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Some estimates on the exponential stability of solutions of nonlinear neutral type systems with periodic coefficients

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Abstract: In this present study, we pay attention to a class of nonlinear neutral type systems (NNSs) with periodic coefficients and construct some assumptions guaranteeing the exponential stability (ES) of the trivial solutions of the system considered. To get specific conditions guaranteeing the ES, we use a modified Lyapunov functional. In conclusion, we get some estimates for the exponential decay of the solutions at infinity with the constructed sufficient conditions. We give two examples to demonstrate the applicability of the results obtained with the constructed assumptions.

Key words: ES, exponential decay, Lyapunov functional, NNS, periodic coefficient

1. Introduction

Differential equation systems with time delays have been of considerable interest for the few decades. Recently, the stability problem of these systems has attracted the attention of many researchers (see, [1–34]). In particular, the interest in stability analysis of various neutral differential systems has attracted attention with the applications in practical fields of many researchers. Current efforts for stability analysis of these systems are examined in two categories, delay independent and delay dependent. The study of ES of solutions of NNSs is one of the important problems. This problem is very important both theoretically and practically in terms of determining the ES criteria of the solutions of delay dependent and delay independent neutral systems. When the related literature is searched, the commonly used method for the stability of delayed differential systems is the Lyapunov stability theory. While defining this theory, it is possible to deal with basically two aspects. The first is to choose an appropriate Lyapunov functional; the second is to reduce the expansion when estimating its derivative. These functionals allow us to examine the qualitative behavior of the differential equations we are considering without determining their roots. For this reason, it is seen that different modifications of Lyapunov functionals are used by considering delayed differential equation systems. These modifications can be constructed in shapes discretized Lyapunov functional [16], augmented Lyapunov functional [18], delayed partitioning Lyapunov functional [34], etc. At the same time, in the case of constant [32] and periodic coefficients in the linear part, a modified Lyapunov functional was suggested and used in [8] to obtain the estimates of exponential decay at the infinity of the solutions to systems of linear and quasi linear time-delay differential equations. In addition, Lyapunov functionals suitable for fractional-order systems have been used to further reduce conservatism [23]. Therefore, it is important to choose a functional suitable for the

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2010 *AMS Mathematics Subject Classification:* 34K20, 34K40 form of the system in order to obtain a less prudent criterion when performing stability analysis of differential systems [25].

In this study motivated by [4, 5, 9, 13, 26–29, 32, 33], we consider the following NNS with periodic coefficients:

$$\begin{aligned}\frac{d}{dt}[u(t)+Du(t-\tau(t))] =& A(t)u(t)+B(t)u(t-\tau(t))+C(t)\frac{d}{dt}u(t-\tau(t)) \\ &+F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t)))\end{aligned} \tag{1.1}$$

where $t \geq 0, u \in \Re^n, D$ is an $n\times n-$ constant matrix, $A(t),\ B(t)$, and $C(t)$ are $n\times n$ matrices with continuous $T-$ periodic entries, $(T>0)$, that is,

$$A(t+T)\equiv A(t),\ B(t+T)\equiv B(t),\ C(t+T)\equiv C(t)$$

and $\tau(t)\in C^1([0,\infty))$,

$$0<\tau_1\leq\tau(t)\leq\tau_2<\infty \tag{1.2}$$

and

$$\tau_3\leq\tau'(t)\leq\tau_4<1. \tag{1.3}$$

where τ_1 and τ_2 are positive constants. In addition, the continuous real value $F(t,\tilde{u},\tilde{v},\tilde{w})$ function satisfies the Lipschitz condition with respect to $\tilde{u}$, such that

$$\|F(t,\tilde{u},\tilde{v},\tilde{w})\|\leq q_1\|\tilde{u}\|+q_2\|\tilde{v}\|+q_3\|\tilde{w}\|,\ t\geq 0,\ \tilde{u},\tilde{v},\tilde{w}\in\Re^n, \tag{1.4}$$

for some constants $q_i\geq 0,\ i=1,2,3.$ Here, the vector norm and dot product mentioned above are defined as follows

$$\langle x,z\rangle=\sum_{j=1}^{n}x_j\bar{z}_j,\ \|x\|=\sqrt{\langle x,x\rangle}\ .$$

For the system (1.1), we deal with the following initial value problem (IVP)

$$\begin{aligned}\frac{d}{dt}[u(t)+Du(t-\tau(t))] =& A(t)u(t)+B(t)u(t-\tau(t))+C(t)\frac{d}{dt}u(t-\tau(t)) \\ &+F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) \\ u(t) =& \vartheta(t), t\in[-\tau_2,0], \\ u(0^+) =& \vartheta(0),\end{aligned} \tag{1.5}$$

where $\vartheta(t)\in C^1([-\tau_2,0])$ is a given real-valued vector-function.

In this research motivated by the above discussions, we consider the NNS (1.1), when the spectrum of the matrix belongs to the unit disk. From this point of view, we can summarize the main purpose and contribution of this research as follows. For the NNS (1.1), our aim is to establish new sufficient conditions for the ES of trivial solution at infinity depending on the norms $\|D\|$ and to obtain some estimates for its exponential decay.

Researchers frequently employ various Lyapunov-Krasovskii functionals to develop stability criteria (see, [6, 15]). However, we cannot derive estimates describing the decay rate of solutions at infinity for every Lyapunov-Krasovskii functional. The research in this area has been progressing rapidly in recent years. We construct a modified Lyapunov functional in proofs to obtain special conditions that guarantee ES and exponential decay of the solutions at infinity. Our theoretical results obtained in this search improve the results of [4, 5, 9, 13, 26–29, 32, 33] and make a contribution to the existing results in the literature from cases without delay to more general cases with time delay. We believe that this research, whose theoretical results are exemplified by numerical simulations, can be useful for researchers working on the qualitative behavior of solutions of NSs.

2. The main results

We first present a result for the ES of the trivial solution of NNS (1.1) with $F(t,u,v,w)\equiv 0$ as follows

$$\frac{d}{dt}[u(t)+Du(t-\tau(t))]=A(t)u(t)+B(t)u(t-\tau(t))+C(t)\frac{d}{dt}u(t-\tau(t)),\ t>0. \tag{2.1}$$

Notation: M^* is the conjugate transpose of M.

