}{n} \right) \geq \left(\frac{\gamma_1^* \gamma_2 \gamma_3 ... \gamma_n + \gamma_2^* \gamma_1 \gamma_3 ... \gamma_n + ... + \gamma_n^* \gamma_1 \gamma_2 ... \gamma_{n-1}}{n \sqrt[n]{(\gamma_1 \gamma_2 ... \gamma_n)^{n-1}}} \right).$$

Considering the partial order relation on dual numbers, we attain $\overline{A}_n \geq_{\mathrm{D}} \overline{G}_n$. □

Now, using the order relation $<_{\mathrm{D}}$, let us show that the topologies can be constructed on D^n. For the dual point $\overline{p} = p + \varepsilon p^* \in \mathrm{D}^n$ and $\overline{r} = r + \varepsilon r^* \in \mathrm{D}^+$, let us take

$$\begin{aligned} \overline{B}(\overline{p}, \overline{r}) &= \{ \overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - p\| < r, \ x^* \in \mathbb{R}^n \} \\ &\quad \cup \left\{ \overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - p\| = r \text{ and } \frac{\langle x - p, x^* - p^* \rangle}{\|x - p\|} < r^* \right\} \\ &= \overline{U}_1 \cup \overline{U}_2 \\ &= \overline{U}_1 \cup \overline{C}_1 \cup ... \cup \overline{C}_l \quad (l \in I = \{1, 2, ...\}) \end{aligned}$$

and

$$\overline{U}_3 = \left\{ \overline{x} \in \mathrm{D}^n \mid x = \text{ constant}, m < x_1^* < n, \ x_{j+1}^* = a_j \in \mathbb{R}, \ m, n \in [-\infty, \infty] \right\},$$

where for all $c_l \in \{ x \in \mathbb{R}^n \mid \|x - p\| = r \}$,

$$\overline{C}_l = \left\{ \overline{x} = c_l + \varepsilon x^* \in \mathrm{D}^n \mid \|c_l - p\| = r \text{ and } \frac{\langle c_l - p, x^* - p^* \rangle}{\|c_l - p\|} < r^* \right\}.$$

Thus, a collection of all the sets $\overline{U}_1, \overline{U}_3, \overline{C}_1, ..., \overline{C}_l$ $(l \in I)$ forms a base on D^n. Let $\overline{\tau}_{\overline{d}}$ denote the topology obtained from this base [4]. Also, assume that d_1 and d_2 are two metrics on $\mathbb{R}^n$ and $\overline{x} = x + \varepsilon x^*, \overline{y} = y + \varepsilon y^* \in \mathrm{D}^n$. In this case, the function

$$\overline{d} : \mathrm{D}^n \times \mathrm{D}^n \to \mathrm{D},$$

$$\overline{d}\left(\overline{x},\overline{y}\right)=d_1\left(x,y\right)+\varepsilon d_2\left(x^*,y^*\right)$$

can be defined. Considering the partial order relation $\leq_{\mathrm{D}}$, the function $\overline{d}$ provides the following properties:

i) For all $\overline{x},\overline{y}\in \mathrm{D}^n,\ \overline{d}\left(\overline{x},\overline{y}\right)\geq_{\mathrm{D}}\overline{0}$.

ii) For all $\overline{x},\overline{y}\in \mathrm{D}^n,\ \overline{d}\left(\overline{x},\overline{y}\right)=\overline{0}\Leftrightarrow \overline{x}=\overline{y}$.

iii) For all $\overline{x},\overline{y}\in \mathrm{D}^n,\ \overline{d}\left(\overline{x},\overline{y}\right)=\overline{d}\left(\overline{y},\overline{x}\right)$.

iv) For all $\overline{x},\overline{y},\overline{z}\in \mathrm{D}^n,\ \overline{d}\left(\overline{x},\overline{y}\right)\leq_{\mathrm{D}}\overline{d}\left(\overline{x},\overline{z}\right)+\overline{d}\left(\overline{z},\overline{y}\right)$.

