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Characterizations of the commutators involving idempotents in certain subrings of $M_2(\mathbb{Z})$

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Abstract: In this paper, we characterize the idempotency, cleanness, and unit-regularity of the commutator $[E_1, E_2] = E_1E_2 - E_2E_1$ involving idempotents E_1, E_2 in certain subrings of $M_2(\mathbb{Z})$.

Key words: Clean (Nil-clean) element, commutator, idempotent, unit element, Nilpotent, unit-regular element

1. Introduction

This study was inspired by an insightful way of Andrica-Călugăreanu [1], involving in solving Pell equations and Wu-Tang-Deng-Zhou [17], fairly a simple way, giving an answer the question whether a nil-clean element is clean. An element in a ring is called nil-clean (clean) if it is a sum of an idempotent and a nilpotent (unit), and the ring is nil-clean (clean) if its every element is nil-clean (clean) ([5] and [14]). In [2], Călugăreanu defined UU-rings as rings with all units unipotent. In [4, 8], a ring R called a UJ-ring if $U(R) = J(R) + 1$ (see also, [9, 10] and [11]). In [17], the authors handled this question by working on the subring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ of $M_2(\mathbb{Z})$ instead of $M_2(\mathbb{Z})$ because the subring contains much less clean elements than $M_2(\mathbb{Z})$, a huge advantage! We also remark that in [7], the author considered this question for unipotent elements, unit-regular elements, nil-clean elements, and clean elements based on Tevfik Pasha's adaptation on the same subrings on $M_2(\mathbb{Z})$. Shortly, he was a well-known mathematician, an army lecturer, a scientist, and a bureaucrat in the last period of the Ottoman Empire. His famous work Linear Algebra (1882) has been used as a basic source for different studies in Algebra [15].

Commutators and anticommutators play an important role in the operator theory and the ring theory. Recall the notions of the commutator $[x, y] := xy - yx$ and the anticommutator $< x, y >:= xy + yx$ for any two elements x, y in a ring R. In particular, commutators and anticommutators of idempotents in rings are very important problems as well as very popular in ring theory. In this direction, we focus on invertible commutators and anticommutators of idempotents in rings. Since idempotents and invertible elements are clean, here we present a direct way to construct a commutator of a pair of idempotents but not clean element in the ring

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$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. We also notice that a criterion is given for a matrix $\begin{pmatrix} 0 & a \\ -s^2b & 0 \end{pmatrix}$ to be a commutator in the ring. Since our topic is related to a product of idempotents, characterizations of products of idempotents (unit-regulars) are also developed. We obtain the following curious statement for a ring R: A product of any two unit-regulars of R is unit-regular if and only if a product of any two idempotents of R is unit-regular. Notice that characterizations of products of idempotents are developed for regular rings which are either unit-regular or right self-injective by many authors. Some of such characterizations generalize the results of Erdos for the case that R is Artinian. For example, in case R is directly finite (here R need not be self-injective, merely simple with the comparability axiom), an element $x \neq 1$ in R is a product of idempotents if and only if x is not a unit. The method for establishing the unit-regularity is by proving that the nonunits in R are products of idempotents. We have used the product of unit-regulars of R idea to prove R has the summand sum property (SSP) in the meantime.

Throughout this paper, R will be an associative ring with identity, $U(R)$ its group of units, $Id(R)$ its set of idempotents of R and $Ureg(R)$ its set of unit-regular elements of R. Here, $\mathbb{Z}$ is the ring of integers, $M_2(\mathbb{Z})$ is the 2×2 matrix ring over $\mathbb{Z}$ whose identity is denoted by I_2.

2. A product of a pair of idempotents (unit-regulars) that is unit-regular

Over the matrix ring $M_2(\mathbb{Z})$, we have the following basic facts:

(1) The units in $M_2(\mathbb{Z})$ are the 2×2 matrices of $det = \mp 1$.

(2) Nontrivial idempotent matrices in $M_2(\mathbb{Z})$ have $rank$ 1.

(3) Nilpotent matrices in $M_2(\mathbb{Z})$ have the characteristic polynomial t^2 and so they have trace and determinant equal to 0.