Theorem 2.1 *Assume that there are matrices $\tilde{H}(t)\in C^1\ [0,\ T],\ \tilde{K}(s)$ and $\tilde{L}(s)\in C^1\ [0,\tau_2]$ such that*

$$\tilde{H}(t)=\tilde{H}^*(t),\ \tilde{H}(t)=\tilde{H}(t+T)>0,\ t\geq 0, \tag{2.2}$$

$$\tilde{K}(s)=\tilde{K}^*(s)>0,\ \frac{d}{ds}\tilde{K}(s)<0,\ s\in[0,\tau_2], \tag{2.3}$$

$$\tilde{L}(s)=\tilde{L}^*(s)>0,\ \frac{d}{ds}\tilde{L}(s)<0,\ s\in[0,\tau_2], \tag{2.4}$$

and for all $t\in[0,T]$, it is presumed that

$$\Phi(t)=\begin{pmatrix}\Phi_{11}(t) & \Phi_{12}(t) & \Phi_{13}(t)\\ \Phi_{12}^*(t) & \Phi_{22}(t) & \Phi_{23}(t)\\ \Phi_{13}^*(t) & \Phi_{23}^*(t) & \Phi_{33}(t)\end{pmatrix}>0 \tag{2.5}$$

with entries

$$\begin{aligned}
\Phi_{11}(t)=&-\frac{d}{dt}\tilde{H}(t)-\tilde{H}(t)A(t)-A^*(t)\tilde{H}(t)-\tilde{K}(0)-A^*(t)\tilde{L}(0)A(t),\\
\Phi_{12}(t)=&-\tilde{H}(t)B(t)+\tilde{H}(t)A(t)D+\tilde{K}(0)D+A^*(t)\tilde{L}(0)A(t)D-A^*(t)\tilde{L}(0)B(t),\\
\Phi_{13}(t)=&-\tilde{H}(t)C(t)-A^*(t)\tilde{L}(0)C(t)+A^*(t)\tilde{L}(0)D,\\
\Phi_{22}(t)=&-D^*\tilde{K}(0)D+(1-\tau_4)\tilde{K}(\tau_2)-D^*A^*(t)\tilde{L}(0)A(t)D+B^*(t)\tilde{L}(0)A(t)D\\
&+D^*A^*(t)\tilde{L}(0)B(t)-B^*(t)\tilde{L}(0)B(t),\\
\Phi_{23}(t)=&D^*A^*(t)\tilde{L}(0)C(t)-B^*(t)\tilde{L}(0)C(t)-D^*A^*(t)\tilde{L}(0)D+B^*(t)\tilde{L}(0)D,\\
\Phi_{33}(t)=&-C^*(t)\tilde{L}(0)C(t)+C^*(t)\tilde{L}(0)D+D^*\tilde{L}(0)C(t)-D^*\tilde{L}(0)D+(1-\tau_3)^{-1}\tilde{L}(\tau_2).
\end{aligned} \tag{2.6}$$

Then the trivial solution of system (2.1) is exponentially stable.

Let us assume that the assumptions of Theorem 2.1 are true. We construct conditions for the ES of the trivial solution to the NNS (1.1). In addition, we introduce some notations to formulate our results. If the matrix $\tilde{H}(t)$ holds the assumptions of Theorem 2.1 then

$$\frac{d}{dt}\tilde{H}(t)+\tilde{H}(t)A(t)+A^*(t)\tilde{H}(t)<-\tilde{K}(0)-A^*(t)\tilde{L}(0)A(t);$$

i.e. for the following Lyapunov differential equation, $\tilde{H}(t)$ is a solution to a special boundary value problem:

$$\begin{aligned}\frac{d}{dt}\tilde{H}+\tilde{H}A(t)+A^*(t)\tilde{H}=&-N(t),\ t\in[0,T],\\ \tilde{H}(0)=&\tilde{H}(T)>0,\end{aligned}$$

where $N(t)=N^*(t)>0$. In this case, $\tilde{H}(t)=\tilde{H}^*(t)>0$ on $[0,T]$. Let us extend the $\tilde{H}(t)$ matrix T-periodically on the whole half-axis $\{t\geq 0\}$, keeping the same notation. Using the matrix $\tilde{H}(t)$ together with the matrices $\tilde{K}(s)$ and $\tilde{L}(s)$ satisfying the conditions of Theorem 2.1, we define the functions

$$\begin{aligned}\beta_1(t)=&2\left\|\tilde{H}(t)\right\|+(2\left\|A(t)\right\|+q_1)\left\|\tilde{L}(0)\right\|,\\ \beta_2(t)=&(2\left\|B(t)\right\|+2\left\|A(t)D\right\|+q_2+q_1\left\|D\right\|)\left\|\tilde{L}(0)\right\|,\\ \beta_3(t)=&(2\left\|C(t)\right\|+2\left\|D\right\|+q_2+q_1\left\|D\right\|)\left\|\tilde{L}(0)\right\|,\end{aligned}\tag{2.7}$$

$$\begin{aligned}\alpha_1(t)=&q_1\beta_1(t)+\frac{q_1\beta_2(t)+(q_2+q_1\left\|D\right\|)\beta_1(t)}{2}+\frac{q_1\beta_3(t)+q_3\beta_1(t)}{2},\\ \alpha_2(t)=&(q_2+q_1\left\|D\right\|)\beta_2(t)+\frac{q_2\beta_1(t)+q_1\beta_2(t)}{2}+\frac{(q_2+q_1\left\|D\right\|)\beta_3(t)+q_3\beta_2(t)}{2},\\ \alpha_3(t)=&q_3\beta_3(t)+\frac{q_3\beta_1(t)+q_1\beta_3(t)}{2}+\frac{q_3\beta_2(t)+(q_2+q_1\left\|D\right\|)\beta_3(t)}{2},\end{aligned}\tag{2.8}$$

and the matrix

$$\Phi^{\alpha}(t)=\Phi(t)-\begin{pmatrix}\alpha_1(t)I & 0 & 0\\ 0 & \alpha_2(t)I & 0\\ 0 & 0 & \alpha_3(t)I\end{pmatrix},\tag{2.9}$$

where I is the unit matrix.

By $\tilde{k},\tilde{l}>0$ and $p_{\min}(t)>0$ the minimal eigenvalue of the matrix $P(t)$,

$$\frac{d}{ds}\tilde{K}(s)+\tilde{k}\tilde{K}(s)\leq 0,\ \frac{d}{ds}\tilde{L}(s)+\tilde{l}\tilde{L}(s)\leq 0,\ s\in[0,\tau_2],\tag{2.10}$$

$$\varepsilon(t)=\min\left\{\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|},\tilde{k},\tilde{l}\right\}.\tag{2.11}$$

We denote by $\tilde{h}_{\min}(t) > 0$ the minimal eigenvalue of $\tilde{H}(t)$. Let

$$\Psi = \max_{t\in[-\tau_2,0]} \|\vartheta(t)\| \ , \tag{2.12}$$

$$\mu = \max_{t\in[0,\ T]} \sqrt{\frac{V(0,\vartheta)}{\tilde{h}_{\min}(t)}}, \tag{2.13}$$

$$\beta(t) = \varepsilon(t)/2 \ \ , \ \beta^+ = \max_{t\in[0,\ T]} \beta(t),\ \ \beta^- = \min_{t\in[0,\ T]} \beta(t). \tag{2.14}$$

Theorem 2.2 *Let the assumptions of Theorem 2.1 be satisfied. Assume that* $q_i \geq 0,\ \ (i = 1, 2, 3)$ *are such that* $\Phi^\alpha(t) > 0$ *for* $t \in [0,\ T]$*. Then the trivial solution of NNS (1.1) is exponentially stable.*

