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A note on half of some MED semigroups of maximal or almost maximal length

Ahmet ÇELİK*

Department of Computer Technologies, Vocational School of Technical Sciences,
Adıyaman University, Adıyaman, Turkey

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Abstract: In this study, we have shown that numerical semigroups $M =< 3, C + 1, C + 2 >$ and $M =< 3, C, C + 2 >$ have maximal or almost maximal length, with conductor C, where $C \equiv 0(3)$ and $C \equiv 2(3)$, respectively. We also examined whether half of these numerical semigroups were of maximal or almost maximal length.

Key words: Length, saturated numerical semigroups, type, conductor

1. Introduction

Numerical semigroups arise in the study of Sylvester which has been known as the Frobenius problem (see [11] for an extensive exposure of this and related problems). Numerical semigroups appear in several areas of mathematics and their theory is connected with algebraic geometry, commutative algebra, and ring theory.

Inspired by the paper (see [1]) of Arf, Lipman introduced and studied Arf rings in his paper (see [9]) where characterizations of those rings via their value semigroups yield Arf numerical semigroups. Besides their importance in algebraic geometry, Arf numerical semigroups have gained lately a particular interest due to their applications to algebraic geometric codes, in particular to one-point algebraic geometric codes when the Weierstrass semigroup of the point is an Arf numerical semigroup [4].

A class of Arf numerical semigroups is the saturated numerical semigroups. The concept of saturated numerical semigroups has emerged by adding the terms of semigroup values of a saturated ring. On the other hand, with the maximal or almost maximal length of the semigroup to which a ring corresponds, the maximal or almost maximal concept of the numerical semigroup has emerged (see [14, 15]).

This work is organized as follows: Firstly, we proved that numerical semigroups $M =< 3, C + 1, C + 2 >$ have maximal length and $M =< 3, C, C + 2 >$ has almost maximal length, by Theorem 2.3 and Theorem 2.4, respectively. Secondly, we will give the half numerical semigroups of these numerical semigroups with conductor C, where $C \equiv 0(3)$ and $C \equiv 2(3)$, respectively, by Theorem 2.5 and Theorem 2.6. Finally, we examine that half of these numerical semigroups are of maximal length or almost maximal length by Theorem 2.9 and Theorem 2.11.

*Correspondence:acelik@adiyaman.edu.tr

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Let $\mathbb{N} = \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{Z}$ be integers set. $\emptyset \neq M \subseteq \mathbb{N}$, M is called a numerical semigroup if it satisfies the following conditions,

$$1)\, 0 \in M$$
$$2)\, x_1 + x_2 \in M,\ \ for\ \ all\ \ x_1, x_2 \in M$$
$$3)\, Card(\mathbb{N} \backslash M) < \infty.$$

For a numerical semigroup M, we define the following integers:

$$f(M) = max\{x \in \mathbb{Z} : x \notin M\};$$
$$m(M) = min\{x \in M : x \neq 0\};$$
$$n(M) = Card(\{0, 1, 2, ...f(M)\} \cap M).$$

$f(M)$ is called the Frobenius number of M, $m(M)$ the multiplicity of M, and $n(M)$ the determine number of M. If M is a numerical semigroup such that,

$$M =< a_1, a_2, ..., a_n >$$

then we observe that,

$$M =< a_1, a_2, ..., a_n >= \{m_0 = 0, m_1, m_2, ..., m_{n-1}, m_n = f(M) + 1, \rightarrow ...\}$$

where $m_j < m_{j+1}$, $n = n(M)$ and the arrow means that every integer greater than $f(M)+1$ belongs to M, for $j = 1, 2, ..., n = n(M)$. Here, we say the number $C = f(M) + 1$ is conductor of M (for details see [2, 12, 13]). For a numerical semigroup M , we give following definitions,

$$\frac{M}{t} = \{a \in \mathbb{N} : ta \in M\}\ \ for\ \ t \in M - \{0\};$$
$$M_j = \{a \in M : a \geq m_j\}\ \ for\ \ j \geq 0, m_j \in M;$$
$$M(j) = \{k \in \mathbb{N} : k + M_j \subseteq M\}.$$

Here, every the set $M(j)$ is a numerical semigroup and we write the following chain:

$$M_n \subset M_{n-1} \subset ... \subset M_1 \subset M_0 = M = M(0) \subset M(1) \subset ... \subset M(n-1) \subset M(n) = \mathbb{N}.$$

The number $t(M) = Card(M(1)\backslash M)$ is called the type of M (for details see [10]). In this case, M has maximal length if and only if $n(M)(t(M) + 1) = f(M) + 1$. Also, M has almost maximal length if and only if $n(M)(t(M) + 1) = f(M) + 2$ (for details see [3, 7]).

