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Metallic-like structures and metallic-like maps

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Abstract: The metallic-like (a,b)-manifold is a manifold endowed with a polynomial structure of second degree which unifies the almost product, complex structures and includes metallic structures. We introduce the metallic-like maps between metallic-like (a,b)-manifolds and we give a criterion for the nonconstancy of these maps. We prove that an almost contact structure on a Riemannian manifold induces a metallic-like (a,b)-structure and we give an example of a nonconstant metallic-like endomorphism of a particular almost contact manifold.

Key words: Contact metric manifold, metallic manifold

1. Introduction

This paper is concerned about metallic-like (a,b)-manifolds and metallic-like maps. One of the main results is a criterion for nonconstancy of the metallic-like maps between two different metallic-like structures on the same base manifold M (these maps are called endomorphisms of M). On the other hand, we provide examples of several classes of metallic-like (a,b)-structures for which all metallic-like endomorphisms are constant.

Another main result states that an almost contact structure on a manifold induces a metallic-like (a,b)-structure with $\Delta = a^2+4b>0$, also giving an example of nonconstant metallic-like endomorphism in the case of a particular almost contact manifold.

To fit these results into context, we defined metallic-like structures as a particular case of polynomial structures on manifolds, which were introduced and studied by Goldberg and Yano in [7]. A polynomial structure F of degree d on a $\mathbf{C}^\infty$ manifold M is, [7], a $(1,1)$-tensor field F satisfying a certain polynomial equation of d degree. Integrability conditions for polynomial structures, involving the Nijenhuis torsion of the structural endomorphism F, are given in [20], under the assumption that the polynomial equation has only simple roots.

Some classes of polynomial structures of the second degree are provided by the almost product structures, (i.e. $F^2 = I_d$), as well as by the almost complex structures, (i.e. $F^2 = -I_d$) for $2n$-dimensional manifolds. Also, an almost contact structure, [3], [5], is an example of the polynomial structure of the third degree since its structural endomorphism satisfies $F^3+F=0$, for a $2n+1$-dimensional manifold. Nevertheless, the metallic structures, (i.e., $F^2 = pF+qI_d$ with nonzero positive integers p, q), are substantially more general polynomial structures of the second degree and they were intensively studied in the last decade, [8], [10], [11], [12], [16], [17]. Recently, a new type of polynomial structures called $f_{a,b}(3,2,1)$-structures (i.e. $F^3 - aF^2 - bF = 0$, with

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a, b real numbers and $b \neq 0$) are introduced in [9]. These structures generalize some polynomial structures of degrees 2 and 3, like metallic structures or almost contact structures. It is an interesting topic to unify these kinds of polynomial structures (product, complex, and metallic), [6].

In this paper, we recall the extension of the metallic structures proposed by us in [14]. We defined the notion of metallic-like (a, b)-structure with two real numbers a, b dropping the constraint that a and b must be integers, only imposing that $\Delta = a^2 + 4b$ does not vanish. Our theory generalizes in a natural way the almost complex and almost product structures on manifolds as well as includes the metallic structure. In [4] the author proved that any almost contact structure on a manifold determines a metallic structure. We obtain a similar statement, using a new argument, for metallic-like structures induced by almost contact structures.

In the last years the metallic maps (similar to holomorphic-like maps) between metallic manifolds were investigated by some authors, [1], [2], [18], [19], which provided conditions for the constancy of some certain metallic maps.

The paper is organized as follows. The notion of metallic-like (a, b)-structures on a Riemannian manifold and some properties of them are given in the second section of the paper. In the third section, we define multiple metallic-like structures on a manifold M endowed with a metallic-like (a, b)-structure, following an idea from [15]. The endomorphisms of M which are metallic-like maps between different metallic-like structures on M are investigated in the fourth section. We obtain one of the main results of this paper, a criterion for the nonconstancy of these endomorphisms (Proposition 4.4). For several particular metallic-like (a, b)-structures we prove that all metallic-like endomorphisms are constant (Propositions 4.6, 4.8, 4.10).