We present below the exponential decay rate estimation of the solution of the IVP (1.5) as $t \to \infty$. Let be defined Lyapunov functional

$$\begin{aligned} V(0,\varphi) = & \left\langle \tilde{H}(0)(\varphi(0) + D\varphi(-\tau(0))),\ (\varphi(0) + D\varphi(-\tau(0)))\right\rangle \\ & + \int_{-\tau(0)}^{0} \left\langle \tilde{K}(-\theta)\varphi(\theta), \varphi(\theta)\right\rangle d\theta + \int_{-\tau(0)}^{0} \left\langle \tilde{L}(-\theta)\frac{d}{d\theta}\varphi(\theta), \frac{d}{d\theta}\varphi(\theta)\right\rangle d\theta, \end{aligned} \tag{2.15}$$

$$\begin{aligned} P(t) = & \Phi_{11}(t) - \alpha_1(t)I - [\Phi_{12}(t) - \Phi_{13}(t)(\Phi_{33}(t) - \alpha_3(t)I)^{-1}\Phi^*_{23}(t)] \\ & \times [\Phi_{22}(t) - \alpha_2(t)I - \Phi_{23}(t)(\Phi_{33}(t) - \alpha_3(t)I)^{-1}\Phi^*_{23}(t)]^{-1} \\ & \times [\Phi_{12}(t) - \Phi_{13}(t)(\Phi_{33}(t) - \alpha_3(t)I)^{-1}\Phi^*_{23}(t)]^* \\ & - \Phi_{13}(t)(\Phi_{33}(t) - \alpha_3(t)I)^{-1}\Phi^*_{13}(t), \end{aligned} \tag{2.16}$$

where the matrices $\Phi_{ij}(t)$ are defined by (2.6). It is not hard to show that $P(t)$ is positive definite if $\Phi^\alpha(t)$ defined in (2.9) is positive definite (see, Lemma 2.6 below).

The main results of this research are expressed in the following theorems.

Theorem 2.3 *Let the assumption given by (2.5) be satisfied and*

$$\|D\| < e^{-\beta^+\tau_2}.$$

Then, for the solution$u(t)$ *of (IVP) (1.5), the below assertion is true:*

$$\|u(t)\| \leq \mu(1 - \|D\| e^{\beta^+\tau_2})^{-1} e^{-\beta^- t} + \|D\|^{\max\{t/\tau_2, 1\}}\Psi, \tag{2.17}$$

where Ψ, μ, β^- *and* β^+ *are defined in (2.12), (2.13) and (2.14), respectively.*

Theorem 2.4 *Let the assumption given by (2.5) be satisfied and*

$$\|D\| = e^{-\beta^+\tau_2}.$$

Then, for the solution $u(t)$ of (IVP) (1.5), the below assertion is true:

$$\|u(t)\| \leq [\mu(t/\tau_1 \ +1)e^{-\beta^- t} + \|D\|^{\max\{t/\tau_2,1\ \}}\Psi, \tag{2.18}$$

where Ψ, μ, β^- and β^+ are defined in (2.12), (2.13) and (2.14), respectively.

Theorem 2.5 *Let the assumption given by (2.5) be satisfied and*

$$e^{-\beta^+\tau_2} < \|D\| < e^{-(\beta^+\tau_2 - \beta^-\tau_1)}.$$

Then, for the solution $u(t)$ of (IVP) (1.5), the below assertion is true:

$$\|u(t)\| \leq \mu(1 - (\|D\|\, e^{\beta^+\tau_2})^{-1})^{-1} (\|D\|\, e^{\beta^+\tau_2 - \beta^-\tau_1})^{t/\tau_1} + \|D\|^{\max\{t/\tau_2,1\ \}}\Psi, \tag{2.19}$$

where Ψ, μ, β^- and β^+ are defined in (2.12), (2.13), and (2.14), respectively.

The next lemmas are needed for the main results of this paper.

Lemma 2.6 *[29] Let*

$$\Xi = \begin{pmatrix} \Xi_{11}(t) & \Xi_{12}(t) & \Xi_{13}(t) \\ \Xi_{12}^*(t) & \Xi_{22}(t) & \Xi_{23}(t) \\ \Xi_{13}^*(t) & \Xi_{23}^*(t) & \Xi_{33}(t) \end{pmatrix} > 0,\ \ t \in [0, T],$$

be a Hermitian matrix. Then

$$\Xi = \begin{pmatrix} I & \tilde{\Xi}_1(t)\tilde{\Xi}_2^{-1}(t) & \Xi_{13}(t)\Xi_{33}^{-1}(t) \\ 0 & I & \Xi_{23}(t)\Xi_{33}^{-1}(t) \\ 0 & 0 & I \end{pmatrix}$$

$$\times \begin{pmatrix} \Xi_{11}(t) - \tilde{\Xi}_1(t)\tilde{\Xi}_2^{-1}(t)\tilde{\Xi}_1^*(t) - \Xi_{13}(t)\Xi_{33}^{-1}(t)\Xi_{13}^*(t) & 0 & 0 \\ 0 & \tilde{\Xi}_2(t) & 0 \\ 0 & 0 & \Xi_{33}(t) \end{pmatrix}$$

$$\times \begin{pmatrix} I & 0 & 0 \\ \tilde{\Xi}_2^{-1}(t)\tilde{\Xi}_1^*(t) & I & 0 \\ \Xi_{33}^{-1}(t)\Xi_{13}^*(t) & \Xi_{33}^{-1}(t)\Xi_{23}^*(t) & I \end{pmatrix},$$

where $\tilde{\Xi}_1(t) = \Xi_{12}(t) - \Xi_{13}(t)\Xi_{33}^{-1}(t)\Xi_{23}^*(t), \tilde{\Xi}_2(t) = \Xi_{22}(t) - \Xi_{23}(t)\Xi_{33}^{-1}(t)\Xi_{23}^*(t)$; moreover, the matrices $\Xi_{11}(t) - \tilde{\Xi}_1(t)\tilde{\Xi}_2^{-1}(t)\tilde{\Xi}_1^*(t) - \Xi_{13}(t)\Xi_{33}^{-1}(t)\Xi_{13}^*(t) > 0,\ \tilde{\Xi}_2(t) > 0$ and $\Xi_{33}(t) > 0$.

Lemma 2.7 *Suppose that the assumptions of Theorem 2.2 are satisfied. Then the trivial solution $u(t)$ of the (IVP) (1.5) holds the below estimate*

$$\|u(t) + Du(t - \tau(t))\| \leq \sqrt{\frac{V(0, \vartheta)}{\tilde{h}_{\min}(t)}} \exp(-\int_0^t \beta(s)ds),\ \ t > 0, \tag{2.20}$$

where $\beta(t)$ and $V(0, \vartheta)$ are defined in (2.14) and (2.15), respectively; $\tilde{h}_{\min}(t) > 0$ is the minimal eigenvalue of the matrix $\tilde{H}(t)$.