An element in a ring is unit-regular if it is a product of an idempotent and a unit, and a ring is unit-regular if its every element is unit-regular. In [12], Khurana and Lam showed that a single unit-regular element in a ring needs not to be clean. Also, a criterion was given for a matrix $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ in $M_2(K)$ over a commutative ring K to be clean. When it is applied to $K = \mathbb{Z}$, the authors of [12] gave many examples of unit-regular matrices in $M_2(\mathbb{Z})$ that are not clean.

The next two examples show that not every product of two unit-regulars (respectively, two nontrivial idempotents) is unit-regular in some subrings of $M_2(\mathbb{Z})$.

Example 2.1 *Let* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

(1) Consider the idempotents $E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} 4 & 1 \\ -12 & -3 \end{pmatrix}$ *in* R*. Then* $E_1E_2 = \begin{pmatrix} -8 & -2 \\ 0 & 0 \end{pmatrix}$ *is not unit-regular. In fact, if* $E_1E_2 = \begin{pmatrix} -8 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -8 & -2 \\ 4a & b \end{pmatrix}$*, then* $8a - 8b = 8(a - b)$ *cannot be* -1 *or* 1 *for any integers* a *and* b*.*

(2) Consider the unit-regulars $A = \begin{pmatrix} 11 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 11 & 1 \\ 32 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 13 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 13 & 5 \\ 8 & 3 \end{pmatrix}$ *in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. *Then* $AB = \begin{pmatrix} 143 & 55 \\ 0 & 0 \end{pmatrix}$ *is not unit-regular. In fact, if* $AB = \begin{pmatrix} 143 & 55 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 143 & 55 \\ 4a & b \end{pmatrix}$, *then* $143b - 220a = 11(13b - 20a)$ *cannot be* -1 *or* 1 *for any integers* a *and* b.

Example 2.2 *Let* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *with* $s \geq 3$.

(1) Consider the idempotents $E_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} 1 & 0 \\ s^2 & 0 \end{pmatrix}$ *in* R *. Then* $E_1E_2 = \begin{pmatrix} 1+s^2 & 0 \\ 0 & 0 \end{pmatrix}$ *is not unit-regular.*

(2) Consider the unit-regulars $A = \begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -25 & -4 \end{pmatrix}$ *and* $B = \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 25 & 6 \end{pmatrix}$ *in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. *Then* $AB = \begin{pmatrix} 24 & 6 \\ 0 & 0 \end{pmatrix}$ *is not unit-regular. In fact, if* $AB = \begin{pmatrix} 24 & 6 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 24 & 6 \\ s^2z & t \end{pmatrix}$, *then* $24t - 6s^2z = 6(4t - s^2z)$ *cannot be* -1 *or* 1 *for any integers* t *and* z.

Proposition 2.3 *The following conditions are equivalent for a ring* R*:*

1. *The product of any two unit-regulars in* R *is unit-regular.*
2. *The product of any two idempotents in* R *is unit-regular.*

Proof (2) $\Rightarrow$ (1) Suppose that the product of two idempotents in R is unit-regular. Let $a = e_1u_1$ and $b = e_2u_2$ be two unit-regulars in R, where $e_1, e_2 \in Id(R)$ and $u_1, u_2 \in U(R)$. It is easy to see that $u_1e_2u_1^{-1}$ is an idempotent and $ab = e_1(u_1e_2u_1^{-1})u_1u_2$. Put $e_3 := u_1e_2u_1^{-1}$. Then we conclude that $ab = e_1e_3u_1u_2$. By the assumption, e_1e_3 is unit-regular and hence ab is unit-regular.

(1) $\Rightarrow$ (2) This implication is clear. □

Example 2.4 *Let* R *be the algebra* $\begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ *over field* F*. Clearly,* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ *are two unit-regular elements of* R *but* $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *is not unit-regular.*

An element a in any ring R has (right) stable range 1 ($sr(a) = 1$) if $aR + bR = R$ (for any $b \in R$) implies that $a + br$ is a unit for some $r \in R$. It is well known that unit-regular elements in any ring have stable range 1.