Let $M =< a_1, a_2, \cdots, a_e >$ be a numerical semigroup. In this case, the integer e is called the embedding dimension of M, and it is denoted by $e(M)$. It is known that $e(M) \leq m(M)$ (for details see [13]). M is called of maximal embedding dimension (MED) if $e(M) = m(M)$.

A numerical semigroup M is Arf if $m_1 + m_2 - m_3 \in M$, for all $m_1, m_2, m_3 \in M$ such that $m_1 \geq m_2 \geq m_3$. It is well known an Arf numerical semigroup is MED. But, a MED numerical semigroup can not an Arf. For example, $M =< 3, 7, 11 >= \{0, 3, 6, 7, 9 \rightarrow ...\}$ is a MED numerical semigroup since $e(M) = m(M) = 3$, but it is not Arf since $7 + 7 - 6 = 8 \notin M$. On the other hand, M is called saturated numerical semigroup if $u + d_M(x) \in M$, for all $u, x \in M - \{0\}$, where $d_M(x) = gcd\{k \in M : k \leq x\}$. It is known that a saturated numerical semigroup is Arf. But, an Arf numerical semigroup can not be saturated. For example, $M =< 5, 12, 14, 16, 18 >$ is Arf but it is not saturated. We note that if $m(M) = 1$ then $M =< 1 >= \mathbb{N}$ and M is both Arf and saturated numerical semigroup (for instance see [6, 8, 14, 16]).

2. Main results

Theorem 2.1 *[14] Let M be a numerical semigroup and $d_M(a) = gcd\{x \in M : x \leq a\}$ Then the following conditions are equalities:*

$$(1)\, M \;\; is \;\; saturated,$$

$$(2)\, a + d_M(a) \in M \;\; for \;\; all \;\; a \in M \backslash \{0\},$$

$$(3)\, a + k.d_M(a) \in M \;\; for \;\; all \;\; a \in M \backslash \{0\} \;\; and \;\; k \in \mathbb{N}.$$

Theorem 2.2 *[6] Let M be a numerical semigroup with $m(M) = 3$ and conductor C. Then, M is saturated if M is one of following numerical semigroups:*

$$(1) M =< 3, C + 1, C + 2 > \;\; for \;\; C \equiv 0(3).$$

$$(2) M =< 3, C, C + 2 > \;\; for \;\; C \equiv 2(3).$$

Theorem 2.3 *Let M be a numerical semigroup with $m(M) = 3$ and conductor $C > 3$ such that $M =< 3, C + 1, C + 2 >$ for $C \equiv 0(3)$. Then, M has maximal length.*

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. Then, we write

$$M =< 3, C + 1, C + 2 >= \{0, 3, 6, 9, ..., C - 3, C, \rightarrow ...\}.$$

Thus, we find

$$n(M) = Card(\{0, 1, 2, ..., f(M)\} \cap M)$$
$$= Card(\{0, 1, 2, ..., C - 1\} \cap M) = Card(\{0, 3, 6, 9, ..., C - 3\}) = \frac{C}{3}.$$

On the other hand, we obtain

$$M_1 = \{a \in M : a \geq 3\} = \{3, 6, 9, ..., C - 3, C, \rightarrow ...\}$$

and

$$M(1) = \{k \in \mathbb{N} : k + M_1 \subseteq M\} = \{0, 3, 6, 9, ..., C - 3, C - 2, C - 1, \rightarrow ...\}.$$