Now, for the dual point $\overline{p}=p+\varepsilon p^*\in \mathrm{D}^n$ and dual number $\overline{r}=r+\varepsilon r^*$, where $r,r^*\in\mathbb{R}^+$, we have

$$\begin{aligned}\overline{\Phi} &= \left\{\overline{x}=x+\varepsilon x^*\in\ \mathrm{D}^n \mid \overline{d}\left(\overline{x},\overline{p}\right)<_{\mathrm{D}}\overline{r}\right\}\\ &= \left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid d_1\left(x,p\right)<r,\ x^*\in\mathbb{R}^n\right\}\\ &\quad\cup\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid d_1\left(x,p\right)=r \text{ and } d_2\left(x^*,p^*\right)<r^*\right\}.\end{aligned}$$

Considering the set $\overline{\Phi}$, we can give the following theorem:

Theorem 3.23 *Suppose that d_1 and d_2 are two metrics on $\mathbb{R}^n$, $\overline{p}=p+\varepsilon p^*\in \mathrm{D}^n$ and $\overline{r}=r+\varepsilon r^*\in \mathrm{D}$, where $r,r^*\in\mathbb{R}^+$. If we take*

$$\overline{\varphi}_1=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid d_1\left(x,p\right)<r,\ x^*\in\mathbb{R}^n\right\}$$

and

$$\overline{\varphi}_2=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid x=\textit{constant},\ d_2\left(x^*,p^*\right)<r^*\right\},$$

then a collection of all the sets $\overline{\varphi}_1$ and $\overline{\varphi}_2$ forms a base $\overline{\Psi}$ on D^n.

Corollary 3.24 *From Theorem 3.23, we have the following topology:*

$$\overline{\tau}=\left\{\underset{i\in I}{\cup}\overline{\Omega}_i \mid \overline{\Omega}_i\in\overline{\Psi}\right\},$$

and it is clear that $\overline{\Phi}\in\overline{\tau}$.

Theorem 3.25 $(\mathrm{D}^n,\overline{\tau})$ *topological space is Hausdorff space.*

Proof For all $\overline{p},\overline{q}\in \mathrm{D}^n$ satisfying the condition $\overline{p}\neq\overline{q}$, there exist two situations. If $p\neq q$, then we get

$$\overline{p}\in\overline{\varphi}_1=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid d_1\left(x,p\right)<r_1,\ x^*\in\mathbb{R}^n,\ r_1\in\mathbb{R}^+\right\}\in\overline{\tau}$$

and

$$\overline{q}\in\overline{\varphi}_2=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid d_1\left(x,q\right)<r_2,\ x^*\in\mathbb{R}^n,\ r_2\in\mathbb{R}^+\right\}\in\overline{\tau}$$

such that $\overline{\varphi}_1\cap\overline{\varphi}_2=\varnothing$. If $p=q$ and $p^*\neq q^*$, then we have

$$\overline{p}\in\overline{\varphi}_1=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid x=p,\ d_2\left(x^*,p^*\right)<r_1^*,\ r_1^*\in\mathbb{R}^+\right\}\in\overline{\tau}$$

and

$$\overline{q}\in\overline{\varphi}_2=\left\{\overline{x}=x+\varepsilon x^*\in \mathrm{D}^n \mid x=q,\ d_2\left(x^*,q^*\right)<r_2^*,\ r_2^*\in\mathbb{R}^+\right\}\in\overline{\tau}$$

such that $\overline{\varphi}_1 \cap \overline{\varphi}_2 = \varnothing$. Thus, the space $(\mathrm{D}^n, \overline{\tau})$ are Hausdorff space. □