Finally, in the last section, we prove another main result, that an almost contact structure on a manifold induces a metallic-like (a, b)-structure with $a^2 + 4b > 0$ (Theorem 5.1). Also, we give an example of nonconstant metallic-like endomorphism of a certain almost contact manifold.

2. Metallic-like structures on manifolds

Let M be an n-dimensional $\mathbf{C}^\infty$-manifold and a, b two real numbers with $\Delta = a^2 + 4b \neq 0$.

We call, [14], a *metallic-like* (a, b)*-structure* on M a polynomial structure of second degree given by a $(1, 1)$-tensor field φ which satisfies the equation,

$$\varphi^2 - a \cdot \varphi - b \cdot I_d = 0, \tag{2.1}$$

where I_d is the identity on the vector fields space $\Gamma(TM)$. In this case the pair (M, φ) will be called a metallic-like (a, b)-manifold. Following an idea from [13], we obtained the following result.

Proposition 2.1 *[14] Let (M, φ) be an n- dimensional metallic-like (a, b)-manifold. If $\Delta = a^2 + 4b < 0$, then the dimension n is an even number.*

Remark 2.2 *For $a = 0$, $b = 1$, φ is an almost product structure. If $a = 0$, $b = -1$, then φ is an almost complex structure. For a, b nonzero positive integers the endomorphism φ is called a metallic structure, [10]. Some authors, [8], [4], consider to be a metallic structure those endomorphisms φ with a, b nonzero integers and the equation $x^2 - ax - b = 0$ has a positive irrational root. Examples of metallic structures are: golden structure ($a = 1$, $b = 1$), silver structure ($a = 2$, $b = 1$), bronze structure ($a = 3$, $b = 1$), nickel structure ($a = 1$, $b = 3$), copper structure ($a = 1$, $b = 2$).*

The main properties of metallic-like (a, b)-structures are given in the following propositions and they are easy to be proved by direct computation.

Proposition 2.3 *[14] To every metallic-like (a,b)-structure φ on M, given by (2.1), we can associate another polynomial structure*

$$J = \frac{2}{\sqrt{|\Delta|}} \cdot \varphi - \frac{a}{\sqrt{|\Delta|}} \cdot I_d. \tag{2.2}$$

If $\Delta > 0$, then $J^2 = I_d$ and J is called the almost product structure associated to φ.

If $\Delta < 0$, then $J^2 = -I_d$ and J is called the almost complex structure associated to φ.

Now, for the above real numbers a, b, let J be a polynomial structure of second degree on M with $J^2 = \epsilon^2 I_d$, where $\epsilon = \sqrt{\frac{\Delta}{|\Delta|}}$. Then we can associate to J the following polynomial structure:

$$\varphi = \frac{a}{2} \cdot I_d + \frac{\sqrt{|\Delta|}}{2} \cdot J. \tag{2.3}$$

By direct computation, we obtain

$$\varphi^2 - a\varphi - bI_d = -\frac{\Delta}{4} \cdot I_d + \frac{|\Delta|}{4} \cdot \epsilon^2 I_d.$$

The above relation shows that:

Proposition 2.4 *[14] Let M be an n-dimensional C^∞ manifold. Then,*

(i) For any real numbers a, b such that $a^2 + 4b > 0$, every almost product structure J on M induces a metallic-like (a,b)-structure on M, given by (2.3).

(ii)For any real numbers a, b such that $a^2 + 4b < 0$, every almost complex structure J on M induces a metallic-like (a,b)-structure on M, given by (2.3).

Let (M, g) be a Riemannian manifold endowed with the metallic-like (a, b)-structure φ. We say that φ is compatible with the metric g and that M is a *metallic-like* (a,b)*-Riemannian manifold* (denoted by (M, g, φ)), if

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{2.4}$$

for every $X, Y \in \Gamma(TM)$. An equivalent condition is

$$g(\varphi X, \varphi Y) = a \cdot g(X, \varphi Y) + b \cdot g(X, Y).$$

Let $\sigma_{a,b}$, $\sigma'_{a,b} = a - \sigma_{a,b}$ be the roots of the equation $x^2 - ax - b = 0$. Analogous to the metallic projections defined in [10], we can consider the projections

$$p_\varphi = \frac{1}{\sqrt{\Delta}} \left(\sigma_{a,b} I_d - \varphi\right), \quad p'_\varphi = \frac{1}{\sqrt{\Delta}} \left(-\sigma'_{a,b} I_d + \varphi\right) \tag{2.5}$$

and two complementary distributions $\mathcal{D}_\varphi = Ker(p_\varphi)$, $\mathcal{D}'_\varphi = Ker(p'_\varphi)$.