So, we have

$$t(M) = Card(M(1) \backslash M) = Card(\{C - 1, C - 2\}) = 2.$$

Therefore, we obtain that M has maximal length from

$$n(M)(t(M) + 1) = \frac{C}{3}(2 + 1) = C = f(M) + 1.$$

□

Theorem 2.4 *Let M be a numerical semigroup with $m(M) = 3$ and conductor $C > 5$ such that $M =< 3, C, C+2 >$ for $C \equiv 2(3)$. Then, M has almost maximal length.*

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor $C > 5$ such that $M =< 3, C, C+2 >$ for $C \equiv 2(3)$. Then, we write

$$M =< 3, C, C+2 >= \{0, 3, 6, 9, ..., C-2, C, \rightarrow ...\}.$$

Thus, we obtain

$$\begin{aligned} n(M) &= Card(\{0, 1, 2, ..., f(M)\} \cap M) \\ &= Card(\{0, 1, 2, ..., C-1\} \cap M) = Card(\{0, 3, 6, 9, ..., C-2\}) = \frac{C+1}{3}. \end{aligned}$$

On the other hand, we find

$$M_1 = \{a \in M : a \geq 3\} = \{3, 6, 9, ..., C-2, C, \rightarrow ...\}$$

and

$$M(1) = \{k \in \mathbb{N} : k + M_1 \subseteq M\} = \{0, 3, 6, 9, ..., C-2, C, C+1, \rightarrow ...\}.$$

So,

$$t(M) = Card(M(1) \backslash M) = Card(\{C-3, C-1\}) = 2.$$

Therefore, we obtain that M has almost maximal length since

$$n(M)(t(M)+1) = C+1 = f(M)+2.$$

□

Theorem 2.5 *Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. Then, we have*

$$\frac{M}{2} = \begin{cases} < 3, \frac{C+1}{2}, \frac{C+5}{2} > & ;\textit{if } C \textit{ is odd number} \\ < 3, \frac{C+2}{2}, \frac{C+4}{2} > & ;\textit{if } C \textit{ is even number.} \end{cases}$$

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. Then,

(1)If C is odd number then $\frac{C+1}{2}$ and $\frac{C+5}{2}$ are even integers. So, we obtain

$$\begin{aligned} x \in < 3, \frac{C+1}{2}, \frac{C+5}{2} > &\Leftrightarrow \exists y_1, y_2, y_3 \in \mathbb{N} \ni x = 3y_1 + (\frac{C+1}{2})y_2 + (\frac{C+5}{2})y_3 \\ &\Leftrightarrow 2x = 3(2y_1) + (C+1)y_2 + (C+5)y_3 \\ &\Leftrightarrow 2x = 3(2y_1 + y_3) + (C+1)y_2 + (C+2)y_3 \\ &\Leftrightarrow 2x \in M \\ &\Leftrightarrow x \in \frac{M}{2}. \end{aligned}$$

(2) If C is an even number then we find

$$
\begin{aligned}
u \in < 3, \frac{C+2}{2}, \frac{C+4}{2} > &\Leftrightarrow \exists k_1, k_2, k_3 \in \mathbb{N} \ni u = 3k_1 + (\frac{C+2}{2})k_2 + (\frac{C+4}{2})k_3 \\
&\Leftrightarrow 2u = 3(2k_1) + (C+2)k_2 + (C+4)k_3 \\
&\Leftrightarrow 2u = 3(2k_1 + k_3) + (C+2)k_2 + (C+1)k_3 \\
&\Leftrightarrow 2u \in M \\
&\Leftrightarrow u \in \frac{M}{2}.
\end{aligned}
$$

□

Theorem 2.6 *Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C, C+2 >$ for $C \equiv 2(3)$. Then, we have*

$$
\frac{M}{2} = \begin{cases} < 3, \frac{C+3}{2}, \frac{C+5}{2} > & ; \textit{if } C \textit{ is odd number} \\ < 3, \frac{C}{2}, \frac{C+2}{2} > & ; \textit{if } C \textit{ is even number.} \end{cases}
$$

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C, C+2 >$ for $C \equiv 2(3)$. Then,