For example, let us take $\overline{d}_1 : \mathrm{D}^n \times \mathrm{D}^n \to \mathrm{D}$, $\overline{d}_1(\overline{x}, \overline{y}) = \|\overline{x} - \overline{y}\|_1 = \|x - y\| + \varepsilon \|x^* - y^*\|$. In this case, a collection of all the sets

$$\overline{\varphi}_1 = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - p\| < r,\ x^* \in \mathbb{R}^n\}$$

and

$$\overline{\varphi}_2 = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid x = \text{constant},\ \|x^* - p^*\| < r^*\}$$

forms a base on D^n. If the topology consisting of this base is denoted by $\overline{\tau}_{\overline{d}_1}$, for $n = 1$, we have $\overline{\tau}_{\overline{d}} = \overline{\tau}_{\overline{d}_1}$ and for $n \geq 2$, we get $\overline{\tau}_{\overline{d}_1} \subseteq \overline{\tau}_{\overline{d}}$. Moreover, the collection of all the sets $\Lambda = \{\overline{x} = x + \varepsilon x^* \in \mathrm{D}^n \mid \|x - p\| < r,\ x^* \in \mathbb{R}^n\}$ found in both topologies also forms a base on D^n. Let $\overline{\tau}_\omega$ denote the topology obtained from this base. Thus, we can write the following relations:

$$\overline{\tau}_\omega \subseteq \overline{\tau}_{\overline{d}_1} \text{ and } \overline{\tau}_\omega \subseteq \overline{\tau}_{\overline{d}}.$$

On the other hand, from the dual distance function $\overline{d}_1$, we conclude that

$$|\overline{\gamma}|_1 = |\gamma| + \varepsilon |\gamma^*|,$$

where $\overline{\gamma} = \gamma + \varepsilon\gamma^* \in \mathrm{D}$. Using this equality instead of dual absolute value function, we can reinterpret the dual Minkowski inequality:

Theorem 3.26 *Let $\{\overline{\gamma}_1, ..., \overline{\gamma}_n\}$ and $\{\overline{\delta}_1, ..., \overline{\delta}_n\}$ be any two sets of dual numbers, where $\overline{\gamma}_i = \gamma_i + \varepsilon\gamma_i^*$, $\overline{\delta}_i = \delta_i + \varepsilon\delta_i^*$ and $1 \leq i \leq n$. Then the following dual inequalities are possible to write*

1) $\sum_{i=1}^{n} \left|\overline{\gamma}_i + \overline{\delta}_i\right|_1 \leq_{\mathrm{D}} \sum_{i=1}^{n} |\overline{\gamma}_i|_1 + \sum_{i=1}^{n} \left|\overline{\delta}_i\right|_1$.
2) *For $\exists\gamma_i \neq 0$, $\exists\delta_i \neq 0$ and $2 \leq p \in \mathbb{N}$, we have*

$$\left(\sum_{i=1}^{n} \left|\overline{\gamma}_i + \overline{\delta}_i\right|_1^p\right)^{\frac{1}{p}} \leq_{\mathrm{D}} \left(\sum_{i=1}^{n} |\overline{\gamma}_i|_1^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} \left|\overline{\delta}_i\right|_1^p\right)^{\frac{1}{p}}$$

$$\overline{I}_1 \leq_{\mathrm{D}} \overline{I}_2 + \overline{I}_3.$$

Besides, this dual inequality is satisfied for all $\gamma_i = \delta_i = 0$. On the other hand, for $\forall\gamma_i = 0$, $\exists\delta_i \neq 0$ and $p \geq 2$, if dual $(\overline{I}_1) \leq$ dual $(\overline{I}_3)$ (for $\forall\delta_i = 0$, $\exists\gamma_i \neq 0$ and $p \geq 2$, if dual $(\overline{I}_1) \leq$ dual $(\overline{I}_2)$), then this dual inequality also holds.