Proposition 2.5 *Let* (M, g, φ) *be a metallic-like* (a,b)*- Riemannian manifold. The metallic-like distributions* $\mathcal{D}_\varphi$ *and* $\mathcal{D'}_\varphi$ *are orthogonal with respect to the compatible metric* g*.*

Proof For every $X \in \mathcal{D}_\varphi$ and $Y \in \mathcal{D'}_\varphi$, we have $\varphi X = \sigma_{a,b}X$ and $\varphi Y = \sigma'_{a,b}Y$.

Relation (2.4) implies

$$(\sigma_{a,b} - \sigma'_{a,b})g(X,Y) = 0,$$

which is equivalent to $\sqrt{\Delta}g(X,Y) = 0$ and, since $\Delta \neq 0$, we obtain $g(X,Y) = 0$. □

For a metallic-like (a,b)-Riemannian manifold (M, g, φ), let J be the associated almost product or almost complex structure from (2.2). We obtain that J is also g-symmetric, namely

$$g(JX, Y) = g(X, JY), \quad \forall X, Y \in \Gamma(TM). \tag{2.6}$$

The integrability of almost product or complex structure J is usually expressed by the vanishing of the Nijenhuis tensor N_J, this is

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y],$$

which expresses the involutivity of eigenbundles of J. From relation (2.3) it is easy to see that the eigenbundles of φ are exactly the eigenbundles of the associated structure J and they are exactly the metallic-like distributions $\mathcal{D}_\varphi$ and $\mathcal{D'}_\varphi$.

Moreover, considering the Nijenhuis tensor of φ

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X, Y],$$

we obtain the following link between N_φ and N_J:

$$N_\varphi = \frac{|\Delta|}{4}N_J + \frac{\Delta}{4}\left(I_d - \frac{|\Delta|}{\Delta}J^2\right) = \frac{|\Delta|}{4}N_J. \tag{2.7}$$

Definition 2.6 *The metallic-like* (a,b)*-structure* φ *is called integrable if* $N_\varphi = 0$*.*

A direct consequence is that

Proposition 2.7 *The metallic-like* (a,b)*-structure* φ *is integrable if and only if the associated structure* J *given by* (2.2) *is integrable. In this case the eigenbundles of* φ *are involutive.*

3. Multiple metallic-like structures

In this section, we will referee to the duality between metallic-like structures and almost product or complex structures, mentioned in Propositions 2.3, 2.4.

Let (M, φ) be a metallic-like (a,b)-manifold and J given by (2.2), the almost product or complex structure induced by φ on M.

We fix two real numbers a', b' with $\Delta' = a'^2 + 4b' \neq 0$ and $\Delta \cdot \Delta' > 0$.

According to Proposition 2.4, we can introduce on the (a,b)-metallic-like manifold M a metallic-like (a',b')-structure φ' by relation

$$\varphi' = \frac{a'}{2} \cdot I_d + \frac{\sqrt{|\Delta'|}}{2} \cdot J. \tag{3.1}$$

If (M,g,φ) is a Riemannian metallic-like (a,b)-manifold, then the above relation and (2.6) show that the Riemannian metric g is φ'-compatible,

$$g(\varphi' X, Y) = g(X, \varphi' Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.2}$$

By direct computation, from relations (2.2) and (3.1) we obtain that

Proposition 3.1 *The link between the initial metallic-like structure and the new one is*

$$\varphi' = \frac{1}{2}\left(a' - \frac{\sqrt{|\Delta'|}}{\sqrt{|\Delta|}}\right) \cdot I_d + \frac{\sqrt{|\Delta'|}}{\sqrt{|\Delta|}} \cdot \varphi. \tag{3.3}$$

Obviously, using now the relation (2.2) for the metallic-like structure φ', we obtain that this new metallic-like structure induces the same almost product J on M.