Proof Let $u(t)$ be a solution of (IVP) (1.5). Using the above matrices $\tilde{H}(t), \tilde{K}(s)$, and $\tilde{L}(s)$, we consider the Lyapunov–Krasovskii functional

$$\begin{aligned} V(t,u) = & \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))),\ (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \int\limits_{t-\tau(t)}^{t} \left\langle \tilde{K}(t-s)u(s), u(s) \right\rangle ds + \int\limits_{t-\tau(t)}^{t} \left\langle \tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s) \right\rangle ds. \end{aligned} \tag{2.21}$$

Positive definiteness of the functional $V(t,u)$ is clear. Differentiating of the functional $V(t,u)$ along solutions of (1.5), we can obtain that

$$\begin{aligned} \frac{d}{dt}V(t,u) = & \left\langle \frac{d}{dt}\tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{H}(t)\frac{d}{dt}(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), \frac{d}{dt}(u(t)+Du(t-\tau(t))) \right\rangle + \left\langle \tilde{K}(0)u(t), u(t) \right\rangle \\ & - (1-\tau'(t)) \left\langle \tilde{K}(\tau(t))u(t-\tau(t)), u(t-\tau(t)) \right\rangle + \int\limits_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s) \right\rangle ds \\ & + \left\langle \tilde{L}(0)\frac{d}{dt}u(t), \frac{d}{dt}u(t) \right\rangle - (1-\tau'(t)) \left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)), \frac{d}{dt}u(t-\tau(t)) \right\rangle \\ & + \int\limits_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s) \right\rangle ds. \end{aligned}$$

We introduce the notation

$$z(t) = A(t)u(t) + B(t)u(t-\tau(t)) + C(t)\frac{d}{dt}u(t-\tau(t)).$$

Considering that $u(t)$ satisfies the system (1.1), we get

$$\begin{aligned} \frac{d}{dt}V(t,u) = & \left\langle \frac{d}{dt}\tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{H}(t)z(t), (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{H}(t)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))), (u(t)+Du(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), z(t) \right\rangle \\ & + \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) \right\rangle \\ & + \left\langle \tilde{K}(0)u(t), u(t) \right\rangle - (1-\tau'(t)) \left\langle \tilde{K}(\tau(t))u(t-\tau(t)), u(t-\tau(t)) \right\rangle \end{aligned} \tag{2.22}$$

$$+\int_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}\tilde{K}(t-s)u(s),u(s)\right\rangle ds+\left\langle \tilde{L}(0)\frac{d}{dt}u(t),\frac{d}{dt}u(t)\right\rangle$$

$$-(1-\tau'(t))\left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle$$

$$+\int_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s),\frac{d}{ds}u(s)\right\rangle ds.$$

Let us consider the expressions

$$(1-\tau'(t))\left\langle \tilde{K}(\tau(t))u(t-\tau(t)),u(t-\tau(t))\right\rangle$$

and

$$(1-\tau'(t))^{-1}\left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle.$$

By (1.3) and the conditions $\tilde{K}(s)=\tilde{K}^*(s)>0$ and $\tilde{L}(s)=\tilde{L}^*(s)>0,\ s\in[0,\tau_2]$, it is clear that

$$(1-\tau'(t))\left\langle \tilde{K}(\tau(t))u(t-\tau(t)),u(t-\tau(t))\right\rangle \geq (1-\tau_4)\left\langle \tilde{K}(\tau(t))u(t-\tau(t)),u(t-\tau(t))\right\rangle$$

and

$$(1-\tau'(t))^{-1}\left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle \geq (1-\tau_3)^{-1}\left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle.$$

Using the assumptions (1.2), (1.3), (2.3), and (2.4), we have $\tilde{K}(\tau(t))\geq\tilde{K}(\tau_2)$ and $\tilde{L}(\tau(t))\geq\tilde{L}(\tau_2)$. Hence, it is obvious that

$$(1-\tau'(t))\left\langle \tilde{K}(\tau(t))u(t-\tau(t)),u(t-\tau(t))\right\rangle \geq (1-\tau_4)\left\langle \tilde{K}(\tau_2)u(t-\tau(t)),u(t-\tau(t))\right\rangle, \tag{2.23}$$

and

$$(1-\tau'(t))^{-1}\left\langle \tilde{L}(\tau(t))\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle \geq (1-\tau_3)^{-1}\left\langle \tilde{L}(\tau_2)\frac{d}{dt}u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))\right\rangle. \tag{2.24}$$

By (2.23) and (2.24), we have

$$\begin{aligned}\frac{d}{dt}V(t,u)\leq&\left\langle \frac{d}{dt}\tilde{H}(t)(u(t)+Du(t-\tau(t))),(u(t)+Du(t-\tau(t)))\right\rangle\\&+\left\langle \tilde{H}(t)A(t)(u(t)+Du(t-\tau(t))),(u(t)+Du(t-\tau(t)))\right\rangle\\&-\left\langle \tilde{H}(t)A(t)Du(t-\tau(t)),(u(t)+Du(t-\tau(t)))\right\rangle\\&+\left\langle \tilde{H}(t)B(t)u(t-\tau(t)),(u(t)+Du(t-\tau(t)))\right\rangle\end{aligned}$$

$$+\left\langle \tilde{H}(t)C(t)\frac{d}{dt}u(t-\tau(t)), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$+\left\langle \tilde{H}(t)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$+\left\langle A^*(t)\tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\left\langle A^*(t)\tilde{H}(t)(u(t)+Du(t-\tau(t))),\ Du(t-\tau(t))\right\rangle$$
$$+\left\langle B^*(t)\tilde{H}(t)(u(t)+Du(t-\tau(t))), u(t-\tau(t))\right\rangle$$
$$+\left\langle C^*(t)\tilde{H}(t)(u(t)+Du(t-\tau(t))), \frac{d}{dt}u(t-\tau(t))\right\rangle$$
$$+\left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t)))\right\rangle$$
$$+\left\langle \tilde{K}(0)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\left\langle \tilde{K}(0)Du(t-\tau(t)), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\left\langle D^*\tilde{K}(0)(u(t)+Du(t-\tau(t))), u(t-\tau(t))\right\rangle$$
$$+\left\langle D^*\tilde{K}(0)Du(t-\tau(t)), u(t-\tau(t))\right\rangle$$
$$-(1-\tau_4)\left\langle \tilde{K}(\tau_2)u(t-\tau(t)), u(t-\tau(t))\right\rangle+\int\limits_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s)\right\rangle ds$$
$$+\left\langle \tilde{L}(0)\frac{d}{dt}(u(t)+Du(t-\tau(t))), \frac{d}{dt}(u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\left\langle \tilde{L}(0)\frac{d}{dt}Du(t-\tau(t)), \frac{d}{dt}(u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\left\langle D^*\tilde{L}(0)\frac{d}{dt}(u(t)+Du(t-\tau(t))), \frac{d}{dt}u(t-\tau(t))\right\rangle$$
$$+\left\langle D^*\tilde{L}(0)D\frac{d}{dt}u(t-\tau(t)), \frac{d}{dt}u(t-\tau(t))\right\rangle$$
$$-(1-\tau_3)^{-1}\left\langle \tilde{L}(\tau_2)\frac{d}{dt}u(t-\tau(t)), \frac{d}{dt}u(t-\tau(t))\right\rangle$$
$$+\int\limits_{t-\tau(t)}^{t}\left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s)\right\rangle ds.$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}V(t,u) \leq & -\left\langle \Phi(t)\begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix} \right\rangle \\
& + \left\langle \tilde{H}(t)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),(u(t)+Du(t-\tau(t))) \right\rangle \\
& + \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) \right\rangle \\
& + < \tilde{L}(0)A(t)(u(t)+Du(t-\tau(t))),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& - < \tilde{L}(0)A(t)Du(t-\tau(t)),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& + < \tilde{L}(0)B(t)u(t-\tau(t)),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& + < \tilde{L}(0)C(t)\frac{d}{dt}u(t-\tau(t)),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& + < A^*(t)\tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),(u(t)+Du(t-\tau(t))) > \\
& - < D^*A^*(t)\tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),u(t-\tau(t)) > \\
& + < B^*(t)\tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),u(t-\tau(t)) > \\
& + < C^*(t)\tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),\frac{d}{dt}u(t-\tau(t)) > \\
& + < \tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& - < \tilde{L}(0)\frac{d}{dt}Du(t-\tau(t)),F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))) > \\
& - D^*\tilde{L}(0)F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t))),\frac{d}{dt}u(t-\tau(t)) > \\
& + \int\limits_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s),u(s) \right\rangle ds + \int\limits_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s),\frac{d}{ds}u(s) \right\rangle ds
\end{aligned}$$