Example 3.27 *Let us take $\overrightarrow{\overline{\gamma}} = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*} \in \mathrm{D}^2$, where $\overrightarrow{\gamma} = (\gamma_1, \gamma_2)$ and $\overrightarrow{\gamma^*} = (\gamma_1^*, \gamma_2^*)$. We know that the dual norm $\left\|\overrightarrow{\overline{\gamma}}\right\|_{\mathrm{D}}$ of $\overrightarrow{\overline{\gamma}} = \overrightarrow{\gamma} + \varepsilon\overrightarrow{\gamma^*}$ is*

$$\left\|\overrightarrow{\overline{\gamma}}\right\|_{\mathrm{D}} = \sqrt{\left\langle \overrightarrow{\overline{\gamma}}, \overrightarrow{\overline{\gamma}} \right\rangle_{\mathrm{D}}} = \begin{cases} \overline{0} & , \overrightarrow{\gamma} = \overrightarrow{0} \\ \|\overrightarrow{\gamma}\| + \varepsilon \dfrac{\left\langle \overrightarrow{\gamma}, \overrightarrow{\gamma^*} \right\rangle}{\|\overrightarrow{\gamma}\|} & , \overrightarrow{\gamma} \neq \overrightarrow{0}. \end{cases}$$

It is seen that $\left\|\overrightarrow{\overline{\gamma}}\right\|_{\mathrm{D}} \in \mathrm{D}^{+} \cup \{\overline{0}\}$. *Now, assume that* $\overline{r} = r + \varepsilon r^{*} \in \mathrm{D}^{+}$ *and* $U = \left\{\overline{x} \in \mathrm{D}^2 \mid \|\overline{x}\|_{\mathrm{D}} <_{\mathrm{D}} \overline{r},\ \overline{r} \in \mathrm{D}^{+}\right\}$.
Let us find a general and specific solution of the set U.
Expanding the set U, *we have*

$$\begin{aligned} U &= \left\{\overline{x} \in \mathrm{D}^2 \mid \|x\| < r,\ x \in \mathbb{R}^2\right\} \\ &\quad \cup \left\{\overline{x} \in \mathrm{D}^2 \mid \|x\| = r \ \textit{and} \ \frac{\langle x, x^{*}\rangle}{\|x\|} < r^{*}\right\} \\ &= U_1' \cup U_2'. \end{aligned}$$

1) *Assume that* $\|x\| < r$. *The solution set for this case is*

$$U_1 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1^2 + x_2^2 < r^2,\ x^{*} \in \mathbb{R}^2\right\}.$$

2) *Assume that* $\|x\| = r$. *In this case, we can write* $x_1 = r\cos t$ *and* $x_2 = r\sin t$, *where* $0 \le t < 2\pi$. *Thus, it is possible to express the following inequality:*

$$x_1^{*}\cos t + x_2^{*}\sin t < r^{*}.$$

Now, we will find the solution set of this case according to the angle t.
i) *Assume that* $t = 0$. *For this case, the solution set is*

$$U_2 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = r,\ x_2 = 0,\ x_1^{*} < r^{*},\ x_2^{*} \in \mathbb{R}\right\}.$$

ii) *Assume that* $0 < t < \frac{\pi}{2}$. *Since* $\cos t > 0$ *and* $\sin t > 0$, *we can write* $\cos t = \lambda_1^2$ *and* $\sin t = \lambda_2^2$, *where* $0 < \lambda_1, \lambda_2 < 1$. *The solution set of this case is as follows:*

$$U_3 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = r\lambda_1^2,\ x_2 = r\lambda_2^2,\ x_1^{*} < \frac{r^{*} - A_1\lambda_2^2}{\lambda_1^2},\ x_2^{*} = A_1 \in \mathbb{R}\right\}.$$

iii) *Assume that* $t = \frac{\pi}{2}$. *The solution set of this case is*

$$U_4 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = 0,\ x_2 = r,\ x_2^{*} < r^{*},\ x_1^{*} \in \mathbb{R}\right\}.$$

iv) *Assume that* $\frac{\pi}{2} < t < \pi$. *It is known that* $\cos t < 0$ *and* $\sin t > 0$. *Thus, it is possible to say that* $\cos t = -\lambda_1^2 < 0$ *and* $\sin t = \lambda_2^2 > 0$, *where* $0 < \lambda_1, \lambda_2 < 1$. *The solution set for this case is expressed as below:*