(1)If C is odd number then $\frac{C+3}{2}$ and $\frac{C+5}{2}$ are even integers. Then, we have

$$
\begin{aligned}
a \in < 3, \frac{C+3}{2}, \frac{C+5}{2} > &\Leftrightarrow \exists n_1, n_2, n_3 \in \mathbb{N} \ni a = 3n_1 + (\frac{C+3}{2})n_2 + (\frac{C+5}{2})n_3 \\
&\Leftrightarrow 2a = 3(2n_1) + (C+3)n_2 + (C+5)n_3 \\
&\Leftrightarrow 2a = 3(2n_1 + n_2 + n_3) + (C)n_2 + (C+2)n_3 \\
&\Leftrightarrow 2a \in M \\
&\Leftrightarrow a \in \frac{M}{2}.
\end{aligned}
$$

(2) If C is an even number then we find

$$
\begin{aligned}
y \in < 3, \frac{C}{2}, \frac{C+2}{2} > &\Leftrightarrow \exists b_1, b_2, b_3 \in \mathbb{N} \ni y = 3b_1 + (\frac{C}{2})b_2 + (\frac{C+2}{2})b_3 \\
&\Leftrightarrow 2y = 3(2b_1) + (C)b_2 + (C+2)b_3 \\
&\Leftrightarrow 2y \in M \\
&\Leftrightarrow y \in \frac{M}{2}.
\end{aligned}
$$

□

Corollary 2.7 *The numerical semigroups $\frac{M}{2}$ given by Theorem 2.5 and Theorem 2.6 are saturated. It is clear from [5].*

Proposition 2.8 *Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. In this case, we have*

$$(1)\, n(\frac{M}{2}) = \frac{C+3}{6} \text{ and } f(\frac{M}{2}) = \frac{C-1}{2} \text{ for } \frac{M}{2} =< 3, \frac{C+1}{2}, \frac{C+5}{2} > \text{ if } C \text{ is odd number.}$$

$$(2)\, n(\frac{M}{2}) = \frac{C}{6} \text{ and } f(\frac{M}{2}) = \frac{C-2}{2} \text{ for } \frac{M}{2} =< 3, \frac{C+2}{2}, \frac{C+4}{2} > \text{ if } C \text{ is even number.}$$

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. In this case,
(1) If C is odd number then

$$\frac{M}{2} =< 3, \frac{C+1}{2}, \frac{C+5}{2} >= \{0, 3, 6, 9, ..., \frac{C-3}{2}, \frac{C+1}{2}, \rightarrow ...\}.$$

Thus we find

$$f(\frac{M}{2}) = \frac{C+1}{2} - 1 = \frac{C-1}{2}$$

and

$$\begin{aligned} n(\frac{M}{2}) &= Card(\{0, 1, 2, ..., f(\frac{M}{2})\} \cap \frac{M}{2}) \\ &= Card(\{0, 1, 2, ..., \frac{C-1}{2}\} \cap \{0, 3, 6, 9, ..., \frac{C-3}{2}, \frac{C+1}{2}, \rightarrow ...\}) \\ &= Card(\{0, 3, 6, 9, ..., \frac{C-3}{2}\}) = \frac{C+3}{6}. \end{aligned}$$

(2) If C is even number then $\frac{M}{2} =< 3, \frac{C+2}{2}, \frac{C+4}{2} >= \{0, 3, 6, 9, ..., \frac{C}{2}, \rightarrow ...\}$.
Thus we obtain

$$f(\frac{M}{2}) = \frac{C-2}{2}$$

and

$$\begin{aligned} n(\frac{M}{2}) &= Card(\{0, 1, 2, ..., f(\frac{M}{2})\} \cap \frac{M}{2}) \\ &= Card(\{0, 1, 2, ..., \frac{C-2}{2}\} \cap \{0, 3, 6, 9, ..., \frac{C-6}{2}, \frac{C}{2}, \rightarrow ...\}) \\ &= Card(\{0, 3, 6, 9, ..., \frac{C-6}{2}\}) = \frac{C}{6}. \end{aligned}$$

□

Theorem 2.9 *Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 0(3)$. In this case,*

$$(1) \text{ if } C \text{ is odd number then } \frac{M}{2} =< 3, \frac{C+1}{2}, \frac{C+5}{2} > \text{ has almost maximal length;}$$

$$(2) \text{ if } C \text{ is even number then } \frac{M}{2} =< 3, \frac{C+2}{2}, \frac{C+4}{2} > \text{ has maximal length.}$$

Proof Let M be a numerical semigroup with $m(M)=3$ and conductor C such that $M=<3,C+1,C+2>$ for $C\equiv 0(3)$. Thus,

(1) If C is odd number then $\frac{M}{2}=<3,\frac{C+1}{2},\frac{C+5}{2}>$, $n(\frac{M}{2})=\frac{C+3}{6}$ and $f(\frac{M}{2})=\frac{C-1}{2}$.