where the matrix $\Phi(t)$ is defined in (2.5) .

Consider the group of the summands containing $F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t)))$ and indicate them by $W(t)$.

Then,

$$\frac{d}{dt}V(t,u) \leq -\left\langle \Phi(t)\begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix} \right\rangle + W(t)$$
$$+ \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s) \right\rangle ds + \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s) \right\rangle ds. \tag{2.25}$$

Obviously,

$$\begin{aligned} W(t) \leq &(2\left\|\tilde{H}(t)\right\| \|u(t)+Du(t-\tau(t))\| \\ &+ 2\left\|\tilde{L}(0)\right\| \left\|A(t)(u(t)+Du(t-\tau(t))) + B(t)u(t-\tau(t)) + C(t)\frac{d}{dt}u(t-\tau(t))\right\| \\ &+ 2\left\|\tilde{L}(0)\right\| \left\|A(t)Du(t-\tau(t)) + D\frac{d}{dt}u(t-\tau(t))\right\| \\ &+ \left\|\tilde{L}(0)\right\| \left\|F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t)))\right\|) \\ &\times \left\|F(t,u(t),u(t-\tau(t)),\frac{d}{dt}u(t-\tau(t)))\right\|. \end{aligned}$$

Using (1.4), we have

$$\begin{aligned} W(t) \leq &\left(\beta_1(t)\|u(t)+Du(t-\tau(t))\| + \beta_2(t)\|u(t-\tau(t))\| + \beta_3(t)\left\|\frac{d}{dt}u(t-\tau(t))\right\|\right) \\ &\times \left(q_1\|u(t)\| + q_2\|u(t-\tau(t))\| + q_3\left\|\frac{d}{dt}u(t-\tau(t))\right\|\right) \\ \leq &\left(\beta_1(t)\|u(t)+Du(t-\tau(t))\| + \beta_2(t)\|u(t-\tau(t))\| + \beta_3(t)\left\|\frac{d}{dt}u(t-\tau(t))\right\|\right) \\ &\times \left(q_1\|u(t)+Du(t-\tau(t))\| + (q_2+q_1\|D\|)\|u(t-\tau(t))\| + q_3\left\|\frac{d}{dt}u(t-\tau(t))\right\|\right), \end{aligned}$$

where $\beta_j(t), j=1,2,3,$ are defined in (2.7). Obviously,

$$W(t) \leq \alpha_1(t)\|(u(t)+Du(t-\tau(t)))\|^2 + \alpha_2(t)\|u(t-\tau(t))\|^2 + \alpha_3(t)\left\|\frac{d}{dt}u(t-\tau(t))\right\|^2, \tag{2.26}$$

where $\alpha_j(t), j=1,2,3,$ are described in (2.8). By (2.26), from (2.25) we get

$$\frac{d}{dt}V(t,u) \leq -\left\langle \Phi^{\alpha}(t)\begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix} \right\rangle$$
$$+ \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s) \right\rangle ds + \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s) \right\rangle ds \tag{2.27}$$

where the matrix $\Phi^{\alpha}(t)$ is given in (2.9).

We apply the Lemma 2.6 from matrix theory for additional transformations. By the Lemma 2.6, for the matrix $\Phi^{\alpha}(t)$, we have

$$\left\langle \Phi^{\alpha}(t) \begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix}, \begin{pmatrix} u(t)+Du(t-\tau(t)) \\ u(t-\tau(t)) \\ \frac{d}{dt}u(t-\tau(t)) \end{pmatrix} \right\rangle$$
$$\geq \langle P(t)u(t)+Du(t-\tau(t)), u(t)+Du(t-\tau(t))\rangle,$$

where $P(t)$ is the positive definite Hermitian matrix given in (2.16). Then

$$\langle P(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\rangle \geq p_{\min}(t)\|u(t)+Du(t-\tau(t))\|^2.$$

Consequently, from (2.27) we obtain

$$\frac{d}{dt}V(t,u) \leq -\langle p_{\min}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\rangle$$
$$+\int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s)\right\rangle ds + \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s)\right\rangle ds.$$

Clearly,

$$\begin{aligned} \tilde{h}_{\min}(t)\|u(t)+Du(t-\tau(t))\|^2 \leq & \left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle \\ \leq & \tilde{H}(t)\|u(t)+Du(t-\tau(t))\|^2. \end{aligned} \tag{2.28}$$

Hence,

$$\frac{d}{dt}V(t,u) \leq -\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|}\left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$+\int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{K}(t-s)u(s), u(s)\right\rangle ds + \int_{t-\tau(t)}^{t} \left\langle \frac{d}{dt}\tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s)\right\rangle ds.$$

Using the condition (2.10), we arrive at

$$\frac{d}{dt}V(t,u) \leq -\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|}\left\langle \tilde{H}(t)(u(t)+Du(t-\tau(t))), (u(t)+Du(t-\tau(t)))\right\rangle$$
$$-\tilde{k}\int_{t-\tau(t)}^{t} \left\langle \tilde{K}(t-s)u(s), u(s)\right\rangle ds - \int_{t-\tau(t)}^{t} \tilde{l}\left\langle \tilde{L}(t-s)\frac{d}{ds}u(s), \frac{d}{ds}u(s)\right\rangle ds.$$