A module M over a ring R is said to be an SSP-module if the sum of two direct summands of the module M is a direct summand of the module M [6].

Theorem 2.5 *The following conditions are equivalent for a ring* R*:*

1. *The product of two idempotents in* R *is unit-regular.*

2. *R_R is an SSP-module.*

Proof

(1) $\Rightarrow$ (2) Let e and f be two idempotents of R. From (1) (and hence from Proposition 2.3), we obtain that $(1-e)f$ is unit-regular. Hence $(1-e)fR$ is a direct summand of R. Let I be a right ideal of R such that $(1-e)fR \oplus I = R$. Then, $(1-e)R = (1-e)fR \oplus [(1-e)R \cap I]$. Inasmuch as $eR + fR = eR \oplus (1-e)fR$, we have

$$R = eR \oplus (1-e)fR \oplus [(1-e)R \cap I] = (eR + fR) \oplus [(1-e)R \cap I].$$

(2) $\Rightarrow$ (1) Let e and f be idempotents of R. Since R satisfies SSP, we get $(1-e)R + fR$ as a direct summand of R, and so efR is a direct summand of R. It follows that ef is regular. Take $a = ef$ and $x \in R$ with $a = axa$. Since all idempotents of R have the right stable range 1, we obtain that $sr(a) = 1$ by [16, Corollary 3.4]. Now, $aR + (1-ax)R = R$. There exists y in R such that $a + (1-ax)y$ is a unit. Let u be a unit of R with $[a + (1-ax)y]u = 1$. Then, we have

$$\begin{aligned} ef &= a = axa = ax[a + (1-ax)y]ua \\ &= [axa + \underbrace{ax(1-ax)}_{0}y]ua \\ &= axaua = aua \end{aligned}$$

which implies that ef is unit-regular. $\square$

Remark 2.6 *If we consider idempotents in Examples 2.2 and 2.3, then we can also see that the rings* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *and* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *do not satisfy SSP.*

Corollary 2.7 *Assume that R_R is an SSP-module for a ring R and $a, b \in Ureg(R)$. The following conditions are equivalent:*

1. *$abR = bR$.*
2. *$ab = bu$ for some $u \in U(R)$.*

Proof This follows from [13, Lemma 3.3] and Theorem 2.5. $\square$

3. Commutators of a pair of idempotents in $M_2(\mathbb{Z})$ and in its certain subrings

In this section, we will investigate the properties of the commutator $C = E_1E_2 - E_2E_1$ and the anticommutator $D = E_1E_2 + E_2E_1$, where E_1 and E_2 are two idempotents in R_i $(i = 1, 2, 3)$.

Lemma 3.1 *[17, Lemma 1] Let $s \in \mathbb{Z}$. Nontrivial idempotents in the ring* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *are matrices* $\begin{pmatrix} \alpha+1 & u \\ sv & -\alpha \end{pmatrix}$ *with $\alpha^2 + \alpha + suv = 0$.*

Theorem 3.2 *For the ring* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, *an element $C \in R$ is a commutator of a pair of idempotents in R if and only if there exist $U \in R$ and $S \in R$ such that $U^2 = I_2$, $UC = -CU$, $US = SU$, $S^2 - C^2 = US$, and $SC - CS = UC$.*

Proof (⇒:) Assume that $C = E_1E_2 - E_2E_1$, where $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} a+1 & b \\ 4c & -a \end{pmatrix}$ with $a^2+a+4bc = 0$. Notice that $E_2^2 = E_2$ or $a^2+a+4bc = 0$ implies $a^2+a = -4bc$. Hence $C = \begin{pmatrix} 0 & b \\ -4c & 0 \end{pmatrix}$ and $C^2 = \begin{pmatrix} -4bc & 0 \\ 0 & -4bc \end{pmatrix} = \begin{pmatrix} a^2+a & 0 \\ 0 & a^2+a \end{pmatrix}$. Define $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}$. Then

$$U^2 = I_2$$

$$UC = -CU$$

$$US = SU$$

$$S^2 - C^2 = US$$

$$SC - CS = UC.$$

(⇐:) For the converse, define E_1 and E_2 by $E_1 = (I_2+U)/2$ and $E_2 = US+UC$, respectively. Clearly, E_1 and E_2 are idempotents and $C = E_1E_2 - E_2E_1$. □