In this case, $\frac{M}{2}=<3,\frac{C+1}{2},\frac{C+5}{2}>$ has almost maximal length since

$$n(\frac{M}{2})(t(\frac{M}{2})+1)=\frac{C+3}{2}=\frac{C-1}{2}+2=f(\frac{M}{2})+2.$$

(2) If C is even number then $\frac{M}{2}=<3,\frac{C+2}{2},\frac{C+4}{2}>$, $n(\frac{M}{2})=\frac{C}{6}$ and $f(\frac{M}{2})=\frac{C-2}{2}$.

Thus, $\frac{M}{2}=<3,\frac{C+2}{2},\frac{C+4}{2}>$ has maximal length since

$$n(\frac{M}{2})(t(\frac{M}{2})+1)=\frac{C}{2}=\frac{C-2}{2}+1=f(\frac{M}{2})+1.$$

□

Proposition 2.10 *Let M be a numerical semigroup with $m(M)=3$ and conductor C such that $M=<3,C,C+2>$ for $C\equiv 2(3)$. In this case, we have*

$$(1)\, n(\frac{M}{2})=\frac{C+1}{6} \text{ and } f(\frac{M}{2})=\frac{C-1}{2} \text{ for } \frac{M}{2}=<3,\frac{C+3}{2},\frac{C+5}{2}> \text{ if } C \text{ is odd number.}$$

$$(2)\, n(\frac{M}{2})=\frac{C-2}{6} \text{ and } f(\frac{M}{2})=\frac{C-4}{2} \text{ for } \frac{M}{2}=<3,\frac{C}{2},\frac{C+2}{2}> \text{ if } C \text{ is even number.}$$

Proof Let M be a numerical semigroup with $m(M)=3$ and conductor C such that $M=<3,C,C+2>$ for $C\equiv 2(3)$.

(1) If C is odd number then

$$\frac{M}{2}=<3,\frac{C+3}{2},\frac{C+5}{2}>=\{0,3,6,9,...,\frac{C-5}{2},\frac{C+1}{2},\rightarrow...\}.$$

Thus we find that

$$f(\frac{M}{2})=\frac{C-1}{2}$$

and

$$\begin{aligned}n(\frac{M}{2})&=Card(\{0,1,2,...,f(\frac{M}{2})\}\cap\frac{M}{2})\\&=Card(\{0,1,2,...,\frac{C-1}{2}\}\cap\{0,3,6,9,...,\frac{C-5}{2},\frac{C+1}{2},\rightarrow...\})\\&=Card(\{0,3,6,9,...,\frac{C-5}{2}\})=\frac{C+1}{6}.\end{aligned}$$

(2) If C is even number then

$$\frac{M}{2}=<3,\frac{C}{2},\frac{C+2}{2}>=\{0,3,6,9,...,\frac{C-8}{2},\frac{C-2}{2},\rightarrow...\}.$$

Thus we find

$$f(\frac{M}{2}) = \frac{C-4}{2}$$

and

$$\begin{aligned} n(\frac{M}{2}) &= Card(\{0,1,2,...,f(\frac{M}{2})\} \cap \frac{M}{2}) \\ &= Card(\{0,1,2,...,\frac{C-4}{2}\} \cap \{0,3,6,9,...,\frac{C-8}{2},\frac{C-2}{2},\rightarrow ...\}) \\ &= Card(\{0,3,6,9,...,\frac{C-8}{2}\}) = \frac{C-2}{6}. \end{aligned}$$