From the definition of $V(t,u)$ in (2.21), we have

$$\frac{d}{dt}V(t,u) + \varepsilon(t)V(t,u) \leq 0$$

where $\varepsilon(t) = \min\left\{\frac{p_{\min}(t)}{\|\tilde{H}(t)\|}, \tilde{k}, \tilde{l}\right\}$. So that, by this inequality, we can reach the below estimate

$$V(t,u) \leq V(0,\vartheta)\exp(-\int_0^t \varepsilon(s)ds),$$

where $V(0,\vartheta)$ is described by (2.15). Taking into account the (2.28) and definition of the functional (2.21), we infer

$$\left\langle \tilde{H}(t)(u(t) + Du(t-\tau(t))), (u(t) + Du(t-\tau(t)))\right\rangle \leq V(t,u)$$

and

$$\|u(t) + Du(t-\tau(t))\| \leq \sqrt{\frac{V(t,u)}{\tilde{h}_{\min}(t)}} \leq \sqrt{\frac{V(0,\vartheta)}{\tilde{h}_{\min}(t)}}\exp(-\frac{1}{2}\int_0^t \varepsilon(s)ds).$$

Hence, we have the required inequality (2.20). This finishes the proof. □

Now, we estimate $\|u(t)\|$. For $t > 0$, let us consider the below functions

$$\begin{aligned} &\eta_0(t) = t, \\ &\eta_1(t) = t - \tau(t), \\ &\ldots \\ &\eta_l(t) = \eta_{l-1}(t) - \tau(\eta_{l-1}(t)), \ l \geq 1 \end{aligned}$$

or equivalently,

$$\begin{aligned} &\eta_0(t) = t, \\ &\eta_1(t) = t - \tau(\eta_0(t)), \\ &\ldots \\ &\eta_l(t) = t - \sum_{j=0}^{l-1}\tau(\eta_j(t)), \ l \geq 1 \end{aligned} \tag{2.29}$$

Assume that $m \in N$ be a minimal number such that

$$\eta_m(t) \in [-\tau_2, 0). \tag{2.30}$$

Lemma 2.8 *Let the assumption given by (2.5) be satisfied. Then the trivial solution $u(t)$ of (IVP) (1.5) holds the below estimate*

$$\|u(t)\| \leq \mu\sum_{j=0}^{m-1}(\|D\|e^{\beta^+\tau_2})^j e^{-\beta^- t} + \|D\|^{\max\{t/\tau_2\ ,1\}}\Psi, \tag{2.31}$$

where Ψ, μ, β^- and β^+ are defined in (2.12), (2.13), and (2.14), respectively.

Proof In the case of the consideration of $\{\eta_l(t)\}_{l\geq 1}$, we can reach the $u(t)$ as follows

$$\begin{aligned} u(t) =& [u(\eta_0(t)) + Du(\eta_1(t))] - D[u(\eta_1(t)) + Du(\eta_2(t))] + ... \\ &+ (-1)^{m-1} D^{m-1}[u(\eta_{m-1}(t)) + Du(\eta_m(t))] + (-1)^m D^m u(\eta_m(t)) \end{aligned}$$

which implies that

$$\begin{aligned} \|u(t)\| \leq & \|u(\eta_0(t)) + Du(\eta_1(t))\| + \|D\| \; \|u(\eta_1(t)) + Du(\eta_2(t))\| + ... \\ &+ \left\|D^{m-1}\right\| \; \|u(\eta_{m-1}(t)) + Du(\eta_m(t))\| + \|D^m u(\eta_m(t))\| \\ =& \|u(\eta_0(t)) + Du(\eta_0(t) - \tau(\eta_0(t)))\| \\ &+ \|D\| \; \|u(\eta_1(t)) + Du(\eta_1(t) - \tau(\eta_1(t)))\| + ... \\ &+ \left\|D^{m-1}\right\| \; \|u(\eta_{m-1}(t)) + Du(\eta_{m-1}(t) - \tau(\eta_{m-1}(t)))\| + \|D^m u(\eta_m(t))\| \; . \end{aligned}$$

It should now be noted that

$$\begin{aligned} \|u(t)\| \leq & \mu e^{-\int\limits_0^{\eta_0(t)} \beta(s)ds} + \|D\| \, \mu e^{-\int\limits_0^{\eta_1(t)} \beta(s)ds} + \dots \\ &+ \left\|D^{m-1}\right\| \mu e^{-\int\limits_0^{\eta_{m-1}(t)} \beta(s)ds} + \|D^m u(\eta_m(t))\| \\ =& \sum_{j=0}^{m-1} \left\|D^j\right\| \; \mu e^{-\int\limits_0^{\eta_j(t)} \beta(s)ds} + \|D^m u(\eta_m(t))\| \\ =& \mu \sum_{j=0}^{m-1} \left\|D^j\right\| \; e^{\int\limits_{\eta_j(t)}^{t} \beta(s)ds} e^{-\int\limits_0^{t} \beta(s)ds} + \|D^m u(\eta_m(t))\| \end{aligned}$$

by (2.13) and (2.20).

We note that (2.29) and (1.2) imply that $\eta_j(t) \geq t - j\tau_2$. Particularly, by (2.30) and $0 > \eta_m(t) \geq t - m\tau_2$, which implies $m > t/\tau_2$ .Thus, we can obtain that

$$\begin{aligned} \|u(t)\| \leq & \mu \sum_{j=0}^{m-1} \left\|D^j\right\| e^{\beta^+(t-\eta_j(t))} e^{-\beta^- t} + \|D\|^{\max\{t/\tau_2, 1\}} \Psi \\ \leq & \mu \sum_{j=0}^{m-1} \left\|D^j\right\| e^{\beta^+ j\tau_2} e^{-\beta^- t} + \|D\|^{\max\{t/\tau_2, 1\}} \Psi. \end{aligned}$$

□

Proof [Proof of Theorems 2.3-2.5] Using the inequality (2.31), it is not hard to prove estimates given by (2.17), (2.18), and (2.19), respectively. Indeed, let $\|D\| < e^{-\beta^+\tau_2}$. By the estimate

$$\sum_{j=0}^{m-1} (\|D\| \, e^{\beta^+\tau_2})^j \leq \sum_{j=0}^{\infty} \left(\|D\| \, e^{\beta^+\tau_2}\right)^j = \left(1 - \|D\| \, e^{\beta^+\tau_2}\right)^{-1}$$

and inequality (2.31), we can arrive at inequality (2.17). This result finishes the proof of Theorem 2.3.

Let $\|D\| = e^{-\beta^+\tau_2}$. By (2.30), we have $\eta_{m-1}(t) \geq 0$. Moreover, by (2.29) and ??eq2), we get $\eta_{m-1}(t) \leq t-(m-1)\tau_1$, which implies

$$m \leq \frac{t}{\tau_1} + 1. \tag{2.32}$$

Hence,

$$\sum_{j=0}^{m-1} (\|D\| e^{\beta^+\tau_2})^j = m \leq \frac{t}{\tau_1} + 1.$$

The last equality above and estimate (2.31) imply (2.18). This result finishes the proof of Theorem 2.4.