Remark 3.3 *For similar conversations for rings* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *with* $s \geq 3$ *and* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *with* $n > 1$, *a commutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *(respectively,* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$*) is of the form* $C = \begin{pmatrix} 0 & b \\ -s^2c & 0 \end{pmatrix}$ *(respectively,* $C = \begin{pmatrix} 0 & b \\ -2^nc & 0 \end{pmatrix}$*).*

Corollary 3.4 *For the ring* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, *assume that* C *is a commutator of a pair of idempotents in* R *and* D *is an anticommutator of a pair of idempotents in* R. *Then there exists a unit* U *in* R *such that* $DU - UD = 2C$.

Proof By Theorem 3.2, for the idempotent matrices $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} a+1 & b \\ 4c & -a \end{pmatrix}$ with $a^2+a+4bc = 0$, an anticommutator of a pair of idempotents in R is of the form $D = E_1E_2 + E_2E_1 = \begin{pmatrix} 2(a+1) & b \\ 4c & 0 \end{pmatrix}$. Notice that $E_2^2 = E_2$ or $a^2+a+4bc = 0$ implies $a^2+a = -4bc$. If we again define $U := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain that $DU - UD = \begin{pmatrix} 0 & 2b \\ -8c & 0 \end{pmatrix} = 2C$. □

Remark 3.5 *For similar conversations for rings* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *with* $s \geq 3$ *and* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *with* $n > 1$, *an anticommutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *(respectively,* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$*) is of the form* $D = \begin{pmatrix} 2(a+1) & b \\ s^2c & 0 \end{pmatrix}$ *(respectively,* $D = \begin{pmatrix} 2(a+1) & b \\ 2^nc & 0 \end{pmatrix}$*).*

Remark 3.6 *Consider the idempotent matrices* $E_1 = \begin{pmatrix} a+1 & b \\ c & -a \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} x+1 & y \\ z & -x \end{pmatrix}$ *in* $M_2(\mathbb{Z})$. *Then* $C = E_1E_2 - E_2E_1 = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}$, *where* $p = -cy + bz$, $q = -b - 2bx + 2ay + y$ *and* $r = c + 2cx - 2az - z$. *Although we have* $C^2 = \begin{pmatrix} -det(C) & 0 \\ 0 & -det(C) \end{pmatrix}$, *to construct* S *for the sufficient implication of Theorem 3.2 looks like impossible.*

It is well known that Jacobson's lemma states the following:

Lemma 3.7 *(Jacobson's Lemma) For any two elements* $a, b \in R$, $1 - ab$ *is a unit if and only if* $1 - ba$ *is a unit.*

Theorem 3.8 *Let* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. *If* $A = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} -35 & 44 \\ 416 & 213 \end{pmatrix}$, *then* $I_2 - AB$ *is a commutator in* R, *but* $I_2 - BA$ *is not a commutator in* R.

Proof Since

$$I_2 - AB = \begin{pmatrix} 36 & -44 \\ -556 & -36 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ -24 & -8 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ 20 & -4 \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ 20 & -4 \end{pmatrix}\begin{pmatrix} 9 & 3 \\ -24 & -8 \end{pmatrix},$$

we get $I_2 - AB$ is a commutator in R.

Assume on the contrary that $C := I_2 - BA = \begin{pmatrix} 212 & -44 \\ 436 & -212 \end{pmatrix}$ is a commutator in R, i.e. $C = E_1E_2 - E_2E_1$, where $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} a+1 & b \\ 4c & -a \end{pmatrix}$ are idempotents in R as in Theorem 3.2. Again by Theorem 3.2, there should exist $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}$ in R such that $UC = -CU$ $US = SU$, $S^2 - C^2 = US$ and $SC - CS = UC$. But $UC \neq -CU$, $S^2 - C^2 \neq US$, and $SC - CS \neq UC$. □

Let $s \in \mathbb{Z}$. Nilpotents in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ are matrices $\begin{pmatrix} \beta & x \\ sy & -\beta \end{pmatrix}$ with $\beta^2 + sxy = 0$ ([1]).