We can also see that

Proposition 3.2 *For a metallic-like manifold (M,φ) with the almost product or complex associated structure J, any metallic-like structure induced by J on M is integrable if and only if the initial metallic-like structure is integrable.*

Proof Using relation (2.7) for the Nijenhuis tensors N_φ and $N_{\varphi'}$, respectively, we have

$$N_\varphi = \frac{\Delta}{\Delta'} N_{\varphi'},$$

which proves the statement. □

Considering now the cases of Golden, silver, bronze, and nickel structures, we can see that:

Proposition 3.3 *Every Golden manifold also carries a silver, a bronze, and a nickel structure. If the Golden structure is integrable, the induced silver, bronze, and nickel structures are integrable, too.*

Proof Let (M,g,φ) be a Golden manifold. Then φ is a metallic-like $(1,1)$-structure and $\Delta = 5$. According to Proposition 3.1, the following polynomial structures are induced by φ:

$\varphi_s = \frac{1}{5}\left[(5-\sqrt{10})I_d + 2\sqrt{10}\varphi\right]$ is a silver structure;

$\varphi_b = \frac{1}{2\sqrt{5}}\left[(3-\sqrt{13})I_d + \sqrt{13}\varphi\right]$ is a bronze structure;

$\varphi_n = \frac{1}{2\sqrt{5}}\left[(1-\sqrt{13})I_d + \sqrt{13}\varphi\right]$ is a nickel structure. The statement of Proposition 3.2 ends the proof. □

Remark 3.4 *The relation (3.3) also proves that if φ is a bronze structure on a manifold M, then $\varphi' = \varphi - I_d$ is a nickel structure on M.*

4. Metallic-like maps between metallic-like manifolds

Let φ_1, φ_2 be metallic-like structures on Riemannian manifolds (M_1, g_1), (M_2, g_2). Inspired by [18], we can give the following definition:

Definition 4.1 *A differentiable map $F : M_1 \to M_2$ is called (φ_1, φ_2)-metallic-like map between metallic-like manifolds (M_1, φ_1) and (M_2, φ_2) if $F_*\varphi_1 = \varphi_2 F_*$, where F_* is the tangent map of F.*

In this section we investigate the properties of a differentiable endomorphism $F : M \to M$ when the manifold M is endowed with a metallic-like (a, b)-structure φ. Let J be the associated almost product or complex structure given by (2.2) and φ' another metallic-like (a', b') structure induced by J, introduced in the previous section.

Proposition 4.2 *The following assertions are equivalent:*

(i) $F_*\varphi = \varphi F_*$; *(ii)* $F_*J = JF_*$; *(iii)* $F_*\varphi' = \varphi' F_*$

Proof Using relations (2.2) we prove the equivalence $(i) \Leftrightarrow (ii)$. Relation (3.1) proves the equivalence $(ii) \Leftrightarrow (iii)$, since the equivalence $(iii \Leftrightarrow (i)$ results from (3.3). □

Remark 4.3 *The above proposition says that a (φ, φ)-metallic-like endomorphism F of a metallic-like manifold (M, φ) is also a (J, J)-metallic-like map and (φ', φ')-metallic-like map for any another metallic-like (a', b') structure φ' induced by the associated almost product or complex structure J.*