$$U_5 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = -r\lambda_1^2,\ x_2 = r\lambda_2^2,\ x_1^{*} > \frac{A_2\lambda_2^2 - r^{*}}{\lambda_1^2},\ x_2^{*} = A_2 \in \mathbb{R}\right\}.$$

v) *Assume that* $t = \pi$. *The solution set is*

$$U_6 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = -r,\ x_2 = 0,\ x_1^{*} > -r^{*},\ x_2^{*} \in \mathbb{R}\right\}.$$

vi) *Assume that* $\pi < t < \frac{3\pi}{2}$. *Since* $\cos t < 0$ *and* $\sin t < 0$, *we can write* $\cos t = -\lambda_1^2$ *and* $\sin t = -\lambda_2^2$, *where* $0 < \lambda_1, \lambda_2 < 1$. *The solution set of this case is*

$$U_7 = \left\{\overline{x} \in \mathrm{D}^2 \mid x_1 = -r\lambda_1^2,\ x_2 = -r\lambda_2^2,\ x_1^{*} > \frac{-A_3\lambda_2^2 - r^{*}}{\lambda_1^2},\ x_2^{*} = A_3 \in \mathbb{R}\right\}.$$

vii) Assume that $t = \frac{3\pi}{2}$ *. The solution set is*

$$U_8 = \left\{ \overline{x} \in \mathrm{D}^2 \mid x_1 = 0,\ x_2 = -r,\ x_2^* > -r^*,\ x_1^* \in \mathbb{R} \right\}.$$

viii) Assume that $\frac{3\pi}{2} < t < 2\pi$ *. Since* $\cos t = \lambda_1^2 > 0$ *and* $\sin t = -\lambda_2^2 < 0$ *, where* $0 < \lambda_1, \lambda_2 < 1$ *, the solution set for this case is*

$$U_9 = \left\{ \overline{x} \in \mathrm{D}^2 \mid x_1 = r\lambda_1^2,\ x_2 = -r\lambda_2^2,\ x_1^* < \frac{r^* + A_4\lambda_2^2}{\lambda_1^2},\ x_2^* = A_4 \in \mathbb{R} \right\}.$$

Thus, the solution set of U *is as follows:*

$$U = U_1 \cup ... \cup U_9.$$

Now, suppose that $\overline{x} = x + \varepsilon x^*$ *, where* $x = (x_1, x_2)$ *and* $x^* = (x_1^*, 0)$ *. Therefore, we will obtain a specific solution of the set* U *and make the geometric modellings of this solution.*

Situation.1. *Assume that* $\|x\| < r$ *. For this case, the solution set is* U_1 *.*

Situation.2. *Let* $\|x\| = r$ *. In this case, we can write the following inequality:*

$$x_1^* \cos t < r^*, \tag{3.3}$$

where $0 \leq t < 2\pi$ *. According to the situations of* r^* *, we will investigate the inequality (3.3).*

Case.1. *Let us consider* $r^* > 0$ *. For* $0 \neq \mu \in \mathbb{R}$ *,* $r^* = \mu^2$ *can be written. For* $0 \leq t < \frac{\pi}{2}$ *and* $\frac{3\pi}{2} < t < 2\pi$ *, it is clear that* $\cos t > 0$ *. Taking* $\cos t = \lambda^2$ *into account, where* $0 < \lambda \leq 1$ *, from (3.3), we have*

$$x_1^* < \frac{\mu^2}{\lambda^2}.$$

For $\frac{\pi}{2} < t \leq \pi$ *and* $\pi \leq t < \frac{3\pi}{2}$ *, it is possible to say that* $\cos t = -\lambda^2 < 0$ *, where* $0 < \lambda \leq 1$ *. Thus, we have*

$$x_1^* > -\frac{\mu^2}{\lambda^2}.$$

If $t = \frac{\pi}{2}$ *and* $t = \frac{3\pi}{2}$ *are taken into consideration, for all* $x_1^* \in \mathbb{R}$ *, the inequality (3.3) is satisfied (see Figure).*