Proposition 3.9 *Let* $R := M_2(\mathbb{Z})$.

(1) Upper triangular commutator matrices of $M_2(\mathbb{Z})$ *are nilpotent.*

(2) If $C := \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ *is a commutator matrix in* R, *then* $x = 0$.

Proof (1) Note that upper triangular idempotents and nilpotents are of the form $\begin{pmatrix} \alpha+1 & u \\ 0 & -\alpha \end{pmatrix}$ with $\alpha^2 + \alpha = 0$ and $\begin{pmatrix} \beta & x \\ 0 & -\beta \end{pmatrix}$ with $\beta^2 = 0$, respectively. Hence we have $\alpha \in \{-1, 0\}$, i.e. upper triangular idempotents are $E_1 = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & u \\ 0 & 1 \end{pmatrix}$. Clearly, $C = E_1E_2 - E_2E_1 = \begin{pmatrix} 0 & 2u \\ 0 & 0 \end{pmatrix}$ is a nilpotent

element in $M_2(\mathbb{Z})$.
(2) Assume $x \neq 0$. Write $C = E_1E_2 - E_2E_1$, where E_1 and E_2 are idempotents in R. Since $E_1CE_1 = 0$ and $E_2CE_2 = 0$, the idempotents E_1 and E_2 cannot be trivial. Let $E_1 = \begin{pmatrix} a+1 & u \\ v & -a \end{pmatrix}$ with $a^2 + a + uv = 0$ and $E_2 = \begin{pmatrix} b+1 & t \\ z & -b \end{pmatrix}$ with $b^2 + b + tz = 0$. Clearly,

$$E_1CE_1 = \begin{pmatrix} a+1 & u \\ v & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} a+1 & u \\ v & -a \end{pmatrix} = \begin{pmatrix} a^2+2a+uvx+1 & au+u-aux \\ av+v-avx & uv+a^2x \end{pmatrix},$$

$$E_2CE_2 = \begin{pmatrix} b+1 & t \\ z & -b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} b+1 & t \\ z & -b \end{pmatrix} = \begin{pmatrix} b^2+2b+tzx+1 & bt+t-btx \\ bz+z-bzx & tz+b^2x \end{pmatrix}$$

and

$$C = E_1E_2 - E_2E_1 = \begin{pmatrix} uz-tv & 2at-2bu+t-u \\ 2bv-2az+v-z & tv-uz \end{pmatrix}$$

which gives that $tv - uz = -1 = x$, and so $a = -1/2 = b$, a contradiction. $\square$

Proposition 3.10 *Not every nilpotent (or invertible) matrix is the commutator of a pair of idempotents in* $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof Consider the nilpotent (or invertible) element $N = \begin{pmatrix} -6 & 9 \\ -4 & 6 \end{pmatrix}$ (respectively, $A = \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}$) in R and assume on the contrary that $N = E_1E_2 - E_2E_1$, where $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} a+1 & b \\ 4c & -a \end{pmatrix}$ are idempotents in R as in Theorem 3.2. Again by Theorem 3.2, there should exist $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}$ in R such that $UN = -NU$, $US = SU$, $S^2 - N^2 = US$, and $SN - NS = UN$.(respectively, $UA = -AU$, $US = SU$, $S^2 - A^2 = US$, and $SA - AS = UA$). But $UN \neq -NU$, $S^2 - N^2 \neq US$ and $SN - NS \neq UN$ (respectively, $UA \neq -AU$, $S^2 - A^2 \neq US$ and $SA - AS \neq UA$). $\square$

Due to Nicholson [14], an element in a ring is called clean if it is a sum of an idempotent and a unit, and the ring is clean if every element is clean.