Proposition 4.4 *The (φ, φ') metallic-like endomorphism F between metallic-like (a, b)-structure φ and metallic-like (a', b')-structure φ' could be nonconstant if*

$$(a - a')^2 b - a(a - a')(b - b') - (b - b')^2 = 0. \tag{4.1}$$

Proof The condition $F_*\varphi = \varphi' F_*$ implies $\varphi' F_*\varphi = \varphi'^2 F_*$, hence it results for every vector field X on M:

$$F_*\varphi^2 X = \varphi'^2 F_* X$$

$$aF_*\varphi X + bF_* X = a'\varphi' F_* X + b' F_* X$$

$$(a - a')F_*\varphi X + (b - b')F_* X = 0. \tag{4.2}$$

Applying again φ', we obtain

$$(a - a')F_*\varphi^2 X + (b - b')F_*\varphi X = 0,$$

$$[(a - a')a + (b - b')F_*\varphi X + (a - a')bF_* X = 0. \tag{4.3}$$

Solving the system (4.2), (4.3) it results

$$[(a - a')^2 b - a(a - a')(b - b') - (b - b')^2]F_* X = 0.$$

So, a sufficient condition for F to be constant is the nonvanishing of expression $(a - a')^2 b - a(a - a')(b - b') - (b - b')^2$. □

Remark 4.5 *A direct consequence of Proposition 4.4 is obtained for two different metallic-like* (a,b), (a',b')*-structures on* M, φ, φ', *with* $a=a'$ *and* $b\neq b'$ *or* $b=b'$ *and* $a\neq a'$. *Then any* (φ,φ') *metallic-like endomorphism* F *of* M *is constant.*

From this remark, we can say that

Proposition 4.6 *Let* F *be a differential endomorphism of a golden manifold* M *with the golden structure* φ_g, *and* φ_s, φ_b, φ_n *the silver, bronze and nickel structures induced by* φ_g *on* M, *respectively. If* F *is a metallic-like map between any two different structures from the mentioned above ones, then* F *is a constant map.*

Proof For F metallic-like map between φ_g and any one of the others the condition (4.1) is not satisfied since $a=a'$ and $b\neq b'$ or $a\neq a'$ and $b=b'$. By direct calculation, relation (4.1) is not verified neither for another combination between φ_s, φ_b, and φ_n. □

Remark 4.7 *The condition (4.1) is equivalent to*

$$(a-a')^2b'-a'(a-a')(b-b')-(b-b')^2=0,$$

since

$$(a-a')^2(b-b')-(a-a')(a-a')(b-b')=0,$$

and it is obviously satisfied for $a=a'$ *and* $b=b'$.

Moreover, two metallic-like (a,b), (a',b')*-structures with* $a\neq a'$ *and* $b\neq b'$ *are satisfying condition (4.1) iff equations*

$$x^2+ax-b=0, \quad x^2+a'x-b'=0,$$

have a common root (which is $\frac{b-b'}{a-a'}$, *obviously).*

Proposition 4.8 *Let* (M,J) *be an almost product manifold and* φ, φ' *two different metallic-like* (p,q), (p',q')*-structures, respectively, induced by* J, *where* p,q,p',q' *are nonzero positive integers such that at least one of the equations* $x^2-px-q=0$, $x^2-p'x-q'=0$ *has irrational roots. Then, any* (φ,φ')*- metallic-like map is constant.*

Proof According to the Remark 4.7, the condition (4.1) is not satisfied since the equations $x^2-px-q=0$ and $x^2-p'x-q'=0$ have irrational roots and $\frac{q-q'}{p-p'}$ is a rational number. □

Remark 4.9 *The metallic-like structures considered in Proposition 4.8 are metallic manifolds. So, this proposition generalizes the result of Proposition 4.6 at the case of metallic manifolds given by a metallic-like* (p,q)*-structure with* p,q *non-zero positive integers such that* $x^2-px-q=0$ *has irrational roots. This proposition also says that there could exist nonconstant* (φ,φ') *metallic-like maps only for metallic manifolds which have metallic* (p,q)*-,* (p',q')*-structures with the same associated almost product and* p^2+4q, p'^2+4q' *both perfect squares.*

Another consequence of the Remark 4.7 is that

Proposition 4.10 *Let* (M, J) *be an almost complex manifold and* (a, b), (a', b') *two different metallic-like structures* φ *and* φ', *respectively, induced by* J. *Any metallic-like map between* φ *and* φ' *is constant.*

Proof The structures φ, φ' induced by the almost complex structure J are satisfying $\varphi^2 - a\varphi - bI_d = 0$, $\varphi'^2 - a'\varphi' - b'I_d = 0$, respectively, and $\Delta = a^2 + 4b$, $\Delta' = a'^2 + 4b'$ are negative. The condition (4.1) is not satisfied since the equations $x^2 + ax - b = 0$ and $x^2 + a'x - b' = 0$ have no real roots and $\frac{b-b'}{a-a'}$ is a real number. □

The above considerations show that there could exist a nonconstant (φ, φ') metallic-like endomorphism only for metallic-like structures φ, φ' which induce almost product structure (that means $\Delta > 0$).