Finally, let $e^{-\beta^+\tau_2} < \|D\| < e^{-(\beta^+\tau_2 - \beta^-\tau_1)}$. By (2.32), we have

$$\begin{aligned}
\sum_{j=0}^{m-1} (\|D\| e^{\beta^+\tau_2})^j &= \sum_{j=0}^{m-1} (\|D\| e^{\beta^+\tau_2})^{m-1-j} \\
&\leq \sum_{j=0}^{m-1} (\|D\| e^{\beta^+\tau_2})^{t/\tau_1 - j} \\
&\leq \sum_{j=0}^{\infty} (\|D\| e^{\beta^+\tau_2})^{-j} (\|D\| e^{\beta^+\tau_2})^{t/\tau_1} \\
&= (1 - (\|D\| e^{\beta^+\tau_2})^{-1})^{-1} (\|D\| e^{\beta^+\tau_2})^{t/\tau_1}.
\end{aligned}$$

By (2.14) and inequality (2.31), we have

$$u(t) \leq \mu(1 - (\|D\| e^{\beta^+\tau_2})^{-1})^{-1} (\|D\| e^{\beta^+\tau_2 - \beta^-\tau_1})^{t/\tau_1} + \|D\|^{\max\{t/\tau_2, 1\}} \Psi.$$

Thus, we can reach the assertion in (2.19). This result finishes the proof of Theorem 2.5. □

3. Numerical examples

Example 3.1 *As a special case of (1.1), we consider the below neutral system with periodic coefficients*

$$\begin{aligned}
\frac{d}{dt}(y(t) + 0.02y(t-\tau(t))) =& (0.1\cos t - 3.2)y(t) + 0.2y(t-\tau(t)) + 0.2\cos t \frac{d}{dt} y(t-\tau(t)) \\
&+ F(t, y(t), y(t-\tau(t)), \frac{d}{dt} y(t-\tau(t))),
\end{aligned} \tag{3.1}$$

for $t > 0$.

We start first by thinking about the linear case

$$F(t, \tilde{u}, \tilde{v}, \tilde{w}) = 0,$$

i.e. $q_i = 0, \quad (i = 1, 2, 3)$. *Let*

$$\tau_1 = 0.1 \leq \tau(t) = (1 + \sin^2 t)/10 \leq 0.2 = \tau_2.$$

In addition, it can be followed that

$$\tilde{H}(t) = 1 - 0.2\sin t,\ \tilde{K}(s) = 0.24e^{-0.45s},\ \tilde{L}(s) = 0.01e^{-0.45s},$$

Obviously, these functions satisfy (2.2-2.4) and (2.10) with

$$\tilde{k}, \tilde{l} = 0.45.$$

Let $\tilde{h}_{\min}(t) > 0$ *be the minimal eigenvalue of* $\tilde{H}(t)$*. Thus, it is can easily calculated that*

$$\tilde{h}_{\min}(t) \geq 0.8\ ,\left\|\tilde{H}(t)\right\| \leq 1.2\ .$$

Therefore, it is simple to verify that the matrix $\Phi(t) > 0$ *for the earlier specific choices. Thus, considering the assumptions of Theorem 2.1, we can say that the trivial solution of (3.1) with* $q_i = 0,\ (i = 1, 2, 3)$ *is exponentially stable.*

Since $q_i = 0,\ (i = 1, 2, 3)$*, it is clear that* $\Phi(t) = \Phi^{\alpha}(t)$*. It is not difficult to show that* $P(t)$ *is positive definite if* $\Phi^{\alpha}(t) > 0$*. It is known that* $p_{\min}(t)$ *is the minimal eigenvalue of* $P(t)$*. The* $p_{\min}(t)$ *value holds* $p_{\min}(t) \geq 0.8562$ *by using MATLAB-Simulink. Therefore* $\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|} \geq 0.7135$ *and* $\varepsilon(t) = \min\left\{\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|}, \tilde{k}, \tilde{l}\right\} = 0.45$*. By (2.14), we establish the following estimate*

$$\|y(t)\| \leq r \max_{-\tau_2 \leq s \leq 0} \|y(s)\|\, e^{-0.225t},\ r > 0,$$

for the solutions to (3.1).

Let us now examine the case of $F(t, \tilde{u}, \tilde{v}, \tilde{w}) \neq 0$ *for the system (3.1). We choose the functions* $\tilde{H}(t), \tilde{K}(s), \tilde{L}(s)$ *and constants* $q_i \geq 0,\ (i = 1, 2, 3)$ *as follows*

$$\tilde{H}(t) = 1 - 0.2\sin t,\ \tilde{K}(s) = 0.12e^{-0.35s},\ \tilde{L}(s) = 0.16e^{-0.35s},$$

and

$$q_1 = 0.01,\ q_2 = 0.02,\ q_3 = 0.06.$$

In this case, it is $\Phi(t) > 0$ *for* $t \in [0,\ 2\pi]$*. Then, by Theorem 2.1, the trivial solution to (3.1) with* $q_i \geq 0,\ (i = 1, 2, 3)$ *is exponentially stable.*

For $t \in [0,\ 2\pi]$*, it is not hard to demonstrate that* $\Phi^{\alpha}(t)$ *described in (2.9) is positive definite. In this case,* $P(t)$ *is positive definite if* $\Phi^{\alpha}(t) > 0$*. By MATLAB-Simulink, the value* $p_{\min}(t)$ *satisfies* $p_{\min}(t) \geq 2.9693$*. Therefore* $\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|} \geq 2.4744$ *and* $\varepsilon(t) = \min\left\{\frac{p_{\min}(t)}{\left\|\tilde{H}(t)\right\|}, \tilde{k}, \tilde{l}\right\} = 0.35$*.*

By (2.14), we have the estimate

$$\|y(t)\| \leq r \max_{-\tau_2 \leq s \leq 0} \|y(s)\|\, e^{-0.175t},\ r > 0,$$

for the solutions to (3.1).

Figure 1 and Figure 2 graphs show the trajectories of solutions of the considered system.