□

Theorem 2.11 *Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 2(3)$. In this case, the numerical semigroup $\frac{M}{2}$ has maximal length where*

$$\frac{M}{2} = \begin{cases} < 3, \frac{C+3}{2}, \frac{C+5}{2} > & ; \textit{if } C \textit{ is odd number} \\ < 3, \frac{C}{2}, \frac{C+2}{2} > & ; \textit{if } C \textit{ is even number.} \end{cases}$$

Proof Let M be a numerical semigroup with $m(M) = 3$ and conductor C such that $M =< 3, C+1, C+2 >$ for $C \equiv 2(3)$. Thus,

(1) If C is odd number then $\frac{M}{2} =< 3, \frac{C+3}{2}, \frac{C+5}{2} >$, $n(\frac{M}{2}) = \frac{C+1}{6}$ and $f(\frac{M}{2}) = \frac{C-1}{2}$. In this case, $\frac{M}{2} =< 3, \frac{C+3}{2}, \frac{C+5}{2} >$ has maximal length since

$$n(\frac{M}{2})(t(\frac{M}{2})+1) = \frac{C+1}{2} = \frac{C-1}{2} + 1 = f(\frac{M}{2}) + 1.$$

(2) If C is even number then $\frac{M}{2} =< 3, \frac{C}{2}, \frac{C+2}{2} >$, $n(\frac{M}{2}) = \frac{C-2}{6}$ and $f(\frac{M}{2}) = \frac{C-4}{2}$. In this case, $\frac{M}{2} =< 3, \frac{C}{2}, \frac{C+2}{2} >$ has maximal length since

$$n(\frac{M}{2})(t(\frac{M}{2})+1) = \frac{C-2}{2} = \frac{C-4}{2} + 1 = f(\frac{M}{2}) + 1.$$

□

Example 2.12 *Let M be a numerical semigroup with $m(M) = 3$ and conductor $C = 9$ such that $M =< 3, 10, 11 >= \{0, 3, 6, 9, \rightarrow ...\}$. Then $f(M) = 8, n(M) = 3$ and*

$$\frac{M}{2} = \{x \in \mathbb{N} : 2x \in M\} = \{0, 3, 5, \rightarrow ...\} =< 3, 5, 7 > .$$

On the other hand, the type of M is $t(M) = 2$:

$$M_1 = \{a \in M : a \geq 3\} = \{3, 6, 9, \rightarrow ...\}$$

and

$$M(1) = \{k \in \mathbb{N} : k + M_1 \subseteq M\} = \{0, 3, 6, \rightarrow ...\}.$$

Thus,

$$t(M) = Card(M(1)\backslash M) = Card(\{7,8\}) = 2.$$

Therefore, we obtain that M *has a maximal length from*

$$n(M)(t(M)+1) = 9 = f(M)+1.$$

On the other hand, we write

$$\frac{M}{2} = \{x \in \mathbb{N} : 2x \in M\} = \{0,3,5,\rightarrow ...\} =< 3,5,7 >, n(\frac{M}{2}) = \frac{C+3}{6} = 2,$$

$$f(\frac{M}{2}) = \frac{C-1}{2} = 4$$

and the type of $\frac{M}{2}$ *is* $t(\frac{M}{2}) = 2:$

$$A = \{a \in \frac{M}{2} : a \geq 3\} = \{3,5,\rightarrow ...\}$$

and

$$A(1) = \{k \in \mathbb{N} : k + A \subseteq \frac{M}{2}\} = \{0,2,3,\rightarrow ...\}.$$

Thus, we find

$$t(\frac{M}{2}) = Card(A(1)\backslash \frac{M}{2}) = Card(\{2,4\}) = 2.$$

Therefore, we obtain that $\frac{M}{2}$ *has almost maximal length from*

$$n\left(\frac{M}{2}\right)\left(t(\frac{M}{2})+1\right) = 6 = f(\frac{M}{2})+2.$$

Example 2.13 *Let* M *be a numerical semigroup with* $m(M) = 3$ *and conductor* $C = 8$ *such that* $M =< 3,8,10 >= \{0,3,6,8,\rightarrow ...\}$. *Then* $f(M) = 7, n(M) = 3$ *and*

$$\frac{M}{2} = \{x \in \mathbb{N} : 2x \in M\} = \{0,3,\rightarrow ...\} =< 3,4,5 > .$$