Case.2. *Suppose that* $r^* < 0$ *. We can write* $r^* = -\mu^2$ *. For* $0 \leq t < \frac{\pi}{2}$ *and* $\frac{3\pi}{2} < t < 2\pi$ *, it is seen that* $\cos t = \lambda^2 > 0$ *, where* $0 < \lambda \leq 1$ *. From (3.3), the following inequality can be written:*

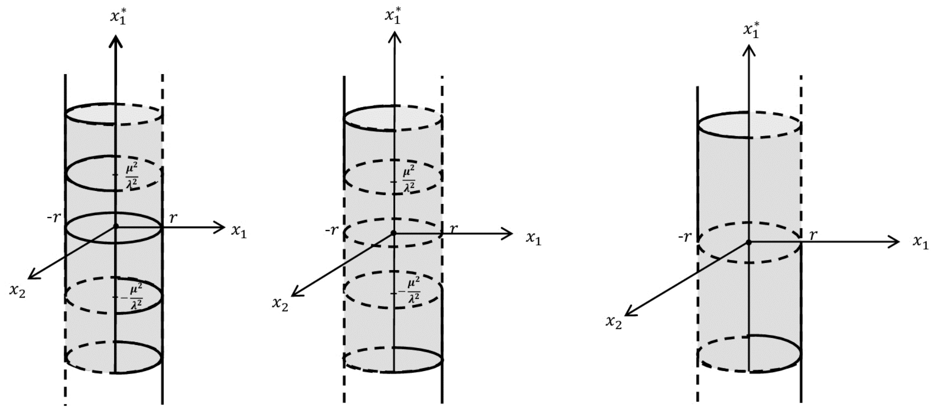
$$x_1^* < -\frac{\mu^2}{\lambda^2}.$$

For $\frac{\pi}{2} < t \leq \pi$ *and* $\pi \leq t < \frac{3\pi}{2}$ *, since* $\cos t = -\lambda^2 < 0$ *, where* $0 < \lambda \leq 1$ *, we get*

$$x_1^* > \frac{\mu^2}{\lambda^2}.$$

Considering $t = \frac{\pi}{2}$ *and* $t = \frac{3\pi}{2}$ *, the solution set is empty. The geometric modelling of this case is shown in Figure.*

Case.3. *Assume that* $r^* = 0$ *. For* $0 \leq t < \frac{\pi}{2}$ *and* $\frac{3\pi}{2} < t < 2\pi$ *, since* $\cos t = \lambda^2$ *, where* $0 < \lambda \leq 1$ *, from (3.3), we get* $x_1^* < 0$ *. For* $\frac{\pi}{2} < t \leq \pi$ *and* $\pi \leq t < \frac{3\pi}{2}$ *, since* $\cos t = -\lambda^2 < 0$ *, where* $0 < \lambda \leq 1$ *, from (3.3), we have* $x_1^* > 0$*. For* $t = \frac{\pi}{2}$ *and* $t = \frac{3\pi}{2}$ *, the solution set is empty (see Figure).*



For the situation 1 *and case* 1. *For the situation* 1 *and case* 2. *For the situation* 1 *and case* 3.

Figure.

4. Conclusion

In this study, firstly, some properties of the dual absolute value function have been examined. Then, the properties of the order relation on dual numbers have been investigated in detail. In the last section, we obtain the topologies on $\mathbb{D}^n$ denoted by $\overline{\tau}, \overline{\tau}_{\overline{d}}$, and $\overline{\tau}_{\omega}$, after then the general and specific solutions of the given example have been analyzed in detail so that the order relation on dual numbers can be understood better, and the geometric modellings of the specific solution have been made.

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