Proposition 4.11 *Let* φ *be a metallic like* (a, b)*-structure on a manifold* M, *with* $\Delta = a^2 + 4b > 0$, *and* J *the almost product structure associated with it. There could exist nonconstant metallic-like maps* $F : M \to M$ *between* φ *and* J *iff* $a = \pm(b - 1)$.

Proof

The associated almost product structure J is a metallic-like $(0, 1)$-structure. F is a metallic-like map if $F_*\varphi = JF_*$ and it could be nonconstant iff the condition (4.1) is verified, where $a' = 0$ and $b' = 1$.

We obtain $a^2b - a^2(b - 1) - (b - 1)^2 = 0$, so $a^2 = (b - 1)^2$, which ends the proof.

□

5. Metallic-like structures on a contact metric manifold

Let M be a $(2n+1)$-dimensional manifold and (φ, ξ, η) a *contact structure* on M. That is, φ is a tensor field of type $(1, 1)$, ξ a vector field, called the *Reeb vector field* on M, and η a 1-form on M, such that

$$\varphi^2 = -I_d + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{5.1}$$

and the $(2n + 1)$-form $\eta \wedge (d\eta)^n$ does not vanish everywhere on M.

A Riemannian metric compatible with the contact structure (φ, ξ, η) is a Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM). \tag{5.2}$$

A manifold M endowed with a contact structure and a Riemannian metric compatible with it is called a *contact metric manifold*.

There are well-known following properties which derive from the conditions (5.1), (5.2):

$$\begin{array}{ccc} (a) \quad \varphi\xi = 0, & (b) \quad \varphi^3 = -\varphi, & (c) \quad \eta \circ \varphi = 0, \\ (d) \quad \eta(X) = g(X, \xi), & (e) \quad d\eta(\xi, X) = 0, & \forall X \in \Gamma(TM), \end{array} \tag{5.3}$$

and

$$d\eta(X, Y) = \Phi(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{5.4}$$

where Φ is the *fundamental* (or *Sasaki*) $2 - form$ on M given by

$$\Phi(X, Y) = g(X, \varphi Y), \quad \forall X, Y \in \Gamma(TM). \tag{5.5}$$

We can consider the *contact distribution* $\mathcal{D}$ defined by the subspaces

$$(D)_x = \{X_x \in T_xM \quad | \quad \eta(X_x) = 0\},$$

which is the transversal distribution to the structural distribution

$$\chi_\xi = \{f\xi \quad | \quad f \in \Omega^0(M)\}.$$

Let $(M, g, \varphi, \xi, \eta)$ be a contact metric manifold. The relation (5.3) (b) implies

$$\varphi^4 = -\varphi^2,$$

so the endomorphism $\zeta = \varphi^2 : TM \to TM$ is a metallic-like $(-1,0)$-structure on M. We can see that $g(\zeta X, Y) = g(-X + \eta(X)\xi, Y) = -g(X,Y) + \eta(X)\eta(Y) = g(X, \zeta Y)$.

It follows that a contact metric manifold carries also a metallic-like $(-1,0)$-structure $\zeta = \varphi^2$ and the metric g is compatible with this structure.