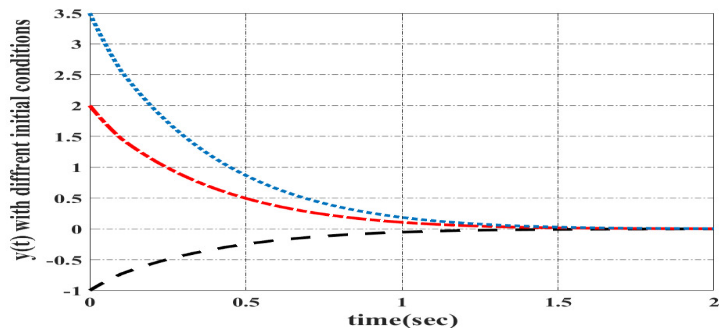

Figure 1. Trajectories of system (3.1) when $F(t,\tilde{u},\tilde{v},\tilde{w})=0,$ for $\tau(t)=\ (1+\sin^2 t)/10.$

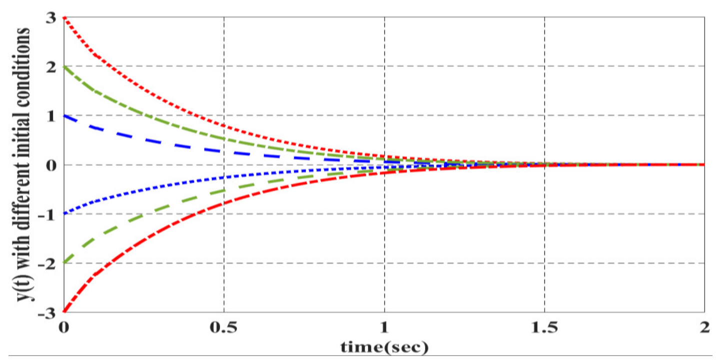

Figure 2. Trajectories of system (3.1) when $F(t,\tilde{u},\tilde{v},\tilde{w})\neq 0,$ for $\tau(t)=\ (1+\sin^2 t)/10$.

Example 3.2 *For $n=2$, as a special subcase of (1.1), we consider the below NNS with periodic coefficients*

$$\begin{aligned}\frac{d}{dt}[y(t)+Dy(t-\tau(t))]=&A(t)y(t)+B(t)y(t-\tau(t))+C(t)\frac{d}{dt}y(t-\tau(t))\\&+F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))\end{aligned}\tag{3.2}$$

where

$A(t)=\begin{pmatrix}-3.8+0.2\cos t & 1-0.4\cos t\\ 1.2 & -3.6\end{pmatrix},\ B(t)=\begin{pmatrix}0.5\sin t & 0\\ -0.6\sin t & 0.1\cos t\end{pmatrix},$

$C(t)=\begin{pmatrix}0.01 & 0.04\\ 0.01 & 0.02\end{pmatrix},\ D=\begin{pmatrix}0.01 & 0.16\\ 0.16 & 0.01\end{pmatrix}$,

$F(t,y(t),y(t-\tau(t)),\frac{d}{dt}y(t-\tau(t)))=\begin{bmatrix}q_1e^{-y_1^2(t)}y_1(t)+q_2e^{-y_1^2(t-\tau(t))}y_1(t-\tau(t))+q_3e^{-y_1^2(t-\tau(t))}\frac{d}{dt}y_1(t-\tau(t))\\ q_1e^{-y_2^2(t)}y_2(t)+q_2e^{-y_2^2(t-\tau(t))}y_2(t-\tau(t))+q_3e^{-y_2^2(t-\tau(t))}\frac{d}{dt}y_2(t-\tau(t))\end{bmatrix},$

$$\tau_1=\ 0.1\leq\ \tau(t)=\ (1+\sin^2 t)/10\ \leq 0.2=\tau_2.$$

Considering the assumption (1.4), for some positive constants $q_1=0.02,\ q_2=0.03$*, and* $q_3=0.004,$

$$\|F(t,\tilde{u},\tilde{v},\tilde{w})\|\leq q_1\|\tilde{u}\|+q_2\|\tilde{v}\|+q_3\|\tilde{w}\|,\ t\geq 0,\ u,v,w\in\Re^2.$$

In addition, we can choose the functions $\tilde{H}(t), \tilde{K}(s)$, *and* $\tilde{L}(s)$ *as follows*

$$\tilde{H}(t) = \begin{pmatrix} 2-0.2\sin t & 1-0.4\sin t \\ 1-0.4\sin t & 4+1.2\sin t \end{pmatrix}, \ \tilde{K}(s) = e^{-0.09s}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$
$$\tilde{L}(s) = e^{-0.09s}\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

For the values selected above, the following calculations can be easily obtained

$$1.5886 \le \tilde{h}_{\min}(t) \le 1.6023 \ ,$$

and

$$4.4214 \le \left\| \tilde{H}(t) \right\| \le 4.4577,$$

where, $\tilde{h}_{\min}(t)$ *is the minimal eigenvalue of the matrix* $\tilde{H}(t)$.

In this case, it is $\Phi(t) > 0$ *for* $t \in [0,\ 2\pi]$. *Thus, considering the assumptions of Theorem 2.1, we can say that the trivial solution of (3.2) is exponentially stable for constants* $q_1 = 0.02,\ q_2 = 0.03$, *and* $q_3 = 0.004$.

For $t \in [0,\ 2\pi]$, *it is not hard to demonstrate that* $\Phi^{\alpha}(t)$ *described in (2.9) is positive definite. In this case,* $P(t)$ *is positive definite if* $\Phi^{\alpha}(t) > 0$. *By MATLAB-Simulink, the value* $p_{\min}(t)$ *satisfies* $p_{\min}(t) \ge 7.7758$. *Therefore* $\frac{p_{\min}(t)}{\|\tilde{H}(t)\|} \ge 1.7443$ *and* $\varepsilon(t) = \min\left\{\frac{p_{\min}(t)}{\|\tilde{H}(t)\|}, \tilde{k}, \tilde{l}\right\} = 0.09$.

By (2.14), we have the estimate

$$\|y(t)\| \le r \max_{-\tau_2 \le s \le 0} \|y(s)\| \, e^{-0.045t}, \ r > 0,$$

for the solutions to (3.2).

Figure 3 graph shows the trajectories of solutions of the considered system.

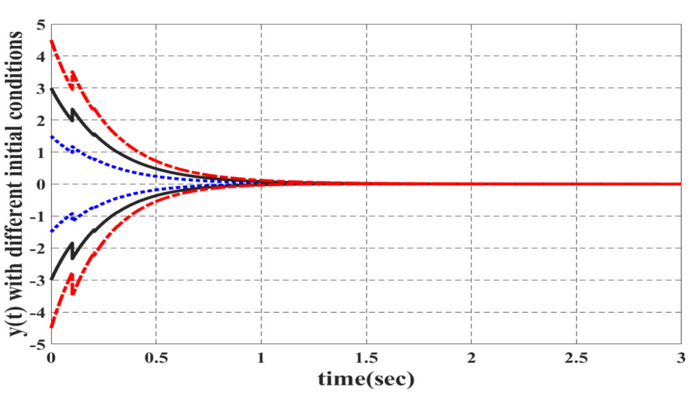

Figure 3. Trajectories of system (3.2) when $F(t,\tilde{u},\tilde{v},\tilde{w}) \ne 0$, for $\tau(t) = (1+\sin^2 t)/10$.

4. Conclusions

This research presents the ES of trivial solutions of nonlinear neutral type systems (NNSs) with periodic coefficients. By applying the Lyapunov functional method, a set of sufficient conditions is obtained that guarantees ES. Two examples with simulations are provided to support the theoretical findings. Our results improve and generalize the results of previous studies on this topic in the literature.

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