On the other hand,the type of M *is* $t(M) = 2:$

$$M_1 = \{a \in M : a \geq 3\} = \{3,6,8,\rightarrow ...\}$$

and

$$M(1) = \{k \in \mathbb{N} : k + M_1 \subseteq M\} = \{0,3,5,\rightarrow ...\}.$$

Thus, we obtain

$$t(M) = Card(M(1)\backslash M) = Card(\{5,7\}) = 2.$$

Therefore, we obtain that M *has an almost maximal length from*

$$n(M)(t(M)+1) = 9 = f(M)+2.$$

On the other hand, we write

$$\frac{M}{2} = \{x \in \mathbb{N} : 2x \in M\} = \{0, 3, \rightarrow ...\} =< 3, 4, 5 >, n(\frac{M}{2}) = \frac{C-2}{6} = 1,$$

$$f(\frac{M}{2}) = \frac{C-4}{2} = 2$$

and the type of $\frac{M}{2}$ *is* $t(\frac{M}{2}) = 2$:

$$B = \{a \in \frac{M}{2} : a \geq 3\} = \{3, \rightarrow ...\}$$

and

$$B(1) = \{k \in \mathbb{N} : k + B \subseteq \frac{M}{2}\} = \{0, 1, \rightarrow ...\}.$$

Thus, we find

$$t(\frac{M}{2}) = Card(B(1) \backslash \frac{M}{2}) = Card(\{1, 2\}) = 2.$$

Therefore, we obtain that $\frac{M}{2}$ *has maximal length from*

$$n\left(\frac{M}{2}\right)\left(t(\frac{M}{2}) + 1\right) = 3 = f(\frac{M}{2}) + 1.$$

References

[1] Arf C. Une interpretation algebrique de la suite de multiplicité d'une branche algébrique. Proceedings of the London Mathematical Society in French 1949; 20: 256-287 (in French).

[2] Barucci V, DE Dobbs, Fontana MT. Maximality properties in numerical semigroups and applications to one dimensional analytically irreducible local domains. Memoirs of the American Mathematical Society 1997; 598: 95.

[3] Brown CW, Curtis F. Numerical semigroups of maximal and almost maximal length. Semigroup Forum 1991; 42: 219-235.

[4] Delgado M, Farrán JI, García-Sánchez PA, Llena D. On the weight hierarchy of codes coming from semigroups with two generators. IEEE Transactions on Information Theory 2014; 60: 282-295.

[5] Dobbs DE, Smith HJ. Numerical semigroups whose fractions are of maximal embedding dimension. Semigroup Forum 2011; 82: 412-422.

[6] İlhan S, Çelik A. L-Sequences of saturated numerical semigroups with multiplicity. Journal of Semigroup Theory and Application 2017; ID 6.

[7] İlhan S, Süer M. Conditions of numerical semigroups having maximal or almost maximal length. International Journal of Physical Sciences 2012; 7: 2549-2553.

[8] İlhan S, Karakaş Hİ. Arf numerical semigroups. Turkish Journal of Mathematics 2017; 41: 1448-1457.

[9] Lipman J. Stable Ideals and Arf Rings. American Journal of Mathematics 1971; 93 (3): 649-685.

[10] Marco D. Type Sequences of numerical semigroups. Semigroup Forum 1998; 56: 1-31.

[11] Ramírez Alfonsín JL. The Diophantine Frobenius Problem. Oxford University Press 2005.

[12] Rosales RC. On numerical semigroups. Semigroup Forum 1996; 52: 307-318.

[13] Rosales JC. García-Sánchez PA. Numerical semigroups. New York, Springer, 2009.

[14] Rosales JC, García-Sánchez PA, Garcia-Garcia JI, Branco MB. Saturated numerical semigroups. Houston Journal of Mathematics 2004; 30 (2): 321-330.

[15] Rosales JC, Vasco P. The Frobenius variety of the saturated numerical semigroups. Houston Journal of Mathematics 2010; 36 (2): 357-365.

[16] Süer M, İlhan S. On a family saturated numerical semigroups. Turkish Journal of Mathematics 2017; 41: 132-137.