According to Proposition 2.3, this metallic-like $(-1,0)$-structure, having $\Delta = 1$, induces an almost product structure on M given by:

$$J = 2\zeta + I_d = 2\varphi^2 + I_d = -I_d + 2\eta \otimes \xi. \tag{5.6}$$

Since the roots of the equation $x^2 + x = 0$ are $\sigma_{-1,0} = 0$, $\sigma'_{-1,0} = -1$, and $\Delta = 1$, the metallic-like projections (2.5) become

$$p_\varphi = -\zeta = (I_d - \eta \otimes \xi), \quad p'_\varphi = (I_d + \zeta) = \eta \otimes \xi. \tag{5.7}$$

The metallic-like distributions are

$$\mathcal{D}_\zeta = \{X \in TM | \ X = \eta(X)\xi\} = \chi_\xi,$$

$$\mathcal{D}'_\zeta = \{X \in TM | \ \eta(X)\xi = 0\} = \mathcal{D},$$

exactly the structural and the contact distributions, respectively.

It is well-known that the contact distribution is not integrable, so the metallic-like $(-1,0)$-structure ζ is not integrable. This fact implies that almost product structure J is not integrable, since the Nijenhuis tensors of φ and J are related by relation (2.7).

Now, taking into account Proposition 2.4, we obtain:

Theorem 5.1 *Let $(M, g, \varphi, \xi, \eta)$ be an almost contact metric manifold. For every real numbers a, b such that $\Delta = a^2 + 4b > 0$, the almost product structure J from (5.6) induces a nonintegrable metallic-like (a,b)-structure*

$$\psi = \frac{a - \sqrt{\Delta}}{2} I_d + \sqrt{\Delta}\eta \otimes \xi \tag{5.8}$$

Theorem 5.1 is a generalization of the fact that every almost contact metric structure induces a metallic structure, [4] (Theorem 4).

Moreover, the metallic-like structure φ^2 satisfies the condition of Proposition 4.11, since it is a metallic-like $(-1,0)$-structure. Then, there could exist a nonconstant endomorphism F of the contact manifold M such that F is a metallic-like map between φ^2 and J.

This endomorphism satisfies condition $F_* \varphi^2 = JF_*$ and, using equations (5.1) and (5.6), this condition is equivalent to $\eta \otimes F_* \xi = 2\eta \circ F_* \otimes \xi$.

Proposition 5.2 *A nonconstant endomorphism F of an almost contact manifold (M, φ, ξ, η) which satisfies $F_* \xi = 0$ and $F_* X \in \mathcal{D}$, $\forall X \in \mathcal{D}$, is a metallic-like map between φ^2 and J, the almost product structure induced by φ^2 on M.*

Example 5.3 *It is well-known, [5] the contact structure of the real $(2n+1)$-dimensional manifold $\mathbb{R}^{2n+1}$ with local coordinates $(x^1, x^2, .., x^n, y^1, y^2, ..., y^n, z)$ given by the global 1-form $\eta = dz - \sum_{i=1}^{n} y^i dx^i$, the global vector field $\xi = \frac{\partial}{\partial z}$. The metallic-like $(-1, 0)$-structure is $\zeta = -I_d + \eta \otimes \xi$ and the associated almost product structure is $J = -I_d + 2\eta \otimes \xi$.*

The structural line distribution χ_ξ is spanned by $\frac{\partial}{\partial z}$ and the contact distribution $\mathcal{D}$ is spanned by

$$\left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial z}, \ \frac{\partial}{\partial y^i} \right\}$$

Now we define a nonconstant differentiable map $F : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ locally given by

$$F(x^1, x^2, ..., x^n, y^1, .., y^n, z) = (0, 0, ..., 0, b_i^1 y^i, b_i^2 y^i, b_i^2 y_i, .., b_i^n y^i, 0), \tag{5.9}$$

where $(b_i^j)_{i,j=\overline{1,n}}$ is a real matrix. By direct computation we obtain

$$\frac{\partial F}{\partial z} = 0, \quad \frac{\delta F}{\delta x^i} = \frac{\partial F}{\partial x^i} = 0, \quad \frac{\partial F}{\partial y^i} = b_i^j \frac{\partial}{\partial y^i}, \tag{5.10}$$

which proves that the tangent map F_* satisfies the conditions of Proposition 5.2, so it is a nonconstant metallic-like map between the metallic-like structures ζ and the associated almost product structure J.

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