Theorem 3.11 *If* $s \geq 3$, *then* $\begin{pmatrix} 0 & s \\ -s^2 & 0 \end{pmatrix}$ *is a commutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, *but not clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof Let $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $C = \begin{pmatrix} 0 & s \\ -s^2 & 0 \end{pmatrix}$ and assume on the contrary that $C = E + (C - E)$ where $E^2 = E \in R$ and $C - E$ is invertible in R. It is clear that $0 \neq E$ and $I_2 \neq E$. So we can get $E = \begin{pmatrix} r+1 & p \\ s^2q & -r \end{pmatrix}$

with $r^2+r+s^2pq=0$. In view of $r^2+r+s^2pq=0$, we have $\mp 1 = det(C-E) = (q-p+s)s^2$. But s must be ∓ 1, a contradiction. $\square$

Example 3.12 *The matrix* $\begin{pmatrix} 4 & 1 \\ -4 & -4 \end{pmatrix}$ *is a clean element in* $M_2(\mathbb{Z})$ *which is a commutator of a pair of idempotents in* $M_2(\mathbb{Z})$.

Proof We see that

$$C := \begin{pmatrix} 4 & 1 \\ -4 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 7 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -11 & -4 \end{pmatrix}$$

is a sum of an idempotent and a unit in $M_2(\mathbb{Z})$. Also if we take idempotents $E_1 := \begin{pmatrix} 1 & 0 \\ -4 & 0 \end{pmatrix}$ and $E_2 := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, one can easily check that $C = E_1E_2 - E_2E_1$ is a commutator of E_1 and E_2. $\square$

Example 3.13 *The matrix* $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ *is a commutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, *but not clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof Let $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ and assume on contrary that $C = E+(C-E)$ where $E^2 = E \in R$ and $C-E$ is invertible in R. It is clear that $0 \neq E$ and $I_2 \neq E$. So we can get $E = \begin{pmatrix} r+1 & p \\ 4q & -r \end{pmatrix}$ with $r^2+r+4pq=0$. In view of $r^2+r+4pq=0$, we have $\mp 1 = det(C-E) = 4+4q-4p$. Then we have equations $4(q-p) = -3$ and $4(q-p) = -5$ which have no integer solutions for p and q. $\square$

Hence, we can conclude the following.

Theorem 3.14 *If* $s \geq 1$, *then not every commutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *is clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Example 3.15 *The matrix* $\begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}$ *is a commutator of a pair of idempotents in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, *but not clean in* $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

Proof Consider a commutator $C := \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}$ in the ring $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Assume on the contrary that $C = E+(C-E)$ where $E^2 = E \in R$ and $C-E$ is invertible in R. It is clear that $0 \neq E$ and $I_2 \neq E$. So we can get $E = \begin{pmatrix} r+1 & p \\ 8q & -r \end{pmatrix}$ with $r^2+r+8pq=0$. In view of $r^2+r+8pq=0$, we have $\mp 1 = det(C-E) = 8+8q-8p$. Then we have $8(p-q) = 9$ or $8(p-q) = 7$ which has no integer solutions for p and q. $\square$

Hence, we can conclude the following.

Theorem 3.16 *If $n \geq 3$, then the matrix $\begin{pmatrix} 0 & 1 \\ 2^n & 0 \end{pmatrix}$ is a commutator of a pair of idempotents in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, but not clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.*

Proof Consider a commutator $C := \begin{pmatrix} 0 & 1 \\ 2^n & 0 \end{pmatrix}$ in the ring $R := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2^3\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Assume on the contrary that $C = E + (C - E)$ where $E^2 = E \in R$ and $C - E$ is invertible in R. It is clear that $0 \neq E$ and $I_2 \neq E$. So we can get $E = \begin{pmatrix} r+1 & p \\ 2^n q & -r \end{pmatrix}$ with $r^2 + r + 2^n pq = 0$. In view of $r^2 + r + 8pq = 0$, we have $\mp 1 = det(C - E) = -2^n + 2^n(p + q)$. Then we have $2^n(p + q) = 2^n + 1$ or $2^n(p + q) = 2^n - 1$ which has no integer solutions for p and q. □

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